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A Class of Functions Defined by Using Hadamard Product

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Abstract

We introduce a class $\boldsymbol{P}_{\alpha}[\beta, \gamma]$ of functions defined by using Hadamard product $f*S_{\alpha}(z)$ of $f(z)$ and $S_{\alpha}(z)=z/(1-z)^{2(1-\alpha)}$. The object of the present paper is to determine extreme points, coefficient inequalities, distortion theorems, and radii of starlikeness and convexity for functions in $\boldsymbol{P}_{\alpha}[\beta, \gamma]$. Further, we give distortion theorems for fractional calculus of functions belonging to the class $\boldsymbol{P}_{\alpha}[\beta, \gamma]$.

1. Introduction

Let $\boldsymbol{A}$ denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_{n}z^{n} \tag{1.1}$$

which are analytic in the unit disk $\boldsymbol{U}=\{z:|z|<1\}$. And let $\boldsymbol{S}$ denote the subclass of $\boldsymbol{A}$ consisting of analytic and univalent functions $f(z)$ in the unit disk $\boldsymbol{U}$. A function $f(z)$ in $\boldsymbol{S}$ is said to be starlike of order α if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha \qquad (z\in\boldsymbol{U}) \tag{1.2}$$

for some $\alpha(0\leqq\alpha<1)$. We denote by $\boldsymbol{S}^{*}(\alpha)$ the class of all starlike functions of order α. Further, a function $f(z)$ in $\boldsymbol{S}$ is said to be convex of order α if

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha \qquad (z\in\boldsymbol{U}) \tag{1.3}$$

for some $\alpha(0\leqq\alpha<1)$. And we denote by $\boldsymbol{K}(\alpha)$ the class of all convex functions of order α. It is well-known that $f(z)\in\boldsymbol{K}(\alpha)$ if and only if $zf'(z)\in\boldsymbol{S}^{*}(\alpha)$, and that $\boldsymbol{S}^{*}(\alpha)\subseteq\boldsymbol{S}^{*}(0)\equiv\boldsymbol{S}^{*}$, and $\boldsymbol{K}(\alpha)\subseteq\boldsymbol{K}(0)\equiv\boldsymbol{K}$ for $0\leqq\alpha<1$.

These classes $\boldsymbol{S}^{*}(\alpha)$ and $\boldsymbol{K}(\alpha)$ were first introduced by Rebertson [9],

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and later were studied by Schild [12], MacGregor [4] and Pinchuk [8].

Now, the function

$$(1.4)\qquad S_\alpha(z)=\frac{z}{(1-z)^{2(1-\alpha)}}$$

is the well-known extremal function for the class $S^*(\alpha)$ (see [9, p. 385], [1]). Setting

$$(1.5)\qquad C(\alpha, n)=\frac{\prod_{k=2}^{n}(k-2\alpha)}{(n-1)!}\qquad (n=2, 3, 4, \dots),$$

$S_\alpha(z)$ can be written in the form

$$(1.6)\qquad S_\alpha(z)=z+\sum_{n=2}^{\infty} C(\alpha, n)z^n.$$

Then we note that $C(\alpha, n)$ is decreasing in α and satisfies

$$(1.7)\qquad \lim_{n\to\infty} c(\alpha, n)=\begin{cases}\infty & (\alpha<1/2)\\ 1 & (\alpha=1/2)\\ 0 & (\alpha>1/2).\end{cases}$$

Let $f*g(z)$ be the convolution or Hadamard product of two functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$(1.8)\qquad g(z)=z+\sum_{n=2}^{\infty} b_n z^n,$$

then

$$(1.9)\qquad f*g(z)=z+\sum_{n=2}^{\infty} a_n b_n z^n.$$

We say that a function $f(z)$ defined by (1. 1) belongs to the class $P_\alpha(\beta, \gamma)$ if $f(z)$ satisfies the following condition

$$(1.10)\qquad \left|\frac{(f*S_\alpha(z))'-1}{(f*S_\alpha(z))'+(1-2\beta)}\right|<\gamma \qquad (z\in U)$$

for $\beta(0\leqq\beta<1)$ and $\gamma(0<\gamma\leqq 1)$.

Let T denote the subclass of A consisting of functions $f(z)$ whose nonzero coefficients, from the second on, are negative. That is, an analytic function $f(z)$ is in the class T if it can be expressed as

$$(1.11)\qquad f(z)=z-\sum_{n=2}^{\infty} a_n z^n \qquad (a_n\geqq 0).$$

And we denote by $\boldsymbol{P}_\alpha[\beta, \gamma]$ the class obtained by taking intersection of $\boldsymbol{P}_\alpha(\beta, \gamma)$ with $\boldsymbol{T}$, that is,

$$\boldsymbol{P}_\alpha[\beta, \gamma]=\boldsymbol{P}_\alpha(\beta, \gamma)\cap\boldsymbol{T}.$$

The class $\boldsymbol{P}_\alpha[\beta, \gamma]$ is the generalization of the class $\boldsymbol{P}^*(\beta, \gamma)$ which was introduced by Gupta and Jain [3]. In particular, $\boldsymbol{P}_{1/2}[\beta, \gamma]=\boldsymbol{P}^*(\beta, \gamma)$ when $\alpha=1/2$. Further we note that many classes defined by using the Hadamard product $f*S_\alpha(z)$ of $f(z)$ and $S_\alpha(z)$ were introduced by Sheil-Small, Silverman and Silvia [13], Silverman and Silvia ([15], [16]), and Ahuja and Silverman [1].

