Hokkaido Mathematical Journal, Vol. 14 (1985), p. 49-73

Idempotent multipliers on the space of analytic singular measures

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(Received March 17, 1983; Revised May 4, 1984)

Introduction.

Let G be a LCA group with dual group $\hat{G}$, and we denote by m_G the Haar measure of G. Let $M(G)$ and $L^1(G)$ be the usual measure algebra and the group algebra respectively. Let $M^+(G)$ be the subset of $M(G)$ consisting of positive measures. Let $M_d(G)$, $M_c(G)$ and $M_s(G)$ denote the subspaces of $M(G)$ consisting of discrete measures, continuous measures and singular measures respectively. " $\wedge$ " and " $\vee$ " denote the Fourier-Stieltjes transform and the inverse Fourier transfomr respectively. For a subset B of $M(G)$, $B^\wedge$ means the set $\{\hat{\mu} : \mu \in B\}$, and for $\mu \in M(G)$, we signify $\|\hat{\mu}\|$ by the total variation norm $\|\mu\|$ of μ. When there is a nontrivial continuous homomorphism ψ from $\hat{G}$ into R (the reals), we say that a measure $\mu \in M(G)$ is of analytic type if $\hat{\mu}(\gamma)=0$ for all $\gamma \in \hat{G}$ with $\psi(\gamma)<0$. We denote by $M^a(G)$ the space of all measures of analytic type, and put $M^a(G)_s = M^a(G) \cap M_s(G)$. The space $M^a(G)_s$ is called the space of analytic singular measures (induced by ψ). In this paper we consider only the case that $M^a(G)_s \neq \{0\}$.

Let A be a closed subspace of $M(G)$ and Φ a function on $\hat{G}$. Φ is called a multiplier (or multiplier function) on A *if* $\Phi\hat{\mu} \in A^\wedge$ for all $\mu \in A$. By the function Φ, there exists a unique bounded linear operator S on A such that $S(\mu)^\wedge = \Phi\hat{\mu}$. We say that S is a multiplier operator (or merely multiplier) on A induced by the function Φ.

As well known, functions Φ on $\hat{G}$ become multipliers on $L^1(G)$ if and only if Φ belong to $M(G)^\wedge$. For each $\nu \in M_d(G)$, $\mu * \nu$ are singular measures for all $\mu \in M_s(G)$. For a compact abelian group G, Doss proved that multiplier operators on $M_s(G)$ are given by convolution with discrete measures ([3]). In [7], Graham and MacLean obtained an analogous result for general LCA groups. By the F. and M. Riesz theorem of Helson

AMS Subject Classification Scheme (1979/1980): 43-XX. 43A05, 43A10, 43A17, 43A22, 43A25.

and Lowdenslager type (cf. [17], Theorem 4.1), we have $M^a(G)=(M^a(G)\cap L^1(G))\oplus M^a(G)_s$. Every multiplier operator on $M^a(G)$ necessarily maps $M^a(G)\cap L^1(G)$ into itslf (cf. [17], Lemma (E)). However it is easy to see that there is a multiplier operator on $M^a(G)$ which does not map $M^a(G)_s$ into itself.

Let Φ be a multiplier on $H^1(R)$ (Hardy space on R). Then $\Phi\circ\psi$ is a multiplier on $M^a(G)$, and the multiplier operator induced by the function $\Phi\circ\psi$ necessarily maps $M^a(G)_s$ into itself (cf. [18], Theorem I). Moreover the author constructed a multiplier operator on $M^a(G)_s$ which is not given by convolution with a bounded regular measure on G ([17], Theorem 2.4). Our purpose in this paper is to characterize idempotent multipliers on $M^a(G)_s$ under the assumption that $\psi(\hat{G})$ is dense in R with respect to the usual topology. More exactly, if $\psi(\hat{G})$ is dense in R, we shall show that each idempotent multiplier operator on $M^a(G)_s$ is necessarily given by convolution with a measure in $M(G)$ of certain type. (If $\psi(\hat{G})$ is not dense in R, there exists an idempotent multiplier operator on $M^a(G)_s$ which is not given by convolution with a measure on G.). Our key methed is to consider a covering group for G and employ the theory of disintegration.

In section 0, we state notations and the main theorem of this paper. In section 1 we shall deal with the disintegration of bounded regular measures on $X_1\times X_2$, where X_1 is a metrizable locally compact Hausdorff space and X_2 is a general locally compact Hausdorff space. In section 2 we characterize convolution operators on $M^a(G)_s$ induced by bounded regular measures on G. We consider special idempotent multipliers on $M^a(G)_s$ in section 3, and we give the proof of our main theorem in section 4.

§ 0 Notations and Main Theorem.

Let G be a LCA group. For a closed subgroup H of G, $H^\perp$ means the annihilator of H. We denote by $\mathrm{Trig}(G)$ the set of all trgonometric polynomials on G. For a subset E of $\hat{G}$, let $M_E(G)$ be the space of measures in $M(G)$ whose Fourier-Stieltjes transforms vanish off E, and let χ_E be the characteristic function of E. We denote by E^o and E^- the interior of E and the closure of E respectively. For $x\in G$, δ_x denotes the point mass at x.

DEFINITION 0.1. Let $\psi:\hat{G}\to R$ be a nontrivial continuous homomorphism, and let $\phi:R\to G$ be the dual homomorphism of ψ (i.e., $(\phi(t),\gamma)=\exp(it\psi(\gamma))$). Put $H_\psi=\ker(\psi)^\perp$.

(i) A function Φ on $\hat{G}$ is called a multiplier (or multiplier function) on $M^a(G)_s$ if $\Phi\hat{\mu}\in M^a(G)_s{}^\wedge$ for all $\mu\in M^a(G)_s$. In this case, there exists a unique bounded linear operator S on $M^a(G)_s$ such that $S(\mu)^\wedge=\Phi\hat{\mu}$. We say that S is a multiplier operator (or merely multiplier) on $M^a(G)_s$ induced by the function Φ. We define a norm $\|\Phi\|$ by $\|\Phi\|=\|S\|$.

(ii) Let Φ be a multiplier on $M^a(G)_s$ and S the multiplier operator on $M^a(G)_s$ induced by Φ. Φ is called an idempotent multiplier if $S^2=S$. We say that S is an idempotent multiplier operator (or merely idempotent multiplier).

(iii) We define subsets $\mathscr{A}_0$ and $\mathscr{A}_1$ of $M(G)$ as follows: If $\ker(\psi)$ is not open, we define $\mathscr{A}_0$ by

$$\mathscr{A}_0=\{\textstyle\sum_{i=1}^n m_i\gamma_i m_H\},$$

where $m_i\in Z$ (the integers), $\gamma_i\in\hat{G}$ and H is a finite subgroup of G. If $\ker(\psi)$ is open, we define $\mathscr{A}_1$ by

$$\mathscr{A}_1=\{\textstyle\sum_{j=1}^m M_j\sigma_j m_{H_0}+(\delta_0-m_{H_\psi})*(\sum_{i=1}^n m_i\gamma_i m_H)\},$$

where m_i, $M_j\in Z$, $\gamma_i\in\hat{G}$, $\sigma_j\in\ker(\psi)$; H is a finite subgroup of G and H_0 is a compact subgroup of G with $H_0\supset H_\psi$ such that $\pi(H_0)$ is a finite subgroup of G/H_ψ, where $\pi: G\to G/H_\psi$ is the natural homomorphism.

In their proof of the Cohen Idempotent Theorem, Ito and Amemiya obtained the following theorem.

THEOREM 0.2 ([11]). *Let μ be a measure in $M(G)$ such that $\hat{\mu}$ is integer-valued. Then μ can be represented as follows :*

$$\mu=\textstyle\sum_{i=1}^n\sum_{j=1}^{l_i} n_{ij}\gamma_{ij}m_{H_i},$$

where $n_{ij}\in Z$, $\gamma_{ij}\in\hat{G}$ and H_i are compact subgroups of G such that $\{\sum_{j=1}^{l_i} n_{ij}\gamma_{ij}m_{H_i}\}$ are mutually singular.

Using Theorem 0.2 and results which shall be obtained in sections 1–3, we characterize idempotent multipliers on $M^a(G)_s$. We state our main theorem which will be proved in section 4.

MAIN THEOREM. *Let G be a LCA group and $\psi:\hat{G}\to R$ a nontrivial continuous homomorphism such that $M^a(G)_s\neq\{0\}$ and $\psi(\hat{G})$ is dense in R. Let Φ be an idempotent multiplier on $M^a(G)_s$ and S the idempotent multiplier operater on $M^a(G)_s$ induced by Φ. Then the following are satisfied :*

(I) *If $\ker(\psi)$ is not open, there exists $\nu\in\mathscr{A}_0$ such that*

$$S(\mu)=\mu*\nu\quad (i.\ e.,\ \Phi\hat{\mu}=\hat{\nu}\hat{\mu})$$

for all $\mu\in M^a(G)_s$.

(II) *If $\ker(\psi)$ is open, there exists $\nu\in\mathscr{A}_1$ such that*

$$S(\mu)=\mu*\nu\quad (i.\ e.,\ \Phi\hat{\mu}=\hat{\nu}\hat{\mu})$$

for all $\mu\in M^a(G)_s$.

Conversely, when $\ker(\psi)$ is not open, let ν be a measure in $\mathscr{A}_0$. Then $\hat{\nu}$ becomes a multiplier on $M^a(G)_s$. When $\ker(\psi)$ is open, let ν be a measure in $\mathscr{A}_1$. Then $\hat{\nu}$ becomes a multiplier on $M^a(G)_s$.

A lemma due to Svensson ([15], Lemma 3.1.1) also plays an important role in the proof of Main Theorem. Before we close this section, we state remarks together with useful theorems.

REMARK 0.3. (i) In Main Theorem we assumed that $M^a(G)_s \neq \{0\}$. A necessary and sufficient condition in order that $M^a(G)_s \neq \{0\}$ is stated in ([17], Remark 2.2, p. 186).

(ii) If $\psi(\hat{G})$ is not dense in R, Main Theorem is not satisfied. We give an example: Let $G=T\oplus K$, and let $\psi: Z\oplus\hat{K}\to Z$ be the projection, where T is the circle group and K is a nondiscrete LCA group. Let $\{a_n\}$ be a sequence of positive integers such that $a_{n+1}/a_n>3$. We put $F=\{a_n : n\in N$ (the natural numbers)$\}$ and define a function Φ on $Z\oplus\hat{K}$ by $\Phi(n, \sigma)=\chi_F(n)$. Then Φ is an idempotent multiplier on $M^a(G)_s$ (see [17], Theorem 2.3). However the idempotent multiplier operator S on $M^a(G)_s$ induced by Φ is not given by convolution with a bounded regular measure on G.

REMARK 0.4. Let ψ be a nontrivial continuous homomorphism from $\hat{G}$ into R such that $M^a(G)_s\neq\{0\}$. Let Φ be a function on $\hat{G}$ that is a multiplier on $M^a(G)_s$. Then Φ is necessarily continuous on $\{\gamma\in\hat{G}: \psi(\gamma)>0\}$. In fact, for $\gamma\in G$ with $\psi(\gamma)>0$, let f be a function in $H^1(R)$ such that $\hat{f}(\psi(\gamma))\neq 0$. Put $\mu=\phi(f)$, where ϕ is the dual homomorphism of ψ. Then μ belongs to $M^a(G)_s$ (see Proposition 2.2 and Remark 2.3 in [17]). Hence we can verify that Φ is continuous at γ. Moreover, if $\ker(\psi)$ is open and noncompact, Φ is also continuous on $\ker(\psi)$. This is obtained from the fact that $\chi_{\ker(\psi)}\in M^a(G)_s{}^\wedge$.

