Hokkaido Mathematical Journal Vol. 15 (1986), p. 47～70

SOLVABILITY OF FINITE GROUPS ADMITTING S_3 AS A FIXED-POINT-FREE GROUP OF OPERATORS

By Hiroshi FUKUSHIMA
(Received November 13, 1984)

1. Introduction

If A is a group of automorphisms of a finite group G, we say that A acts fixed-point-freely on G if $C_G(A)=1$ ($C_G(A)$ is the set of elements of G fixed by every element of A). An important theorem of Thompson states that, in this situation, if A has prime order then G is nilpotent. R. P. Martineau has shown that G must be solvable if A is any elementary abelian group. Mrs. E. W. Ralston has shown that G must be solvable if A is cyclic of order rs, r and s distinct primes. Here we prove the following result without using the Feit-Thompson theorem on the groups of odd order.

THEOREM. *Let G be a finite group admitting a fixed-point-free group of automorphisms A, where A is isomorphic to the symmetric group of degree* 3 *and* $(|G|,|A|)=1$. *Then G is solvable.*

We now discuss the proof of the theorem. We assumed that the theorem is false and take a counterexample G to the theorem of least order.

To fix ideas, set $A=<\sigma,\tau\,|\,\sigma^3=\tau^2=1,\tau^{-1}\sigma\tau=\sigma^{-1}>$. By Lemma 2. 1(iv), G has only one A-invariant Sylow p-subgroups of G for each prime p that divides $|G|$. Let P be the A-invariant Sylow p-subgroup of G.

In section 4, we prove that if $C_P(\sigma)=1$, then $C_G(\tau)$ has a normal p-complement. This result is important in the proof of the theorem.

In section 5, 6, 7, and 8, we prove that if P, Q be the A-invariant Sylow p-, q-subgroups, then $PQ=QP$. By P. Hall's characterization of solvable groups, G is solvable. This shows that G does not exist.

All groups considered in this paper are assumed finite. Our notation corresponds to that of Gorenstein [2]. For a prime p, we let $Syl_p(G)$ denote the set of Sylow p-subgroups of G.

2. Some preliminary results

We first quote some frequently used results.

LEMMA 2.1 *Let G be a group admitting the coprime operator group V.*

(i) *If N is a normal V-invariant subgroup of G, then*

$$C_{G/N}(V)=C_G(V)N/N.$$

(ii) $G=C_G(V)[G, V]$ *where* $[G, V]=<g^{-1}g^v| g \in G, v \in V>$ *and* $[[G, V], V]=[G, V]\lhd G$. *Furthermore, if G is abelian, then* $G=C_G(V)\times[G, V]$.

(iii) *Let S be a subset of G, and set $\psi=\{S^g|g\in G\}$. If ψ is V-invariant, then there exists $S_1\in\psi$ such that S_1 is V-invariant.*

(iv) *For each $p\in\pi(G)$ there exists at least one V-invariant Sylow p-subgroup of G and any two such Sylow p-subgroups are conjugate by an element of $C_G(V)$. Moreover, every V-invariant p-subgroup of G is contained in at least one V-invariant Sylow p-subgroup of G.*

(v) *Suppose G is solvable, and let $\pi\subseteq\pi(G)$. Then G possesses at least one V-invariant Hall π-subgroup and every V-invariant π-subgroup of G is contained in some V-invariant Hall π-subgroup.*

PROOF. (i) and (iv) follow from Theorem 6. 2. 2 of [3], and (ii) follows from (i) and Corollary 5. 2. 5 of [3]. (iii) is proved as [2] Corollary 1 of Theorem 4. Theorem 6. 4. 1 of [3] and (iii) yield (v).

LEMMA 2.2 [3, p. 341]. *Let G be a group of odd order which admits an automorphism ϕ of order 2. Set $F=C_G(\phi)$ and I be the subset of elements of G transformed into their inverses by ϕ. Then the following conditions hold:*

(i) $G=FI=IF$, $I\cap F=1$, *and* $|I|=|G:F|$.

