Hokkaido Mathematical Journal Vol. 13 (1984) p. 133-144

Kernels associated with cylindrical measures on locally convex spaces

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(Received August 8, 1983)

§ 1. Introduction

The present paper contains some results concerning kernels of cylindrical measures on locally convex spaces. The notion of kernel has been introduced by C. Borell [3].

Let E be a locally convex space, E^* be its topological dual space, μ be a cylindrical measure on E and $L: E^*\rightarrow L^0(\Omega, \Sigma, P)$ be a random linear functional associated with μ. The inverse image of the topology of the convergence in probability on $L^0(\Omega, \Sigma, P)$ under L is called the topology associated with μ and denoted by τ_μ; τ_μ is a linear topology on E^*. The topological dual of (E^*, τ_μ) is called the kernel of μ and denoted by K_μ. Let τ be a linear topology on E^*. The cylindrical measure μ is called of type 0 with respect to τ if the random linear functional $L: (E^*, \tau)\rightarrow L^0(\Omega, \Sigma, P)$ is continuous, and μ is called of type p (for $p>0$) with respect to τ if the image of E^* under L is contained in $L^p(\Omega, \Sigma, P)$ and $L: (E^*, \tau)\rightarrow L^p(\Omega, \Sigma, P)$ is continuous. Then our main results are stated as follows.

Let E and F be locally convex spaces, T be a continuous linear mapping of F into E, τ be a linear topology on E^* and τ_k be the Mackey topology on F^*. Then it is shown that the adjoint mapping $T^*: (E^*, \tau)\rightarrow(F^*, \tau_k)$ can be factored through a subspace of $L^0(\Omega, \nu)$ for some probability space (Ω, ν) if and only if there exists a cylindrical measure μ on E of type 0 with respect to τ such that K_μ contains $T(F)$. In this case, if F is quasi-complete or barrelled, then τ_k can be replaced by the strong topology $b(F^*, F)$. As a special case, we can give a characterization of L^0-imbeddable spaces, which is similar to the results of S. Chevet [4] and Y. Okazaki [8]. For $p>0$, it is also shown that if there exists a cylindrical measure μ on E of type p with respect to τ such that K_μ contains $T(F)$, then $T^*: (E^*, \tau)\rightarrow(F^*, \tau_k)$ can be factored through a subspace of $L^p(\Omega, \nu)$ for some probability space (Ω, ν). In this case, if $p=2$, then the converse is also true. Here we are

The research by this author was partially supported by Grant-in-Aid for Scientific Research (No. 57540086), Ministry of Education.

very interested in the case when K_μ contains E.

Now suppose that the topology τ is stronger than the weak*-topology $\sigma(E^*, E)$ and weaker than the Mackey topology $\tau_k(E^*, E)$. Then it is shown that if there exists a cylindrical measure μ on E of type p with respect to τ such that K_μ contains E, then (E^*, τ) is isomorphic to a subspace of $L^p(\Omega, \nu)$ for some probability space (Ω, ν). In particular, taking $p=2$, we obtain that (E^*, τ) is isomorphic to a pre-Hilbert space if and only if there exists a cylindrical measure μ on E of type 2 with respect to τ such that K_μ contains E. These results generalize the works of W. Linde [7] and the author [11].

Some results contained in this paper have been announced without the proofs in our previous paper [12].

§ 2. Preliminaries

Let E be a locally convex space, E^* be its topological dual space, μ be a cylindrical measure on E and $L: E^* \to L^0(\Omega, \Sigma, P)$ be a random linear functional associated with μ. (For the details of random linear functionals; see R. M. Dudley [6].) As in Section 1, denote by τ_μ the topology on E^* associated with μ and denote by K_μ the kernel of μ, respectively. It is clear that τ_μ does not depend on the choice of L. If for each positive integer n, we put

$$U_n(\mu) = \left\{x^* \in E^* ;\ \mu\left\{x \in E;\ |\langle x^*, x\rangle| \geqq \frac{1}{n}\right\} \leqq \frac{1}{n}\right\},$$

then $\{U_n(\mu)\}$ forms a fundamental system of neighborhoods of zero for the topology τ_μ. τ_μ is also defined by the following translation-invariant semi-metric d_μ;

$$d_\mu(x^*, y^*) = \int_E \frac{|\langle x^* - y^*, x\rangle|}{1 + |\langle x^* - y^*, x\rangle|} d\mu(x) \qquad \text{for} \quad x^*, y^* \in E^* .$$