REMARK 0.5. (i) Suppose $M^a(G)_s\neq\{0\}$ and Φ is an idempotent multiplier on $M^a(G)_s$. Then, for $\gamma\in\hat{G}$ with $\psi(\gamma)>0$, $\Phi(\gamma)=0$ or 1. In fact, let f be a function in $H^1(R)$ with $\hat{f}(\psi(\gamma))\neq 0$. Then, as same as in the previous remark, $\mu=\phi(f)$ belongs to $M^a(G)_s$. Hence $\Phi(\gamma)^2\hat{\mu}(\gamma)=\Phi(\gamma)\hat{\mu}(\gamma)$, which yields $\Phi(\gamma)=0$ or 1 since $\hat{\mu}(\gamma)=\hat{f}(\psi(\gamma))\neq 0$. Moreover, if $\ker(\psi)$ is open and noncompact, $\Phi(\gamma)=0$ or 1 on $\ker(\Phi)$ because $\Phi|_{\ker(\psi)}$ becomes an idempotent multiplier on $M_s(G/\ker(\psi)^\perp)$. However, if $\ker(\psi)$ is not open, or open and compact, Φ does not need assuming the values 0 or 1 on $\ker(\psi)$ since $\hat{\mu}=0$ on $\ker(\psi)$ for each $\mu\in M^a(G)_s$ (cf. [18], Lemma 1.2).

(ii) When $\ker(\psi)$ is not open, let ν be a measure in $\mathscr{A}_0$ such that $\hat{\nu}=0$ or 1 no $\psi^{-1}((0,\infty))$. Then $\hat{\nu}$ becomes an idempotent multiplier on $M^a(G)_s$. When $\ker(\psi)$ is open and compact, let ν be a measure in $\mathscr{A}_1$ such that $\hat{\nu}=0$ or 1 on $\psi^{-1}((0,\infty))$. Then $\hat{\nu}$ becomes an idempotent multiplier on $M^a(G)_s$. When $\ker(\psi)$ is open and noncompact, let ν be a measure in $\mathscr{A}_1$ such that $\hat{\nu}=0$ or 1 on $\psi^{-1}([0,\infty))$. Then $\hat{\nu}$ becomes an idempotent multiplier on $M^a(G)_s$.

The following theorem, which will be frequently used later on, is obtained from ([7], Theorems 1 and 2).

THEOREM 0.6. *Let G be a nondiscrete LCA group. Let μ be a measure in $M_c(G)$. Then there exists a probability measure $\nu \in M_s(G)$ such that $\mu * \nu \in L^1(G)$.*

We get the following theorem from the above theorem.

THEOREM 0.7. *Let G be a nondiscrete LCA group and Φ a function on $\hat{G}$ that is a multiplier on $M_s(G)$. Then $\Phi \in M_d(G)^\wedge$.*

When G is compact, Theorem 0.7 was obtained by Doss ([3], Theorem 2).

§1 Disintegration of measures.

This section is devoted to the disintegration of measures on the product space $X_1 \times X_2$ of a metrizable locally compact Hausdorff space X_1 and a locally compact Hausdorff space X_2. We employ Edgar's idea ([5]). For a locally compact Hausdorff space X, let $C_0(X)$ be the Banach space of continuous functions on X which vanish at infinity and $M(X)$ the Banach space of complex-valued bounded regular measures on X with the total variation norm. Then, by the Riesz representation theorem, the dual space of $C_0(X)$ coincides with $M(X)$. Let $(M(X), \mathrm{w}^*)$ be the space $M(X)$ with weak*-topology. Then $(M(X), \mathrm{w}^*)$ becomes a locally convex topological vector space, and its dual space is $C_0(X)$. We denote by $\mathscr{B}(X)$ the σ-algebra of Borel sets in X.

DEFINITION 1.1. Let V be a locally convex topological vector space, and let $(S, \mathscr{F}, \lambda)$ be a finite measure space.

(a) A function $m : \mathscr{F} \to V$ is called a V-valued measure if, for every sequence $\{E_n\}$ of disjoint members in $\mathscr{F}$, we have $m(\bigcup_1^\infty E_n) = \sum_{n=1}^\infty m(E_n)$, where the series converges in the topology of the space V.

(b) The measure m is said to be absolutely continuous with respect to λ if $m(E)=0$ for each $E \in \mathscr{F}$ with $\lambda(E)=0$.

(c) The average range of m is the set $\{m(E)/\lambda(E) : E \in \mathscr{F}, \lambda(E)>0\}$. The measure m is said to have relatively compact average range if the average range of m is relatively compact.

The following lemma is due to Edgar.

LEMMA 1.2 ([5], 2.1 Theorem). *Let V be a locally convex topological vector space, and let $(S, \mathscr{F}, \lambda)$ be a finite measure space. Let $m : \mathscr{F} \to V$ be a measure. Assume (1) m is absolutely continuous with respect to λ, and (2) m has relatively compact average range. Then there exists a function ϕ: $S \to V$ with range contained in the closure of the average range of m such that, for $E \in \mathscr{F}$, we have $\int_E \phi(s)\, d\lambda(s) = m(E)$, i. e., for all continuous* *linear functionals h on V, the map* $s \to h(\phi(s))$ *is* λ*-integrable, and* $\int_E h(\phi(s))d\lambda(s) = h(m(E))$.

In general, for locally compact Hausdorff spaces X_1 and X_2, $\mathscr{B}(X_1)\times\mathscr{B}(X_2)$ is included in $\mathscr{B}(X_1\times X_2)$. However, if X_1 is σ-compact, metrizable locally compact space, the following lemma is satisfied.

LEMMA 1. 3. *Let* X_1 *be a* σ*-compact, metrizable locally compact space and* X_2 *a locally compact Hausdorff space. Then* $\mathscr{B}(X_1)\times(\mathscr{B}(X_2) = \mathscr{B}(X_1\times X_2)$.

PROPOSITION 1. 4. *Let* X_1 *be a* σ*-compact, metrizable locally compact space and* X_2 *a locally compact Hausdorff space. Let* $\pi_{X_2} : X_1\times X_2 \to X_2$ *be the projection. Let* μ *be a positive measure in* $M(X_1\times X_2)$*, and put* $\eta = \pi_{X_2}(\mu)$*. Then there exists a family* $\{\lambda_s\}_{s\in X_2}$ *of positive measures in* $M(X_1\times X_2)$ *with the following properties :*

(1) $s \to \lambda_s(f)$ *is a* η*-measurable function for each bounded Borel function f on* $X_1\times X_2$ *;*

(2) $\|\lambda_s\| \leq 1$ *;*

(3) $supp(\lambda_s) \subset X_1\times\{s\}$ *;*

(4) $\mu(f) = \int_{X_2}\lambda_s(f)d\eta(s)$ *for each bounded Borel measurable function f on* $X_1\times X_2$.

PROOF. Put $V = (M(X_1), \mathrm{w}^*)$, and we define a set function $m : \mathscr{B}(X_2) \to V$ by $m(E)(B) = \mu(B\times E)$ for $B\in\mathscr{B}(X_1)$ and $E\in\mathscr{B}(X_2)$. Then m is a V-valued measure, and

(5) m is absolutely continuous with respect to η.

For $E\in\mathscr{B}(X_2)$ with $\eta(E)>0$, we have $\|m(E)/\eta(E)\| = 1$, hence the average range of m is included in the unit ball of $M^+(X_1)$. Since the unti ball of $M(X_1)$ is weak*-compact, m has relatively compact average range. Hence, by Lemma 1. 2, there exists a function $\phi : X_2 \to V$ with the range contained in the closure of the average range of m such that

(6) $s \to \int_{X_1} h(x)d\phi(s)(x)$ is η-integrable for each $h\in C_0(X_1)$, and

(7) $\int_E\int_{X_1} h(x)d\phi(s)(x)d\eta(s) = \int_{X_1} h(x)dm(E)(x)$ for $h\in C_0(X_1)$ and $E\in\mathscr{B}(X_2)$.

Since the average range of m is contained in the unit ball of $M^+(X_1)$, we have

(8) $\phi(X_2) \subset \{\omega\in M^+(X_1) : \|\omega\| \leq 1\}$,

hence we may replace (6) by

(6)′ $s \to \int_{X_1} h(x)d\phi(s)(x)$ is a bounded η-measurable function for each $h\in C_0(X_1)$.

Since X_1 metrizable and σ-compact, the usual montonicity arguments show that (6)$'$ and (7) continue to hold for bounded Borel functions. That is,

(9) $s \to \int_{X_1} h(x) d\phi(s)(x)$ is a bounded η-measurable function for each bounded Borel function h on X_1

and

(10) $\int_E \int_{X_1} h(x) d\phi(s)(x) d\eta(s) = \int_{X_1} h(x) dm(E)(x)$ for each bounded Borel h on X_1 and $E \in \mathscr{B}(X_2)$.

We define positive measures $\lambda_s \in M(X_1 \times X_2)$ $(s \in X_2)$ by

$$d\lambda_s(x, y) = d\phi(s)(x) \times d\delta_s(y)$$

Then by (8) we have

(11) $\|\lambda_s\| \leq 1$;

(12) $\mathrm{supp}(\lambda_s) \subset X_1 \times \{s\}$.

Moreover, by the similar argument in the proof of ([5], Theorem 3.1), the following is satisfied:

(13) $s \to \lambda_s(F)$ is η-measurable and $\mu(F) = \int_{X_2} \lambda_s(F) d\eta(s)$

for $F \in \mathscr{B}(X_1) \times \mathscr{B}(X_2)$. Hence by (11)-(13) and Lemma 1.3, we can verify that $\{\lambda_s\}_{s \in X_2}$ satisfies (1)-(4). This completes the proof.

COROLLARY 1.5. *Let X_1 be a metrizable locally compact space and X_2 a locally compact Havsdorff space. Let μ be a positive measure in $M(X_1 \times X_2)$, and put $\eta = \pi_{X_2}(\mu)$, where $\pi_{X_2} : X_1 \times X_2 \to X_2$ is the projection. Then there exists a family $\{\lambda_s\}_{s \in X_2}$ of positive measures in $M(X_1 \times X_2)$ which satisfies (1)-(4) in Proposition* 1.4.

PROOF. Since μ is bounded and regular, there exists a σ-compact subset Y_1 of X_1 such that μ is concentrated on $Y_1 \times X_2$. Hence, by Proposition 1.4, the corollary is easily obtained.

COROLLARY 1.6. *Let X_1, X_2 and π_{X_2} be as in the previous corollary. For $\mu \in M(X_1 \times X_2)$, put $\eta = \pi_{X_2}(|\mu|)$. Then there exists a family $\{\lambda_s\}_{s \in X_2}$ of measures in $M(X_1 \times X_2)$ with the following properties :*

(1) *$s \to \lambda_s(f)$ is η-measurable for each bounded Borel measurable function f on $X_1 \times X_2$;*

(2) $\|\lambda_s\| \leq 1$;

(3) *$supp(\lambda_s) \subset X_1 \times \{s\}$;*

(4) *$\mu(f) = \int_{X_2} \lambda_s(f) d\eta(s)$ for each bounded Borel function f on $X_1 \times X_2$.*

PROOF. By Corollary 1.5, there exists a family $\{\lambda'_s\}_{s \in X_2}$ of positive measures in $M(X_1 \times X_2)$ which satisfies (1)-(4) in Proposition 1.4 with respect to $|\mu|$. Let g be a unimodular Borel function on $X_1 \times X_2$ such that

$\mu=g|\mu|$. We define measures $\lambda_s\in M(X_1\times X_2)$ by $d\lambda_s(x,y)=g(x,y)d\lambda'_s(x,y)$. Then we can verify that $\{\lambda_s\}_{s\in X_2}$ satisfies (1)-(4) of the corollary, and the proof is complete.