(ii) *I is invariant under F.*

(iii) *If H is a subset of F such that $H^x\subseteq F$ for x in I, then x centralizes H.*

(iv) *If H is a subgroup of I, then H is abelian.*

LEMMA 2.3. (*Clifford* [2, *Theorem* 6.4.1]). *Let U/F be an irreducible G-module and let H be a normal subgroup of G. Then U is the direct sum of H-invariant subspaces U_i, $1\leq i\leq r$, which satisfy the following conditions:*

(i) $U_i=X_{i1}\oplus \dots \oplus X_{it}$, *where each X_{ij} is an irreducible H-submodule, $1\leq i\leq r$, t is independent of i, and X_{ij}, $X_{i'j'}$ are isomorphic H-modules if and only if $i=i'$.*

(ii) *For x in G, the mapping $\pi(x): U_i\rightarrow U_ix$, $1\leq i\leq r$, is a permutation of the set $S=\{U_1, \dots, U_r\}$ and π induces a transitive permutation representation of G on S.*

LEMMA 2.4 (*Shult* [7, *Theorem A*]). *Let $G=NQP$ with $N \lhd G$, $Q \lhd QP$, $|P|$ is a prime, $|Q|$ is an odd and $(|Q|, |P|)=1$, $(|N|, |Q|)=1$. Assume further that $C_N(P)=1$. Then $[P, Q] \subseteq C_Q(N)$.*

LEMMA 2.5 [4]. *A p-group which admits a fixed-point-free automorphism of order 3 has class at most 2.*

LEMMA 2.6 [2, p. 218]. *If G is solvable, then $C_G(F(G)) \subseteq F(G)$. In particular, if $O_{p'}(G)=1$, then $C_G(O_p(G)) \subseteq O_p(G)$.*

Suppose p is an odd prime and P is an Sylow p-subgroup of G. A normal subgroup T of P is said to control strong fusion in P if T has the following property.

Whenever $W \subseteq P$, $g \in G$, and $W^g \subseteq P$, then there exists $c \in C_G(W)$ and $n \in N_G(T)$ such that $cn=g$.

Define the quadratic group for the prime p to be the semi-direct product $Qd(p)$ of a two dimentional vector space V over $GF(p)$ by the special linear group $SL(V)$ on V. Let $F(p)$ be the normalizer of some Sylow p-subgroup of $Qd(p)$.

LEMMA 2.7 (*Glauberman* [1]). *If $F(p)$ is not involved in $N_G(Z(J(P)))$, then $Z(J(P))$ controls strong fusion in P with respect to G.*

3. Finite groups which admits a fixed-point-free group of automorphisms which is isomorhic to the symmetric group S_3

For the remainder of this paper, we are concerned with the following situation.

HYPOTHESIS 3.1. *Let G be a finite group which admits a fixed-point-free group of automorphisms A, where A is isomorphic to the symmetric group of degree 3 and $(|G|, |A|)=1$.*

We fix notation as in this hypothesis and set $A=<\sigma, \tau \mid \sigma^3=1=\tau^2, \tau^{-1}\sigma\tau=\sigma^{-1}>$

LEMMA 3.1. *τ, $\sigma\tau$ and $\sigma^2\tau$ invert every element of $C_G(\sigma)$. In particular $C_G(\sigma)$ is abelian and for each $a \in C_G(\sigma)$, $<a>$ is an A-invariant subgroup of G.*

PROOF. As A acts fixed-point-freely on $C_G(\sigma)$, τ, $\sigma\tau$ and $\sigma^2\tau$ invert every element of $C_G(\sigma)$ and so $C_G(\sigma)$ is abelian and for each $a \in C_G(\sigma)$, $<a>$ is an A-invariant subgroup of G.

LEMMA 3.2. *If* $C_G(\sigma)=1$, *then* $G=C_G(\tau)C_G(\sigma\tau)$.

PROOF. By (3.1) of [8], $|G|=|C_G(\tau)||C_G(\sigma\tau)|$. Since A acts fixed-point-freely on G, $C_G(\tau)\cap C_G(\sigma\tau)=1$ and so $|C_G(\tau)C_G(\sigma\tau)|=|C_G(\tau)||C_G(\sigma\tau)|=|G|$. Hence $G=C_G(\tau)C_G(\sigma\tau)$.

