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On separation points of solutions to Prandtl boundary layer problem

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1. Introduction

The equations for 2-dimensional stationary boundary layer theory of incompressible fluid past a rigid wall are

$$(1.1)\qquad \begin{aligned} &uu_x+vu_y=\nu u_{yy}-p_x\,,\\ &u_x+v_y=0 \end{aligned}$$

in the domain $D_A=\{(x, y)\,;\ 0<x<A,\ 0<y<\infty\}$ (see [1], [8], [9], [10] and [11]). Here the subscripts x and y denote the partial differentiation with respect to the corresponding variable, (x, y) are orthogonal coordinates in the boundary layer with x representing the length along the wall and y the perpendicular distance from the wall, $u=u(x, y)$ and $v=v(x, y)$ are the corresponding unknown velocity components. The constant ν is a viscous coefficient. Finally $p=p(x)$ is a pressure function. Let $U=U(x)$ be an exterior streaming speed; we assume that $p(x)$ and $U(x)$ satisfy the Bernoulli law and the origin $(0,0)$ is not a stagnation point, i.e.,

$$(1.2)\qquad \begin{aligned} &U(x)\,U_x(x)+p_x(x)=0\,,\\ &U(0)>0\,. \end{aligned}$$

The appropriate boundary conditions are

$$(1.3)\qquad \begin{aligned} &u=v=0 \quad \text{for} \quad y=0 \quad \text{and} \quad u(x, y)\longrightarrow U(x) \quad \text{as } y\rightarrow\infty\,,\\ &\text{uniformly in } x \text{ on any compact subset of } [0,\ A)\,. \end{aligned}$$

In order to obtain a well-set problem, we suppose that at an initial position, say $x=0$, an initial datum $u_0(y)$ is assigned to the velocity component u, i.e.,

$$(1.4)\qquad u(0, y)=u_0(y) \qquad (0\leq y<\infty)\,.$$

In this paper we study the existence of the separation point of the flow deterministically.

Hereafter, unless otherwise provided, we assume that the datum $u_0(y)$ belongs to $I^{2+\alpha}=I^{2+\alpha}(\nu, U)$ (for notations see Section 2) and that the speed $U(x)$ and the pressure gradient $p_x(x)$ have following properties:

(1. 5) $U(X_0)=0$ for some point $X_0(0<X_0<\infty)$ and $U(x)>0$ for $0\leq x<X_0$,

(1. 6) The pressure $p(x)$ is sufficiently smooth, and if $p_x(c)=0$ at a point $x=c$, then a certain N-th derivative does not vanish at this point.

Now we mention our theorems.

THEOREM 1. *For the problem* (1. 1), (1. 3) *and* (1. 4) *there exists a solution* $(u, v)\in P^2([0, s))$ *such that the point* $(s, 0)$ *is its separation point and the inequality* $0<s<X_0$ *holds.*

If we put $S(u_0)=s$ in Theorem 1, then we obtain the mapping $S(u_0)$ from $I^{2+\alpha}$ to $(0, X_0)$.

THEOREM 2. (i) *For the fixed viscosity, no separation point exists near the point* $(X_0, 0)$:

$$\sup\left\{S(u_0)\,;\ u_0\in I^{2+\alpha}\right\}<X_0 \tag{1. 7}$$

(*see* [6]).

(ii) *For the viscosity* ν *tending to zero, if the pressure gradient* p_x *is monotone non increasing, there exist* $u_0^{(\nu)}\in I^{2+\alpha}(\nu, U)$ *and* $u_0\in I^{2+\alpha}$ (*without the compatibility condition*) *such that*

$$\begin{aligned} &u^{(\nu)}\to u_0 \quad \textit{in} \quad B^1([0,\infty)) \quad \textit{and} \\ &S(u_0^{(\nu)})\to 0 \quad \textit{as} \quad \nu\to 0\,. \end{aligned} \tag{1. 8}$$

In physical or numerical experiments ([2], [4], [6], [12] and [13]) it is always assumed that the initial datum $u_0(y)$ is the constant $U(0)$ which does not belong to the class $I^{2+\alpha}$ and so that the separation point is independent of the viscosity ν.

On the other hand, O. A. Oleinik [9] proved the local existence and the uniqueness of a solution (u, v) in some domain D_{A_0} to the problem (1. 1), (1. 3) and (1. 4) with a certain initial datum. Theorem 1 means that this local solution can be continued to the separation point. Recently Liu-Lee have tried to prove such a result. Their arguments do not appear plausible (their inequality (18. C). [7]). To prove Theorem 1, we use essentially Lemma 1, 4 and 5 described below and to obtain (1. 7) we compare our solution with the Blasius'.

In Section 2 we give a suitable definition of a separation point, notations mentioned above and the results on Oleinik's local solution with remarks about it. In Section 3 we prepare some lemmas and in Section 4 we prove Theorem 1 and 2.

2. Preliminaries

For an interval $[a, A]$ let $B^0([a, A]\times[0, \infty))$ be the Banach space of uniformly bounded continuous functions defined over $[a, A]\times[0, \infty)$ with supermum norm. For α $(0<\alpha\leq 2/3)$ and $y_0>0$ let $C^\alpha([a, A]\times[y_0, \infty))$ be the set of continuous functions $u(x, y)\in C^0([a, A]\times[y_0, \infty))$ which satisfy

$$\left|u(x_1, y_1)-u(x_2, y_2)\right|\leq M\left\{|x_1-x_2|^{1/2}+|y_1-y_2|\right\}^\alpha$$

for $(x_i, y_i)\in[a, A]\times[y_0, \infty)$ $(i=1, 2)$ and $M=M(y_0, u)$, and let $C^\alpha([a, A]\times(0, \infty))=\bigcap_{y_0>0} C^\alpha([a, A]\times[y_0, \infty))$. Furthermore let $B^\alpha([a, A]\times(0, \infty))=B^0([a, A]\times[0, \infty))\cap C^\alpha([a, A]\times(0, \infty))$. We also define $C^\alpha((0, \infty))$, $B^0((0, \infty))$ and $B^{2+\alpha}((0, \infty))$ by the analogous way.
Then we define the space of the initial data:

$$\begin{aligned}I^{2+\alpha}(\nu, U)=\Big\{&u(y)\in C^2\big([0, \infty)\big)\cap B^{2+\alpha}\big((0, \infty)\big);\ u(0)=0,\\ &u_y(0)>0,\ u_y(y)\geq 0 \text{ for } y\geq 0,\ u\to U(0) \text{ as } y\to\infty \text{ and}\\ &\nu u_{yy}(y)-p_x(0)=O(y^2) \text{ as } y\to 0\Big\}.\end{aligned}$$

The condition in $I^{2+\alpha}$

$$\nu u_{yy}(y)-p_x(0)=O(y^2) \quad \text{as} \quad y\to 0 \tag{2.1}$$

is a strong compatibility condition.