PROPOSITION 1.7. *Let X_1 be a σ-compact, metrizable locally compact space and X_2 a locally compact Hausdorff space. Let η be a nonzero positive measure in $M(X_2)$, and let $\{\lambda_s^1\}_{s\in X_2}$ and $\{\lambda_s^2\}_{s\in X_2}$ be families of measures in $M(X_1\times X_2)$ with the following properties:*

(1) *$s\to\lambda_s^i(f)$ are η-integrable functions for each bounded Borel function f on $X_1\times X_2$ $(i=1,2)$;*

(2) *$supp(\lambda_s^i)\subset X_1\times\{s\}$ $(i=1,2)$;*

(3) $\int_{X_2}\lambda_s^1(f)d\eta(s)=\int_{X_2}\lambda_s^2(f)d\eta(s)$ *for $f\in C_o(X_1\times X_2)$.*

Then we have

$$\lambda_s^1=\lambda_s^2\quad \eta-a.\ a.\ s\in X_2.$$

PROOF. Since X_1 is σ-compact and metrizable, $C_0(X_1)$ contains a countable dense set $\mathscr{A}=\{f_n\}_1^\infty$. By (2), there exist measures $\nu_s^i\in M(X_1)$ such that

$$d\lambda_s^i(x,y)=d\nu_s^i(x)\times d\delta_s(y).$$

Then it follows from (1) that

(4) $s\to\nu_s^i(h)$ is a η-integrable function for each bounded Borel function h *on* X_1.

For $f_n\in\mathscr{A}$, we have by (3)

$$\int_{X_2}g(s)\nu_s^1(f_n)d\eta(s)=\int_{X_2}g(s)\nu_s^2(f_n)d\eta(s)$$

for all $g\in C_0(X_2)$, hence (4) yields

$$\nu_s^1(f_n)=\nu_s^2(f_n)\quad \eta-\text{a. a. } s\in X_2.$$

Since $\mathscr{A}$ is a countable dense set in $C_0(X_1)$, we have

$$\nu_s^1=\nu_s^2\quad \eta-\text{a. a. } s\in X_2,$$

which shows $\lambda_s^1=\lambda_s^2$ $\eta-$a. a. $s\in X_2$. This completes the proof.

§ 2 Convolution operators on $M^a(G)_s$ induced by bounded regular measures on G.

In this section we first state several properties of measures on the direct product group of a σ-compact metrizable LCA group and a general LCA group by using the theory of disintegration (Lemmas 2.1–2.3). Next we characterize convolution operators on $M^a(G)_s$ induced by bounded regular measures on G. From Lemma 2.1 through Lemma 2.3, we shall assume that G_1 is a σ-compact metrizable LCA group, G_2 is a general LCA group and π_{G_2}: $G_1\oplus G_2\to G_2$ is the projection.

LEMMA 2.1. *Let E_1 be a closed set in $\hat{G}_1$. For $\mu\in M_{E_1\times\hat{G}_2}(G_1\oplus G_2)$, put*

$\eta=\pi_{G_2}(|\mu|)$. *Let* $\{\lambda_S\}_{S\in X_2}$ *be the family of measures in* $M(G_1\oplus G_2)$ *which satisfies* (1)-(4) *in Corollary* 1.6. *Put* $d\lambda_h(x, y)=d\nu_h(x)\times d\delta_h(y)$, *where* $\nu_h\in M(G_1)$ *and* δ_h *is the point mass at* $h\in G_2$. *Then we have* $\nu_h\in M_{E_1}(G_1)$ η*-a. a.* $h\in G_2$.

PROOF. Since G_1 is σ-compact and metrizable, there exists a countable dense set $\mathscr{A}=\{f_m\}_1^\infty$ in $C_0(G_1)$. Let F_m be a function on $G_1\oplus G_2$ such that $F_m(x, y)=f_m(x)$. Then, since $h\to\nu_h(f_m)=\lambda_h(F_m)$ is η-measurable, it follows from Lusin's theorem that there exists a compact set K_m in G_2 such that

(1) $\eta(G_2\backslash K_m)<\dfrac{1}{m}$

and

(2) $h\to\nu_h(f_n)$ is continuous on K_m for all $f_n\in\mathscr{A}$.

Since η is regular, we may assume that K_m satisfies

(3) $\eta(V\cap K_m)>0$

for each $x\in K_m$ and neighborhood V of x. In fact, put

$$F=\left\{x\in K_m:\ \begin{array}{l}\text{there exists an open set } V_x \text{ containing } x \text{ such that}\\ \eta(V_x\cap K_m)=0.\end{array}\right\}.$$

Then $F=K_m\cap\bigcup\{V:\text{open set in } G_2',\ V\cap K_m\neq\emptyset,\ \eta(V\cap K_m)=0\}$. By regularity of η, for $\varepsilon>0$, there exists a compact set S contained in F such that $\eta(F\backslash S)<\varepsilon$. Hence there exist open sets $V_1,\cdots, V_l$ in G_2 with $V_i\cap K_m\neq\emptyset$, $\eta(V_i\cap K_m)=0$ such that $S\subset\bigcup_{i=1}^{l}V_i$. Then $\eta(S)\leq\sum_{i=1}^{l}\eta(V_i\cap K_m)=0$, so that $\eta(F)=\eta(F\backslash S)+\eta(S)<\varepsilon$. Since $\varepsilon>0$ is arbitrary, we have $\eta(F)=0$. We employ $K_m\backslash F$ instead of K_m if necessary. Then $K_m\backslash F$ is compact and it satisfies (1)-(3).

By (2), (3) and the method used in ([16], Lemma 3, Claim 1), we have

(4) $\nu_h\in M_{E_1}(G_1)$ for all $h\in K_m$.

Hence, by (1) and (4), we have $\nu_h\in M_{E_1}(G_1)$ η-a. a. $h\in G_2$ and the proof is complete.

LEMMA 2.2. *Under the assumption in the previous lemma, the following are satisfied*:

(I) *If* $\eta\in L^1(G_2)$ *and* $\nu_h\in L^1(G_1)$ η*-a. a.* $h\in G_2$, *then* $\mu\in L^1(G_1\oplus G_2)$;

(II) *If* $\mu\in M_S(G_1\oplus G_2)$ *and* $\nu_h\in L^1(G_1)$ η*-a. a.* $h\in G_2$, *the* $\eta\in M_S(G_2)$.

PROOF. (I): Let E be a compact set in $G_1\oplus G_2$ with $m_{G_1\oplus G_2}(E)=0$. Then there exists a Borel set A_2 in G_2 with $m_{G_2}(A_2)=0$ such that

$m_{G_1}(E_y)=0$ for $y\in A_2$,

where $E_y=\{x\in G_1:(x, y)\in E\}$. Then we have

$$\begin{aligned}\mu(E)&=\int_{G_2}\lambda_h(E)d\eta(h)\\&=\int_{G_2}\nu_h(E_h)d\eta(h)\\&=\int_{A_2}\nu_h(E_3)d\eta(h)+\int_{G_2\setminus A_2}\nu_h(E_h)d\eta(h)\\&=0.\end{aligned}$$

Hence, by regularity of μ, we get $\mu(F)=0$ for all Borel sets F in $G_1\oplus G_2$ with $m_{G_1\oplus G_2}(F)=0$, which shows $\mu\in L^1(G_1\oplus G_2)$.

(II): Let $\eta=\eta_a+\eta_s$ be the Lebesgue's decomposition of η with respect to . We define measures μ_1, $\mu_2\in M(G_1\oplus G_2)$ by

$$\mu_1(f)=\int_{G_2}\lambda_h(f)d\eta_a(h),$$

$$\mu_2(f)=\int_{G_2}\lambda_h(f)d\eta_s(h)$$

for $f\in C_0(G_1\oplus G_2)$. Then $\mu=\mu_1+\mu_2$ and $\mu_2\in M_s(G_1\oplus G_2)$. Moreover we have $\mu_1\in L^1(G_1\oplus G_2)$ as same as in (I), hence $\mu=\mu_2$. Therefore, since $\|\lambda_h\|\leq 1$ η-a. a. $h\in G_2$, we get

$$\begin{aligned}\|\eta_a\|+\|\eta_s\|&=\|\eta\|\\&\leq\|\eta_s\|,\end{aligned}$$

which shows $\eta=\eta_s\in M_s(G_2)$. This completes the proof.

LEMMA 2.3. *Let μ_i be measures in $M(G_1\oplus G_2)$, and put $\eta_i=\pi_{G_2}(|\mu_i|)$ $(i=1, 2)$. Let $\{\lambda_h^i\}_{h\in G_2}=\{\nu_h^i\times\delta_h\}_{h\in G_2}$ be families of measures in $M(G_1\oplus G_2)$ which satisfy (1)-(4) in Corollary 1.6 with respect to μ_i. Then the following is satisfied:*

(I) *$(h_1, h_2)\to\lambda_{h_1}^1*\lambda_{h_2}^2(f)=\{(\nu_{h_1}^1*\nu_{h_2}^2)\times\delta_{h_1+h_2}\}(f)$ is a $(\eta_1\times\eta_2)$-measurable function for each bounded Borel function f on $G_1\oplus G_2$.*

In particular, we can define a measure $\xi\in M(G_1\oplus G_2)$ by

$$\xi(f)=\int_{G_2}\int_{G_2}\{(\nu_{h_1}^1*\nu_{h_2}^2)\times\delta_{h_1+h_2}\}(f)d\eta_1(h_1)d\eta_2(h_2)$$

for $f\in C_0(G_1\oplus G_2)$. Then the following are satisfied:

(II) $\xi=\mu_1*\mu_2$;

(III) *If $\eta_1*\eta_2\in L^1(G_2)$ and $\nu_{h_1}^1*\nu_{h_2}^2\in L^1(G_1)$ $(\eta_1\times\eta_2)$-a. a. $(h_1, h_2)\in G_1\oplus G_2$, then $\mu_1*\mu_2$ belongs to $L^1(G_1\oplus G_2)$.*

PROOF. (I): We note that $h\to\nu_h^i(f)$ is a η_i-measurable function for each bounded Borel function f on G_1. Hence

(1) $(h_1, h_2)\to\nu_{h_1}^1*\nu_{h_2}^2(f)$ is a $(\eta_1\times\eta_2)$-measurable function for each bounded Borel function f on G_1.

In fact, for f_1, $f_2\in C_0(G_1)$, we define a function $f\in C_0(G_1\oplus G_1)$ by $f(x_1, x_2)=f_1(x_1)f_2(x_2)$. Then

(2) $(h_1, h_2)\to(\nu_{h_1}^1\times\nu_{h_2}^2)(f)$ is $(\eta_1\times\eta_2)$-measurable.