LEMMA 3.3. *Let* P *be an* A*-invariant Sylow* p*-subgroup of* G. *Then* $C_P(\tau)$ *is a Sylow* p*-subgroup of* $C_G(\tau)$.

PROOF. Let P^* be a Sylow p-subgroup of $C_G(\tau)$. Since P is a τ-invariant Sylow p-subgroup of G, $P^{*g}\subseteq P$ for some $g\in C_G(\tau)$ by Lemma 2.1(iv). Hence $P^{*g}\subseteq C_P(\tau)$. This implies that $C_P(\tau)$ is a Sylow p-subgroup of $C_G(\tau)$.

LEMMA 3.4. *If* Y *is a subgroup of* $C_G(\sigma)$, *then* Y *is an* A*-invariant subgroup of* G *and* $N_G(Y)=C_G(Y)$. *In particular, if* G *is a* p*-group and* $[G,\sigma]\neq 1$, *then* $[C_G(C_G(\sigma)),\sigma]\neq 1$.

PROOF. By Lemma 3.1, Y is A-invariant. Since $[\sigma, Y, N_G(Y)]=[Y, N_G(Y),\sigma]=1$, $[N_G(Y),\sigma, Y]=1$ by the three subgroup lemma. Hence $[N_G(Y),\sigma]\subseteq C_G(Y)$. Then since $N_G(Y)=[N_G(Y),\sigma]\ (N_G(Y)\cap C_G(\sigma))$ and $Y\subseteq C_G(\sigma)$ is abelian by Lemma 3.1, $N_G(Y)=C_G(Y)$.

If G is a p-group and $[G,\sigma]\neq 1$, $N_G(C_G(\sigma))\supsetneqq C_G(\sigma)$. Since $N_G(C_G(\sigma))=C_G(C_G(\sigma))$, $[C_G(C_G(\sigma)),\sigma]\neq 1$.

LEMMA 3.5. *Let* P *be an* A*-invariant Sylow* p*-subgroup of* G. *If* $[P,\sigma]=1$, *then* G *has a normal* p*-complement.*

PROOF. Let Y be a subgroup of P. By Lemma 3.4, $N_G(Y)/C_G(Y)=1$. Hence G has a normal p-complement.

LEMMA 3.6. *If* G *is cyclic, then* $[G,\sigma]=1$.

PROOF. By Lemma 2.1(iii), there exists the A-invariant Sylow p-subgroup of G for each $p\in\pi(G)$. Since the group of automorphisms of a cyclic group is abelian, σ centralizes a Sylow p-subgroup of G. Hence σ centralizes G.

LEMMA 3.7. *If* G *is a* p*-group and* $C_G(\sigma)=1$, *then* $G'=C_G(\tau)'C_G(\sigma\tau)'$. *In particular, if* $C_G(\tau)$ *is abelian, then* G *is abelian.*

PROOF. Let x, y be elements of G. Since $C_G(\sigma)=1$, class $G\leq 2$ by Lemma 2.5 and so $yy^{\sigma}y^{\sigma^2}=1=[x,y]\ [x,y]^{\sigma}[x,y]^{\sigma^2}$ by Lemma 1.1, p. 334 of [3]. Hence $1=[x, yy^{\sigma}y^{\sigma^2}]=[x,y]\ [x,y^{\sigma}]\ [x,y^{\sigma^2}]$ and so $[x,y^{\sigma}]\ [x,y^{\sigma^2}]=$