Let $\hat{\mu}$ denote the characteristic functional of μ defined on E^*. Then the following result is very useful in our ensuing discussions.

LEMMA 2.1. (cf. V. N. Sudakov and A. M. Veršik [10]). *Let $\{x_n^*\}$ be a sequence in E^*. Then $\{x_n^*\}$ converges to zero for the topology τ_μ if and only if $\hat{\mu}(x_n^*)$ converges to* 1.

It is clear that K_μ is contained in E if τ_μ is weaker than the Mackey topology $\tau_k(E^*, E)$, K_μ contains E if and only if τ_μ is stronger than the weak*-topology $\sigma(E^*, E)$, and τ_μ is separated if the set $K_\mu \cap E$ is dense in E. (For the details of the topology τ_μ and the kernel see S. Chevet [4, 5].)

Now let τ be a linear topology on E^*. It is easy to see that the cylindrical measure μ is of type 0 with respct to τ if and only if $\hat{\mu}$ is τ-continuous. For $p>0$, the cylindrical measure μ is called of weak p-th order if for each $x^*\in E^*$, the inequality

$$||x^*||_p=\left(\int_E\left|\langle x^*, x\rangle\right|^p d\mu(x)\right)^{\frac{1}{p}}<\infty$$

holds. Then it is clear that the cylindrical measure μ is of type p with respect to τ if and only if it is of weak p-th order and the quasi-seminorm $||\cdot||_p$ is τ-continuous.

Finally we shall introduce a negative definite function, which is very useful to characterize L^0-imbeddable spaces. (For the general definition of a negative definite function and related results; see C. Berg and G. Forst [1, Chapter II].)

A function f from a linear space X into the non-negative reals is called negative definite if $f(0)=0$, $f(x)=f(-x)$ for every $x\in X$ and

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j f(x_i-x_j)\leqq 0$$

for every n, every $x_1, \cdots, x_n\in X$ and every real numbers $\alpha_1, \cdots, \alpha_n$ such that $\sum\alpha_i=0$.

It is known that if f is negative definite, then $f/(1+f)$ is also negative definite (cf. [1, Exercise 7. 10]) and $\exp(-f)$ is positive definite (cf. [1, Theorem 7. 8]). Let d_μ be the semi-metric on E^* associated with the cylindrical measure μ defined as before. Then the function: $x^*\to d_\mu(x^*, 0)$ is clearly negative definite. For $p>0$, let $||\cdot||_p$ be the quasi-seminorm on E^* associated with the cylindrical measure μ of weak p-th order. If $0<q\leqq p\leqq 2$, then the function: $x^*\to||x^*||_p^q$ is also negative definite.

§ 3. L^0-imbeddable spaces

In this section we shall study kernels of cylindrical measures of type 0 and obtain a factorization theorem through a subspace of a L_0-space. As a special case we shall give a characterization of L^0-imbeddable spaces.

First we shall prepare the following elementary lemma.

LEMMA 3. 1. *Let E and F be locally convex spaces, T be a continuous linear mapping of F into E and τ be a semi-metrizable linear topology on E^*. Then the following three statements are equivalent.*

(1) *$(E^*, \tau)^*$ contains $T(F)$.*

(2) *τ is stronger than the weak*-topology $\sigma(E^*, T(F))$.*

(3) *The adjoint mapping $T^*: (E^*, \tau)\to(F^*, \tau_k)$ is continuous, where τ_k*

denotes the Mackey topology $\tau_k(F^*, F)$.