Since $\{\sum_{i=1}^{n} f_{1i}(x)f_{2i}(y) : f_{1i}, f_{2i} \in C_0(G_1)\}$ is dense in $C_0(G_1 \oplus G_1)$, (2) holds for all $f \in C_0(G_1 \oplus G_1)$. Hence, since G_1 is σ-compact and metrizable, (2) holds for every bounded Borel function f on $G_1 \oplus G_1$. Let $\tau_1(x, y) = x+y$ for $(x, y) \in G_1 \oplus G_1$. Then, for each bounded Borel function g on G_1, we have $\nu^1_{h_1} * \nu^2_{h_2}(g) = (\nu^1_{h_1} \times \nu^2_{h_2})(g \circ \tau_1)$ and (1) follows.

Let $E \in \mathscr{B}(G_1)$ and $F \in \mathscr{B}(G_2)$, and put $A = E \times F$. Then

(3) $(h_1, h_2) \to \{(\nu^1_{h_1} * \nu^2_{h_2}) \times \delta_{h_1+h_2}\}(A)$ is $(\eta_1 \times \eta_2)$-measurable.

In fact, let $\tau_2(u, v) = u+v$ for $(u, v) \in G_2 \oplus G_2$. Then $\{(\nu^1_{h_1} * \nu^2_{h_2}) \times \delta_{h_1+h_2}\}(A) = \nu^1_{h_1} * \nu^2_{h_2}(E)\chi_{\tau_2^{-1}(F)}(h_1, h_2)$, hence (3) in obtained from (1).

Now let $\mathscr{F}$ be the collection of all subsets A of $G_1 \oplus G_2$ which satisfy (3). Then by (3) we have

(4) $\bigcup_{i=1}^{n} E_i \times F_i \in \mathscr{F}$

for $E_i \in \mathscr{B}(G_1)$ and $F_i \in \mathscr{B}(G_2)$ such that $\{E_i \times F_i\}$ are pairwise disjoint. Moreover $\mathscr{F}$ is closed under monotone limits, hence it follows from Lemma 1.3 that $\mathscr{F} \supset \mathscr{B}(G_1 \oplus G_2)$. Thus (I) is satisfied.

(II) is obtained by considering their Fourier-Stieltjes transforms (cf, [16], Theorem 1, Claim 3). We can prove (III) as same as in the proofs of Lemma 2.2 (I) and ([16], lines 1–13 on p. 275). This completes the proof.

Let G be a LCA group and $\psi : \hat{G} \to R$ a nontrivial continuous homomorphism. We assume that there exists $\chi_0 \in \hat{G}$ such that $\psi(\chi_0) = 1$ for convenience' sake. Let $\phi : R \to G$ be the dual homomorphism of ψ, and let Λ be the discrete subgroup of $\hat{G}$ generated by χ_0. Let K be the annihilator of Λ. We define a continuous homomorphism $\alpha : R \oplus K \to G$ by

(2.1) $\alpha(t, u) = \phi(t) + u$ for $(t, u) \in R \oplus K$.

Then α is onto and satisfies the following (cf. [17], Proposition 2.2):

(2.2) $\alpha(L_1(R \oplus K)) \subset L^1(G)$;

(2.3) $\alpha(M_s(R \oplus K)) \subset M_s(G)$.

For $0 < \varepsilon < \frac{1}{6}$, let Δ_ε be a function on $R \oplus K$ defined by $\Delta_\varepsilon(t, \sigma) = \max(1 - \frac{1}{\varepsilon}|t|, 0)$ if $\sigma = 0$ and $\Delta_\varepsilon(t, \sigma) = 0$ if $\sigma \neq 0$. For $\mu \in M(G)$, we define a function $\Phi^\varepsilon_\mu(t, \sigma)$ on $R \oplus K$ by

(2.4)(*) $\Phi^\varepsilon_\mu(t, \sigma) = \sum_{\gamma \in \hat{G}} \hat{\mu}(\gamma) \Delta_\varepsilon((t, \sigma) - (\psi(\gamma), \gamma|_K))$.

Then the following are satisfied (cf. [19], Lemma 3.3):

(2.5) $\Phi^\varepsilon_\mu \in M(R \oplus K)^\wedge$ and $\|\Phi^\varepsilon_\mu\| = \|\mu\|$;

(*) In [17], we define $\Phi^\varepsilon_\mu (= \Phi_\mu)$ by $\Phi^\varepsilon_\mu(t, \sigma) = \sum_{\gamma \in \hat{G}} \hat{\mu}(\gamma) \Delta^2_\varepsilon((t, \sigma) - (\psi(\gamma), \gamma|_K))$. However we may take Δ_ε instead of Δ^2_ε (cf. [19], Lemma 3.3).

(2.6) $\Phi_\mu^\varepsilon \in L_1(R\oplus K)^\wedge$ if $\mu\in L^1(G)$;

(2.7) $\Phi_\mu^\varepsilon \in M_s(R\oplus K)^\wedge$ if $\mu\in M_s(G)$.

Moreover by ([19], Lemma 4.1) we have

(2.8) $\alpha((\Phi_\mu^\varepsilon)^\vee)=\mu$.

We define an isometry(**) $T_\psi^\varepsilon : M(G)\to M(R\oplus K)$ by

(2.9) $T_\psi^\varepsilon(\mu)^\wedge=\Phi_\mu^\varepsilon$.

DEFINITION 2.4. Let $\psi:\hat{G}\to R$ be a nontrivial continuous homomorphism and $\phi : R\to G$ the dual homomorphism of ψ. Let H_ψ be the annihilator of $\ker(\psi)$.

(i) Fer $\nu\in M(G)$, we define a bounded linear operator S_ν on $M(G)$ by $S_\nu(\mu)=\nu*\mu$.

(ii) If $\ker(\psi)$ is not open, we define a subspace $\mathscr{B}_0$ of $M(G)$ by
$\mathscr{B}_0=\{\sum_{n=1}^\infty u_n*\delta_{x_n}\}$,
where $x_n\in G$ and $u_n\in\phi(M(R))$ with $\sum_{n=1}^\infty \|u_n\|<\infty$.

(iii) If $\ker(\psi)$ is open, we define a subspace $\mathscr{B}_1$ of $M(G)$ by
$\mathscr{B}_1=\{m_{H_\psi}*\rho+(\delta_0-m_{H_\psi})*(\sum_{n=1}^\infty u_n*\delta_{x_n})\}$,
where $x_n\in G$, $\rho\in M_d(G)$ and $u_n\in\phi(M(R))$ with $\sum_{n=1}^\infty \|u_n\|<\infty$.

Before we state our theorem, we note that K is not discrete if $M^a(G)_s\neq\{0\}$ (cf. [17], Remark 2.3).

THEOREM 2.5. *Let G be a LCA group and $\psi:\hat{G}\to R$ a nontrivial continuous homomorphism. We assume $M^a(G)_s\neq\{0\}$ and there exists $\chi_0\in\hat{G}$ with $\psi(\chi_0)=1$. For $\nu\in M(G)$, suppose $S_\nu(M^a(G)_s)\subset M_a(G)_s$. Then the following are satisfied:*

(I) *If $\ker(\psi)$ is not open, there exists $\xi\in\mathscr{B}_0$ such that*
$S_\nu(\mu)=\mu*\xi$ *for all* $\mu\in M^a(G)_s$;

(II) *If $\ker(\psi)$ is open, there exists $\xi\in\mathscr{B}_1$ such that*
$S_\nu(\mu)=\mu*\xi$ *for all* $\mu\in M^a(G)_s$.

Conversely, when $\ker(\psi)$ is not open, let ν be a measure in $\mathscr{B}_0$. Then S_ν maps $M^a(G)_s$ into itself. When $\ker(\psi)$ is open, let ν be a measure in $\mathscr{B}_1$. Then S_ν also maps $M^a(G)_s$ into itself.

PROOF. Put $\eta=\pi_K(|T_\psi^\varepsilon(\nu)|)$, where $\pi_K: R\oplus K\to K$ is the projection. Then by Corollary 1.6 there exists a family $\{\lambda_h\}_{h\in k}=\{\nu_h\times\delta_h\}_{h\in k}$ of measures in $M(R\oplus K)$ with the following properties:

(1) $h\to\lambda_h(f)$ is η-measurable for each bounded Borel function f on $R\oplus K$;

(**) J. Inoue recently constructed lifting operators with properties (2.5)−(2.7) under a general circumustance ([10]). However, in general, it is not known whether the representation as (2.4) is satisfied or not.

(2) $\|\lambda_h\| \leq 1$;

(3) $T_\psi^\varepsilon(\nu)(f)=\int_K \lambda_h(f)d\eta(h)$ for each bounded Borel function f on $R\oplus K$.

Let $\eta=\eta_1+\eta_2(\eta_1\in M_d(K),\ \eta_2\in M_c(K)$, and we define measures $\omega_i\in M(R\oplus K)(i=1,2)$ by

(4) $\omega_i(f)=\int_K \lambda_h(f)d\eta_i(h)$

for bounded Borel functions f on $R\oplus K$. Then we have

(5) $\alpha(\omega_1)*M^a(G)_s\subset M^a(G)_s$.

In fact, let μ be a measure in $M^a(G)_s$ and put $\eta_\mu=\pi_K(|T_\psi^\varepsilon(\mu)|)$. Let $\{\nu_h^\mu\times\delta_h\}_{h\in k}$ be a family of measures in $M(R\oplus K)$ which satisfies (1)-(3) with respect to $T_\psi^\varepsilon(\mu)$ and η_μ. By (2.4) we note

$T_\psi^\varepsilon(\mu)^\wedge(t,\sigma)=0$ for $t<-\varepsilon$,

hence by Lemma 2.1 and the F. and M. Riesz theorem we have

(6) $\nu_h^\mu\in L^1(R)$ η_μ-a. a. $h\in K$.

Thus (2.7) and Lemma 2.2 (II) yield $\eta_\mu\in M_s(K)$. Hence by Lemma 2.3 (II) and the fact fact that $\eta_1\in M_d(K)$, we can verify that $T_\psi^\varepsilon(\mu)*\omega_1\in M_s(R\oplus K)$. Thus by (2.3) and (2.8) we get $\mu*\alpha(\omega_1)=\alpha(T_\psi^\varepsilon(\mu)*\omega_1)\in M^a(G)_s$, and (5) follows.

Next we claim the following.

(7) If $\alpha(\omega_2)^\wedge(\gamma_0)\neq 0$ for some $\gamma_0\in\hat{G}$ with $\psi(\gamma_0)>0$, there exists a measure $\mu\in M^a(G)_s$ such that $0\neq\alpha(\omega_2)*\mu\in L^1(G)$.

In fact, since $\eta_2\in M_c(K)\cap M^+(K)$, it follows from Theorem 0.6 that there exists a positive measure $\eta_3\in M_s(K)$ such that $0\neq\eta_2*\eta_3\in L^1(K)$. Put $\sigma_0=\gamma_0|_K$, and let g be a function in $H^1(R)$ such that $\hat{g}(\psi(\gamma_0))\neq 0$. We define a measure $\zeta\in M_s(R\oplus K)$ by $\zeta=g\times(\sigma_0\eta_3)$. Then $\alpha(\zeta)\in M^a(G)_s$ and $\alpha(\zeta)^\wedge(\gamma_0)=\hat{\zeta}(\psi(\gamma_0),\ \gamma_0|_K\neq 0$. We note that ζ can be represented as follows:

$$\zeta(f)=\int_K\{\sigma_0(h)g\times\delta_h\}(f)d\eta_3(h)$$

for all bounded Borel functions f on $R\oplus K$. Hence we can verify that $\omega_2*\zeta$ belongs to $L^1(R\oplus K)$(cf. Lemma 2.3 (III)), so that $\alpha(\omega_2)*\alpha(\zeta)\in L^1(G)$. Thus $\mu=\alpha(\zeta)$ is the desired one, and (7) follows. Since $T_\psi^\varepsilon(\nu)=\omega_1+\omega_2$, it follows from (2.8) that $\nu=\alpha(\omega_1)+\alpha(\omega_2)$. Hence, by (5), (7) and the hypothesis that $S_\nu(M^a(G)_s)\subset M^a(G)_s$, we have

(8) $\hat{\nu}(\gamma)=\alpha(\omega_1)^\wedge(\gamma)$ on $\{\gamma\in\hat{G}:\psi(\gamma)>0\}$.