$[x, y]^{\sigma}[x, y]^{\sigma^2}=[x^{\sigma}, y^{\sigma}]\ [x^{\sigma^2}, y^{\sigma^2}]$. Since $[x^{-\sigma}, y^{\sigma}]\ [x^{\sigma}, y^{\sigma}]=[x^{-\sigma}x^{\sigma}, y^{\sigma}]$ $=1$ and $[x, y^{\sigma^2}]\ [x^{-1}, y^{\sigma^2}]=[xx^{-1}, y^{\sigma^2}]=1$, $[x^{-\sigma}, y^{\sigma}]\ [x, y^{\sigma}]=[x^{\sigma^2}, y^{\sigma^2}]$ $[x^{-1}, y^{\sigma^2}]$. Hence $[x^{-\sigma}x, y^{\sigma}]=[x^{\sigma^2}x^{-1}, y^{\sigma^2}]$. Set $z=x^{-\sigma}x$, then $z^{\sigma^2}=x^{-1}x^{\sigma^2}$. Hence $[z, y^{\sigma}]=[z^{\sigma^2}, y^{\sigma^2}]=[z, y]^{\sigma^2}$. Since $C_G(\sigma)=1$, every element of G can be expressed in the form $x^{-\sigma}x$ for suitable x in G and so $[z, y^{\sigma}]$ $=[z, y]^{\sigma^2}$ for every element z of G.

Let $a\in C_G(\tau)$ and $b\in C_G(\sigma\tau)$. Since $b^{\sigma^{-1}}\in C_G(\tau)$, $[a, b]=[a, (b^{\sigma^{-1}})^{\sigma}]=$ $[a, b^{\sigma^{-1}}]^{\sigma^2}\in C_G(\sigma^2\tau)'$. This implies $[C_G(\tau), C_G(\sigma\tau)]\subseteq C_G(\sigma^2\tau)'$. Similarly, we have $[C_G(\sigma\tau), C_G(\sigma^2\tau)]\subseteq C_G(\tau)'$ and $[C_G(\sigma^2\tau), C_G(\tau)]\subseteq C_G(\sigma\tau)'$.

Let $c\in C_G(\tau)$ and $d\in C_G(\sigma\tau)$. Since $C_G(\sigma)=1$, $cc^{\sigma}c^{\sigma^2}=1$ and so $c=$ $(c^{-1})^{\sigma^2}(c^{-1})^{\sigma}$. Then, since class $G\leq 2$ and $G=C_G(\tau)C_G(\sigma\tau)$ by lemma 3. 2, $G'=C_G(\tau)'C_G(\sigma\tau)'[C_G(\tau), C_G(\sigma\tau)]$. Hence $G'=C_G(\tau)'C_G(\sigma\tau)'$.

If $C_G(\tau)$ is abelian, then $C_G(\tau)'=1=(C_G(\tau)')^{\sigma}=(C_G(\tau)^{\sigma})'=C_G(\sigma\tau)'$. Thus $G'=1$ and so G is abelian.

LEMMA 3. 8. *If G is solvable, then G' is nilpotent. Furthermore, let P be a Sylow p-subgroup of G, then $G=O_{p'}(G)N_G(P)$.*

PROOF. See Corollary 2. 1 of [7] and Lemma 5. 4, p. 350 of [3].

LEMMA 3. 9. *Assume that $G=HV\triangleright V$, where V is an A-invariant elementary abelian p-group, p a prime, and H is an A-invariant abelian p'-group. We consider V to be a vector space over the field Z_p with p elements, and so we regard V as a HA-module. Then $C_H(C_V(\tau))\subseteq C_H(V)$.*

PROOF. Suppose false. We may assume that $C_H(V)=1$. Since $C_H(C_V(\tau))$ contains an element x of order r for some prime r distinct from p, we may assumed that H is an elementary abelian r-group. Moreover, since V is a completely reducible HA-module, there exists an irreducible HA-submodule of V on which x acts non-trivially. Hence we may assume that HA acts irreducibly on V.

Let W be a Wedderburn component of V with respect to H. We now consider three cases as $V=W$, $V=W\oplus W^{\tau}$ and $V=W\oplus W^{\sigma}\oplus W^{\sigma^2}$.

CASE I. $V=W$

Since H is abelian, H is represented by scalor multiplication on V. Then $[A, H]=1$ since $C_H(V)=1$. Thus $1\neq x\in H=C_H(A)=1$, a contradiction.

CASE II. $V=W\oplus W^{\tau}$

Let $a\in W$. Since $a+a^{\tau}\in C_V(\tau)$, $(a+a^{\tau})^x=a+a^{\tau}$ and so $a=a^x$ and $(a^{\tau})^x=a^{\tau}$. Hence $x\in C_H(V)=1$, a contradiction.

