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CONVERGENCE OF CONVEX FUNCTIONS AND DUALITY

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Introduction

Let f_n (n=1, 2,...) be a sequence of convex functions which converges pointwise to a proper function f which is convex as a consequence.

In the case that sequence of convex functions is monotonically decreasing, then the sequence of conjugate functions f_n^* of f_n converges to f^* (the conjugate function of f). But, we don't have the convergence of the sequence of conjugate functions f_n^* in general case. In this note, we shall discuss this problem.

Although we can consider this problem generally for convex functions defined on any finite-dimensional vector spaces, the fundamental tools of proof of the theorems are almost same in the case of functions defined on 1-dimensional space $\boldsymbol{R}$ i. e. (the space of real numbers). So, we only deal with cases of convex functions defined on $\boldsymbol{R}$ in this paper.

Our results show that if a sequence of convex functions $f_n(x)$ converges pointwise to a proper convex function f with domain of non-void interior, then the sequence of conjugate functions $f_n^*(y)$ of f_n converges to the conjugate function $f^*(y)$ of f except an exceptional set of y which has at most two point.
In this note, we shall show the fundamental theorem (Theorem 2) and applications of this theorem.

1. The space of convex functions

A function f : $\boldsymbol{R}\rightarrow\boldsymbol{R}\cup\{+\infty\}$ is called convex if for each $x,\ y,\ a,\ b\in\boldsymbol{R}$ with $a,\ b\in[0,1]$ and $a+b=1$

$$f(ax+by)\leqq af(x)+bf(y).$$

The effective domain of f is defined as $D(f)=\{x\,;\,f(x)<+\infty\}$. A convex function f is called *proper* if the effective domain $D(f)$ of f is not empty. The conjugate function of f is defined as follows:

$$f^*(y)=\sup_{x\in\boldsymbol{R}}\{yx-f(x)\}\ \text{ for } y\in\boldsymbol{R}.$$

Generally, the conjugate function f^* are defined on the dual space X^* if f is defined on a locally convex topological vector space X. But, in Euclidean spaces X or Hilbert spaces the dual X^* of X is isomorphic to X. So, conjugate functions are defined on the same space as original space.

Let $\boldsymbol{C}$ be a space of all proper convex functions defined on $\boldsymbol{R}$.
We shall discuss the structures of $\boldsymbol{C}$ as an ordered set. For f, $g \in \boldsymbol{C}$, $f \leqq g$ is by definition iff

$$f(x) \leqq g(x) \text{ for all } x \in \boldsymbol{R}.$$

Note that $\boldsymbol{C}$ is a semi-ordered set in this order relation.

LEMMA 1 *Let $\{f_\lambda\}_{\lambda \in \Lambda}$ be a collection of functions in $\boldsymbol{C}$. Then, $f(x) = \sup_\lambda f_\lambda(x) \in \boldsymbol{C}$ iff there exists x_0 with $\sup_\lambda f_\lambda(x_0) < +\infty$.*

Proof of Lemma is very easy, so it is omitted.
A collection $\{f_\lambda\}_{\lambda \in \Lambda}$ of $\boldsymbol{C}$ is called lower-directed if for any finite collection $f_{\lambda_1}, \dots, f_{\lambda_n}$ from $\{f_\lambda\}_{\lambda \in \Lambda}$, there exists f_λ with $f_{\lambda_k} \geqq f_\lambda$ for $k = 1, 2, \dots, n$.
The set of all interior points in a set A is denoted by Int A.

LEMMA 2 *Let $\{f_\lambda\}_{\lambda \in \Lambda}$ be a lower-directed collection of functions in $\boldsymbol{C}$, and let $f(x) = \inf_\lambda f_\lambda(x)$. Suppose $\mathrm{Int}\{x : f(x) < +\infty\} \neq \phi$. Then $f(x) \in \boldsymbol{C}$ iff there exists $x_0 \in \mathrm{Int}\{x : f(x) < +\infty\}$ with $\inf_\lambda f_\lambda(x_0) > -\infty$.*

PROOF $f(x) = \inf_\lambda f_\lambda(x_0)$ is a convex function of $x \in \boldsymbol{R}$ since $\{f_\lambda\}_{\lambda \in \Lambda}$ is lower directed. Hence, $f(x) = -\infty$ for all $x \in \mathrm{Int}\{x : f(x) < +\infty\}$ or $f(x) > -\infty$ for all $x \in \boldsymbol{R}$ by virtue of convexity of f. Since there exists x_0 with $\inf_\lambda f_\lambda(x_0) > -\infty$, we have Lemma 2.

We must remark that $\boldsymbol{C}$ is not a lattice with the order as a simple example show below.

For each f, $g \in \boldsymbol{C}$, we shall denote by

$$f \vee g$$

the least upper bound in $\boldsymbol{C}$ for f and $g \in \boldsymbol{C}$ by the order of $\boldsymbol{C}$, and also we shall denote

$$f \wedge g$$

the greatest lower bound in C for f and $g \in \boldsymbol{C}$ by the order of $\boldsymbol{C}$.
It is easy to see that $f \vee g$ exists in $\boldsymbol{C}$ iff $D(f) \cap D(g) \neq \phi$. For example, if we define f and g as follows:

$$f(x)=\begin{cases}0 & x=0\\ +\infty & \text{otherwise}\end{cases} \quad \text{and } g(x)=\begin{cases}0 & x=1\\ +\infty & \text{otherwise},\end{cases}$$

there is no $f\vee g$ in $\boldsymbol{C}$.