The space of the solutions to the problem (1.1), (1.3) and (1.4) is given as follows:
Let $P^2([a, A])$ be a set of all functions (u, v) such that
(i) u, u_x, u_y, u_{yy}, v and $v_y\in C^0([a, A]\times[0, \infty))$,
(ii) $u(x, y)>0$ in $[a, A]\times[0, \infty)$,
(iii) $u_y(x, 0)>0$ for $x\in[a, A]$,
(iv) u, u_y, $u_{yy}\in B^0([a, A]\times[0, \infty))$.
Then we define the space by

$$P^2\big([0, A)\big)=\bigcap_{0<A'<A} P^2\big([0, A']\big).$$

Now we define the separation point of a solution to the problem (1.1), (1.3) and (1.4):

DEFINITION. *A point $(s, 0)$ is a separation one of a solution (u, v) to our problem in the domain D_s, if the solution (u, v) belongs to $P^2([0, s))$ and for some sequence (x_n, y_n) in $[0, s)\times[0, \infty)$*

$$(x_n, y_n)\to(s, 0) \quad \textit{and} \quad u_y(x_n, y_n)\to 0 \quad \textit{as } n\to\infty.$$

Here we note that physicists call $(s, 0)$ a separation point if $u_y(s, 0)=0$ for a solution (u, v) (see [4], [12]). Our definition of the separation point is the same one as the above for a solution $(u, v)\in P^2([0, s))$ with u, $u_y\in C^0([0, s]\times[0, \infty))$. But to our knowledge, the extstence of such an exact solution is not yet proved under our assumptions.

We consider the transformation of the independ variables in the system (1.1) of the form

(2.2) $$x=x, \quad \psi=\psi(x, y),$$

where

$$u=\psi_y(x, y), \quad v=-\psi_x(x, y), \quad \psi(x, 0)=0.$$

If we put $w(x, \psi)=u^2(x, y)$, then the transformation (2.2) reduces the problem (1.1), (1.3) and (1.4) to the Von Mises' form:

(2.3) $$L(w)=\nu\sqrt{w}\,w_{\psi\psi}-w_x=2p_x \quad \text{in} \quad G_A$$

with the conditions

(2.4) $$w(x, 0)=0, \ w(x, \psi)\to U^2(x) \quad \text{as } \psi\to\infty$$
uniformly in x on $[0, A)$,

(2.5) $$w(0, \psi)=w_0(\psi)$$

where

$$G_A=\left\{(x, \psi);\ 0<x<A,\ 0<\psi<\infty\right\},$$

$$w_0\left(\int_0^y u_0(t)\,dt\right)=u_0^2(y).$$

Let $I_M^{2+\alpha}=\left\{w(\psi);\ w\left(\int_0^y u(t)\,dt\right)=u^2(y),\ u\in I^{2+\alpha}\right\}$.

Then $w(\psi)\in I_M^{2+\alpha}$ if and only if

$$w,\ w_\psi,\ \sqrt{w}\,w_{\psi\psi}\in B^0\big([0, \infty)\big)\cap C^\alpha\big((0, \infty)\big),$$

$$w(0)=0,\ w_\psi(0)>0,\ w_\psi(\psi)\geq 0 \text{ for } \psi\geq 0,$$

$$w(\psi)\to U^2(0) \quad \text{as} \quad \psi\to\infty \quad \text{and}$$

(2.6) $$\mu(\psi)=\nu\sqrt{w(\psi)}\,w_{\psi\psi}(\psi)-2p_x(0)=0(\psi) \quad \text{as} \quad \psi\to 0.$$

Let $P_M^{2+\alpha}\big([0, A]\big)=\Big\{w(x, \psi);$

$$w,\ w_x\ w_\psi,\ \sqrt{w}\,w_{\psi\psi}\in B^0(\bar{G}_A)\cap C^\alpha\big([0, A]\times(0, \infty)\big),$$

$$|w_x|\leq K\psi^{1-\beta} \text{ and } w_\psi\geq m \text{ in } [0, A]\times[0, \psi_1] \text{ and}$$

$$w(x, \psi)>l \text{ for } \psi>\psi_1\Big\},$$

where positive constants ψ_1, m, l depend on w. Furthermore for any β $(0<\beta<1/2)$ the positive constant K depends on ψ_1, β and w.

Let $P_M^{2+\alpha}\bigl([0, A)\bigr)=\bigcap_{0<A'<A} P_M^{2+\alpha}\bigl([0, A']\bigr)$.

Here we summarize the results and remarks on Oleinik's local solutions (see [9] or [10]):

(I) A solution $(u, v)\in P^2([0, A])$ of the problem (1. 1), (1. 3) and with the initial datum $u_0\in I^{2+\alpha}$ (without $u_{0_y}(y)\geq 0$ for $y>0$) exists, if a solution $w\in P_M^{2+\alpha}([0, A])$ of the problem (2. 3), (2. 4) and (2. 5) exists.

(II) Oleinik assumes that the initial datum u_0 belongs to $B^{2+\alpha}([0, \infty))$, but our initial datum u_0 belongs to $C^2([0, \infty))\cap B^{2+\alpha}((0, \infty))$ and this assumption of regularity is enough for the existence of Oleinik's local solution.

Here we note that $I_M^{2+\alpha}$ contains a sufficiently large number of elements. In fact let $w_0(\psi)$ be $\{m_0\psi+B_1\psi^{3/2}-B_2\psi^2\}\,\chi(\psi)+(1-\chi(\psi))\,U^2(0)$, where $m_0>0$, $B_1=8p_x(0)/3\nu\sqrt{m_0}$, $B_2=4p_x^2(0)/3\nu^2 m_0^2$ and $\chi(\psi)$ is a function $\in C^\infty([0, \infty))$ such that $\chi_\psi\leq 0$, $0\leq\chi\leq 1$, $\chi=1$ for $\psi\ll 1$, and $\sup\,(\operatorname{supp}\chi)\ll 1$. Then it is easy to see that w_0 belongs to $I_M^{2+\alpha}$.