Furthermore, if we assume that F is quasi-complete or barrelled, then the above three statements are equivalent to the following.

(4) *The adjoint mapping* $T^*: (E^*, \tau) \to (F^*, b)$ *is continuous, where b denotes the strong topology* $b(F^*, F)$.

PROOF. The equivalence of (1) and (2) is clear. Hence in order to prove the first assertion, it suffices to show that the implication (2)$\Rightarrow$(3) holds. Suppose that the statement (3) does not hold. Then there exists a neighborhood U of zero in (F^*, τ_k) such that $(T^*)^{-1}(U)$ does not contain any neighborhood of zero in (E^*, τ). Since the topology τ is semi-metrizable, there exists a sequence $\{x_n^*\}$ in E^* such that the sequence $\{nx_n^*\}$ converges to zero with respect to τ and $T^*(x_n^*) \notin U$ for every n. It follows from Theorem 36. 2 of [14] that the statement (2) does not hold. This proves the first assertion.

Next we shall prove the second assertion. Since the implication (4)$\Rightarrow$(1) clearly holds, it suffices to show that the implication (1)$\Rightarrow$(4) holds. Suppose that F is quasi-complete and the statement (1) holds. Let B be any bounded subset of F, and let F_B be the linear subspace of F spanned by B. Since F is quasi-complete, we may assume that B is bounded convex balanced complete. Hence F_B forms a complete seminormed space with the closed unit ball B and the natural injection of F_B into F is continuous. Let $\{U_n\}$ be a countable basis of neighborhoods of zero in (E^*, τ), and for each n let U_n^0 denote the polar of U_n in E. Then by the assumption it holds $T(F) \subset \cup U_n^0$, so that we have $F_B \subset \cup T^{-1}(U_n^0)$. Since F_B is of the second category, it follows from the Baire category theorem that the statement (4) certainly holds. On the other hand, suppose that F is barrelled. Then the assertion follows from the Banach-Steinhaus theorem. This completes the proof.

Now we shall prove our main theorem of this section.

THEOREM 3. 2. *Let E and F be locally convex spaces, T be a continuous linear mapping of F into E and* τ *be a linear topology on* E^*. *Then the following three statements are equivalent.*

(1) *There exists a probability space* (Ω, ν) *such that the adjoint mapping* $T^*: (E^*, \tau) \to (F^*, \tau_k)$ *can be factored through a subspace of* $L^0(\Omega, \nu)$.

(2) *There exists a cylindrical measure* μ *on E of type* 0 *with respect to* τ *such that* K_μ *contains* $T(F)$.

(3) *There exists a translation-invariant semi-metric d on* E^* *being compatible with the linear structure such that the topology on* E^* *defined by the semi-metric d is stronger than* $\sigma(E^*, T(F))$ *and the function*: $x^* \to d(x^*, 0)$ *is negative definite and* τ*-continuous.*

Furthermore, if we assume that F is quasi-complete or barrelled, then in the statement (1), τ_k *can be replaced by the strong topology* $b(F^*, F)$.

PROOF. First we shall prove (1)$\Rightarrow$(2). Suppose (1) holds. Then the adjoint mapping $T^*: (E^*, \tau)\rightarrow(F^*, \tau_k)$ can be factored as follows;

$$(E^*, \tau)\xrightarrow[J]{} G \xrightarrow[K]{} (F^*, \tau_k)$$

where $T^*=K\cdot J$, G is a linear subspace of $L^0(\Omega, \nu)$, and J and K are continuous linear mappings, respectively. Since $J: E^*\rightarrow L^0(\Omega, \nu)$ is linear, there corresponds a cylindrical measure μ on E such that

$$\mu(Z)=\nu\left\{\omega\in\Omega\,;\,\left(J(x_1^*)(\omega), \cdots, J(x_n^*)(\omega)\right)\in B\right\}$$

for every $x_1^*, \cdots, x_n^*\in E^*$, and for every cylindrical set $Z=\{x\in E\,;\,(\langle x_1^*, x\rangle, \cdots, \langle x_n^*, x\rangle)\in B\}$, where B is a Borel set of R^n. Then the continuity of J implies that μ is of type 0 with respect to τ. Let τ_μ be the topology on E^* associated with μ. Since $J: (E^*, \tau_\mu)\rightarrow G$ is continuous, $T^*: (E^*, \tau_\mu)\rightarrow (F^*, \tau_k)$ is also continuous, so that K_μ contains $T(F)$. Thus (2) holds.