If we set

$$f(x)=x \qquad \text{and} \qquad g(x)=-x,$$

then we can not find the existence of $f\wedge g$ in $\boldsymbol{C}$. So, even if $D(f)=D(g)=\boldsymbol{R}$, there exists an example $f\wedge g$ does not exist.

For a collection $\{f_\lambda\}_{\lambda\in\Lambda}\subset\boldsymbol{C}$, we denote

$$\bigvee_\lambda f_\lambda$$

if there exists the least upper bound of the collection $\{f_\lambda\}_{\lambda\in\Lambda}$ by the order of $\boldsymbol{C}$. Similarly, we denote

$$\bigwedge_\lambda f_\lambda$$

if there exists the greatest lower bound of the collection $\{f_\lambda\}_{\lambda\in\Lambda}$ by the order of $\boldsymbol{C}$.

LEMMA 3 *If the collection* $\{f_\lambda\}_\lambda\}_{\lambda\in\Lambda}\subset\boldsymbol{C}$ *is upper bounded (lower bounded) i. e. there exists an* $f\in\boldsymbol{C}$ *with* $f_\lambda\leqq f$ $(f_\lambda\geqq f)$ *for all* $\lambda\in\Lambda$, *then* $\bigvee_{\lambda\in\Lambda} f_\lambda$ $(\bigwedge_{\lambda\in\Lambda} f_\lambda)$ *exists in* $\boldsymbol{C}$. *Furthermore*

$$(\bigvee_{\lambda\in\Lambda} f_\lambda)(x)=\sup_{\lambda\in\Lambda} f_\lambda(x).$$

PROOF Since $g(x)=\sup_{\lambda\in\Lambda} f_\lambda(x)\leqq f(x)$ for all $x\in\boldsymbol{R}$ and $g(x)$ is a convex function, we have $g\in\boldsymbol{C}$. It is easy to see that g is the lower upper bound for the collection $\{f_\lambda\}_{\lambda\in\Lambda}\subset\boldsymbol{C}$. For $\bigwedge_{\lambda\in\Lambda} f_\lambda$, we shall consider the set A of elements g of $\boldsymbol{C}$ for which $g\leqq f_\lambda$ for all $\lambda\in\Lambda$. Then, $\bigwedge_{\lambda\in\Lambda} f_\lambda=\bigvee_{\lambda\in A} g$. Note that we don't have the relation $\bigwedge_{\lambda\in\Lambda} f_\lambda(x)=\inf_{\lambda\in\Lambda} f_\lambda(x)$ for some collection $\{f_\lambda\}_{\lambda\in\Lambda}$.

LEMMA 4 *Let* $\{f_n\}$ *be a sequence of convex functions and converges pointwise to f. Then f is an element of* $\boldsymbol{C}$ *iff there exist* x_0, x_1, $x_2\in\boldsymbol{R}$ *with* $x_1<x_0<x_2$ *and* c, $d\in\boldsymbol{R}$ *with* $c\leqq f_n(x_0)\leqq d$ *and* $c\leqq f_n(x_1)$, $f_n(x_2)$ *for large n.*

PROOF.

$$f(ax+by)=\lim_{n\to\infty} f_n(ax+by)$$

$$\leqq a\lim_{n\to\infty} f_n(x)+b\lim_{n\to\infty} f_n(x)=af(x)+bf(x)$$

for a, $b \in [0, 1]$ with $a+b=1$ and $x \in \boldsymbol{R}$. Hence, $f \in \boldsymbol{C}$ iff there exists x_0 with $f(x_0) < +\infty$ and $f(x) > -\infty$ for all x. This means that there exists c and d$\in \boldsymbol{C}$ with

$$c \leqq f_n(x_0) \leqq d \text{ for large } n.$$

Next example show shat exen if there exist x_0 and $x_1 \in \boldsymbol{R}$ $(x_0 < x_1)$ with $f_n(x_0) = f_n(x_1)$, f does not necessary belonging to $\boldsymbol{C}$.

$$f_n(x) = \begin{cases} nx - n & \text{if } x > 0 \\ -nx - n & \text{if } x \leq 0 \end{cases}$$

Lemma 5 *Let f and $f_\lambda (\lambda \in \Lambda)$ be elements of $\boldsymbol{C}$. Then*

$$f \wedge (\bigvee_{\lambda \in \Lambda} f) = \bigvee_{\lambda \in \Lambda} (f \wedge f_\lambda)$$

and

$$f \vee (\bigwedge_{\lambda \in \Lambda} f_\lambda) = \bigwedge_{\lambda \in \Lambda} (f \vee f_\lambda)$$

if right or left side of above equalities exist in $\boldsymbol{C}$.
Proof of this Lemma is quite easy, so that it is omitted.

Let X be a locally convex space and X^* be the dual space of X i. e. X^* is the set of all continuous linear functional of X. In the case that X is $\boldsymbol{R}$, then $\boldsymbol{R}^*$ is isomorphic to $\boldsymbol{R}$. We have already defined conjugate function f^* of convex function f. f^* is a convex function defined on X^*. Since $X \subset X^{**}$ we can define a convex function on X as follows:

$$f^{**}(x) = \sup\{<x^*, x> - f^*(x^*)\} \text{ for } x \in X.$$

From now on, we shall restrict ourselves X is $\boldsymbol{R}$.
We shall state here the Fenchel-Moreau's theoren.