(III) For small $\varepsilon>0$, let $G_A^\varepsilon=\{(x, \psi)\,;\ 0<x<A,\ 0<\psi<1/\varepsilon\}$. Then for some positive constant A_0 and any sufficiently small ε, there exists the approximate positive solution $w_\varepsilon(x, \psi)$ defined on $G_{A_0}^\varepsilon$, which satisfies (2. 3) with boundary conditions:

$$(2.7)\qquad \begin{aligned} &w_\varepsilon(0, \psi)=w_0(\varepsilon+\psi),\ \ w_\varepsilon(x, 0)=w_0(\varepsilon)\exp\bigl\{\mu(\varepsilon)\,x/w_0(\varepsilon)\bigr\},\\ &w_\varepsilon(x, 1/\varepsilon)=w_0(\varepsilon+1/\varepsilon)\exp\bigl\{\mu(\varepsilon+1/\varepsilon)\,x/w_0(\varepsilon+1/\varepsilon)\bigr\}. \end{aligned}$$

Then the solution $w_\varepsilon(x, \psi)$ belongs to $P_M^{2+\alpha}(\overline{G_{A_0}^\varepsilon})$, where the last functional space is defined as in $P_M^{2+\alpha}([0, A])$, but we must replace $B^0(\bar{G}_{A_0})$ and $C^\alpha([0, A_0]\times(0, \infty))$ by $B_0(\overline{G_{A_0}^\varepsilon})$ and $C^\alpha([0, A_0]\times(0, 1/\varepsilon])$ respectively.

Here we must mention that the solution $w_\varepsilon(x, \psi)$ has the following properties:

(2. 8) All constants in $P_M^{2+\alpha}$ (i. e., α, m, ψ_1, K, l) and supremum norms over $\overline{G_{A_0}^\varepsilon}$ of the functions w_ε, w_{ε_x}, w_{ε_ψ}, $\sqrt{w_\varepsilon}\,w_{\varepsilon_{\psi\psi}}$, $w_\varepsilon^{\beta-1}w_{\varepsilon_x}$ do not depend on ε.

Finally the unique local solution $w\in P_M^{2+\alpha}([0, A_0])$ is obtained as follows: For a subsequence $\{w_{\varepsilon'}\}$ and any $N\gg 1$

$$w_{\varepsilon'}\to w \quad \text{as} \quad \varepsilon'\to 0 \quad \text{in} \quad C^0\bigl([0, A_0]\times[0, N]\bigr)$$

and $w_{\varepsilon'_x}$, $w_{\varepsilon'_\psi}$, $\sqrt{w_{\varepsilon'}}\,w_{\varepsilon'_{\psi\psi}}$ converge the corresponding functions with respect to w as $\varepsilon' \to 0$ in $C^0([0, A_0] \times [1/N, N])$, provided $\alpha \leq 2/3$. Here we note that $w(x, \psi) \to U^2(x)$ as $\psi \to \infty$ uniformly in x on $[0, A_0]$.

Now by the transformation (2.2), we have $u_y = \frac{1}{2} w_\psi$. Therefore we may define the separation point of the solution w to the problem (2.3), (2.4) and (2.5) by the same way as above, i.e., a point $(s, 0)$ is the separation one of a solution $w(x, \psi)$ to our problem, if the solution $w(x, \psi)$ belongs to $P_M^{2+\alpha}([0, s))$ and for some sequence (x_n, ψ_n) in $[0, s) \times [0, \infty)$

$$(x_n, \psi_n) \to (s, 0) \quad \text{and} \quad w_\psi(x_n, \psi_n) \to 0 \quad \text{as } n \to \infty .$$

We also denote this separation point by $S(w_0)$. Then, in fact, we obtain that $S(u_0) = S(w_0)$ for u_0 corresponding to $w_0 \in I_M^{2+\alpha}$ by (2.2).

3. Lemmas

LEMMA 1. *Let $w(x, \psi)$ be Oleinik's local solution. Then there exist positive constants M_1 and λ such that*

$$|w_x| \leq M_1 \psi \quad \textit{for} \quad 0 \leq x \leq A_0 \quad \textit{and} \quad 0 \leq \psi \leq \lambda . \tag{3.1}$$

PROOF. Let $w_\varepsilon(x, \psi)$ be the approximate solution given in Section 2.

We put $E = \{(x, \psi)\,;\ 0 \leq x \leq A_0,\ 0 \leq \psi \leq \lambda\}$, where the small positive constant λ will be determined latter. From the fact (III), if we show

$$|w_\varepsilon^{-1} w_{\varepsilon_x}| \leq M_1 \qquad \text{in } E$$

for some M_1, λ and sufficiently small $\varepsilon > 0$, then we obtain (3.1). Hence it is sufficient to show the above inequality.

Since $w_\varepsilon > 0$ (see (III)), we may put $g_\varepsilon(x, \psi) = \ln (w_\varepsilon(x, \psi))$, then from (2.3) g_ε satisfies the following equation:

$$\nu g_{\varepsilon_{\psi\psi}} e^{3g_\varepsilon/2} + \nu (g_{\varepsilon_\psi})^2 e^{3g_\varepsilon/2} - e^{g_\varepsilon} g_{\varepsilon_x} = 2p_x . \tag{3.2}$$

Differentiating (3.2) with respect to x, we obtain an equation for $g_{\varepsilon_x} = h_\varepsilon$:

$$\nu h_{\varepsilon_{\psi\psi}} e^{3g_\varepsilon/2} + \frac{3}{2} \nu e^{3g_\varepsilon/2} g_{\varepsilon_{\psi\psi}} h_\varepsilon + \frac{3}{2} \nu e^{3g_\varepsilon/2} (g_{\varepsilon_\psi})^2 h_\varepsilon +$$
$$2\nu e^{3g_\varepsilon/2} g_{\varepsilon_\psi} h_{\varepsilon_\psi} - e^{g_\varepsilon} (h_\varepsilon)^2 - e^{g_\varepsilon} h_{\varepsilon_x} = 2p'' .$$