Next we shall prove (2)$\Rightarrow$(3). Suppose (2) holds. As mentioned in Section 2, if we denote by d_μ the semi-metric on E^* associated with the cylindrical measure μ, then the semi-metric d_μ is translation-invariant and compatible with the linear structure of E^*. Since the topology on E^* defined by d_μ is identical with τ_μ, it is stronger than $\sigma(E^*, T(F))$. Also since μ is of type 0 with respect to τ, the function: $x^*\rightarrow d_\mu(x^*, 0)$ is negative definite and τ-continuous. Thus (3) holds.

Finally we shall prove (3)$\Rightarrow$(1). Suppose (3) holds. Since the function: $x^*\rightarrow d(x^*, 0)$ is negative definite and τ-continuous, if we put

$$f(x^*)=\exp\left(-d(x^*, 0)\right) \qquad \text{for every} \quad x^*\in E^*,$$

then f is positive definite, τ-continuous and $f(0)=1$ (cf. [1, Theorem 7.8]). By the Bochner's theorem (cf. [2]), there exists a probability measure ν on (Ω, Σ) such that

$$f(x^*)=\int_\Omega e^{i\langle x^*, \omega\rangle}d\nu(\omega) \qquad \text{for every} \quad x^*\in E^*,$$

where Ω denotes the algebraic dual space of E^* and Σ denotes the minimal σ-albegra of Ω which makes every element of E^* measurable. Let τ_ν be the topology on E^* associated with the measure ν. Then by Lemma 2.1, τ_ν is identical with the topology on E^* defined by the semi-metric d, and hence it is stronger than $\sigma(E^*, T(F))$ and weaker than τ. Since τ_ν is semi-metrizable, it follows from Lemma 3.1 that $T^*: (E^*, \tau_\nu)\rightarrow(F^*, \tau_k)$ is continuous.

Let G denote the associated Hausdorff space $(E^*, \tau_\nu)/\{\bar{0}\}$ equipped with the quotient topology, where $\{\bar{0}\}$ is the closure of $\{0\}$ in (E^*, τ_ν). Then it is clear that G is isomorphic to a subspace of $L^0(\Omega, \nu)$ and $T^* : (E^*, \tau) \to (F^*, \tau_k)$ can be factored through G since $\mathrm{Ker}\, T^* = \{x^* \in E^* ;\ T^*(x^*) = 0\}$ contains $\{\bar{0}\}$. Thus (1) holds.

The remainder part of the assertion follows from Lemma 3. 1. This completes the proof.

As a special case we have the following result, which gives a characterization of L^0-imbeddable spaces.

THEOREM 3. 3. *Let E be a locally convex space and τ be a linear topology on E^* which is stronger than the weak*-topology $\sigma(E^*, E)$ and weaker than the Mackey topology $\tau_k(E^*, E)$. Then the following three statements are equivalent.*

(1) *There exists a probability space (Ω, ν) such that (E^*, τ) is isomorphic to a subspace of $L^0(\Omega, \nu)$, that is, it is L^0-imbeddable.*

(2) *There exists a cylindrical measure μ on E of type 0 with respect to τ such that K_μ contains E.*

(3) *There exists a translation-invariant metric d on E^* such that the function: $x^* \to d(x^*, 0)$ is negative definite and the topology on E^* defined by the metric d is identical with τ.*

REMARK 3. 1. Theorem 3. 3 says that (E^*, τ_k) is L^0-imbeddable if and only if there exists a cylindrical measure μ on E of type 0 with respect to τ_k such that K_μ coincides with E. Furthermore, if we assume that E is quasi-complete or barrelled, then it is shown that (E^*, b) is L^0-imbeddable if and only if there exists a cylindrical measure μ on E of type 0 with respect to b such that K_μ contains E, where b denotes the strong topology $b(E^*, E)$. This generalizes a result of W. Linde [7].