2. Coefficient Inequalities

THEOREM 1. *Let the function $f(z)$ be defined by* (1.11). *Then $f(z)$ is in the class $\boldsymbol{P}_\alpha[\beta, \gamma]$* if and only if

$$\sum_{n=2}^{\infty} n(1+\gamma)C(\alpha, n)a_n \leqq 2\gamma(1-\beta). \tag{2.1}$$

The result is sharp.

PROOF. We employ the same technique as used by Ahuja and Silverman [1]. Let $f(z)$ be in the class $\boldsymbol{P}_\alpha[\beta, \gamma]$. Then we have

$$\left|\frac{(f*S_\alpha(z))'-1}{(f*S_\alpha(z))'+(1-2\beta)}\right| = \left|\frac{-\sum_{n=2}^{\infty} nC(\alpha, n)a_n z^{n-1}}{2(1-\beta)-\sum_{n=2}^{\infty} nC(\alpha, n)a_n z^{n-1}}\right| < \gamma \tag{2.2}$$

for all $z\in\boldsymbol{U}$. Since the denominator in (2.2) is positive for small positive values of z and, consequently, for all $z\,(0<z<1)$, we let $z\to 1^-$ to obtain

$$\sum_{n=2}^{\infty} nC(\alpha, n)a_n \leqq \gamma\{2(1-\beta)-\sum_{n=2}^{\infty} nC(\alpha, n)a_n\}, \tag{2.3}$$

which is equivalent to (2.1).

For the converse, let the inequality (2.1) hold. Then we obtain that

$$\begin{aligned} &|(f*S_\alpha(z))'-1|-\gamma|(f*S_\alpha(z))'+(1-2\beta)| \\ &=|-\sum_{n=2}^{\infty} nC(\alpha, n)a_n z^{n-1}|-\gamma|2(1-\beta)-\sum_{n=2}^{\infty} nC(\alpha, n)a_n z^{n-1}| \\ &\leqq \sum_{n=2}^{\infty} n(1+\gamma)C(\alpha, n)a_n-2\gamma(1-\beta)\leqq 0. \end{aligned} \tag{2.4}$$

Hence, by the maximum modulus theorem, we can see that $f(z)$ is in the class $\boldsymbol{P}_\alpha[\beta, \gamma]$.

Finally the result is sharp, with the extremal function being of the form

$$f(z)=z-\frac{2\gamma(1-\beta)}{n(1+\gamma)C(\alpha, n)}z^n \tag{2.5}$$

for some $n \geqq 2$.

COROLLARY 1. *Let the function $f(z)$ defined by* (1.11) *be in the class $\boldsymbol{P}_\alpha[\beta, \gamma]$. Then*

$$a_n \leqq \frac{2\gamma(1-\beta)}{n(1+\gamma)C(\alpha, n)} \tag{2.6}$$

for $n \geqq 2$. The equality is attained by the function $f(z)$ in (2.5).

In view of Theorem 1, it follows that $\boldsymbol{P}_\alpha[\beta, \gamma]$ is closed under convex linear combinations. We shall now show that the extreme points of the closed convex hull of $\boldsymbol{P}_\alpha[\beta, \gamma]$ are those that maximize the coefficients.

THEOREM 2. *Let*

$$f_1(z)=z \tag{2.7}$$

and

$$f_n(z)=z-\frac{2\gamma(1-\beta)}{n(1+\gamma)C(\alpha, n)}z^n \qquad (n \geqq 2). \tag{2.8}$$

Then $f(z)$ is in the class $\boldsymbol{P}_\alpha[\beta, \gamma]$ if and only if it can be expressed in the form

$$f(z)=\sum_{n=1}^{\infty}\lambda_n f_n(z), \tag{2.9}$$

where $\lambda_n \geqq 0$ for $n \in \boldsymbol{N}=\{1, 2, 3, \dots\}$ and

$$\sum_{n=1}^{\infty}\lambda_n=1. \tag{2.10}$$

PROOF. Assume that

$$\begin{aligned} f(z)=\sum_{n=1}^{\infty}\lambda_n f_n(z) &= z-\sum_{n=2}^{\infty}\frac{2\gamma(1-\beta)}{n(1+\gamma)C(\alpha, n)}\lambda_n z^n \\ &= z-\sum_{n=2}^{\infty}a_n z^n, \end{aligned} \tag{2.11}$$

where

$$(2.12)\qquad a_n=\frac{2\gamma(1-\beta)}{n(1+\gamma)C(\alpha,\ n)}\lambda_n.$$

Then we observe that

$$(2.13)\qquad \sum_{n=2}^{\infty} n(1+\gamma)C(\alpha,\ n)a_n=2\gamma(1-\beta)\sum_{n=2}^{\infty}\lambda_n$$
$$=2\gamma(1-\beta)(1-\lambda_1)\leqq 2\gamma(1-\beta).$$