Replacing $\nu g_{\varepsilon_{\psi\psi}} \exp 3g_\varepsilon/2$ by its value from (3.2), we can write the equation for h_ε in the form:

$$\nu e^{3g_\varepsilon/2} h_{\varepsilon_{\psi\psi}} + C_\varepsilon h_\varepsilon + 2\nu e^{3g_\varepsilon/2} g_{\varepsilon_\psi} h_{\varepsilon_\psi} - e^{g_\varepsilon} h_{\varepsilon_x} = 2p'' , \tag{3.3}$$

where

$$C_\varepsilon = \frac{1}{2} h_\varepsilon e^{g_\varepsilon} + 3p' = \frac{1}{2} w_{\varepsilon_x} + 3p' .$$

From (2. 8), there exists a positive constant M_2 independent of ε such that

(3. 4) $\quad |C_\varepsilon| \leq M_2 \quad$ in $\overline{G^\varepsilon_{A_0}}$.

Therefore (3. 3) implies that $j_\varepsilon = h_\varepsilon \exp\{-(M_2+1)x\}$ satisfies

(3. 5) $\quad \nu e^{3g_\varepsilon/2} j_{\varepsilon_{\psi\psi}} + (C_\varepsilon - M_2 - 1) j_\varepsilon + 2\nu e^{3g_\varepsilon/2} g_{\varepsilon_\psi} j_{\varepsilon_\psi} - e^{g_\varepsilon} j_{\varepsilon_x} = 2p'' e^{-(M_\varepsilon+1)x}$.

Now if j_ε attains its maximum in E without the lines $x=0$, $\psi=0$ and $\psi=\lambda$, then at this point we have

$$\nu e^{3g_\varepsilon/2} j_{\varepsilon_{\psi\psi}} - e^{g_\varepsilon} j_{\varepsilon_x} \leq 0 \quad \text{and} \quad j_{\varepsilon_\psi} = 0 .$$

Hence from (3. 4) and (3. 5) we deduce

(3. 6) $\quad j_\varepsilon \leq \dfrac{2p'' \exp\{-(M_2+1)x\}}{C_\varepsilon - M_2 - 1} \leq M_3$ at the maximum point,

where the positive constant M_3 does not depend on $\varepsilon > 0$.

By the same argument, at the minimum point of j_ε in the same domain of E we have

(3. 7) $\quad j_\varepsilon > -M_3$.

If ε and λ are sufficiently small, from (2. 6) it follows that $h_\varepsilon(0, \psi) = w_0(\varepsilon+\psi)^{-1}\mu(\varepsilon+\psi) = O(1)$ and $h_\varepsilon(x, 0) = w_0(\varepsilon)^{-1}\mu(\varepsilon) = O(1)$. Furthermore the fact (III) in Section 2 implies $h_\varepsilon(x, \psi) = O(1)$.

Thus from (3. 6) and (3. 7) we conclude that

$$|h_\varepsilon| \leq M_4 \quad \text{in } E ,$$

where the positive constant M_4 does not depend on ε. This proves Lemma 1. q. e. d.

It is easy to see that (2. 3) and (3. 1) imply the following Corollary.

COROLLARY. *Let $w(x, \psi)$ be as in Lemma* 1. *Then the section $w(x, \cdot)$ satisfies the compatibility condition* (2. 6) *for any x* $(0 \leq x \leq A_0)$.

LEMMA 2. ([8]). *Let $w(x, \psi)$ be a solution to the problem* (2. 3), (2. 4) *and* (2. 5) *belonging to $P_M^{2+\alpha}([0, A])$. Then $w(x, \psi)$ is monotone nondecreasing with respect to ψ in $\bar{G}_A$.*

PROOF. We use the methods of the proof in [5] here.

Since $w(x, \psi)$ belongs to $P_M^{2+\alpha}$, for sufficiently small $\delta > 0$ there exist positive constants m and M such that

(3. 8) $\quad m \leq w \leq M$ in $\bar{G}_A \cap \{\psi \geq \delta\}$,

(3.9) $|w_\psi| \leq M$ in $\bar{G}_A$ and

(3.10) $w_\psi \geq m$ in $\bar{G}_A \cap \{0 \leq \psi \leq \delta\}$.

Differentiating the equation (2.3) with respect to ψ, we obtain an equation for $z = w_\psi$:

(3.11) $$P(z) = \nu\sqrt{w}\, z_{\psi\psi} + \frac{\nu}{2} w^{-1/2} w_\psi z_\psi - z_x = 0 .$$

We put $g(x, \psi) = R^{-2} M(\psi^2 + Kx)\, e^{Kx} + z(x, \psi)$ and $E_{r,R} = \{(x, \psi)\,;\ 0 \leq x \leq A,\ r \leq \psi \leq R\}$ for $R \gg r > 0$, where a positive constant K will be determined latter.

From (3.11) we obtain an equation for g:

$$P(g) = R^{-2} M e^{Kx} (2\nu\sqrt{w} + \nu w^{-1/2} w_\psi \psi - K - K\psi^2 - K^2 x) .$$

If $K > \nu(2M^{1/2} + m^{-1/2} M)$, taking account of (3.8), (3.9) and (3.10), we have

$$P(g) \leq R^{-2} M e^{Kx} (2\nu M^{1/2} \psi^2 + \nu m^{-1/2} M \psi^2 - K\psi^2) < 0 \text{ in } E_{1,R}$$

and

$$P(g) \leq R^{-2} M e^{Kx} (2\nu M^{1/2} + \nu m^{-1/2} M - K) < 0 \text{ in } E_{\delta,1} .$$

Hence we see $P(g) < 0$ in $E_{\delta,R}$. Furthermore on the lines $x=0$, $\psi=\delta$ and $\psi = R$ we have the following: Since $w_0 \in I_M^{2+\alpha}$ and $w_\psi = 2u_y$, $g(0, \psi) = R^{-2} M\psi^2 + z(0, \psi) = R^{-2} M\psi^2 + w_{0_\psi}(\psi) \geq 0$ and (3.10) implies that $g(x, \delta) \geq z(x, \delta) = w_\psi(x, \delta) \geq 0$. Finally $g(x, R) \geq M + z(x, R) = M + w_\psi(x, R) \geq 0$ by (3.9).