Now we shall investigate more details.

PROPOSITION 3. 4. *Let E be a locally convex Hausdorff space. Then the following two statements are equivalent.*

(1) *E can be represented as a countable union of finite dimensional spaces.*

(2) *There exists a cylindrical measure μ on E of type 0 with respect to $\sigma(E^*, E)$ such that K_μ contains E.*

PROOF. Suppose (1) holds. Then (E^*, σ) is a nuclear metrizable space, and hence it is L^0-imbeddable (cf. Y. Okazaki [8]). By Theorem 3. 3, it holds (2). On the other hand, suppose (2) holds. Then it follows from Theorem 3. 3 that (E^*, σ) is L^0-imbeddable and so it is metrizable. Let

$\{U_n\}$ be a basis of neighborhoods of zero in (E^*, σ). For each n, U_n^0 denotes the polar of U_n in E. Then it holds $E = \cup U_n^0$. Since each U_n^0 is contained in the convex balanced hull of a finite set, it holds (1). This completes the proof.

Let E be a Banach space with the norm $\|\cdot\|$. Then E is said to be of cotype 2 if there exists a positive constant C such that for every finite set $\{x_1, \cdots, x_n\}$ in E, the inequality

$$\sum_{i=1}^n \|x_i\|^2 \leqq C\left(\int_\Omega \left\|\sum_{i=1}^n \varepsilon_i(\omega)\, x_i\right\|^2 dP(\omega)\right)$$

holds, where (Ω, P) is a probability space with a symmetric Bernoulli sequence $\{\varepsilon_i(\omega)\}$.

PROPOSITION 3.5. *Let E be a locally convex Hausdorff space of the second category. Suppose that there exists a cylindrical measure μ on E such that K_μ contains E. Then the following three statements holds.*

(1) *If μ is of type 0 with respect to the compact convergence topology $c(E^*, E)$, then E is finite dimensional.*

(2) *If μ is of type 0 with respect to $\tau_k(E^*, E)$, then E is isomorphic to a reflexive Banach space, τ_k is identical with $b(E^*, E)$ and (E^*, b) is isomorphic to a reflexive Banach space of cotype 2.*

(3) *If μ is of type 0 with respect to $b(E^*, E)$, then E is normable and (E^*, b) is isomorphic to a Banach space of cotype 2.*

PROOF. Let τ_μ be the topology on E^* associated with the cylindrical measure μ, and let $\{U_n\}$ be a basis of neighborhoods of zero in (E^*, τ_μ). For each n, if we denote by U_n^0 the polar of U_n in E, then it is clear that each U_n^0 is bounded convex balanced closed and by the assumption, it holds $E = \cup U_n^0$. Since E is of the second category, there exists a positive number n such that U_n^0 contains a neighborhood of zero in E. This means that E is normable. Now we shall prove the three statements. For (1). Since τ_μ is weaker than $c(E^*, E)$, the set U_n^0 is pre-compact. This means E is locally pre-compact, so that it must be finite dimensional. For (2). Since τ_μ is weaker than $\tau_k(E^*, E)$, the set U_n^0 is weakly compact. This means that E is isomorphic to a reflexive Banach space and τ_k is identical with $b(E^*, E)$. It follows from Theorem 3.3 that the Banach space (E^*, b) is L^0-imbeddable, so that it must be of cotype 2 (cf. H. Shimomura [9, Theorem 4.1]). For (3). Since E is barrelled, it follows from Remark 3.1 that the Banach space (E^*, b) is L^0-imbeddable, so that it must be of cotype 2 (cf. H. Shimomura [9, Theorem 4.1]). This completes the proof.