This shows that $f(z)\in \boldsymbol{P}_\alpha[\beta,\ \gamma]$ with the aid of Theorem 1.

Conversely, assume that $f(z)$ is in the class $\boldsymbol{P}_\alpha[\beta,\ \gamma]$ for $0\leqq\alpha<1$, $0\leqq\beta<1$, and $0<\gamma\leqq 1$. Remembering the formula

$$\sum_{n=2}^{\infty}\frac{n(1+\gamma)C(\alpha,\ n)}{2\gamma(1-\beta)}a_n\leqq 1,$$

from Theorem 1, we may set

$$(2.14)\qquad \lambda_n=\frac{n(1+\gamma)C(\alpha,\ n)}{2\gamma(1-\beta)}a_n \qquad\qquad (n\geqq 2)$$

and we have

$$\sum_{n=2}^{\infty}\lambda_n\leqq 1.$$

Setting

$$(2.15)\qquad \lambda_1=1-\sum_{n=2}^{\infty}\lambda_n,$$

we have the representation (2.9). Thus we have the theorem.

3. Distortion Theorems

With the aid of Theorem 2, we may now find bounds on the modulus of $f(z)$ and $f'(z)$ for $f(z)\in \boldsymbol{P}_\alpha[\beta,\ \gamma]$.

THEOREM 3. *If the function* $f(z)$ *defined by* (1.11) *is in the class* $\boldsymbol{P}_\alpha[\beta,\ \gamma]$, $0\leqq\beta<1$, $0<\gamma\leqq 1$, *and either* $0\leqq\alpha\leqq 5/6$ *or* $|z|\leqq 3/4$, *then*

$$(3.1)\qquad |f(z)|\geqq\max\left\{0,\ |z|-\frac{\gamma(1-\beta)}{2(1-\alpha)(1+\gamma)}|z|^2\right\}$$

and

$$(3.2)\qquad f(z)|\leqq|z|+\frac{\gamma(1-\beta)}{2(1-\alpha)(1+\gamma)}|z|^2.$$

The bounds are sharp.

PROOF. By virtue of Theorem 2, we note that

$$|f(z)| \geqq \max\left\{0, |z| - \max_{n\in N-\{1\}} \frac{2\gamma(1-\beta)}{n(1+\gamma)C(\alpha, n)}|z|^n\right\} \tag{3.3}$$

and

$$|f(z)| \leqq |z| + \max_{n\in N-\{1\}} \frac{2\gamma(1-\beta)}{n(1+\gamma)C(\alpha, n)}|z|^n \tag{3.4}$$

for $z\in U$. Hence it suffices to deduce that

$$G(\alpha, \beta, \gamma, |z|, n) = \frac{2\gamma(1-\beta)}{n(1+\gamma)C(\alpha, n)}|z|^n \tag{3.5}$$

is a decreasing function of $n(n\geqq 2)$. Since

$$C(\alpha, n+1) = \frac{n+1-2\alpha}{n} C(\alpha, n), \tag{3.6}$$

we can see that, for $|z|\neq 0$,

$$G(\alpha, \beta, \gamma, |z|, n) \geqq G(\alpha, \beta, \gamma, |z|, n+1) \tag{3.7}$$

if and only if

$$H(\alpha, |z|, n) = (n+1)(n+1-2\alpha) - n^2|z| \geqq 0. \tag{3.8}$$

It is easy to see that $H(\alpha, |z|, n)$ is a decreasing function of α for fixed $|z|$. Consequently it follows that

$$H(\alpha, |z|, n) \geqq H(5/6, |z|, n) = n^2(1-|z|) + \frac{1}{3}(n-2) \geqq 0 \tag{3.9}$$

for $0\leqq\alpha\leqq 5/6$, $z\in U$, and $n\geqq 2$.

Further, since $H(\alpha, |z|, n)$ is decreasing in $|z|$ and increasing in n, we obtain that

$$H(\alpha, |z|, n) > H(1, |z|, n) \geqq H(1, 3/4, 2) = 0 \tag{3.10}$$

for $0\leqq\alpha<1$, $|z|\leqq 3/4$, and $n\geqq 2$. Thus

$$\max_{n\in N-\{1\}} G(\alpha, \beta, \gamma, |z|, n)$$

is attained at $n=2$, and the proof is complete.