Therefore by the maximum principle we obtain $g \geq 0$ in $E_{\delta,R}$. Then if $R \to \infty$, we have $w_\psi(x, \psi) \geq 0$ for $\psi \geq \delta$. Taking account of (3.10), we conclude that our assertion for $\psi < \delta$ is valid. q. e. d.

LEMMA 3. *There exists a point $x_0 > 0$ such that the pressure gradient $p_x(x)$ is monotone decreasing in* $[x_0, X_0]$.

PROOF. From (1.5) and the Bernoulli law (1.2) we have $p_x(X_0) = 0$. Taking account of (1.6), we have that for some constant $a \neq 0$ and x near X_0

(3.12) $$p_x(x) = a(X_0 - x)^N + O\Big((X_0 - x)^{N+1}\Big) .$$

Now if the constant a is negative, then $p_x(x) < 0$ near X_0. Therefore an inequality $U^2(x) = 2(p(X_0) - p(x)) < 0$ holds for such x, since (1.2) and (1.5) are assumed. This contradiction means that the pressure gradient p_x is non-negative and monotone decreasing on some interval $[x_0, X_0]$. q. e. d.

LEMMA 4. *Suppose that the pressure gradient p_x is monotone non increasing on* $[0, 1]$ *and the point X_0 in* (1.5) *equlas to* 1. *Then for any positive constant k there exist positive constants γ and $A < 1$ such that if*

$w(x, \psi)$ is a solution to the problem (2. 3), (2. 4) *and* (2. 5) *belonging to* $P_M^{2+\alpha}([0, A))\cap C^0([0, A]\times[0, \infty))$ *and if*

$$(3.13)\qquad w_0(\psi)\leq k\psi\int_0^1 p_x(t)\,dt \quad \textit{for}\quad 0\leq\psi\leq 2/k,$$

then

$$(3.14)\qquad \begin{aligned} w(x, \psi) &\leq F(x, \psi)\\ &\equiv k\psi\left\{2\int_{x/A}^1 p_x(t)\,dt(1-\gamma\psi)+2\gamma\psi\int_x^1 p_x(t)\,dt\right\}\end{aligned}$$

for $0\leq x\leq A$, $0\leq\psi\leq 1/\gamma$.

PROOF. Since the initial datum $w_0\in I_M^{2+\alpha}$, we may find a positive constant k such that (3. 13) is valid.

Now the above inequality implies that

$$(3.15)\qquad w_0(\psi)\leq 2(k\psi-k^2\psi^2/4)\int_0^1 p_x(t)\,dt \quad \text{for}\quad 0\leq\psi\leq 2/k.$$

Furthermore for an arbitrary constant $A<1$ we have

$$(3.16)\qquad w(x, \psi)\leq 2\int_x^1 p_x(t)\,dt \quad \text{in}\quad [0, A]\times[0, \infty),$$

if a solution w is well defined in $[0, A]\times[0, \infty)$ as above. In fact, (1. 2) and (1. 5) imply that $w(0, \psi)=w_0(\psi)\leq U^2(0)=U^2(1)+2p(1)-2p(0)=2\int_0^1 p_x(t)\,dt$ and $w(x, 0)=0\leq 2\int_x^1 p_x(t)\,dt$ and $L(w)-L\left(2\int_x^1 P_x dt\right)=0$. Hence by the maximum principle (3. 16) holds (see the proof of Theorem 3 in [9]).

Moreover we define $H(x, \psi)$ as follows:

$$H(x, \psi)=2(k\psi-k^2\psi^2/4)\int_x^1 p_x(t)\,dt \quad \text{for}\quad 0\leq\psi\leq 2/k.$$

Then it follows from (3. 15) and (3. 16) that an inequality $w(x, \psi)\leq H(x, \psi)$ holds on the lines $x=0$, $\psi=0$ and $\psi=2/k$. Furthermore by simple calculation we obtain

$$L(H)-L(w)\leq 0 \quad \text{in}\quad [0, A)\times[0, 2/k]$$

for any $A<1$.

Hence by the maximum principle we have

$$(3.17)\qquad w(x, \psi)\leq H(x, \psi) \quad \text{in}\quad [0, A]\times[0, 2/k].$$

Now we prove the inequality (3. 14) for some γ and $A<1$. From (3. 13) and (3. 17), the inequality (3. 14) holds on the lines $x=0$, $\psi=0$ and $\psi=1/\gamma$

for $2\gamma>k$. Furthermore, from the inequalities $A<1$, $k<2\gamma$, the monotoneity of p_x and (1. 5), we deduce the following:

$$
\begin{aligned}
(3.18) \qquad L(F) &= \nu\sqrt{F}F_{\psi\psi}-F_x \\
&\leq 4\nu k\gamma p_x(x)(x/A-x)\left(2k\gamma^{-1}\int_0^1 p_x(t)\,dt\right)^{1/2}+p_x(x)\frac{2k}{A}\psi \\
&\leq 2ck^{3/2}\gamma^{1/2}(1-A)\,p_x(x)+2kp_x/\gamma A\,, \\
&\text{for}\quad 0\leq\psi\leq 1/\gamma,\ 0\leq x\leq A\,,
\end{aligned}
$$

where $c^2=8\nu^2\int_0^1 p_x(t)\,dt$.

Hence an inequality $L(F)\leq 2p_x(x)$ holds for $0\leq x\leq A$ and $0\leq\psi\leq 1/\gamma$ if an inequality

$$ck^{3/2}\gamma^{1/2}(1-A)\,A+k/\gamma\leq A$$

holds.

Therefore considering the quadratic inequality in A, we may find $A<1$ for $\gamma>k$. Then from (3. 18) it follows that $L(F)-L(w)\leq 0$ for $0\leq x<A$ and $0\leq\psi\leq 1/\gamma$. Thus by the maximum principle we have the inequality (3. 14) q. e. d.