REMARK 3.2. In general, if E is not of the second category, then the statements (1), (2) and (3) of Proposition 3.5 do not hold even in the case of complete barrelled spaces. For example, it is the case when E is a topological inductive limit of a properly increasing sequence of finite dimensional spaces. However, if E is barrelled and there exists a cylindrical measure μ on E of weak p-th order (for $p>0$) such that K_μ contains E, then E is normable (cf. [13]). In this case, if we assume furthermore that μ is of type 0 with respect to $b(E^*, E)$, then the Banach space (E^*, b) is of cotype 2.

§ 4. L^p-imbeddable spaces (for $p>0$)

In this section we shall study kernels of cylindrical measures of type p. Throughout this section, let p be a positive number.

THEOREM 4.1. *Let E and F be locally convex spaces, T be a continuous linear mapping of F into E and τ be a linear topology on E^*. Suppose that there exists a cylindrical measure μ on E of type p with respect to τ such that K_μ contains $T(F)$. Then there exists a probability space (Ω, ν) such that the adjoint mapping $T^*: (E^*, \tau)\to(F^*, \tau_k)$ can be factored through a subspace of $L^p(\Omega, \nu)$, where τ_k denotes the Mackey topology $\tau_k(F^*, F)$.*

Furthermore, if we assume that F is quasi-complete or barrelled, then τ_k can be replaced by the strong topology $b(F^*, F)$.

PROOF. Let $L: E^*\to L^0(\Omega, \nu)$ be a random linear functional associated with μ. Since μ is of type p with respect to τ, $L: (E^*, \tau)\to L^p(\Omega, \nu)$ is continuous. If we put

$$||x^*||_p=\left(\int_E\left|\langle x^*, x\rangle\right|^p d\mu(x)\right)^{\frac{1}{p}} \qquad \text{for every} \quad x^*\in E^*,$$

then $||\cdot||_p$ is a continuous quasi-seminorm on (E^*, τ) and it clearly holds $||x^*||_p=||L(x^*)||_{(L^p)}$ for every $x^*\in E^*$. Let G denotes the associated Hausdorff space $(E^*, ||\cdot||_p)/\text{Ker}\,||\cdot||_p$ equipped with the quotient topology, where $\text{Ker}\,||\cdot||_p=\{x^*\in E^*;\ ||x^*||_p=0\}$. Then it is easy to see that G is linearly isometric to a subspace of $L^p(\Omega, \nu)$. Now we shall show that $T^*: (E^*, \tau)\to (F^*, \tau_k)$ can be factored through G. Since $(E^*, ||\cdot||_p)^*$ contains K_μ, it also contains $T(F)$. Hence it follows from Lemma 3.1 that $T^*: (E^*, ||\cdot||_p)\to (F^*, \tau_k)$ is continuous, so that it can be factored through G since $\text{Ker}\,||\cdot||_p$ is contained in $\text{Ker}\,T^*=\{x^*\in E^*;\ T^*(x^*)=0\}$. Thus the first assertion holds.

The remainder part of the assertion follows from Lemma 3.1. This completes the proof.

In particular, taking $p=2$, we have the following.

THEOREM 4.2. *Let E and F be locally convex spaces, T be a continu-*

ous linear mapping of F into E and τ be a linear topology on E^*. Then the following two statements are equivalent.

(1) *The adjoint mapping $T^*: (E^*, \tau)\to(F^*, \tau_k)$ can be factored through a pre-Hilbert space.*

(2) *There exists a cylindrical measure μ on E of type 2 with respect to τ such that K_μ contains $T(F)$.*

Furthermore, if we assume that F is quasi-complete or barrelled, then in the statement (1), *τ_k can be replaced by the strong topology $b(F^*, F)$.*