Theorem 1 *Let f be a convex function belonging to $\boldsymbol{C}$. $f(x) = f^{**}(x)$ for $x \in \boldsymbol{R}$ iff f is lower-semicontinuous at $x \in \boldsymbol{R}$. The set $\{x ; f(x) \neq f^{**}(x)\}$ is at most two point.*

Lemma 6 i) *$f \geqq g$ implies $f^* \leqq g^*$*

ii) $f \geqq f^{**}$

iii) $(\bigwedge_\lambda f_\lambda)^*(y) = (\bigvee_\lambda f_\lambda^*)(y)$ *for all* $y \in \boldsymbol{R}^* \cong \boldsymbol{R}$

iv) $(\bigvee_\lambda f_\lambda)^*(y) \leqq (\bigwedge_\lambda f_\lambda^*)(y)$ *for all* $y \in \boldsymbol{R}^* \cong \boldsymbol{R}$

Proof It is known that f is proper convex iff f^* is proper convex (see [5] Chap. 6). i) and ii) are deduced easily from the definition and for

iii) we put

$$g(x)=(\bigwedge_{\lambda} f_{\lambda})(x).$$

Then $g^*(y)\geqq(\bigvee_{\lambda} f_{\lambda}^*)(y)$ for all $y\in\boldsymbol{R}$ by i) since $g(x)\leqq f_{\lambda}(x)$ for $x\in\boldsymbol{R}$ and $\lambda\in\Lambda$. Let $h(x)$ be an element of $\boldsymbol{C}$ such that

$$h(y)\leqq f_{\lambda}^*(y) \text{ for all } y\in\boldsymbol{R} \text{ and } \lambda\in\Lambda.$$

Then, $h^*(x)\leqq f^{**}(x)\leqq f_{\lambda}(x)$ for all $x\in\boldsymbol{R}$ and $\lambda\in\Lambda$. Hence,

$$h^*\leqq g \text{ and so } h\geqq h^{**}\geqq g^*$$

by virtue of i) and ii).
This means that

$$(\bigwedge_{\lambda} f_{\lambda})^*=g^*=\bigvee_{\lambda} f_{\lambda}^*.$$

The proof of iv) is easy, so it is omitted. Next example shows that the equality of iv) does not hold in general.
Let

$$f_n(x)=\begin{cases} -x & (x\leqq 0) \\ -\dfrac{1}{n}x & (0\leqq x\leqq n) \\ -1 & (x\geqq n) \end{cases} \text{ and } f(x)=\begin{cases} -x & (x<0) \\ 0 & (x\geqq 0). \end{cases}$$

Then, $f_n\uparrow f$ and $f=\bigvee_{n} f_n$. But, $\bigwedge_{n} f_n^*$ is not f^* in this case, since

$$f_n^*(0)=1 \text{ and } f^*(0)=0.$$

But, we have the following lemma.

LEMMA 7 *Let a sequence of $f_n\in\boldsymbol{C}$ be non-decreasing and convergent pointwise to $f\in\boldsymbol{C}$. Then $\lim_{n\to\infty} f_n^*(y)=f^*(y)$ except at most two points of $y\in\boldsymbol{R}$.*

PROOF Since $f_n^*(y)\geqq f_{n+1}^*(y)$ for $y\in\boldsymbol{R}$, for the function $g(y)=\lim_{n\to\infty} f_n^*(y)$, we have

$$\sup f_n^{**}(x)=g^*(x) \text{ for all } x\in\boldsymbol{R}$$

by iii) of Lemma 6.
Since $f_n^{**}(x)=f_n(x)$ except at most two point of x, $f_n(x)\to g^*(x)$ except at

most countable point of $x \in \boldsymbol{R}$.
Hence,

$$f(x)=g^*(x) \text{ except at most a countable set of points in } \boldsymbol{R}.$$

Since f and g^* are convex functions, f and g^* are continuous except at most two point. We know that f is continuous on Int $D(f)$: the interior of its effective domain $D(f)$. Hence, Int $D(f)=$Int $D(g^*)$ i. e. $f(x)$ and $g^*(x)$ are equal except the boundary of $D(f)$ and $D(g^*)$. In a convex subset I of $\boldsymbol{R}$, the boundary of I consists of at most two point. Hence, $f(x)=g^*(x)$ except two point of $x \in \boldsymbol{R}$.

LEMMA 8 *Let f be a convex function which is finite-valued on a closed interval $[a, b]$ and $g(x)$ be a linear function (affine function). If $(f(a)-g(a))(f(b)-g(b))<0$, then $f(x)=g(x)$ for some exactly one point $x \in \boldsymbol{R}$.*

PROOF Since the function $h(x)=f(x)-g(x)$ is a convex function with $h(a)h(b)<0$ and hence continuous on the closed interval $[a, b]$, the equation $h(x)=0$ has a unique solution x in $\boldsymbol{R}$. Hence we have the assertion.

PROPOSITION 1 *Let $f_n \in \boldsymbol{C}$ and $\lim_{n\to\infty} f_n(x)=f(x)$ and a closed interval $[a, b] \subset Int\ D(f)$. Then, f_n is uniformly convergent to f in $[a, b]$.*

PROOF If the assertion of Proposition 1 is false, then there exist a positive number $\varepsilon>0$ and $a_n \in [a, b]$ such that

$$|f(a_n)-f_n(a_n)| \geqq \varepsilon>0$$

without loss of generality.
Since the interval $[a, b]$ is compact, we can assume that the sequence a_n converges to some number $a_0 \in [a, b]$.