We note that the above quadtatic inequality is valid for

$$(3.19) \qquad \gamma=(1+D^{1/2})^2/c_1k^3 \quad\text{and}\quad A=1-2/(1+D^{1/2})\,,$$

where $D=1+c_2k^4$, provided $c_1=3^2c^2$ and $c_2=3^3c^2$. Furtheremore the argument described below and (3. 19) imply that for the separation point $(s, 0)$

$$(3.20) \qquad 0<s\leq A=1-2/(1+D^{1/2})\,.$$

Corollary. *There exists no solution $w(x,\psi)$ to the problem* (2. 3), (2. 4) *and* (2. 5) *in G_{X_0}, which belongs to the class $P_M^{2+\alpha}([0, X_0))$ and whose section $w(x,\cdot)$ belongs to $I_M^{2+\alpha}$ for any x $(0\leq x<X_0)$.*

Proof. Assume that the above corollary is not valid. Let x_0 be the point in Lemma 3. Then, from our assumption, we may consider the line $x=x_0$ as the initial position and the section $w(x_0,\cdot)$ as the initial datum. Therefore by a coordinates transformation, setting $x_0=0$ and $X_0-x_0=1$, we may consider that the solution $w(x,\psi)$ satisfies the assumptions in Lemma 4. Then from $w(x, 0)=0$ and Lemma 4, we have

$$\frac{w(x,\psi)-w(x,0)}{\psi}\leq 2k\int_{x/A}^1 p_x(t)\,dt\,(1-\gamma\psi)+2\gamma k\psi\int_x^1 p_x(t)\,dt$$

for $0\leq x\leq A$ and $0\leq\psi\ll 1$.

Therefore if $\psi\downarrow 0$ and $x\uparrow A$, then we obtain

$$w_\psi(A, 0) \leq 0 .$$

This proves the corollary. q. e. d.

For a point $X_1(0<X_1<X_0)$, let k_1 and k_2 be $\min_{0\leq x\leq X_1} U^2(x)$ and $\max_{0\leq x\leq X_1} |2p_x(x)|$ respectively. Furthermore let W be the subset of $I_M^{2+\alpha}$ such that for $w_0(\psi) \in W$

$$\inf\left\{w_{0_\psi}(\psi) \,;\, 0\leq\psi\leq\psi_0\right\}\geq k$$

where ψ_0 and k are, *a* priori, sufficiently small positive numbers.

Then we have

LEMMA 5. *For any $w_0\in W$, the constant A_0 in Section* 2-(III) *may be chosen depending only on ψ_0, k, k_1 and k_2, but independent of w_0.*

PROOF. Let $w_\varepsilon(x, \psi)$ be the approximate solution mentioned in (III) with respect to the initial datum w_0. Then by the Oleinik's theory [9] it is sufficient to prove the following: For the constants a, δ, A_0, $\cdots$ and A_5 which depend only on ψ_0, k, k_1 and k_2, there exists a sequence $\{\varepsilon_n\}$ depending on $w_0(\psi)$ and tending to zero as $n\to\infty$ such that for any n

$$\begin{aligned}(3.21)\qquad w_{\varepsilon_n}(x, \psi) &\geq V(x, \psi)\\ &\equiv w_{\varepsilon_n}(x, 0)+f(\psi)\,(1+e^{-ax}) \quad\text{in}\quad G_{A_0}^{\varepsilon_n}.\end{aligned}$$

Here $f(\psi)=A_1\psi^{4/3}+A_2\psi$ for $\psi\leq\delta$, $A_3\geq f(\psi)\geq f(\delta)$, $|f_\psi(\psi)|\leq A_4$ and $|f_{\psi\psi}(\psi)|\leq A_5$ for $\psi\geq\delta$.

To show this, let B_0, B_1 and B_2 be sufficiently small constants such that

$$B_2>B_0 ,$$

$$(3.22)\qquad \frac{1}{2}k_1\exp\{-3k_2X_1/k_1\}\geq B_2 .$$

Moreover we take sufficiently small constants δ, A_1, A_2 and A_3 as follows:

$$\delta<\psi_0 ,$$

$$(3.23)\qquad \min(B_2-B_0, k\delta)\geq 2A_3>4(A_1\delta^{1/3}+A_2)\,\delta>0 ,$$

$$(3.24)\qquad \nu\frac{4}{9}\sqrt{A_2}A_1\geq\delta^{1/6}(k_2+B_1) .$$

Then from (3.23) we may determine the desired monotone nondecreasing function $f\in C^2([0, \infty))$, constants A_4 and A_5.

Now let $w_0\in W$. From the condition

$$w_0(\psi)\uparrow U^2(0) \quad\text{as}\quad \psi\uparrow\infty ,$$

we find a sequence $\{\psi_n\}$ such that $w_{0_{\psi\psi}}(\psi_n)\to 0$ as $n\to\infty$. Therefore we have

$$\mu(\psi_n)\,x\geq -3k_2X_1$$

for any sufficiently large n and for $x\in[0, X_1]$. Accordingly by (3. 22) we can find a sequence $\{\varepsilon_n\}$ as follows:

$$\varepsilon_n+\delta<\psi_0\,,$$

for $x\in[0, X_1]$

$$\begin{aligned}&\Big(w_0\exp\{\mu x/w_0\}\Big)(\varepsilon_n)\leq B_0\,,\\ (3.\,25)\qquad&\Big(\mu\exp\{\mu x/w_0\}\Big)(\varepsilon_n)\leq B_1\,,\\ &\Big(w_0\exp\{\mu x/w_0\}\Big)\Big(\varepsilon_n+\frac{1}{\varepsilon_n}\Big)\geq B_2\,.\end{aligned}$$

Because $w_0(\varepsilon_n)$, $\mu(\varepsilon_n)\downarrow 0$ as $\varepsilon_n\downarrow 0$ and $\mu(\varepsilon_n)/w_0(\varepsilon_n)$ is bounded by (2. 6).

Now we can show the validity of (3. 21) on the boundary of $G_{X_1}^{\varepsilon_n}$ except the line $x=X_1$. On the line $x=0$, (3. 23) implies that for $\psi\leq\delta$

$$\begin{aligned}w_0(\varepsilon_n+\psi)-w_0(\varepsilon_n)&=\int_0^1 w_0'(\varepsilon_n+\theta\psi)\,d\theta\cdot\psi\geq k\psi\\ &\geq 2(A_1\delta^{1/3}+A_2)\,\psi\geq 2(A_1\psi^{4/3}+A_2\psi)\,.\end{aligned}$$

For $\psi\geq\delta$, $w_0(\varepsilon_n+\psi)-w_0(\varepsilon_n)\geq w_0(\varepsilon_n+\delta)-w_0(\varepsilon_n)=\delta\int_0^1 w_0'(\varepsilon_n+\theta\delta)\,d\theta\geq 2A_3\geq 2f(\varepsilon_n+\psi)$ by the definition of f and also by (2. 23).