Finally, since the functions $f_n(z)(n\geqq 2)$ defined in Theorem 2 are the extreme points of the class $\boldsymbol{P}_\alpha[\beta, \gamma]$, we can see that the bounds of the theorem are attained by the function $f_2(z)$ in (2.8), i.e.

$$(3.11) \qquad f_2(z)=z-\frac{\gamma(1-\beta)}{2(1-\alpha)(1+\gamma)}z^2.$$

COROLLARY 2. *Let the function* $f(z)$ *defined by* (1.11) *be in the class* $\boldsymbol{P}_\alpha[\beta,\gamma]$ *with* $0\leqq\alpha\leqq 5/6$, $0\leqq\beta<1$, *and* $0<\gamma\leqq 1$. *Then* $f(z)$ *is included in a disk with its center at the origin and radius* r *given by*

$$(3.12) \qquad r=1+\frac{\gamma(1-\beta)}{2(1-\alpha)(1+\gamma)}.$$

REMARK 1. The extremal function $f(z)$ given by (3.11) is equal to zero when $z=2(1-\alpha)(1+\gamma)/\gamma(1-\beta)$. Letting $z\to 1^-$, it follows that $\alpha\to(2+\gamma+\beta\gamma)/2(1+\gamma)$. We thus have

$$(3.13) \qquad |f(z)|\geqq|z|-\frac{\gamma(1-\beta)}{2(1-\alpha)(1+\gamma)}|z|^2$$

for all $z\in\boldsymbol{U}$ if and only if $0\leqq\alpha\leqq(2+\gamma+\beta\gamma)/2(1+\gamma)$.

Theorem 3 leaves open the question of an upper bound for $|f(z)|$ when $5/6<\alpha<1$ and $3/4<|z|<1$. That is, given n_0 fixed ($n_0=3, 4, 5, \ldots$) we are interested to find the vaules of α, β, γ and $|z|$ for which $|f_{n_0}(z)|$ gives the extremal values of $|f(z)|$ whenever $f(z)\in\boldsymbol{P}_\alpha[\beta,\gamma]$.

THEOREM 4. *Let* $r(n_0,\alpha)=(n_0+1)(n_0+1-2\alpha)/n_0^2$. *Further let the function* $f(z)$ *defined by* (1.11) *be in the class* $\boldsymbol{P}_\alpha[\beta,\gamma]$ *for* $r(n_0,\alpha)<|z|<1$ *with* $0\leqq\beta<1$, $0<\gamma\leqq 1$, *and*

$$(3.14) \qquad \frac{2n_0+1}{2(n_0+1)}<\alpha\leqq\frac{2n_0+3}{2(n_0+2)} \qquad (n_0\geqq 2).$$

Then

$$(3.15) \qquad |f(z)|\leqq|z|+\frac{2\gamma(1-\beta)}{(n_0+1)(1+\gamma)C(\alpha, n_0+1)}|z|^{n_0+1}.$$

Equality holds for the function given by

$$(3.17) \qquad f(z)=z-\frac{2\gamma(1-\beta)}{(n_0+1)(1+\gamma)C(\alpha, n_0+1)}z^{n_0+1}.$$

PROOF. Let us find a condition under which $G(\alpha,\beta,\gamma,|z|,n)$ in (3.5) is maximized for $n=n_0+1>2$. The maximum will occur if $H(\alpha,|z|,n)$ in (3.8) is negative for $n=n_0$ and positive for $n=n_0+1$. For fixed n_0, it follows from (3.8) that both conditions will be satisfied when

$$(3.18) \qquad r(n_0,\alpha)<|z|<r(n_0+1,\alpha).$$

Here, $r(n_0, \alpha)<1$ if and only if $\alpha>(2n_0+1)/2(n_0+1)$ and $r(n_0+1, \alpha)\geqq 1$ for $\alpha\leqq(2n_0+3)/2(n_0+2)$. Hence, the maximum of $G(\alpha, \beta, \gamma, |z|, n)$ in (3.5) occurs at $n=n_0+1$ for $r(n_0, \alpha)<|z|<1$ and α as given by (3.14). This completes the proof of the theorem.

THEOREM 5. *If the function $f(z)$ defined by* (1.11) *is in the class $\boldsymbol{P}_\alpha[\beta, \gamma]$, $0\leqq\beta<1$, $0<\gamma\leqq 1$, and either $0\leqq\alpha\leqq 1/2$ or $|z|\leqq 1/2$, then*

$$(3.19)\qquad 1-\frac{\gamma(1-\beta)}{(1-\alpha)(1+\gamma)}|z|\leqq|f'(z)|\leqq 1+\frac{\gamma(1-\beta)}{(1-\alpha)(1+\gamma)}|z|.$$

The bounds are sharp.