Let $f(a_n) \geqq f_n(a_n)+\varepsilon$ and $a_0<a_n$ for infinitely many n, and let choose c_1 with $c_1 \in$Int $D(f)$ and $a_0<c_1$. Now, we consider the straight line l from the point $(c_1, f(c_1)+\varepsilon)$ to the point $(a_0, f(a_0)-\varepsilon)$ is the plane. Then, this straight line l meets with the graph of the convex function $y=f(x)$ at only one point by Lemma 8. This point will be denoted by $(c_2, f(c_2))$ with $a_0<c_2<c_1$.

Since $f_n(c_1)<f(c_1)+\varepsilon$ and $f_n(a_n) \leqq f(a_n)-\varepsilon$ for large n by assumption, the graph of the function $y=f_n(x)(x \in [a_n, c_1])$ is below under the straight line l for large n. Hence, if we take $c_3=(a_0+c_2)/2$, we find a positive number $d>0$ with

$$f_n(c_3) \leqq \frac{f(a_0)+f(c_2)-\varepsilon}{2}=f(c_3)-d$$

for large n.
But this is a contradiction to

$$\lim_{n\to\infty} f_n(c_3)=f(c_3).$$

We shall consider the case that

$$f(a_n)\geqq f_n(a_n)+\varepsilon$$

and $a_0>a_n$ for infinitely many n. We take an arbitrary number $c_1<a_0$ so that $c_1\in\text{Int } D(f)$. The straight line from the point $(a_0, f(a_0)-\varepsilon)$ to the point $(c_1, f(c_1)+\varepsilon)$ meet at only one point $(c_2, f(c_2))$ with the graph of the convex function $y=f(x)$ in the plane. By the same discussion as written above, for $c_3=(a_0+c_2)/2$, we have

$$f_n(c_3)\leqq f(c_3)-\varepsilon/2 \text{ for large } n.$$

But this is also a contradiction.
Next, if we have

$$f(a_n)\leqq f_n(a_n)-\varepsilon \text{ and } a_n>a_0$$

for infinitely many n,
we have hence

$$f_n(c_1)\to\infty \text{ for large } n \text{ and for } c_1\in\text{Int } D(f) \text{ with } c_1>a_0.$$

But, this is a contradiction.
If we have

$$f(a_n)\leqq f_n(a_n)-\varepsilon \text{ and } a_n<a_0$$

for infinitely many n, we have also

$$f_n(c_1)\to\infty \text{ for large } n \text{ and for } c_1\in\text{Int } D(f) \text{ with } a_0>c_1.$$

This is also a contradiction. Hence, we have the assertion of Proposition 1.

The sequence of functions f_n $(n=1, 2, \dots)$ is called uniformly Lipschitzian in the closed interval $[a, b]$ if there exists $K>0$ and positive integer N such that

$$|f_n(x)-f_n(y)|\leqq K|x-y|$$

whenever $n\geqq N$ and $x, y\in[a, b]$.

PROPOSITION 2 *Let f_n $(n=1, 2, \dots)\in \boldsymbol{C}$ and $\lim_{n\to\infty} f_n(x)=f(x)$ and the closed interval $[a, b]\subset \text{Int } D(f)$. Then, the sequence of functions f_n is* *uniformly Lipschnitzian in the closed interval* $[a, b]$.

PROOF If there exist positive numbers $\delta>0$ and $K>0$ such that $|x-y|\leqq\delta$ implies $|f_n(x)-f_n(y)|\leqq K|x-y|$ for sufficient large n, then we have

$$|f_n(x)-f_n(y)|\leqq K|x-y| \text{ for } x,\ y\in[a,\ b].$$

Assume now that $\{f_n\}$ is not uniformly Lipschitzian in the interval $[a, b]\subset$ Int $D(f)$. Then, for each natural number n, there exist a sequence of increasing numbers $K_n\uparrow\infty$ and increasing positive integers

$$m_1<m_2<\ldots<m_n<\ldots$$

with

$$|f_{m_n}(X_n)-f_{m_n}(Y_n)|>K_n|x_n-y_n|$$

and

$$|x_n-y_n|\leqq 1/n$$

for some x_n, $y_n\in[a, b]$. Since the closed interval $[a, b]$ is compact, we can assume that the sequence of points x_n converges to some real number x_0 so that

$$\lim_{n\to\infty} x_n=\lim_{n\to\infty} y_n=x_0\in[a,\ b].$$

Since $[a, b]\subset$Int $D(f)$, there exist real numbers a_1 and a_2 with $a_1<x_0<a_2$ and a_1, $a_2\in$Int $D(f)$. By the former proposition 1, we know that the sequence of convex functions f_n converges uniformly to f on the closed interval $[a_1, a_2]$. Hence, we can assume that

$$f(x)+1\geqq f_n(x)\geqq f(x)-1 \text{ for } x\in[a_1,\ a_2].$$

Since the points $(x_n, f_{m_n}(x_n))$ and $(Y_n, f_{m_n}(Y_n))$ in the plane are between two graphs of functions $y=f(x)+1$ and $y=f(x)-1$ in the plane, if x_n and y_n are very close to x_0, by virtue of convexity of functions f_{m_n}, we must have $f_{m_n}(x_n)$ or $f_{m_n}(y_n)$ converges to $+\infty$ for large n. But, this is a contradiction.