PROOF. First we shall prove (1)$\Rightarrow$(2). Suppose (1) holds. Then there exists a continuous Hilbertian seminorm $\|\cdot\|$ on (E^*, τ) such that $T^*: (E^*, \|\cdot\|)\to(F^*, \tau_k)$ is continuous. If we put $f(x^*)=\exp(-\|x^*\|^2/2)$ for every $x^*\in E^*$, then there corresponds a Gaussian cylindrical measure γ on E such that $f(x^*)=\hat{\gamma}(x^*)$ for every $x^*\in E^*$. Let τ_γ be the topology on E^* associated with γ. Since the topology on E^* defined by the Hilbertian seminorm $\|\cdot\|$ is identical with τ_γ (cf. Lemma 2.1), $T^*: (E^*, \tau_\gamma)\to(F^*, \tau_k)$ is continuous. Hence K_γ contains $T(F)$. Since γ is of type 2 with respect to $\|\cdot\|$ (cf. W. Linde [7]), it is also of type 2 with respect to τ. Thus it holds (2). On the other hand, the remainder part of the assertion follows from Theorem 4.1. This completes the proof.

As corollaries to these theorems, we have the following results.

COROLLARY 4.3. *Let E be a locally convex space and τ be a linear topology on E^* which is stronger than $\sigma(E^*, E)$ and weaker than $\tau_k(E^*, E)$. Suppose that there exists a cylindrical measure μ on E of type p with respect to τ such that K_μ contains E. Then there exists a probability space (Ω, ν) such that (E^*, τ) is isomorphic to a subspace of $L^p(\Omega, \nu)$, that is, it is L^p-imbeddable.*

Furthermore, if we assume that E is quasi-complete or barrelled, then τ_k can be replaced by $b(E^*, E)$.

COROLLARY 4.4. *Let E be a locally convex space and τ be a linear topology on E^* which is stronger than $\sigma(E^*, E)$ and weaker than $\tau_k(E^*, E)$. Then (E^*, τ) is isomorphic to a pre-Hilbert space if and only if there exists a cylindrical measure μ on E of type 2 with respect to τ such that K_μ contains E.*

Furthermore, if we assume that E is quasi-complete or barrelled, then τ_k can be replaced by $b(E^*, E)$.

REMARK 4.1. These corollaries generalize the results of W. Linde [7] and the author [11]. On the other hand, H. Shimomura's result [9, Theorem 4.6] says that (E^*, τ) is isomorphic to a pre-Hilbert space if and only if there exists a quasi-invariant cylindrical measure μ on E of type 0 with respect to τ. Corollary 4.4 is closely related to his result. However, in our case, we can not replace by the assumption of type 0 instead of type 2.

As a similar result to Corollary 4.4, we shall give a characterization of countably pre-Hilbert spaces. Before stating the result, we shall introduce partially admissible shifts of cylindrical measures. (For the details of partially admissible shifts; see [11].)

For a cylindrical measure μ on a locally convex space E, an element x of E is called a partially admissible shift of μ if there exist an $\varepsilon>0$ and a $\delta>0$ such that the inequality $\mu(Z)<\delta$ implies $\mu(Z-x)<1-\varepsilon$ for every cylindrical set Z of E. The set of all partially admissible shifts of μ will be denoted by $\widetilde{M}_{\mu}$.

It is known that K_{μ} contains $\widetilde{M}_{\mu}$, but in general, K_{μ} does not coincide with $\widetilde{M}_{\mu}$ (cf. [11, Proposition 3.1]).

THEOREM 4.5. *Let E be a locally convex space and τ be a linear topology on E^* which is stronger than $\sigma(E^*, E)$ and weaker than $\tau_k(E^*, E)$. Then the following three statements are equivalent.*

(1) *(E^*, τ) is isomorphic to a countably pre-Hilbert space.*

(2) *There exists a sequence $\{\mu_n\}$ consisting of cylindrical measures on E of type 2 with respect to τ such that $\cup \widetilde{M}_{\mu_n}$ coincides with E.*

(3) *There exists a sequence $\{\mu_n\}$ consisting of cylindrical measures on E of type 2 with respect to τ such that $\cup K_{\mu_n}$ contains E.*