Furthermore on the line $\psi=1/\varepsilon_n$, (3. 23) and (3. 25) imply that

$$\begin{aligned}&\Big(w_0\exp\{\mu x/w_0\}\Big)\Big(\varepsilon_n+\frac{1}{\varepsilon_n}\Big)-\Big(w_0\exp\{\mu x/w_0\}\Big)(\varepsilon_n)\\ &\geq B_2-B_0\geq 2A_3\quad\text{for}\quad x\in[0, X_1]\,.\end{aligned}$$

Since (3. 21) is trivial on the line $\psi=0$, we obtain the assertion above mentioned.

Finally we consider

$$L(V)=\nu\sqrt{V}f''(\psi)\,(1+e^{-ax})-w_\varepsilon(x, 0)_x+af(\psi)\,e^{-ax}\,.$$

For $\psi\leq\delta$, from (3. 24) and (3. 25) it follows that

$$\begin{aligned}L(V)&\geq\nu\frac{4}{9}\sqrt{A_2}A_1\delta^{-1/6}-\sup_{0\leq x\leq X_1}\Big(\mu\exp\{\mu x/w_0\}\Big)(\varepsilon_n)\\ &\geq k_2\geq 2p_x(x)\,.\end{aligned}$$

By the monotoneity of f we can find a constant d such that $f(\psi)\geq d>0$ for $\psi\geq\delta$. Thus there exist constants A_0 and a which satisfy

$$0<A_0<X_1,\ aA_0=1,$$
$$ade^{-aA_0}\geq k_2+\nu\sqrt{k_1}A_5(1+e^{-aA_0})+B_1.$$

Then we obtain that $L(V)\geq 2p_x(x)$ for $x\in[0, A_0]$ and for $\psi\geq\delta$.

Thus we conclude that $L(w_{\varepsilon_n})-L(V)\leq 0$ in $G_{A_0}^{\varepsilon_n}$ and by the maximum principle (3. 21) is valid. q. e. d.

4. Proofs of Theorems

(I) First, we shall prove Theorem 1. Let $w(x, \psi)$ be Oleinik's local solution in G_{A_0}. From Corollary of Lemma 1 and Lemma 2 the section $w(A_0, \cdot)$ belongs to $I_M^{2+\alpha}$. Hence, from the facts (II) and (III) in Section 2, there exists a continued solution of Oleinik's local solution (By $w(x, \psi)$ we also denote this continuation). Furthermore by the same arguments as Lemma 1 this continuation belongs the class $P_M^{2+\alpha}([0, A])$ for some $A>A_0$, and its section $w(x, \cdot)$ belongs to $I_M^{2+\alpha}$ for any x $(0\leq x\leq A)$.

Let $s=\sup\{A$; the above continuation exists in $G_A\}$. Then we obtain the continued solution $w(x, \psi)$ belonging to $P_M^{2+\alpha}([0, s))$. Furthermore, from Colorllary of Lemma 4, an inequality $s<X_0$ holds.

Assume that the point $(s, 0)$ is not the separation point. That is, for a sequence $A_n<s$ with $A_n\to s$ (as $n\to\infty$) there exist positive constants m_0, ψ_0 and a natural number n_0 which do not depend on n and satisfy the following:

$$\text{(4. 1)}\qquad \inf\left\{w_\psi(A_n, \psi)\,;\ 0\leq\psi\leq\psi_0\right\}\geq m_0\quad\text{for}\quad n\geq n_0.$$

For an arbitrary but fixed $n\geq n_0$, if we consider the line $x=A_n$ as an initial position and the section $w(A_n, \cdot)$ as an initial datum, then there exists the solution of the problem (2. 3), (2. 4) and (2. 5) in the domain G_{A_n+B} for some constant B, which belongs to the class $P_M^{2+\alpha}([A_n, A_n+B])$. On the other hand from (4. 1) and Lemma 5 the constant B does not depend on $n\geq n_0$. Hence, for sufficiently large n_1 an inequality $B>s-A_{n_1}$ holds. Thus we get a solution of the problem (2. 3), (2. 4) and (2. 5) in $G_{A_{n_1}+B}$ as a continuation of the solution $w(x, \psi)\in P_M^{2+\alpha}([0, s))$, which is contrary to the definition of s. This proves Theorem 1.

We note that, by the same way as above, if $U(x)$ does not vanish and if there exists no separation point, then Oleinik's local solution can be continued to the infinity.

(II) To prove Theorem 2, we show a relation between the solution obtained in Theorem 1 and the Blasius solution, which is given as follows (see [3]).

Let $f=f(\eta)$ be the solution of the problem:

$$f'''+ff''=0 \quad \text{in} \quad [0, \infty),$$

with conditions

$$f(0)=f'(0)=0 \quad \text{and} \quad f'(\eta)\to 1 \quad \text{as} \quad \eta\to\infty.$$

If we put

$$\eta=\left(\frac{U(x_0)+c}{2\nu x}\right)^{1/2} y,$$

$$(4.2) \qquad u(x, y)=\big(U(x_0)+c\big)f'(\eta),$$

$$v(x, y)=\nu^{1/2}\left(\frac{U(x_0)+c}{2x}\right)^{1/2}\big(\eta f'(\eta)-f(\eta)\big)$$

for an arbitrary but fixed positive constant c and the point x_0 given in Lemma 3, then the function (u, v) satisfies

$$uu_x+vu_y=\nu u_{yy} \quad \text{and} \quad u_x+v_y=0 \quad \text{in} \quad (0, \infty)\times(0, \infty)$$

with conditions

$$u=v=0 \quad \text{for} \quad y=0 \quad \text{and} \quad x>0.$$

Using the transformation (2.2) and putting $w_B(x, \psi)=u^2(x, y)$, by definitions we obtain

$$(4.3) \qquad L(w_B)=0 \quad \text{in} \quad (0, \infty)\times(0, \infty),$$

$$w_B(x, 0)=0 \quad \text{for} \quad x>0,$$

$$(4.4) \qquad w_B(x, \psi)\to\big(U(x_0)+c\big)^2 \text{ as } \psi\to\infty \text{ pointwise in } x>0,$$

$$w_B(x, \psi)\to\big(U(x_0)+c\big)^2 \text{ as } x\to 0 \text{ pointwise in } \psi>0.$$

Now let $w(x, \psi)$ be the solution in Theorem 1 with $S(w_0)>x_0$. Then we obtain

$$(4.5) \qquad w(x, \psi)\leq w_B(x-x_0, \psi) \quad \text{in} \quad [x_0, S(w_0))\times[0, \infty),$$

where the point x_0 is given in Lemma 3.