COROLLARY 1 *Every convex function f is Lipschitzian in every interval $[a, b]\subset Int\ D(f)$; i. e. there exists $K>0$ such that*

$$|f(x)-f(y)|\leqq K|x-y| \text{ for } x,\ y\in[a,\ b].$$

LEMMA 9 *Let f_n $(n=1, 2, \ldots)$ be a sequence of convex functions on* $\boldsymbol{R}$

and $\lim_{n\to\infty} f_n(x)=f(x)$ *with* $f\in\boldsymbol{C}$. *If* f *is not a linear function (affine function) and* $x_0\in Int\ D(f)$, *then for every* $\varepsilon>0$ *there exists a convex function* $g(x)$ *such that*

$$g(x_0)=f(x_0)-\varepsilon$$

and there exist n *with*

$$f_m(x)\geqq g(x) \text{ for all } m\geqq n \text{ and } x\in\boldsymbol{R}.$$

PROOF (1) We assume first $f(x_0)=\inf f(x)$ and the function $y=f(x)$ is not constant on any interval I where $I=[a,\infty)$ or $(-\infty, b]$ for some $a, b\in\boldsymbol{R}$.
Then, there exist a_1 and $a_2\in\boldsymbol{R}$ such that $a_1\leqq a_2$ and

$$f(a)=f(x_0) \text{ for } a_1\leqq a\leqq a_2$$

and

$$f(a)>f(x_0) \text{ for } a<a_1 \text{ or } a>a_2.$$

Hence, we can find $b_1<a_1$, $a_2<b_2$ and $\varepsilon>0$ with

$$f(b)>f(x_0)+\varepsilon \text{ for } b<b_1 \text{ or } b>b_2.$$

Define a constant function g with

$$g(x)=f(x_0)-\varepsilon.$$

Since f_n is uniformly convergent to f in $[b_1, b_2]$, we have

$$f(x)+\varepsilon/2\geqq f_n(x)\geqq f(x)-\varepsilon/2 \text{ for } x\in[b_1, b_2] \text{ and for large } n.$$

If there exists $x_n\in[b_1, b_2]$ with $f_n(x_n)\leqq f(x_0)-\varepsilon$ for some n, by virtue of convexity of f_n, we have

$$f_n(b_1)\leqq f(x_0)+\varepsilon/2 \quad \text{or} \quad f_n(b_2)\leqq f(x_0)+\varepsilon/2.$$

Hence, we know that the number of such $x_n\in[b_1, b_2]$ is finite. That is, the constant function $g(x)=f(x_0)-\varepsilon$ has desired properties.
(2) Let $f(x)<+\infty$ for all $x\in\boldsymbol{R}$ and $f(x_0)=\inf f(x)$. Suppose that f is not constant function but constant on some interval I where I is $[b,\infty)$ or $(-\infty, b]$ for some $b\in\boldsymbol{R}$. If $f(x)$ is constant on some interval $[b,\infty)$, then there exists $a_0\leqq x_0$ with $f(a)>f(a_0)$ for $a<a_0$ and $f(x_0)=f(a_0)$. Consider the eqigraph of f i. e. $EP(f)=\{(x, y); y\geqq f(x)\}$ in the plane and the point $(x_0, f(x_0)-\varepsilon/2)\in EP(f)$.
Since $EP(f)$ is convex in the plane and $(x_0, f(x_0)-\varepsilon/2)\in EP(f)$, by using

the Hahn-Banach theorem, there exists a linear function (affine function) $g_1(x)=ax+b$ $(a\neq 0)$ which is through the point $(x_0, f(x_0)-\varepsilon/2)$ in the plane and places $EP(f)$ in upper side i. e.

$g_1(x)\leqq f(x)$ for all $x\in\boldsymbol{R}$ and satisfies the condition

$$\lim_{x\to\pm\infty}(f(x)-g_1(x))=+\infty$$

Then, the linear function g defined by $g(x)=g_1(x)-(1/2)\varepsilon$ has the desired properties.

(3) We assume $f(x)<+\infty$ for all $x\in\boldsymbol{R}$ and $f(x_0)>\inf f(x)$ and f is not linear function. Consider the epigraph of f : $EP(f)=\{(x, y)\,;\, y\geqq f(x)\}$ in the plane. Since $EP(f)$ is convex, there exists some line l therough the point $(x_0, f(x_0))$ by which $EP(f)$ is placed in upper side. Let l be represented as the function $h(x)=ax+b$. Since $f(x_0)>\inf$ f(x), we have $a\neq 0$.

Considering the convex function $f_1(x)=f(x)-h(x)$ instead of f, we have $f_1(x_0)=\inf f_1(x)$ and $\lim_{n\to\infty}(f_n-h)(x)=f_1(x)$. Applying the former results (1) and (2) to this case, for each $\varepsilon>0$ we have a convex function g_1 with $g_1(x_0)=f_1(x_0)-\varepsilon$ and there exists n with $f_m(x)-h(x)\geqq g_1(x)$ for all integer $m\geqq n$ and for all $x\in\boldsymbol{R}$.

Hence, we have the desired convex function $g(x)=g_1(x)+h(x)$.

(4) We shall consider the case that $D(f)=\{x\,;f(x)<+\infty\}\neq\boldsymbol{R}$, $x_0\in$ Int $D(f)$ and $f(x_0)=\inf f(x)$. Let a_0 be a boundary point of $D(f)$ in $\boldsymbol{R}$ i. e. $a_0\in D(f)^-\cap D(f)^c)^-$. We can choose the point a_0 with $x_t<a_0$ and also some point $b_0\in$Int $D(f)$ with $b_0<x_0$ since x_0 since $x_0\in$Int $D(f)$.

