Hokkaido Mathematical Journal Vol. 13 (1984) p. 89-91

A short proof to Brauer's third main theorem

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(Received February 16, 1983; Revised July 18, 1983)

Introduction

In this note we present a short proof which uses Brauer's first main theorem, Nagao's lemma and some basic results from block thoery.

Acknowledgements

I am indebted to the referee for his useful suggestions. This work was written while I was a Feinberg Fellow in the Department of Pure Mathematics, The Weizmann Institute of Science, Rehovot. I am grateful to the Foundation for the grant and to the Department for its generous hospitality.

1. Notation and basic results

In this paper G is a finite group of order $|G|$, K is a field of characteristic $p>0$ which is algebraically closed (however see the remark at the end) and KG is the group-algebra. Let H be a subgroup of G. $C_G(H)$ and $N_G(H)$ stand for the centralizer and the normalizer of H in G, respectively. Following R. Brauer, we shall call a block b of H admissible if b has a defect group D such that $C_G(D)\subseteq H$.

We recall some definitions and results for convenience.

(a) A block b of H is called the principal block if b contains the trivial representation of H.

(b) The defect groups of the principal block of H and the vertices of the trivial representation are Sylow p-subgroups of H.

(c) ([2, sec. 6]) If B is a block of G with a defect group D and $DC_G(D)\subseteq H$, then H has a block b with a defect group D which satisfies $b^G=B$.

(d) ([2, sec. 6]) Let b_1 and b_2 be admissible blocks of H satisfying $b_1^G=b_2^G$. If H is normal in G then b_1 is conjugate to b_2 in G.

(e) ([1, 57.4, 58.3]) Let b be a block of H with a defect group D. If b^G is defined then it has a defect group which contains D.

(f) Brauer's first main theorem ([1, 65.4]). Let D be a p-subgroup of G and let $H=N_G(D)$. There exists a one to one correspondence between the blocks of G with a defect group D and the blocks of H with defect group D. This correspondence is given by $b \leftrightarrow b^G$.

(g) Nagao's lemma ([1, 56. 4]). Let Q be a p-subgroup of G and assume that H is a subgroup of G which satisfies $QC_G(Q) \subseteq H \subseteq N_G(Q)$. Let M be a KG-module in a block B of G. Then every component V of M_H with a vertex $U \geqq Q$ belongs to a block b of H with $b^G = B$.

For unexplained terms see [1].

2. Preparatory results

(a) LEMMA. *Let b be an admissible block of H. If b is the principal block of H then b^G is the principal block B_0 if G.*

PROOF. For a subgroup X of G, let 1_X be the trivial one dimensional representation of X. By 1 (a) 1_X belongs to the principal block of X. Let D be a defect group of b with $C_G(D) \subseteq H$. If $H \subseteq N_G(D)$ then 1 (g) and 1 (b) with $M = 1_G$ and $V = 1_H$ implies $b^G = B_0$. If $H \not\subseteq N_G(D)$ then let $N = N_G(D) \cap H$ and let β be the Brauer correspondent of b in N. Thus $\beta^H = b$. (See 1). Now 1 (b), 1 (g) with $M = 1_H$ and $V = 1_N$ implies that β is the principal block of N. By the above argument $\beta^G = B_0$. But $\beta^G = (\beta^H)^G = b^G$, hence b^G is the principal block of G.

(b) LEMMA. *Let H be a normal subgroup of G and b_0 the principal block of H. Then for every $g \in G$, $b_0^g = b_0$.*

PROOF. $1_H \otimes_H g \cong 1_H$ as kH-modules. On the other hand $1_H \otimes_H g$ belongs to b_0^g. Hence the result follows by 1 (a).

(c) LEMMA. *Let H be a subgroup of G and b an admissible block of H with a defect group Q. Then there exists a block β of $QC_G(Q)$ with defect group Q satisfying $\beta^H = b$ and a unique block $\alpha(b)$ of $N_G(Q)$ such that $\alpha(b) = \beta^{N_G(Q)}$ and $\alpha(b)^G = b^G$. If $\alpha(b)$ is the principal block of $N_G(Q)$ then b is the principal block of H.*

PROOF. By 1 (c) there exists a β as required. If β' is another block of $QC_G(Q)$ with $\beta'^H = b$ then $\beta'^{N_H(Q)} = \beta^{N_H(Q)}$ by Brauer's first main theorem, hence certainly $\beta^{N_G(Q)} = \beta'^{N_G(Q)}$. So $\alpha(b)$ is well defined and $\alpha(b)^G = \beta^G = (\beta^H)^G = b^G$. Finally, if $\alpha(b)$ is the principal block of $N_G(Q)$ then $\beta_0^{N_G(Q)} = \alpha(b)$, where β_0 is the principal block of $QC(Q)$, by lemma 2 (a). Hence $\beta_0 = \beta^g$ for some $g \in G$ by 1 (d). Consequently $\beta = \beta_0$ by Lemma 2 (b), hence $b = \beta^H$ is the principal block of H by Lemma 2 (a), as required.

Brauer's thrid main theorem [1, 65. 4]

Let b be an admissible block of H. Then b^G is the principal block of G if and only if b is the principal block of H.

REMARK. In view Lemma 2(a) we have to show that if b^G is the principal block of G then b is necessarily the principal block of H. We proof this below.

PROOF. Let b be an admissible block of H with a defect group Q and let P be a Sylow p-subgroup G which contains Q. We prove the theorem by induction on $|P:Q|$. Assume $P=Q$. Then $\alpha(b)$ of Lemma 2(c) must have defect group P, by 1(b) and Lemma 2(a). On the other hand if f is the principal block of $N_G(Q)$ then $f^G=B_0$ by Lemma 2(a) and f also has defect group Q by 1(b). But then $\alpha(b)$ is the principal block of $N_G(Q)$, hence b is the principal block of H, by Lemma 2(c), as required. Assume now that $|P:Q|>1$. Then $\alpha(b)$ has defect group $>Q$ by Brauer's first main theorem and 1(b). Since $\alpha(b)$ is an admissible block of $N_G(Q)$, $\alpha(b)$ is the principal block of $N_G(Q)$ by the induction hypothesis. But then b is the principal block of H by Lemma 2(c) and the theorem is proved.

REMARK. No restriction on the field K is really needed, for the above proof can be adopted to a purely module-theoretic context.

References

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[2] G. MICHLER: Blocks and centres of group-algebras; Lecture Notes in Math. Springer, Berlin, 429–563.

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