PROOF. By means of Theorem2, we note that

$$(3.20)\qquad |f'(z)|\geqq 1-\max_{n\in N-\{1\}}\frac{2\gamma(1-\beta)}{(1+\gamma)C(\alpha, n)}|z|^{n-1}$$

and

$$(3.21)\qquad |f'(z)|\leqq 1+\max_{n\in N-\{1\}}\frac{2\gamma(1-\beta)}{(1+\gamma)C(\alpha, n)}|z|^{n-1}.$$

It suffices to deduce that

$$(3.22)\qquad G_1(\alpha, \beta, \gamma, |z|, n)=\frac{2\gamma(1-\beta)}{(1+\gamma)C(\alpha, n)}|z|^{n-1}$$

is a decreasing function of $n(n\geqq 2)$. But we can see that, for $|z|\neq 0$,

$$(3.23)\qquad G_1(\alpha, \beta, \gamma, |z|, n)\geqq G_1(\alpha, \beta, \gamma, |z|, n+1)$$

if and only if

$$(3.24)\qquad H_1(\alpha, |z|, n)=n+1-2\alpha-n|z|\geqq 0.$$

Since $H_1(\alpha, |z|, n)$ is decreasing in $|z|$, it follows that

$$(3.25)\qquad H_1(\alpha, |z|, n)\geqq H_1(\alpha, 1, n)=1-2\alpha\geqq 0$$

for $0\leqq\alpha\leqq 1/2$. Further, $H_1(\alpha, |z|, n)$ is decreasing in α, we have

$$(3.26)\qquad \begin{aligned} H_1(\alpha, |z|, n)&\geqq H_1(1, |z|, n)=n-1-n|z|\geqq H_1(1, 1/2, n)\\ &\geqq H_1(1, 1/2, 2)=0\end{aligned}$$

for $|z|\leqq 1/2$.

Finally the bounds of the theorem are attained by the function $f_2(z)$ given by (3.11).

REMARK 2. In order to see that $|z|\leqq 1/2$ is best possible, we note that

(3.27) $\quad H_1(1, |z|, 2)=1-2|z|<0$

for $|z|>1/2$.

REMARK 3. The bound of α for which (3.19) still holds for all $|z|$ can't be improved. To show this, let us find $n\geqq 3$ for which the maximum in (3.20) or (3.21) attains. This means to show the inequality

$$(3.28)\qquad \max_{n\in N-\{1\}} \frac{2\gamma(1-\beta)}{(1+\gamma)C(\alpha, n)}|z|^{n-1}>\frac{\gamma(1-\beta)}{(1-\alpha)(1+\gamma)}|z|$$

for each α $(1/2<\alpha<1)$. And this is equivalent to showing that there is an n for which

$$(3.29)\qquad C(\alpha, n)<2(1-\alpha)|z|^{n-2}.$$

Putting $|z|=1-1/(n-2)$, the right hand side of (3.29) approaches $2(1-\alpha)e^{-1}>0$ as $n\to\infty$. Further, for $1/2<\alpha<1$, we found in (1.7) that

$$\lim_{n\to\infty} C(\alpha, n)=0.$$

Consequently, for each α $(1/2<\alpha<1)$, there exists surely an integer n and a real number $|z|<1$ for which (3.29) holds.

4. Radii of Starlikeness and Convexity

In this section, we determine the radii of starlikeness and convexity of functions $f(z)$ in $\boldsymbol{P}_\alpha[\beta, \gamma]$.

We say that the function $f(z)$ is starlike of order α in the disk $|z|<r$ if it satisfies (1.2) for $|z|<r$, and that the function $f(z)$ is convex of order α in the disk $|z|<r$ if it satisfies (1.3) for $|z|<r$.

THEOREM 6. *$\boldsymbol{P}_\alpha[\beta, \gamma]$ is a subclass of $\boldsymbol{S}$ if and only if $0\leqq\alpha\leqq 1/2$.*

PROOF. Note that the function $f(z)$ defined by (1.1) is in the class $\boldsymbol{S}$ if

$$(4.1)\qquad \sum_{n=2}^{\infty} n|a_n|\leqq 1 \qquad\qquad \text{(cf. [14])}.$$

Hence it suffices to prove that

$$(4.2)\qquad (1+\gamma)C(\alpha, n)\geqq 2\gamma(1-\beta)$$

for $0\leqq\alpha\leqq 1/2$ and $n\geqq 2$ by means of Theorem 1. Since $C(\alpha, n)\geqq C(1/2, n)=1$ for $0\leqq\alpha\leqq 1/2$, we can see that, for $0\leqq\alpha\leqq 1/2$, $0\leqq\beta<1$, and $0<\gamma\leqq 1$,

$$(4.3)\qquad (1+\gamma)C(\alpha, n)-2\gamma(1-\beta)\geqq 1-\gamma+2\beta\gamma\geqq 0.$$

Conversely, if we assume $\alpha>1/2$, then

$$\lim_{n\to\infty} C(\alpha, n)=0.$$

Taking the function $f_n(z)$ given by (2.8), we have

$$f'_n(z)=1-\frac{2\gamma(1-\beta)}{(1+\gamma)C(\alpha, n)}z^{n-1}=0 \tag{4.4}$$

for

$$z^{n-1}=\frac{(1+\gamma)C(\alpha, n)}{2\gamma(1-\beta)}, \tag{4.5}$$

which is less than one for n sufficiently large. Thus $f_n(z)$ is not univalent for $\alpha>1/2$ and $n=n(\alpha)$ sufficiently large.