Consider the line l through the point $(b_0, f(b_0)+\varepsilon)$ *and point* $(a_0, f(X_0)-\varepsilon)$ in the plane. This line l will meet with the line of a constant function $y=f(x_0)-\varepsilon/2$ at the point $(a, f(x_0)-\varepsilon/2)$ in the plane for some real number a with $x_0<a<a_0$. Since $[b_0, a]\subset$Int $D(f)$, f_n converges uniformly to f on $[b_0, a]$ by proposition 1.

Suppose that there exist infinitely many n and $x_n\in[a, a_0]$ with $f_n((x)\leqq f(x_0)-\varepsilon$. Then,

$$f_n(a)\leqq f(a)-\varepsilon/2$$

for infinitely many n by convexity of functions f_n.

But, this is a contradiction to

$$\lim_{n\to\infty}f_n(a)=f(a).$$

Since we can prove also

$$f_n(x) \geqq f(x_0) - \varepsilon$$

For all $x \in [b_0, a_0]^c$ and for large n, the function g defined by

$$g(x) = f(x_0) - \varepsilon \quad \text{(constant function)}$$

has the desired property on $[b_0, a_0]$.

(5) We shall consider the case that $x_0 \in \text{Int } D(f)$ and $f(x_0) > \inf f(x)$. Since $x_0 \in \text{Int } D(f)$, we can construct the line l in the plane such that l is through the point $(x_0, f(x_0))$ and $EP(f)$ is placed in upper side of l. l will be defined by the function $h(x) = cx + d$ $(c \neq 0)$. The convex function $f(x) - h(x)$ has the same property as the function in (4). By (4), we can easily define a desired function g. q. e. d.

We shall state the following two lemmas without proof, since proofs are quite similar to the proof of Lemma 9.

LEMMA 10 *Let f_n $(n=1, 2, \ldots)$ be a sequence of convex functions on $\boldsymbol{R}$ and $\lim_{n\to\infty} f_n(x) = f(x)$ with $f \in \boldsymbol{C}$. If $x_0 \in \text{Int } D(f)^c$, then for every positive number K there exists a convex function $g(x)$ such that*

$$g(x_0) \geqq K$$

and there exists a natural number n_0 with

$$f_n(x) \geqq g(x)$$

for all $n \geqq n_0$ and $x \in \boldsymbol{R}$.

LEMMA 11 *Let f_n $(n=1, 2, \ldots)$ be a sequence of convex functions on $\boldsymbol{R}$ and $\lim_{n\to\infty} f_n(x) = f(x)$ with $f \in \boldsymbol{C}$ and* Int $D(f) \neq \phi$. *If $x_0 \in D(f)^- \cap (D(f)^c)^-$, then for every positive number $\varepsilon > 0$, there exists a convex function $g(x)$ such that*

$$g(x_0) = f^{**}(x_0) - \varepsilon$$

and there exists a natural number n_0 with

$$f_n(x) \geqq g(x)$$

for all $n \geqq n_0$ and $x \in \boldsymbol{R}$.

2. Main theorem

Now, we shall state the main theorem:

THEOREM 2 *Let f_n $(n=1, 2, \ldots)$ be a sequence of convex functions on $\boldsymbol{R}$*

and $\lim_{n\to\infty} f_n(x)=f(x)$ *with* $f\in \boldsymbol{C}$. *If Int* $D(f)\neq\phi$, *then the sequence of conjugate functions* f_n^* *of* f_n *converges pointwise to the conjugate function* f^* *of* f *except at most two points of* $\boldsymbol{R}$.

Proof At first we assume that f is not a linear (affine) function. By Lemma 9, the set of convex functions $\{f_n;\ n\geqq m\}$ of $\boldsymbol{C}$ is lower bounded for large m. By Lemma 3, we find the greatest lower bound of the family $\{f_m, f_{m+1}, \dots\}$ in $\boldsymbol{C}$ which is denoted by $h_m=\bigwedge_{n\geq m} f_n\in\boldsymbol{C}$ for large m. By definition

$$h_m\leqq h_{m+1}\leqq\dots\dots\leqq f.$$

By Lemma 9 again and Lemma 11, we find that for $x\in\text{Int } D(f)$,

$$f^{**}(x)\leqq\lim_{m\to\infty} h_m(x)=f_0(x)\leqq f(x).$$

By the Fenchel's duality theorem, $f(x)$ differs from $f^{**}(x)$ only at the exceptional set which consists at most two points. Hence

$$f_0(x)=f(x) \text{ except at most two points of } x\in\boldsymbol{R}.$$

On the other hand, the family $\{f_m, f_{m+1}, \dots\}$ is upper bounded for large m. Hence, by Lemma 3, we find the least upper bound of $\{f_m, f_{m+1}, \dots\}$ which is denoted by

$$k_m(x)=\bigvee_{n\geqq m} f_n(x)=\sup_{n\geqq m} k_n(x).$$

Since the sequence of convex function k_m is monotonically decreasing and converges pointwise to f, we have

$$\lim_{m\to\infty} k_m^*(y)=f^*(y) \text{ for all } y\in\boldsymbol{R}.$$

By Lemma 7,

$$\lim_{m\to\infty} h_m^*(y)=f_0^*(y)=f^*(y) \text{ for } y\in\boldsymbol{R} \text{ except at most two points of } y\in\boldsymbol{R}.$$

Since $h_m\leqq f_m\leqq k_m$ and $h_m^*\geqq f_m^*\geqq k_m^*$, we have

$$\lim_{n\to\infty} f_n^*(y)=f^*(y)$$

except at most two points of $y\in\boldsymbol{R}$.