PROOF. We shall prove (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(1). Suppose (1) holds. Then the topology τ is defined by a family of Hilbertian seminorms $\|\cdot\|_n$ $(n=1, 2, \cdots)$. For each n, we denote by H_n the topological dual space of $(E^*, \|\cdot\|_n)$. Note that each H_n is a Hilbert subspace of E and $E=\cup H_n$. Let γ_n be the canonical Gaussian cylindrical measure on H_n, and let μ_n be the image of γ_n under the inclusion mapping of H_n into E. Then it is easy to see that for each n, μ_n is a cylindrical measure on E of type 2 with respect to τ and $\widetilde{M}_{\mu_n}=H_n$. Thus it holds (2). The implication (2)$\Rightarrow$(3) is clear (cf. [11, Proposition 3.1]). Suppose (3) holds. For each n, if we put

$$\|x^*\|_n=\left(\int_E |\langle x^*, x\rangle|^2 d\mu(x)\right)^{\frac{1}{2}} \qquad \text{for every} \quad x^*\in E^*,$$

then $\|\cdot\|_n$ is a continuous Hilbertian seminorm on (E^*, τ) since μ_n is of type 2 with respect to τ. Let τ_2 denote the topology on E^* defined by the family of Hilbertian seminorms $\|\cdot\|_n$ $(n=1, 2, \cdots)$. Since τ_2 is weaker than τ, to prove (1), it is enough to show that τ_2 is stronger than τ. Note that the topological dual of $(E^*, \|\cdot\|_n)$ contains K_{μ_n}. Since $\cup K_{\mu_n}$ contains E, τ_2 is stronger than $\sigma(E^*, E)$. Hence it follows from Lemma 3. 1 that τ_2 is stronger than $\tau_k(E^*, E)$, so that it is stronger than τ. Thus it holds (1). This completes the proof.

The following example shows that H. Shimomura's result [9, Theorem 4. 6] can not be extended to partially admissible shifts of cylindrical measures.

EXAMPLE 4. 1. Let E be a locally convex space and τ be a linear topology on E^* which is stronger than $\sigma(E^*, E)$ and weaker than $\tau_k(E^*, E)$. Suppose that (E^*, τ) is a countably pre-Hilbert space. Then there exists a cylindrical measure μ on E of type 0 with respect to τ such that $\widetilde{M}_\mu$ coincides with E.

PROOF. It follows from Theorem 4. 5 that there exists a sequence $\{\mu_n\}$ consisting of cylindrical measures on E of type 2 with respect to τ such that $\cup \widetilde{M}_{\mu_n}$ coincides with E. If we put $\mu=\sum 2^{-n}\mu_n$, then it is clear that μ is a cylindrical measure on E of type 0 with respect to τ. Now we shall prove that $\widetilde{M}_\mu$ coincides with E. Let $x\in E$ be given. Then there exists a positive integer n such that $x\in\widetilde{M}_{\mu_n}$. Hence there exist an $\varepsilon_n>0$ and a $\delta_n>0$ such that the inequality $\mu_n(Z)<\delta_n$ implies $\mu_n(Z-x)<1-\varepsilon_n$ for every cylindrical set Z of E. Here we take positive numbers ε and δ as $\varepsilon<2^{-n}\varepsilon_n$ and $\delta<2^{-n}\delta_n$, respectively. Then for every cylindrical set Z, the inequality $\mu(Z)<\delta$ implies $\mu(Z-x)<1-\varepsilon$. Thus it holds $x\in\widetilde{M}_\mu$. This completes the proof.

REMARK 4. 2. In Example 4. 1, if (E^*, τ) is nuclear, then by Minlos' theorem μ is a Borel probability measure on E. Here we are very interested in the converse. It is shown that (E^*, τ) is a nuclear countably pre-Hilbert space if and only if there exists a Borel probability measure μ on E of type 0 with respect to τ such that $\widetilde{M}_\mu$ coincides with E. The author will discuss the measure case in another paper.

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