REMARK 4. Since a function $f(z)\in \boldsymbol{T}$ is starlike if and only if it is univalent [14], and $\boldsymbol{P}_\alpha[\beta, \gamma]\subset \boldsymbol{T}$, it follows from the above theorem that the functions in $\boldsymbol{P}_\alpha[\beta, \gamma]$, $0\leqq\alpha\leqq 1/2$ are all starlike.

In view of Remark 4, we now determine the largest disk in which functions in $\boldsymbol{P}_\alpha[\beta, \gamma]$, $0\leqq\alpha\leqq 1/2$, are starlike of order $\delta(0\leqq\delta<1)$.

THEOREM 7. *Let the function $f(z)$ defined by (1.11) be in the class $\boldsymbol{P}_\alpha[\beta, \gamma]$ with $0\leqq\alpha\leqq 1/2$, $0\leqq\beta<1$, and $0<\gamma\leqq 1$. Then $f(z)$ is starlike of order $\delta(0\leqq\delta<1)$ in the disk $|z|<r_1$, where*

$$r_1=\inf_{n\in \boldsymbol{N}-\{1\}}\left\{\frac{n(1+\gamma)(1-\delta)C(\alpha, n)}{2\gamma(1-\beta)(n-\delta)}\right\}^{1/(n-1)}. \tag{4.6}$$

PROOF. Note that

$$\left|\frac{zf'(z)}{f(z)}-1\right|\leqq\frac{\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}a_n|z|^{n-1}}\leqq 1-\delta \tag{4.7}$$

if and only if

$$\sum_{n=2}^{\infty}\left(\frac{n-\delta}{1-\delta}\right)a_n|z|^{n-1}\leqq 1. \tag{4.8}$$

By virtue of Theorem 1, we need only to find values of $|z|$ for which the inequality

$$\left(\frac{n-\delta}{1-\delta}\right)|z|^{n-1}\leqq\frac{n(1+\gamma)C(\alpha, n)}{2\gamma(1-\beta)} \tag{4.9}$$

is valid for all $n \geqq 2$, which will be true when $|z|<r_1$. This completes the proof of the theorem.

THEOREM 8. *Let the function $f(z)$ defined by* (1.11) *be in the class $\boldsymbol{P}_\alpha[\beta,\gamma]$ with $0 \leqq \alpha \leqq 1/2$, $0 \leqq \beta<1$, and $0<\gamma \leqq 1$. Then $f(z)$ is convex of order $\delta (0 \leqq \delta<1)$ in the disk $|z|<r_2$, where*

$$(4.10) \qquad r_2=\inf_{n \in \boldsymbol{N}-\{1\}}\left\{\frac{(1+\gamma)(1-\delta) C(\alpha, n)}{2 \gamma(1-\beta)(n-\delta)}\right\}^{1/(n-1)}.$$

PROOF. Since $f(z)$ is convex of order δ if and only if $zf'(z)$ is starlike of order δ, we have the theorem by replacing a_n with na_n in Theorem 7.

5. Order of Starlikeness

In view of Remark 4, it is of interest to determine the order of starlikeness for functions $f(z)$ *in the class* $\boldsymbol{P}_\alpha[\beta,\gamma]$ with $0 \leqq \alpha \leqq 1/2$, $0 \leqq \beta<1$, and $0<\gamma \leqq 1$.

THEOREM 9. *Let the function $f(z)$ defined by* (1.11) *be in the class $\boldsymbol{P}_\alpha[\beta,\gamma]$ with $0 \leqq \alpha \leqq 1/2$, $0 \leqq \beta<1$, and $0<\gamma \leqq 1$. Then $f(z)$ is starlike of order λ, where*

$$(5.1) \qquad \lambda=\frac{2(1-\alpha)-2 \gamma(\alpha-\beta)}{2(1-\alpha)+\gamma(1-2 \alpha+\beta)}.$$