Secondly, we assume that $f(x)$ is a constant function i. e. $f(x)=c$ for some real number c.

For every positive number $\varepsilon>0$, and for all positive integer N, by Proposition 1,

$$|f_n(x)-f(x)|<\varepsilon \text{ for } |x|\leqq N$$

for large n.
If $y\neq 0$, we have

$$f_n^*(y)=\sup_{x\in\boldsymbol{R}}\{yx-f_n(x)\}\geqq \sup_{x\in[-N,\,N]}\{yx-c-\varepsilon\}$$
$$=N|y|-c-\varepsilon.$$

It follows that $\lim_{n\to\infty} f_n^*(y)=+\infty$ for $y\neq 0$.
On the other hand $f^*(y)=+\infty$ for $y\neq 0$. Hence, we have the assertion of Theorem 2 in this case.
Let $f(x)$ is a linear function, say $f(x)=ax+b$ $(a\neq 0)$.
We set

$$h_n(x)=f_n(x)-(ax+b).$$

Then $h_n(x)\to 0$ for all $x\in\boldsymbol{R}$.
Since

$$h_n^*(y)=\sup_x\{yx-f_n(x)+ax+b\}$$
$$=\sup_x\{(y+a)x-f_n(x)\}+b$$
$$=f_n^*(y+a)+b,$$

we have $f_n^*(y)\to\infty$ as $n\to\infty$ for $y\neq a$.
On the other hand,

$$f^*(y)=\begin{cases} b & \text{if } y=a \\ -\infty & \text{if } y\neq a. \end{cases}$$

Hence, we have the assertion of Theorem 2. q. e. d.

REMARK 1 Next example shows that in the case $\lim_{n\to\infty} f_n(x)=f(x)$ and Int $D(f)=\phi$, Theorem 2 is not true in general.
Let

$$f_n(x)=\begin{cases} n^2x & \text{if } x\geq -1/n \\ -n^2x-2n & \text{if } x\leq -1/n. \end{cases}$$

Then, $\lim_{n\to\infty} f_n(x)=f(x)$
where

$$f(x)=\begin{cases} 0 & \text{if } x=0 \\ +\infty & \text{if } x\neq 0. \end{cases}$$

But,

$$f_n^*(y) = \begin{cases} -(1/n)y+n & \text{if } |y| \leqq n^2 \\ +\infty & \text{if } |y| > n^2 \end{cases}$$

and

$$f^*(y) = 0 \quad \text{for all } y \in \boldsymbol{R}.$$

In this case, $\bigvee_{n \geqq m} f_n = f$ and $\bigwedge_{n \geqq m} f_n$ does not exists for all positive integer m.

REMARK 2 Theorem 2 is true in the case that $f(x)$ is a linear function as is shown in proof. But we don't have the existence of $\bigwedge_{n \geqq m} f_n$ in $\boldsymbol{C}$ for all m in general.

3. Applications

Let Ω be a finite measure space with measure μ and let $P(\Omega)$ be a set of all measurable functions on Ω assuming values in $\boldsymbol{R} \cup \{+\infty\}$. Then $P(\Omega)$ is a convex set in the sppace $U(\Omega)$ of all measurable functions on Ω assuming values in $\boldsymbol{R} \cup \{-\infty\} \cup \{+\infty\}$. Let $S(\Omega)$ be a set of all measurable functions on Ω assuming values in $\boldsymbol{R}$. We shall identify f and g of $U(\Omega)$ if they differ only on a set of μ-measure zero. In [3], we stated the following Lemma;

LEMMA 12 *Let F be a convex operator from $\boldsymbol{R}$ into $P(\Omega)$ such that there exists $\alpha_0 \in \boldsymbol{R}$ with $F(\alpha_0) \in S(\Omega)$. Then, there exist a subset A of Ω of measure zero and a function $F(\alpha, t)$ defined on $\boldsymbol{R} \times \Omega$ such that for each fixed $t \equiv A$, $\boldsymbol{R} \ni \alpha \to F(\alpha, t)$ is a convex function on $\boldsymbol{R}$ and for each fixed $\alpha \in \boldsymbol{R}$, $\Omega \ni t \to F(\alpha, t)$ is a measurable function on Ω which is identified with $F(\alpha)$ as an element of $P(\Omega)$.*

We shall consider the average convex function of F. Let us consider the integration function $I(F)(x)$ defined on $x \in \boldsymbol{R}$ with

$$I(F)(x) = \int_\Omega F(x, t)\, d\mu(t)$$

if the integral has a sense.