We note $\alpha(\omega_1)\in\mathscr{B}_0$. Thus, if $\ker(\psi)$ is not open, (8) says that (I) is satisfied since $\{\gamma\in\hat{G}:\psi(\gamma)<0\}^-=\{\gamma\in\hat{G}:\psi(\gamma)\leq 0\}$.

Next we consider (II). When $\ker(\psi)$ is compact, $\hat{\mu}$ vanishes on ker

(ψ) for each $\mu\in M^a(G)_s$ (cf. [18], Lemma 1.2). Put $\xi=(\delta_0-m_{H_\psi})*\alpha(\omega_1)$. Then $\xi\in\mathscr{B}_1$, and by (8) we can verify that ξ satisfies (II). If ker(ψ) is not compact, then $M_s(G/H_\psi)\neq\{0\}$ and $\hat{\nu}|_{\ker(\psi)}$ becomes a multiplier on $M_s(G/H_\psi)$. Hence it follows from Theorem 0.7 that there exists $L\in M_d(G/H_\psi)$ such that $\hat{L}(\gamma)=\hat{\nu}|_{\ker(\psi)}(\gamma)$ on $\ker(\psi)$. We note that there exists $\rho\in M_d(G)$ such that $(m_{H_\psi}*\rho)^\wedge(\gamma)=\hat{L}(\gamma)$ for $\gamma\in\ker(\psi)$. Put $\xi=m_{H_\psi}*\rho+(\delta_0-m_{H_\psi})*\alpha(\omega_1)$. Then, in this case, $\xi\in\mathscr{B}_1$ and ξ satisfies (II) by virtue of (8). Thus (II) is obtained. Finally we prove the converse. For $u_n\in\phi(M(R))$, put $u_n=\phi(A_n)(A_n\in M(R))$. Then $\hat{u}_n=\hat{A}_n\circ\psi$, and the converse follows from ([17], Theorem 2.3) and ([18], Lemma 1.2). This completes the proof.

REMARK 2.6. In Theorem 2.5 we can remove the assumption that there exists $\chi_0\in\hat{G}$ such that $\psi(\chi_0)=1$. This is obtained from the following fact: Let $\psi:\hat{G}\to R$ be a nontrivial continuous homomorphism. For $\beta>0$, we define $\psi_\beta:\hat{G}\to R$ by $\psi_\beta(\gamma)=\beta\psi(\gamma)$. Then the dual homomorphism ϕ_β of ψ_β is given by $\phi_\beta(t)=\phi(\beta t)$, and so $\phi_\beta(M(R))=\phi(M(R))$.

REMARK 2.7. The operator T_ψ^ε, which is defined in (2.8), becomes a positive operator (i.e., $T_\psi^\varepsilon(\mu)\geq 0$ if $\mu\geq 0$). This is obtained from ([8], A. 7.1 Theorem (ii), p. 421) and the construction of Φ_μ^ε (see the proof of ([17], Theorem 2.1) or ([19], Lemma 3.3)).

§ 3 Special idempotent multipliers on $\mathbf{M}^a(\mathbf{G})_s$.

Helson and Lowdenslager obtained the following useful result.

LEMMA 3.1 (cf. [14], 8.2.3 Theorem (b), p. 200).
Let G be a compact abelian group such that $\hat{G}$ is ordered. Let μ be a measure in $M_s(G)$ that is of analytic type (i.e., $\hat{\mu}(\gamma)=0$ for $\gamma<0$). Then $\hat{\mu}(0)=0$.

On the other hand, the author extended the above lemma as follows:

LEMMA 3.2 (cf. [18], Lemma 1.2). *Let G be a LCA group and P an open semigroup in $\hat{G}$ such that $P\cup(-P)=\hat{G}$. Let H be the annihilator of $P\cap(-P)$. Then we have $m_H*\{M_P(G)\cap M_s(G)\}\subset M_P(G)\cap M_s(G)$.*

In orther words, Lemma 3.2 claims that the characteristic function of $P\cap(-P)$ becomes a multiplier on $M_P(G)\cap M_S(G)$. In this section, when there is a nontrivial continuous homomorphism ψ from $\hat{G}$ into R, we consider whether characteristic functions of cosets of $\ker(\psi)$ become multipliers on $M^a(G)_s$ or not. We state our result.

THEOREM 3.3. *Let G be a LCA group and $\psi:\hat{G}\to R$ a nontrivial continuous homomorphism. We assume that $\ker(\psi)$ is noncompact and open. Let γ_0 be an element in $\hat{G}$ with $\psi(\gamma_0)<0$. Then $\chi_{\gamma_0+\ker(\psi)}$ is a multiplier on $M^a(G)_s$ if and only if $\psi(\hat{G})$ is isomorphic to the integer group*

Z.

PROOF. Suppose $\psi(\hat{G})$ is isomorphic to Z. Then we have $\hat{G}\cong Z\oplus F$ and $\{\gamma\in\hat{G}: \psi(\gamma)\geq 0\}=\{n, \sigma)\in Z\oplus F : n\geq 0\}$, where $F=\ker(\psi)$. Let ν be a measure in $M^a(G)_s$, and put $\eta=\pi_{\hat{F}}(|\nu|)$, where $\pi_{\hat{F}}$ is the projection from $T\oplus\hat{F}$ onto $\hat{F}$. Then, by Corollary 1. 6, there exists a family $\{\lambda_h\}_{h\in\hat{F}}$ of measures in $M(T\oplus\hat{F})$ with the following properties:

(1) $h\rightarrow\lambda_h(f)$ is a η-measurable function for each bounded Borel function f on $T\oplus\hat{F}$;

(2) $\|\lambda_h\|\leq 1$;

(3) $\mathrm{supp}(\lambda_h)\subset T\times\{h\}$;

(4) $\nu(f)=\int_{\hat{F}}\lambda_h(f)d\eta(h)$ for each bounded Borel measurable function f on $T\oplus\hat{F}$.

By (3) we have

(5) $d\lambda_h(x, y)=d\nu_h(x)\times d\delta_h(y)$,

where $\nu_h\in M(T)$ and δ_h is the point mass at h. Since $\hat{\nu}(n, \sigma)=0$ for $n<0$, it follows from Lemma 2. 1, Lemma 2. 2 and the F. and M. Riesz theorem that

(6) $\eta\in M_s(\hat{F})$.

Put $n_0=\psi(\gamma_0)$. Then we have

$$\begin{aligned}\hat{\nu}(n_0, \sigma)&=\int_{\hat{F}}(\nu_h\times\sigma_h)(e^{in_0\cdot}(-\bullet, \sigma)d\eta(h)\\&=\int_{\hat{F}}\hat{\nu}_h(n_0)(-h, \sigma)d\eta(h)\\&=\int_{\hat{F}}(-h, \sigma)d\eta^{\#}(h),\end{aligned}$$

where $\eta^{\#}$ is a measure in $M_s(\hat{F})$ defined by $d\eta^{\#}(h)=\hat{\nu}_h(n_0)d\eta(h)$. Then $\chi_{\gamma_0+ker(\psi)}(n, \sigma)\hat{\nu}(n, \sigma)=((e^{in_0\cdot}m_T)\times\eta^{\#})^{\wedge}(n, \sigma)$, so that becomes a multiplier on $M^a(G)_s$.

Conversely suppose $\psi(\hat{G})$ is not isomorphic to Z. We may assume that there exists $\chi_0\in\hat{G}$ such that $\psi(\chi_0)=1$ without loss of generality. Let Λ be the discrete subgroup of $\hat{G}$ generated by χ_0, and let K be the annihilator of Λ. Let F_1 be the open subgroup of $\hat{G}$ generated by $\ker(\psi)$ and Λ, and let K_1 be the annihilator of F_1. Then K_1 is an infinite compact subgroup of K, so that m_{K_1} is a continuous measure. Hence it follows from Theorem 0.7 that there exists a positive measure $\xi\in M_s(K)$ such that $0\neq\xi * m_{K_1}\in L^1(K)$. Let g and k be functions in $H^1(R)$ satisfying the following:

(7) $\hat{g}(\psi(\gamma_0))\neq 0$ and $\mathrm{supp}(\hat{g})\subset(\psi(\gamma_0)-\varepsilon, \psi(\gamma_0)+\varepsilon)$;
$\hat{k}(\psi(\gamma_0))=1$ and $\mathrm{supp}(\hat{k})\subset(\psi(\gamma_0)-\varepsilon, \psi(\gamma_0)+\varepsilon)$,

where $0<\varepsilon<\min(\psi(\gamma_0), \frac{1}{6})$. Moreover we define measures μ, $\nu\in M(R\oplus$

K) by

(8) $d\mu(x, y)=g(x)dx\times(y, \gamma_0|_K)d\xi(y)$;

$d\nu(x, y)=k(x)dx\times(y, \gamma_0|_K)dm_{K_1}(y)$.

Then we have $\mu\in M_s(R\oplus K)$ and $\mu*\nu\in L^1(R\oplus K)$. Let $\alpha: R\oplus K\rightarrow G$ be the continuous homomorphism defined in (2.1). Then (2.2) yields $\alpha(\mu)*\alpha(\nu)=\alpha(\mu*\nu)\in L^1(G)$. Since $\alpha(\mu)^\wedge(\gamma_0)\alpha(\nu)^\wedge(\gamma_0)=\hat{\mu}(\psi(\gamma_0), \gamma_0|_K)\neq 0$, we have $\alpha(\mu)*\alpha(\nu)\neq 0$. On the other hand, by (2.2) and the construction of μ, we get $\alpha(\mu)\in M^a(G)_s$. Moreover, noting $\psi^{-1}((\psi(\gamma_0)-\varepsilon, \psi(\gamma_0)+\varepsilon))\cap(\ker(\psi)+\Lambda+\gamma_0)=\gamma_0+\ker(\psi)$, we have $\alpha(\nu)^\wedge=\chi_{\gamma_0+ker(\psi)}$. Thus $\chi_{\gamma_0+ker(\psi)}$ is not a multiplier on $M^a(G)_s$ and the proof is complete.

Corollary 3.4. *Let G be a LCA group and $\psi:\hat{G}\rightarrow R$ a nontrivial continuous homomorphism with $M^a(G)_s\neq\{0\}$. We assume that $ker(\psi)$ is open and $\psi(\hat{G})$ is dense in R. Let γ_0 be an element in $\hat{G}$ with $\psi(\gamma_0)>0$. Then, for each nonzero function Φ on $\hat{G}$ with $supp(\Phi)\subset\gamma_0+ker(\psi)$, Φ does not become a multiplier on $M^a(G)_s$.*

Proof. We consider the corollary by dividing two cases that $\ker(\psi)$ is compact or not.

Case 1. $\ker(\psi)$ is compact.