PROOF. It is known [14] that a function

$$f(z)=z-\sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqq 0)$$

in $\boldsymbol{T}$ is starlike of order λ if and only if

$$\sum_{n=2}^{\infty}(n-\lambda) a_n \leqq 1-\lambda.$$

Therefore, in view of Theorem 1, it suffices to show that

$$(5.2) \qquad \sum_{n=2}^{\infty} \frac{n(1+\gamma) C(\alpha, n)}{2 \gamma(1-\beta)} a_n \leqq 1$$

implies that

$$(5.3) \qquad \sum_{n=2}^{\infty}\left(\frac{n-\lambda}{1-\lambda}\right) a_n \leqq 1.$$

This will be true if

$$(5.4)\qquad G_2(\alpha, \beta, \gamma, n)=\frac{n(1+\gamma)C(\alpha, n)(1-\lambda)}{2\gamma(1-\beta)(n-\lambda)}\geqq 1$$

for $n\geqq 2$. For β and γ fixed, $G_2(\alpha, \beta, \gamma, n)$ can be shown to be a decreasing function of α $(0\leqq\alpha\leqq 1/2)$, and an increasing function of $n(n\geqq 2)$, so that

$$(5.5)\qquad G_2(\alpha, \beta, \gamma, n)\geqq G_2(1/2, \beta, \gamma, 2)=1$$

for $0\leqq\alpha\leqq 1/2$ and $n\geqq 2$. Thus we have the theorem.

6. Application of the Fractional Calculus

Many essentially equivalent definitions of the fractional calculus, that is, the fractional derivatives and the fractional integrals, have been given in the literature (cf., e.g., [2, Chapter 13], [5], [6], [10], [11], and [17, p. 28 et seq.]). We find it to be convenient to recall here the following definitions which were used recently by Owa [7] (and by Srivastave and Owa [18]).

DEFINITION 1. The fractional integral of order λ is defined by

$$(6.1)\qquad D_z^{-\lambda}f(z)=\frac{1}{\Gamma(\lambda)}\int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}}\,d\zeta,$$

where $\lambda>0$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$.

DEFINITION 2. The fractional derivative of order λ is defined by

$$(6.2)\qquad D_z^{\lambda}f(z)=\frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}}\,d\zeta,$$

where $0\leqq\lambda<1$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$.

DEFINITION 3. Under the hypotheses of Definition 2, the fractional derivative of order $(n+\lambda)$ is defined by

$$(6.3)\qquad D_z^{n+\lambda}f(z)=\frac{d^n}{dz^n}D_z^{\lambda}f(z),$$

where $0\leqq\lambda<1$ and $n\in\boldsymbol{N}\cup\{0\}$.

THEOREM 10. *Let the function* $f(z)$ *defined by* (1.11) *be in the class* $\boldsymbol{P}_\alpha[\beta, \gamma]$ *with* $0\leqq\alpha\leqq 1/2$, $0\leqq\beta<1$, *and* $0<\gamma\leqq 1$. *Then*

$$(6.4)\qquad |D_z^{-\lambda}f(z)|\geqq\frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)}\left\{1-\frac{\gamma(1-\beta)}{(2+\lambda)(1-\alpha)(1+\gamma)}|z|\right\}$$

and

$$(6.5)\qquad |D_z^{-\lambda}f(z)|\leqq\frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)}\left\{1+\frac{\gamma(1-\beta)}{(2+\lambda)(1-\alpha)(1+\gamma)}|z|\right\}$$

for $\lambda>0$ *and* $z\in\boldsymbol{U}$. *The bounds are sharp.*

PROOF. For $f(z)\in\boldsymbol{P}_\alpha[\beta,\gamma]$, it is easily known that

$$D_z^{-\lambda}f(z)=\frac{1}{\Gamma(2+\lambda)}z^\lambda\left\{z-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)}a_nz^n\right\}.$$

Now, we consider the function

$$(6.6)\qquad F(z)=z-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)}a_nz^n$$

for $\lambda>0$. We note that

$$(6.7)\qquad 0<\frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)}\leqq\frac{2}{2+\lambda}$$

for $\lambda>0$ and $n\geqq2$, and $C(\alpha,n+1)\geqq C(\alpha,n)$ for $0\leqq\alpha\leqq1/2$ and $n\geqq2$. Hence, by using Theorem 1, we obtain that

$$\begin{aligned}(6.8)\qquad |F(z)|&\geqq|z|-|z|^2\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)}a_n\\&\geqq|z|-\left(\frac{2}{2+\lambda}\right)|z|^2\sum_{n=2}^{\infty}a_n\\&\geqq|z|-\frac{\gamma(1-\beta)}{(2+\lambda)(1-\alpha)(1+\gamma)}|z|^2\end{aligned}$$

which gives (6.4) and

$$\begin{aligned}(6.9)\qquad |F(z)|&\leqq|z|+|z|^2\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)}a_n\\&\leqq|z|+\left(\frac{2}{2+\lambda}\right)|z|^2\sum_{n=2}^{\infty}a_n\\&\geqq|z|+\frac{\gamma(1-\beta)}{(2+\lambda)(1-\alpha)(1+\gamma)}|z|^2\end{aligned}$$

which gives (6.5).