In fact from (1.5) and Lemma 3 we have

$$(4.6) \qquad p_x(x)\geq 0 \quad \text{for} \quad x_0\leq x\leq X_0.$$

Furthermore from Theorem 8.1 in [3] it follows that

$$(4.7) \qquad f''(\eta)>0 \quad \text{for} \quad \eta>0.$$

Hence, from (2.2), (4.2) and (4.7), we have

(4. 8) $$w_{B_\psi}(x, \psi)=\frac{2(U(x_0)+c)^{3/2}}{(2\nu x)^{1/2}}f''(\eta)>0\,,$$

since $\dfrac{\partial}{\partial\psi}=2\dfrac{\partial}{\partial y}$ is valid.

Therefore from (4. 4) and (4. 8) we see that for a fixed positive number δ there exists a positive constant ε_0 such that

$$w_B(\varepsilon, \psi)-w(x_0, \psi)\geq 0 \quad \text{for} \quad \delta\leq\psi<\infty \quad \text{and} \quad 0<\varepsilon<\varepsilon_0\,.$$

Furthermore from (4. 8) it follows that

$$\begin{aligned} &w_B(\varepsilon, \psi)-w(x_0, \psi)\\ &\quad=\psi\int_0^1\left(\frac{\partial w_B}{\partial\psi}(\varepsilon, t\psi)-\frac{\partial w}{\partial\psi}(x_0, t\psi)\right)dt\geq 0 \quad \text{for} \quad 0<\psi\leq\delta\,, \end{aligned}$$

provided $\varepsilon\ll 1$.

Hence we have

$$w_B(\varepsilon, \psi)-w(x_0, \psi)\geq 0 \quad \text{for} \quad 0\leq\psi<\infty\,.$$

Moreover from (4. 3), (4. 4) and (4. 6) we deduce that

$$\begin{aligned} &w_B(x-x_0+\varepsilon, 0)-w(x, 0)=0 \quad \text{for} \quad x_0\leq x<S(w_0)\,,\\ &L\big(w_B(x-x_0+\varepsilon, \psi)\big)-L(w)\leq 0 \quad \text{in} \quad [x_0, S(w_0))\times[0, \infty)\,. \end{aligned}$$

Thus by the maximum principle we conclude that for any sufficiently small $\varepsilon>0$

$$w(x, \psi)\leq w_B(x-x_0+\varepsilon, \psi) \quad \text{in} \quad [x_0, S(w_0))\times[0, \infty)\,,$$

which implies (4. 5).

(III) We shall prove Theorem 2. Let $x=x_0$ be given in Lemma 3. As in the proof of Corollary of Lemma 4, we may consider the line $x=x_0+\varepsilon$ as an initial position for sufficiently small $\varepsilon>0$. Furthermore from (4. 5) we obtain

$$w(x_0+\varepsilon, \psi)\leq w_B(\varepsilon, \psi)\leq k\psi\int_0^1 p_x(t)\,dt$$

for $0\leq\psi\leq 2/k$, if the constant k is sufficiently large. Then the constant k in (3. 13) can be chosen independently of the section $w(x_0+\varepsilon, \cdot)$ for any solution with $S(w_0)>x_0$. Hence after a certain coordinates transformation the constant A in (3. 19) can be also chosen independently of $w(x_0+\varepsilon, \cdot)$. Then from (3. 20) the inequality (1. 7) holds. The first part of Theorem 2 is proved.

Finally we shall prove the second assertion of Theorem 2. We define $w_0^{(\nu)} \in I_M^{2+\alpha}(\nu, U)$ as follows:

$$w_0^{(\nu)} = m\psi\chi(\psi) + B_1\psi^{3/2}\chi(\nu^{-5/2}\psi) - B_2\psi^2\chi(\nu^{-5/2}\psi) + \left(1-\chi(\psi)\right) U^2(0),$$

where B_1, B_2 and χ are defined in Section 2-(II).

By simple calculations, we have

$$w_0^{(\nu)} \to m\psi\chi(\psi) + \left(1-\chi(\psi)\right) U^2(0) \quad \text{in} \quad B^1\left([0, \infty)\right),$$

$\nu\sqrt{w_0^{(\nu)}}\dfrac{d^2 w_0^{(\nu)}}{d\psi^2} \to 0$ in $B^0([\psi_0, \infty))$ for any $\psi_0 > 0$ as $\nu \to 0$, $\left\{\nu\sqrt{w_0^{(\nu)}}\dfrac{d^2 w_0^{(\nu)}}{d\psi^2}\right\}$ is contained in a bounded subset of $B^0([0, \infty))$.

Let $u_0^{(\nu)}(y)$ be the function $\in I^{2+\alpha}(\nu, U)$ which corresponds to $w_0^{(\nu)}(\psi)$ by the Von Mises' transformation restricted on $x=0$. i.e.,

$$u_0\left(\int_0^\psi \frac{dt}{\sqrt{w_0(t)}}\right) = \sqrt{w_0(\psi)}.$$

Then we obtain $u_0^{(\nu)} \to u_0$ in $B^1([0, \infty))$ as $\nu \to 0$ for some u_0. On the other hand, since $\{w_0^{(\nu)}\}$ is convergent in $B^1([0, \infty))$, the constant k in (3. 13) with respect to $w_0^{(\nu)}$ is bounded and the sequence of the numbers D in (3. 19) tends to 1 as $\nu \to 0$. Hence we have $S(w_0^{(\nu)}) \to 0$ as $\nu \to 0$ by (3. 20). This proves the second assertion of Theorem 2.

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