In this case, the corollary follows from the fact that measures, whose Fourier-Stieltjes transforms have compact supports, belong to $L^1(G)$.

Case 2. $\ker(\psi)$ is not compact.

Suppose Φ is a multiplier on $M^a(G)_s$. Then it is easy to see that

(1) $\Phi\in M^a(G)_s{}^\wedge$.

By Theorem 3.3, there exists $\mu\in M^a(G)_s$ such that $0\neq\hat{\mu}\chi_{\gamma_0+ker(\psi)}\in\{L^1(G)\cap M^a(G)\}^\wedge$ and $\Phi\hat{\mu}\neq 0$. Then by (1) we have

$$\Phi\hat{\mu}=\Phi\hat{\mu}\chi_{\gamma_0+ker(\psi)}$$
$$\in\{L^1(G)\cap M^a(G)\}^\wedge.$$

Hence we have a contradiction, and the proof is complete.

§4 Proof of Main Theorem.

Our purpose in this section is to prove Main Theorem. We denote by $B(\hat{G})$ the set $M(G)^\wedge$.

Definition 4.1 Let G be a LCA group and $\psi:\hat{G}\rightarrow R$ a nontrivial continuous homomorphism. Let Φ and Φ' be functions on $\hat{G}$. We write $\Phi\approx\Phi'$ if $\Phi\hat{\mu}=\Phi'\hat{\mu}$ for all $\mu\in M^a(G)_s$.

The following lemma is due to [20].

Lemma 4.2 (cf. [20], Main Theorem).

Let G be a LCA group and ψ a nontrivial continuous homomorphism from $\hat{G}$ into R. Let Ω be a bounded open interval in R. Suppose $\hat{\mu}\in B(\hat{G})$ is *integer-valued on* $\psi^{-1}(\Omega)$. *Then there exists an integer-valued* $\hat{\nu}\in B(\hat{G})$ *such that* $\hat{\nu}(\gamma)=\hat{\mu}(\gamma)$ *on* $\psi^{-1}(\Omega)$.

LEMMA 4.3. *Let G be a LCA group, and let* $\phi: R\rightarrow G$ *be an one-to-one, continuous homomorphism. Let F be a compact subgroup of G contained in* $\phi(R)$. *Then* $F=\{0\}$.

PROOF. Put $\Gamma=\phi^{-1}(F)$. Then Γ is a closed subgroup of R, so that $\Gamma\cong\{0\}$, Z or R. By ([9], (6.29) Theorem, p. 42), $\phi|_{\Gamma}:\Gamma\rightarrow F$ is an open continuous homomorphism, so that $\phi|_{\Gamma}$ is a homeomorphism. Since F is compact, we have $F=\{0\}$ and the proof is complete.

LEMMA 4.4. *Let G be a LCA group and* $\psi:\hat{G}\rightarrow R$ *a nontrivial continuous homomorphism. We assume that* $\psi(\hat{G})$ *is dense in R. Let* $\phi: R\rightarrow G$ *be the dual homomorphism of* ψ *and H a compact subgroup of G. Suppose*

$$m_H=\sum_{k=1}^{\infty}\phi(\nu_k)*\delta_{x_k},$$

where $\nu_k\in M(R)$ *with* $\sum_{k=1}^{\infty}\|\nu_k\|<\infty$ *and* δ_{x_k} *is the point mass at* $x_k\in G$. *Then H is a finite group.*

PROOF. We note that ϕ is one-to-one since $\psi(\hat{G})$ is dense in R. Suppose H is an infinite group. Since m_H is concentrated on $\bigcup_{k=1}^{\infty}(\phi(R)+x_k)$, we have $m_H(\phi(R)+x_i)>0$ for some i. Let y_i be an element in $H\cap(\phi(R)+x_i)$. Then

$$\begin{aligned}0&<m_H(\phi(R)+x_i)\\&=m_H((\phi(R)+x_i)\cap H-y_i)\\&=m_H(\phi(R)\cap H),\end{aligned}$$

which shows that $\phi(R)\cap H$ is an open subgroup of H. In particular, $\phi(R)\cap H$ is a compact subgroup of G, hence it follows from Lemma 4.3 that $\phi(R)\cap H=\{0\}$. Thus we have

(1) $(\phi(R)+x_k)\cap H$ is a single point if $(\phi(R)+x_k)\cap H\neq\{0\}$.

Hence by (1) we have $m_H(\bigcup_{k=1}^{\infty}(\phi(R)+x_k))=0$, which yields a contradiction. This completes the proof.

LEMMA 4.5. *Let G be a LCA group and* $\psi:\hat{G}\rightarrow R$ *a nontrivial continuous homomorphirm such that* $\psi(\hat{G})$ *is dense in R. We assume that there exists* $\chi_o\in\hat{G}$ *with* $\psi(\chi_o)=1$, *and let K be the annihilator of the discrete subgroup of* $\hat{G}$ *generated by* χ_o. *Then for infinite compact subgroups H of G,* $\pi_K(T_\psi^\varepsilon(m_H))$ *belong to* $M_c(K)\cap M^+(K)$, *where* $\pi_K: R\oplus K\rightarrow K$ *is the projection and* $T_\psi^\varepsilon: M(G)\rightarrow M(R\oplus K)$ *is the operator defined in* (2.9).

PROOF. We first note that $T_\psi^\varepsilon(m_H)$ is a positive measure (cf. Remark 2.7). Put $\eta=\pi_K(T_\psi^\varepsilon(m_H))$ and $\eta=\eta_d+\eta_c$, where $\eta_d\in M_d(K)$ and $\eta_c\in M_c$

(K). Suppose $\eta_d \neq 0$. Then

$$\eta_d = \sum_{n=1}^{\infty} a_n \delta_{y_n},$$

where $a_n \geq 0$, $0 < \sum_{n=1}^{\infty} a_n < \infty$ and $y_n \in K$. By Proposition 1. 4, there exists a family $\{\lambda_h\}_{h\in K} = \{\nu_h \times \delta_h\}_{h\in K}$ of measures in $M^+(R\oplus K)$ with the following properties:

(1) $h \to \lambda_h(f)$ is a η-measurable function for each bounded Borel function f on $R \oplus K$;

(2) $\|\lambda_h\| \leq 1$;

(3) $T_\psi^\varepsilon(m_H)(f) = \int_K \lambda_h(f)\, d\eta(h)$ for each bounded Borel function f on $R \oplus K$.

Since $\|\lambda_h\| = 1$ η-a. a. $h \in K$, we note

(4) $\nu_{y_n} \times \delta_{y_n} \neq 0$ if $a_n > 0$.

Let $\alpha : R \oplus K \to G$ be the homomorphism defined in (2. 1). Then, since $\alpha(T_\psi^\varepsilon(m_H)) = m_H$, we have

(5) $m_H = \sum_{n=1}^{\infty} a_n \phi(\nu_{y_n}) * \delta_{y_n} + \alpha(\int_K \nu_h \times \delta_h d\eta_c(h))$,

where $\int_K \nu_h \times \delta_h d\eta_c(h)$ is the measure defined by $(\int_K \nu_h \times \delta_h d\eta_c(h))(f) = \int_K (\nu_h \times \delta_h)(f)\, d\eta_c(h)$ for bounded Borel functions f on $R \oplus K$. Since $\sum_{n=1}^{\infty} a_n \phi(\nu_{y_n}) * \delta_{y_n} \neq 0$, it follows from (5) that there exists a natural number n such that $a_n > 0$ and $(\phi(R) + y_n) \cap H \neq 0$. Then, since m_H is H-invariant, we have

$$\begin{aligned} m_H(\phi(R) \cap H) &= m_H((\phi(R) + y_n) \cap H) \\ &= m_H(\phi(R) + y_n) \\ &\geq a_n \phi(\nu_{y_n}) * \delta_{y_n}(\phi(R) + y_n) \\ &= a_n \nu_{y_n}(R) \\ &> 0. \qquad \text{(by (4))} \end{aligned}$$

Hence $\phi(R) \cap H$ is an open subgroup of H. Since $\alpha(R \oplus K) = G$ and H is compact, there exist $x_1, \cdots, x_l \in K$ such that $H \subset \bigcup_{m=1}^{l} (\phi(R) + x_m)$. Then, noting $\ker(\alpha) = \{(2\pi n, -\phi(2\pi n)) : n \in Z\}$, we have

$$\begin{aligned} \alpha(\int_K \nu_h \times \delta_h d\eta_c(h))(H) &= \int_K (\nu_h \times \delta_h)(\alpha^{-1}(H))\, d\eta_c(h) \\ &\leq \int_K (\nu_h \times \delta_h)(\alpha^{-1}(\bigcup_{m=1}^{l} (\phi(R) + x_m)))\, d\eta_c(h) \\ &\leq \sum_{n\in Z} \int_K (\nu_h \times \delta_h)(\bigcup_{m=1}^{l} R \times \{x_m\} + (2\pi n, -\phi(2\pi n)))\, d\eta_c(h) \\ &= 0. \qquad (\eta_c \in M_c(K)) \end{aligned}$$

Hence we have $m_H = \sum_{n=1}^{\infty} a_n \phi(\nu_{y_n}) * \delta_{y_n}$. Then it follows from Lemma 4. 4 that H is finite, which contradicts the hypothesis. Hence we have $\eta_d = 0$ and the proof is complete.

The following lemma is due to [19].

LEMMA 4.6 (cf. [19], Proposition 2.7, p. 149).
Let G be a LCA group and $\psi:\hat{G}\rightarrow R$ a nontrivial continuous homomorphism. We assume that there exists $\chi_0\in\hat{G}$ with $\psi(\chi_0)=1$, and let Λ be the discrete subgroup of $\hat{G}$ generated by χ_0. Put $K=\Lambda^{\perp}$, and we define a homomorphism $\tau:\hat{G}\rightarrow R\oplus\hat{K}$ by $\tau(\gamma)=(\psi(\gamma),\gamma|_K)$. Then $\tau(\hat{G})$ is a closed subgroup of $R\oplus\hat{K}$ and $\tau:\hat{G}\rightarrow\tau(\hat{G})$ is a topological isomorphism.

The following lemma is due to Svensson.

LEMMA 4.7 (cf. [15], Lemma 3.1.1, p. 124).
Let $a_1,\cdots,a_m\in C$ (complex numbers), and let $\Delta_1,\cdots,\Delta_m$ be cosets of subgroups of $R^n\oplus\Gamma$, where Γ is a LCA group. Let Ω be a nonempty open subset of R^n. Suppose

(1) $CLP(\Delta_i)=R^n (1\leq i\leq m)$;

(2) $\sum_{i=1}^m a_i\chi_{\Delta i}(\gamma)=0$ *for all* $\gamma\in\Omega\times\Gamma$,

where $CLP(\Delta_i)$ means the closure of the projection of Δ_i into R^n. Then $\sum_{i=1}^m a_i\chi_{\Delta_i}(\gamma)=0$ for all $\gamma\in R^n\times\Gamma$.

LEMMA 4.8. *Let G be a LCA group and $\psi:\hat{G}\rightarrow R$ a nontrivial continuous homomorphism such that $\psi(\hat{G})$ is dense in R. Let $a_i\in C$, and let H_i be compact subgroups of G such that $\psi(H_i^{\perp})$ are dense in R $(1\leq i\leq n)$. Let Ω be a nonempty open set in R. For $\gamma_i\in\hat{G}$ $(1\leq i\leq n)$, suppose*

(1) $\{\sum_{i=1}^n a_i\gamma_i m_{H_i}\}^{\wedge}(\gamma)=0$ *on* $\psi^{-1}(\Omega)$.