Further, taking the function $f(z)$ defined by (3.11), we can see that the bounds of the theorem are sharp.

THEOREM 2. *Let the function* $f(z)$ *defined by* (1.11) *be in the class* $\boldsymbol{P}_\alpha[\beta,\gamma]$ *with* $0\leqq\alpha<1$, $0\leqq\beta<1$, *and* $0<\gamma\leqq1$. *Then*

$$(6.10)\qquad |D_z^\lambda f(z)| \geqq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)}\left\{1-\frac{\gamma(1-\beta)}{(2-\lambda)(1-\alpha)(1+\gamma)}|z|\right\}$$

and

$$(6.11)\qquad |D_z^\lambda f(z)| \leqq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)}\left\{1+\frac{\gamma(1-\beta)}{(2-\lambda)(1-\alpha)(1+\gamma)}|z|\right\}$$

for $0 \leqq \lambda < 1$ *and* $z \in \boldsymbol{U}$. *The bounds are sharp.*

PROOF. We consider the function

$$(6.12)\qquad P(z)=\Gamma(2-\lambda)z^\lambda D_z^\lambda f(z)=z-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}a_n z^n$$

for $0 \leqq \lambda < 1$. In view of Theorem 1, we have

$$(6.13)\qquad (1+\gamma)C(\alpha,2)\sum_{n=2}^{\infty} na_n \leqq \sum_{n=2}^{\infty} n(1+\gamma)C(\alpha,n)a_n \leqq 2\gamma(1-\beta)$$

which implies that

$$(6.14)\qquad \sum_{n=2}^{\infty} na_n \leqq \frac{\gamma(1-\beta)}{(1-\alpha)(1+\gamma)}.$$

Further we note that

$$(6.15)\qquad 1 \leqq \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} \leqq \frac{n}{2-\lambda}$$

for $0 \leqq \lambda < 1$ and $n \geqq 2$. Consequently we find that

$$(6.16)\qquad \begin{aligned} |P(z)| &\geqq |z|-|z|^2\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}a_n \\ &\geqq |z|-\left(\frac{1}{2-\lambda}\right)|z|^2\sum_{n=2}^{\infty} na_n \\ &\geqq |z|-\frac{\gamma(1-\beta)}{(2-\lambda)(1-\alpha)(1+\gamma)}|z|^2 \end{aligned}$$

which implies (6.10), and

$$(6.17)\qquad \begin{aligned} |P(z)| &\leqq |z|+|z|^2\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}a_n \\ &\leqq |z|+\left(\frac{1}{2-\lambda}\right)|z|^2\sum_{n=2}^{\infty} na_n \\ &\leqq |z|+\frac{\gamma(1-\beta)}{(2-\lambda)(1-\alpha)(1+\gamma)}|z|^2 \end{aligned}$$

which implies (6.11).

Finally we can see that the bounds of the theorem are sharp for the function $f(z)$ given by (3.11). This completes the proof of the theorem.

THEOREM 12. *Let the function* $f(z)$ *defined by* (1.11) *be in the class* $\boldsymbol{P}_\alpha[\beta, \gamma]$ *with* $0 \leqq \alpha < 1$, $0 \leqq \beta < 1$, *and* $0 < \gamma \leqq 1$. *Then*

$$(6.18) \qquad |D_z^{1-\lambda} f(z)| \leqq \frac{|z|^\lambda}{\Gamma(2+\lambda)} \left\{ 1+\lambda+\frac{\gamma(1-\beta)}{(1-\alpha)(1+\gamma)}|z| \right\}$$

for $\lambda > 0$ *and* $z \in \boldsymbol{U}$. *The result is sharp.*

PROOF. Let the function $F(z)$ *be defined by* (6.6). *Then, by using* (6.7) *and* (6.14), *we prove that*

$$(6.19) \qquad \begin{aligned} |F'(z)| &\leqq 1+|z| \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)} n a_n \\ &\leqq 1+\left(\frac{2}{2+\lambda}\right)|z| \sum_{n=2}^{\infty} n a_n \\ &\leqq 1+\frac{2\gamma(1-\beta)}{(2+\lambda)(1-\alpha)(1+\gamma)}|z| \end{aligned}$$

which implies that

$$(6.20) \qquad \begin{aligned} &|\Gamma(2+\lambda) z^{-\lambda} D_z^{1-\lambda} f(z)| \\ &\leqq \lambda \Gamma(2+\lambda)|z|^{-1-\lambda} |D_z^{-\lambda} f(z)|+1 \\ &\qquad +\frac{2\gamma(1-\beta)}{(2+\lambda)(1-\alpha)(1+\gamma)}|z|. \end{aligned}$$

Hence we have the inequality (6.18) with the aid of (6.5). Further, taking the function $f(z)$ defined by (3.11), we can show that the result of the theorem is sharp.

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