We assume now that $I(F)(x) > -\infty$ for all $x \in \boldsymbol{R}$. Then $I(F)(\alpha)$ is a convex function defined on $\alpha \in \boldsymbol{R}$. We shall consider the conjugate function $I(F)^*$ of the convex function $I(F)$. Let f and g be convex functions from $\boldsymbol{R}$ to $\boldsymbol{R} \cup \{+\infty\}$. We define the infimal convolution

$$(f \oplus g)(x) = \inf\{f(x_1) + g(x_2)\,;\ x_1 + x_2 = x\}.$$

We know the following theorem (see [1] p. 178). Let $f_1, \ldots, f_n$ be (convex) functions on $\boldsymbol{R}$. Then

THEOREM 3

$$(f_1 \oplus f_2 \oplus \dots \oplus f_n)^* = f_1^* + f_2^* + \dots + f_n^*$$
$$(f_1 + f_2 + \dots + f_n)^* \leqq f_1^* \oplus f_2^* \oplus \dots \oplus f_n^*.$$

If $f_1, f_2, \dots, f_n$ are proper convex functions and if their effective domains contain a common point at which all these functions except possibly one are continuous, then

$$(f_1 + f_2 + \dots + f_n)^* = f_1^* \oplus f_2^* \oplus \dots \oplus f_n^*.$$

We shall state some Lemma:

LEMMA 13 *Let F be a convex operator such that there exist at least two $\alpha_1, \alpha_2 \in \boldsymbol{R}$ with $F(\alpha_1, t)$, $F(\alpha_2, t) \in L^1(d\mu)$, where $F(\alpha, t)$ is defined in Lemma* 12. *Then $I(F)$ is a convex function defined on $\boldsymbol{R}$ with values $\boldsymbol{R} \cup \{+\infty\}$ with* $\mathrm{Int}(D(I(F))) \neq \phi$. *If $[a, b] \subset D(I(F))$, then for every positive number $\varepsilon > 0$ there exists a measurable set A with $\mu(A) < \varepsilon$ such that*

$F(x, t)$ is bounded on $(x, t) \in [a, b] \times (\Omega \backslash A)$.

PROOF Since $F(\alpha_1, \mathrm{t})$ and $F(\alpha_2, \mathrm{t})$ as functions of $\mathrm{t} \in \Omega$ are elements of $\mathrm{L}^1(\mathrm{d}\mu)$, there exists a measurable set A_1 with $\mu(\mathrm{A}_1) < \frac{1}{2}\varepsilon$ such that $F(\alpha_1, \mathrm{t})$ and $F(\alpha_2, \mathrm{t})$ are bounded for $\mathrm{t} \in \Omega \backslash \mathrm{A}_1$. Set $\alpha_3 = \frac{\alpha_1 + \alpha_2}{2}$. Since $F(\alpha_3, \mathrm{t})$ as a function of $\mathrm{t} \in \Omega$ is also an element of $\mathrm{L}^1(\mathrm{d}\mu)$, there exists a measurable set A_2 with $\mu(\mathrm{A}_2) < \frac{1}{2}\varepsilon$ such that $F(\alpha_3, \mathrm{t})$ is bounded for $\mathrm{t} \in \Omega \backslash \mathrm{A}_2$.
If $|F(\alpha_1, \mathrm{t})|$, $|F(\alpha_2, \mathrm{t})|$, $|F(\alpha_3, \mathrm{t})| \leqq \mathrm{N}$ for some $\mathrm{t} \in \Omega$, then $|F(\alpha, \mathrm{t})| \leqq 3\mathrm{N}$ for $\alpha \in [\alpha_1, \alpha_2]$ by virtue of convexity of $F(\alpha, \mathrm{t})$.
Hence, $F(\alpha, \mathrm{t})$ is bounded for all $\alpha \in [\alpha_1, \alpha_2]$ and for all $\mathrm{t} \in \Omega \backslash \mathrm{A}$ where $\mathrm{A} = \mathrm{A}_1 \cup \mathrm{A}_2$ with $\eta(\mathrm{A}) < \varepsilon$.

THEOREM 4 *Let F be a convex operator from $\boldsymbol{R}$ into $p(\Omega)$ such that there exist α_1 and $\alpha_2 (\alpha_1 < \alpha_2)$ with $F(\alpha_1, t)$ and $F(\alpha_2, t)$ as functions of $t \in \Omega$ are in $L^1(d\mu)$ where $F(x, t)$ is a function of $(x, t) \in \boldsymbol{R} \times \Omega$ defined in Lemma* 12. *Then, there exists a sequence of $G_n(x, t)$ of form*:

$$G_n(x) = \sum_{k=1}^{k(n)} \lambda_k F(x, t_k)$$

where there exists a decomposition of Ω of measurable sets $\{A_k\}$ with $\lambda_k =$

$\mu(A_k)$, $t_k \in A_k$ such that

$$\lim_{n\to\infty} G_n(x) = I(F)(x)$$

where $I(F)$ is the average convex function of $F(x, t)$. Furthermore,

$$\lim_{n\to\infty} G_n^*(y) = I(F)^*(y)$$

except at most two points of $y \in \boldsymbol{R}$, i. e.

$$\lim_{n\to\infty} \sum_{k-1}^{k(n)} \oplus (\lambda_k F(\cdot, t_k))^*(y) = I(F)^*(y)$$

except at most two points of $y \in \boldsymbol{R}$.

PROOF If a closed interval $[a, b] \subset \mathrm{Int}(D(I(F)))$, then by Lemma 13 for every positive integer n there exists a measurable subset A of Ω with $\mu(A) < 1/n$ such that $F(x, t)$ is bounded for all $x \in [a, b]$ and $t \equiv A$.
Hence, we can find

$$G_n(x) = \sum_{k=1} \lambda_k F(x, t_k)$$

such that

$$\lim_{n\to\infty} G_n(x) = \int_\Omega F(x, t) d\mu(t)$$

and $F(x, t_k)$ is always finite for all $x \in D(I(F))$.
Hence, the assertion of Theorem 4 is an easy consequence of Theorem 2.

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