Then $\sum_{i=1}^n a_i\gamma_i m_{H_i}=0$.

PROOF. For $\beta>0$, let $\psi_\beta:\hat{G}\rightarrow R$ be a continuous homomorphism defined by $\psi_\beta(\gamma)=\beta\psi(\gamma)$. By considering ψ_β instead of ψ if necessary, we may assume that there exists $\chi_0\in\hat{G}$ *with* $\psi(\chi_0)=1$. Let K and τ be as in Lemma 4.6. Then it follows from Lemma 4.6 that $\tau(H_i^{-})$ are closed subgroups of $R\oplus\hat{K}$. Put $\Delta_i=\tau(\mathrm{H}_i^{\perp})+\tau(\gamma_i)$. Then $\mathrm{CLP}(\Delta_i)=R$ since $\psi(H_i^{\perp})$ are dense in R. By (1) we have $\sum_{i=1}^n a_i\chi_{\Delta i}(x,\sigma)=0$ on $\Omega\times\hat{K}$. Hence by Lemma 4.7 we have

$$\sum_{i=1}^n a_i\chi_{\Delta i}(x,\sigma)=0 \text{ for all } (x,\sigma)\in R\oplus\hat{K}.$$

In particular,

$$\sum_{i=1}^n a_i\chi_{\Delta_i}(\psi(\gamma),\gamma|_K)=0 \text{ for all } \gamma\in\hat{G},$$

which shows that

$$\{\sum_{i=1}^n a_i\gamma_i m_{H_i}\}^{\wedge}(\gamma)=\sum_{i=1}^n a_i\chi_{H_i^{\perp}+\gamma_i}(\gamma)=0$$

for all $\gamma\in\hat{G}$. This completes the proof.

LEMMA 4.9. *Let G and ψ be as in the previous lemma. Moreover we assume that $M^a(G)_s\neq\{0\}$. Let Φ be a multiplier on $M^a(G)_s$ that is integer-valued on $\psi^{-1}((0,\infty))$, and let $[a,b)$ be a bounded right-half open interval in R contained in $(0,\infty)$. Suppose $\Phi|_{\psi^{-1}((a,b))}\neq0$. Then we* *have*

$$\Phi(\gamma)=\{\sum_{i=1}^{n} m_i\gamma_i m_H\}^\wedge(\gamma)+\{\sum_{j=1}^{p}\sum_{k=1}^{q_j} M_{jk}\sigma_{jk}m_{L_j}\}^\wedge(\gamma)$$

for all $\gamma\in\psi^{-1}((a, b))$, *where* γ_i, $\sigma_{jk}\in\hat{G}$, m_i, $M_{jk}\in Z$; H *is a finite subgroup of* G *and* L_j *are compact subgroups of* G *such that* $\psi(L_j^{\perp})$ *are not dense in* R.

PROOF. We may assume that there exists $\chi_o\in\hat{G}$ with $\psi(\chi_o)=1$ as in Lemma 4.8. Let K, α and T_ψ^ε be as in section 2. Let f be a function in $H^1(R)$ such that $\hat{f}(x)=1$ on (a, b). We note $\phi(f)$ belongs to $M^a(G)_s$, where $\phi: R\rightarrow G$ is the dual homomorphism of ψ. Hence, by Lemma 4.2, there exists $\nu\in M(G)$ such that

(1) $\hat{\nu}$ is integer-valued on $\hat{G}$

and

(2) $\hat{\nu}(\gamma)=\Phi(\gamma)\hat{f}(\psi(\gamma))=\Phi(\gamma)$ for $\gamma\in\psi^{-1}((a, b))$.

Then by Theorem 0.2 we have

$$\nu=\sum_{i=1}^{m}\sum_{j=1}^{l_i} m_{ij}\gamma_{ij}m_{H_i},$$

where $m_{ij}\in Z$, $\gamma_{ij}\in\hat{G}$ and H_i are compact subgroups of G such that $\{\sum_{j=1}^{l_i} m_{ij}\gamma_{ij}m_{H_i}\}_{i=1}^{m}$ are mutually singular. We define subsets I_1 and I_2 of $\{1, \cdots, m\}$ as follws:

$$I_1=\{i : \psi(H_i^{\perp}) \text{ is dense in } R\} ;$$
$$I_2=\{i : \psi(H_i^{\perp}) \text{ is not dense in } R\}.$$

Then we have

(3) $\nu=\sum_{i\in I_1}\sum_{j=1}^{l_i} m_{ij}\gamma_{ij}m_{H_i}+\sum_{i\in I_2}\sum_{j=1}^{l_i} m_{ij}\gamma_{ij}m_{H_i}$.

We define subsets I_{1I} and I_{1F} of I_1 as follows:

$$I_{1I}=\{i\in I_1 : H_i \text{ is an infinite compact subgroup of } G\} ;$$
$$I_{1F}=\{i\in I_1 : H_i \text{ is a finite subgroup of } G\}.$$

Claim 1. $\alpha(\sum_{i\in I_{1I}}\sum_{j=1}^{l_i} m_{ij}\tau_{ij}T_\psi^\varepsilon(m_{H_i}))=\sum_{i\in I_{1I}}\sum_{j=1}^{l_i} m_{ij}\gamma_{ij}m_{H_i}$,

where $\tau_{ij}\in R\oplus\hat{K}$ such that $\tau_{ij}(x, u)=\exp(i\psi(\gamma_{ij})x)(\gamma_{ij}|_K, u)$.

In fact, it is sufficient to show that $\alpha(\tau_{ij}T_\psi^\varepsilon(m_{H_i}))=\gamma_{ij}m_{H_i}$ since α is linear.

$$\begin{aligned}\alpha(\tau_{ij}T_\psi^\varepsilon(m_{H_i}))^\wedge(\gamma)&=(\tau_{ij}T_\psi^\varepsilon(m_{H_i}))^\wedge(\psi(\gamma), \gamma|_K)\\&=T_\psi^\varepsilon(m_{H_i})^\wedge(\psi(\gamma)-\psi(\gamma_{ij}), \gamma|_K-\gamma_{ij}|_K)\\&=\Phi_{mH_i}^\varepsilon(\psi(\gamma)-\gamma_{ij}, (\gamma-\gamma_{ij})|_K)\\&=m_{H_i}{}^\wedge(\gamma-\gamma_{ij})\\&=(\gamma_{ij}m_{H_i})^\wedge(\gamma).\end{aligned}$$

Thus the claim is obtained.

Claim 2. $\sum_{i\in I_{1I}}\sum_{j=1}^{l_i} m_{ij}\gamma_{ij}m_{H_i}=0$.

In fact, suppose $\sum_{i\in I_{1I}}\sum_{j=1}^{l_i} m_{ij}\gamma_{ij}m_{H_i}\neq 0$. Then, by Lemma 4.8, $\{\psi(\gamma)\in(a, b) : (\sum_{i\in I_{1I}}\sum_{j=1}^{l_i} m_{ij}\gamma_{ij}m_{H_i})^\wedge(\gamma)\neq 0\}$ is dense in (a, b), hence there exists an open interval (a_1, b_1) included in (a, b) such that

(4) $\{\sum_{i\in I_2}\sum_{j=1}^{l_i} m_{ij}\gamma_{ij}m_{H_i}\}^\wedge(\gamma)=0$ on $\psi^{-1}((a_1, b_1))$

and

(5) $\{\sum_{i\in I_{1l}}\sum_{j=1}^{l_i} m_{ij}\gamma_{ij}m_{H_i}\}^\wedge|_{\psi^{-1}((a_1,\ b_1))}\neq 0.$

We define measures η, $\eta_*\in M^+(K)$ as follows:

(6) $\eta=\sum_{i\in I_{1l}}\pi_K(T_\psi^\varepsilon(m_{H_i}))$;

$\eta_*=\pi_K(|\sum_{i\in I_{1l}}\sum_{j=1}^{l_i} m_{ij}\tau_{ij}T_\psi^\varepsilon(m_{H_i})|)$,

where $\pi_K : R\oplus K\rightarrow K$ is the projection. Then $0\neq\eta_*<<\eta$. Moreover, by Lemma 4.5, we have $\eta\in M_c(K)$. We put

$\xi=\sum_{i\in I_{1l}}\sum_{j=1}^{l_i} m_{ij}\tau_{ij}T_\psi^\varepsilon(m_{H_i})$

for convenience' sake. Then, by Corollary 1.6, there exists a family $\{\lambda_h\}_{h\in K}$ of reasures in $M(R\oplus K)$ with the following properties:

(7) $h\rightarrow\lambda_h(f)$ is a η_*-measurable function for each bounded Borel function f on $R\oplus K$;

(8) $\|\lambda_h\|\leq 1$;

(9) $\mathrm{supp}(\lambda_h)\subset R\times\{h\}$;

(10) $\xi(f)=\int_K\lambda_h(f)d\eta_*(h)$ for each bounded Borel function f on $R\oplus K$.

By (8) and (9), there exist measures $\nu_h\in M(R)$ with $\|\nu_h\|\leq 1$ such that

(11) $d\lambda_h(x, y)=d\nu_h(x)\times d\delta_h(y)$.

Since $0\neq\eta_*\in M^+(K)\cap M_c(K)$, it follows from Theorem 0.6 that there exists $\eta_0\in M^+(K)\cap M_s(K)$ such that

(12) $\eta_0*\eta_*\neq 0$ and $\eta_0*\eta_*\in L^1(K)$.

By (5) and Claim 1, there exists $\gamma_*\in\psi^{-1}((a_1,\ b_1))$ such that

(13) $\alpha(\xi)^\wedge(\gamma_*)=\{\sum_{i\in I_{1l}}\sum_{j=1}^{l_i} m_{ij}\gamma_{ij}m_{H_i}\}^\wedge(\gamma_*)$

$\neq 0.$

We choose $\sigma_0\in\hat{K}$ so that

(14) $(\sigma_0\eta_0)^\wedge(\gamma_*)\neq 0$.

Let f_* be a function in $H^1(R)$ such that

(15) $\mathrm{supp}(\hat{f}_*)\subset(a_1,\ b_1)$ and $\hat{f}_*(\psi(\gamma_*))\neq 0$,

and we define a measure $\mu'\in M(R\oplus K)$ by

(16) $d\mu'(x, y)=f_*(x)dx\times(y,\ \sigma_0)d\eta_0(y)$.

Then we have

(17) $\alpha(\mu')\in M^a(G)_s$ and $\mathrm{supp}(\alpha(\mu')^\wedge)\subset\psi^{-1}((a_1,\ b_1))$.

Moreover, by (13)-(15), we have

(18) $\alpha(\mu')*\alpha(\xi)\neq 0$.

We note that μ' can be represented as follows:

$$\mu'=\int_K(\sigma_0(h)f_*)\times\delta_h d\eta_0(h).$$

Then it follown from (12) and Lemma 2.3 (III) that $\mu'*\xi\in L^1(R\oplus K)$, hence (2.2) and (18) yield

(19) $0\neq\alpha(\mu')*\alpha(\xi)\in L^1(G)$.

On the other hand, by (2)-(4), (17) and Claim 1, we have

$$\begin{aligned}\Phi(\gamma)\alpha(\mu')^{\wedge}(\gamma)&=\{\sum_{i\in I_1}\sum_{j=1}^{l_i}m_{ij}\gamma_{ij}m_{H_i}\}^{\wedge}(\gamma)\alpha(\mu')^{\wedge}(\gamma)\\&=\{\sum_{i\in I_{1F}}\sum_{j=1}^{l_i}m_{ij}\gamma_{ij}m_{H_i}\}^{\wedge}(\gamma)\alpha(\mu')^{\wedge}(\gamma)+\\&\qquad\{\sum_{i\in I_{1I}}\sum_{j=1}^{l_i}m_{ij}\gamma_{ij}m_{H_i}\}^{\wedge}(\gamma)\alpha(\mu')^{\wedge}(\gamma)\\&=F(\gamma)+\alpha(\xi)^{\wedge}(\gamma)\alpha(\mu')^{\wedge}(\gamma)\quad(\gamma\in\hat{G}),\end{aligned}$$

where $F(\gamma)=\{\sum_{i\in I_{1F}}\sum_{j=1}^{l_i}m_{ij}\gamma_{ij}m_{H_i}\}^{\wedge}(\gamma)\alpha(\mu')^{\wedge}(\gamma)$. If $i\in I_{1F}$, m_{H_i} are discrete measures, so that $F\in M^a(G)_s{}^{\wedge}$. Hence by (19) we have

$$\begin{aligned}\Phi\cdot\alpha(\mu')^{\wedge}&=F+\alpha(\xi)^{\wedge}\alpha(\mu')^{\wedge}\\&\notin M^a(G)_s{}^{\wedge},\end{aligned}$$

which contradicts the hypothesis that Φ is a multiplier on $M^a(G)_s$. Hence we have $\sum_{i\in I_{1I}}\sum_{j=1}^{l_i}m_{ij}\gamma_{ij}m_{H_i}=0$ and the claim follows. We note that there exist $\gamma_i\in\hat{G}$, $m_i\in Z$ and a finite subgroup H of G such that

$$\sum_{i=1}^{n}m_i\gamma_i m_H=\sum_{i\in I_{1F}}\sum_{j=1}^{l_i}m_{ij}\gamma_{ij}m_{H_i}.$$

Hence the lemma is obtained from (2), (3) and Claim 2. This completes the proof.

LEMMA 4.10. *Under the assumption in the previous lemma, we have*

$$\Phi(\gamma)=\{\sum_{i=1}^{n}m_i\gamma_i m_H\}^{\wedge}(\gamma)\quad on\quad \psi^{-1}((a,b)),$$

where $m_i\in Z$, $\gamma_i\in\hat{G}$ and H is a finite subgroup of G.

PROOF. By Lemma 4.9, there exist m_i, $M_{jk}\in Z$ and γ_i, $\sigma_{jk}\in\hat{G}$ such that

$$\Phi(\gamma)=\{\sum_{i=1}^{n}m_i\gamma_i m_H\}^{\wedge}(\gamma)+\{\sum_{j=1}^{p}\sum_{k=1}^{q_j}M_{jk}\sigma_{jk}m_{L_j}\}^{\wedge}(\gamma)$$

for $\gamma\in\psi^{-1}((a,b))$, where H is a finite subgroup of G and L_j are compact subgroups of G such that $\psi(L_j{}^{\perp})$ are not dense in R. Since m_H is a discrete measure, $\Phi'=\Phi-\{\sum_{i=1}^{n}m_i\gamma_i m_H\}^{\wedge}(\gamma)$ becomes a multiplier on $M^a(G)_s$. Suppose there exists $\gamma_*\in\psi^{-1}((a,b))$ such that $\{\sum_{j=1}^{p}\sum_{k=1}^{q_j}M_{jk}\sigma_{jk}m_{L_j}\}^{\wedge}(\gamma_*)\neq 0$. Then there exists a positive real number δ with $(\psi(\gamma_*)-\delta,\psi(\gamma_*)+\delta)\subset(a,b)$ such that

$$\begin{aligned}\{\psi(\gamma)\in(\psi(\gamma_*)-\delta,\psi(\gamma_*)+\delta):\{\sum_{j=1}^{p}\sum_{k=1}^{q_j}M_{jk}\sigma_{jk}m_{L_j}\}^{\wedge}(\gamma)\neq 0\}\\=\{\psi(\gamma_*)\}.\end{aligned}$$

We may assume that $\ker(\psi)$ is open. Let f be a function in $H^1(R)$ such that $\hat{f}(\psi(\gamma_*))=1$ and $\mathrm{supp}(\hat{f})\subset(\psi(\gamma_*))-\delta,\psi(\gamma_*)+\delta)$. Let $\phi:R\to G$ be the dual homomorphism of ψ. Then $\phi(f)^{\wedge}=\hat{f}\circ\psi$ is a multiplier on $M^a(G)_s$ (cf. [17], Theorem 2.3, p. 188), hence $\Phi'\phi(f)^{\wedge}$ becomes a multiplier on $M^a(G)_s$ and $(\Phi'\phi(f)^{\wedge})|_{\gamma_*+ker(\psi)}\neq 0$. This contradicts Corollary 3.4 since $\mathrm{supp}(\Phi'\phi(f)^{\wedge})\subset\gamma_*+\ker(\psi)$, and the proof is complete.

Let $\mathscr{A}_0$ and $\mathscr{A}_1$ be subsets of $M(G)$ defined in Definition 0.1.

THEOROM 4.11. *Let G be a LCA group, and let $\psi:\hat{G}\to R$ be a nontrivial continuous homomorphism such that $M^a(G)_s\neq\{0\}$ and $\psi(\hat{G})$ is dense in R. Let Φ be a multiplier on $M^a(G)_s$ that is integer-valued on* $\{\gamma\in$

$\hat{G}: \psi(\gamma) \geq 0\}^0$. *Then the following are satisfied :*

(I) *If* $ker(\psi)$ *is not open, there exists a measure* $\nu \in \mathscr{A}_0$ *such that* $\Phi \approx \hat{\nu}$;

(II) *If* $ker(\psi)$ *is open, there exists a measure* $\nu \in \mathscr{A}_1$ *such that* $\Phi \approx \hat{\nu}$.

Conversely, when $ker(\psi)$ *is not open, let* ν *be a measure in* $\mathscr{A}_0$. *Then* $\hat{\nu}$ *becomes a multiplier on* $M^a(G)_s$. *When* $ker(\psi)$ *is open, let* ν *be a measure in* $\mathscr{A}_1$. *Then* $\hat{\nu}$ *becomes a multiplier on* $M^a(G)_s$.

PROOF. We first show that there exist $m_i \in Z$, $\gamma_i \in \hat{G}$ and a finite subgroup H of G such that

(1) $\Phi(\gamma) = \{\sum_{i=1}^{n} m_i \gamma_i m_H\}^{\wedge}(\gamma)$ on $\psi^{-1}((0, \infty))$.

In fact, if Φ vanishes on $\psi^{-1}((0, \infty))$, (1) is trivial. Hence we may assume that there exists $\gamma_0 \in \hat{G}$ with $\psi(\gamma_0) > 0$ such that $\Phi(\gamma_0) \neq 0$. We choose positive real numbers a and b so that $a < \psi(\gamma_0) < b$, and put $\Omega_n = (\frac{a}{n}, nb)$ $(n = 1, 2, 3, \cdots)$. Then, for each $n \in N$, it follows from Lemma 4.10 that there exists a measure $\xi_n = \sum_{i=1}^{l_n} m_{i,n} \gamma_{i,n} m_{H_n}$ such that

(2) $\hat{\xi}_n(\gamma) = \Phi(\gamma)$ on $\psi^{-1}(\Omega_n)$,

where $m_{i,n} \in Z$, $\gamma_{i,n} \in Z$ and H_n is a finite subgroup of G. Since H_n is a finite subgroup of G, $\psi(H_n^{\perp})$ is dense in R because $\psi(\hat{G})$ is dense in R. Hence by (2) and Lemma 4.8 we have

(3) $\xi_1 = \xi_n$ $(n = 1, 2, 3, \cdots)$,

hence

(4) $\Phi(\gamma) = \hat{\xi}_1(\gamma)$ on $\psi^{-1}((0, \infty))$.

Thus (1) follows from (4). If $\ker(\psi)$ is not open, $\{\gamma \in \hat{G}: \psi(\gamma) \geq 0\}^0$ coincides with $\psi^{-1}((0, \infty))$. Hence (I) follows from (1). Next we prove (II). We consider (II) by dividing two cases that $\ker(\psi)$ is compact or not.

Case 1. $\ker(\psi)$ is compact.

We note by Lemma 3.2 that

(5) $\hat{\mu}(\gamma) = 0$ on $\ker(\psi)$

for all $\mu \in M^a(G)_s$. Hence, in this case, (II) follows from (1) and (5).

Case 2. $\ker(\psi)$ is not compact.

Put $H_\psi = \ker(\psi)^{\perp}$, and let $\pi: G \rightarrow G/H_\psi$ be the natural homomorphism. Then, in this case, G/H_ψ is not discrete and $\Phi|_{ker(\psi)}$ becomes a multiplier on $M_s(G/H_\psi)$. Hence, by Theorems 0.2 and 0.7, there exist $M_j \in Z$ and $\sigma_j \in \ker(\psi)$ such that

(6) $\Phi|_{ker(\psi)}(\gamma) = \{\sum_{j=1}^{m} M_j \sigma_j m_{H_0}\}^{\wedge}(\gamma)$ on $\ker(\psi)$,

where H_0 is a compact subgroup of G *with* $H_0 \supset H_\psi$ such that $\pi(H_0)$ is a finite subgroup of G/H_ψ. Hence, in this case, (II) follows from (1) and (6). The converse is obtained from Lemma 3.2 and the fact that Fourier-Stieltjes transforms of discrete measures become multipliers on $M^{a}(G)_{s}$. This completes the proof.

Now we prove Main Theorem. By Remark 0.5, $\Phi(\gamma)=0$ or 1 on $\{\gamma\in\hat{G}:\psi(\gamma)>0\}$. If $\ker(\psi)$ is not open, then $\{\gamma\in\hat{G}:\psi(\gamma)\geq 0\}^{0}$ coincides with $\{\gamma\in\hat{G}:\psi(\gamma)>0\}$. Hence (I) follows from Theorem 4.11 (I). Next we consider (II) by dividing two cases that $\ker(\psi)$ is compact or not.

Case 1. $\ker(\psi)$ is compact.

In this case, $\hat{\mu}$ vanish on $\ker(\psi)$ for all $\mu\in M^{a}(G)_{s}$ (cf. Lemma 3.2). We define a function Φ' on $\hat{G}$ by $\Phi'(\gamma)=0$ on $\ker(\psi)$ and $\Phi'(\gamma)=\Phi(\gamma)$ on $\hat{G}\backslash\ker(\psi)$. Then $\Phi\approx\Phi'$ and $\Phi'(\gamma)$ is integer-valued on $\{\gamma\in\hat{G}:\psi(\gamma)\geq 0\}^{0}$. Thus, in this case, (II) is obtained from Theorem 4.11 (II).

Case 2. $\ker(\psi)$ is not compact.

In this case, G/H_{ψ} is not discrete and $\Phi|_{ker(\psi)}$ becomes an idempotent multiplier on $M_{s}(G/H_{\psi})$. In particular, $\Phi(\gamma)=0$ or 1 on $\{\gamma\in\hat{G}:\psi(\gamma)\geq 0\}$, and so (II) follows from Theorem 4.11 (II). Thus (II) is obtained. The converse has already been proved in Theorem 4.11, and the proof of Main Theorem is complete.

Finally the author wishes to express his thanks to the referee for his valuable advice.

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