CASE III. $V=W\oplus W^{\sigma}\oplus W^{\sigma^2}$

Since $H/C_H(W)$ is cyclic, rank of $H\leq 3$. Let $z\in C_H(\sigma)$ and $b\in W$. Then, since $b+b^{\sigma}+b^{\sigma^2}\in C_G(\sigma)$, $(b+b^{\sigma}+b^{\sigma^2})^z=b+b^{\sigma}+b^{\sigma^2}$ by Lemma 3.1, and so $b^z=b$, $(b^{\sigma})^z=b^{\sigma}$ and $(b^{\sigma^2})^z=b^{\sigma^2}$. Hence $z\in C_H(V)=1$. Thus $C_H(\sigma)=1$. Since $H=C_H(\tau)\times C_H(\sigma\tau)$ by Lemma 3.2 and $C_H(\tau)$ is isomorphic to $C_H(\sigma\tau)$, H is an elementary abelian r-group of order r^2. Since A acts on a set $S=\{W, W^{\sigma}, W^{\sigma^2}\}$, τ fixes an element of S. Hence we may assume that $W=W^{\tau}$. Then $(W^{\sigma})^{\tau}=W^{\sigma\tau}=W^{\tau\sigma^2}=W^{\sigma^2}$ and so $1\neq x\in C_H(W^{\sigma}\oplus W^{\sigma^2})$. Hence $|C_H(W^{\sigma}\oplus W^{\sigma^2})|=r$ or r^2. If $|C_H(W^{\sigma}\oplus W^{\sigma^2})|=r^2$, then $H=C_H(W^{\sigma}\oplus W^{\sigma^2})$ and so $C_V(H)\neq 1$. Since $C_V(H)$ is HA-invariant, $V=C_V(H)$, a contradiction. Hence $|C_H(W^{\sigma}\oplus W^{\sigma^2})|=r$. Similarly, we obtain that $|C_H(W^{\sigma})|=|C_H(W^{\sigma^2})|=r$. Hence $C_H(W^{\sigma})=C_H(W^{\sigma}\oplus W^{\sigma^2})=C_H(W^{\sigma^2})$. Thus $C_H(W^{\sigma})$ is A-invariant and cyclic and so $C_H(W^{\sigma})\subseteq C_H(\sigma)=1$, a contradiction.

4. Properties of a minimal counterexample

For the remainder of this paper G denotes a counterexample of minimal order to the theorem stated in Section 1.

LEMMA 4.1. *G is a non-abelian simple group.*

PROOF. By Lemma 2.1(i), G does not possess any non-trivial proper A-invariant normal subgroups. Hence G is the direct product of isomorphic non-abelian simple groups by Theorem 2.1.4 of [3]. If G is not simple, since A acts fixed-point-freely on G, $G=G_1\times G_1^{\sigma}\times G_1^{\sigma^2}$ or $G=G_2\times G_2^{\tau}$, where G_i are simple, $i=1, 2$.

Suppose $G=G_1\times G_1^{\sigma}\times G_1^{\sigma^2}$. Let $F=\{xx^{\sigma}x^{\sigma^2}\mid x\in G_1\}$. As $G=G_1\times G_1^{\sigma}\times G_1^{\sigma^2}$, we deduce that $F\simeq G_1$ and $F\subseteq C_G(\sigma)$. But $C_G(\sigma)$ is abelian by Lemma 3.1, in contradiction with the simplicity of G_1.

Now suppose $G=G_2\times G_2^{\tau}$. If $C_G(\sigma)=1$, then G is nilpotent, a contradiction. Hence $C_G(\sigma)\neq 1$. Since G_2 and G_2^{τ} are σ-invariant, we may assume that $C_{G_2}(\sigma)\neq 1$. Then $1\neq C_{G_2}(\sigma)\subseteq G_2\cap G_2^{\tau}=1$ since τ inverts every element of $C_G(\sigma)$ by Lemma 3.1, a contradiction.

LEMMA 4.2. *Let P be a unique A-invariant Sylow p-subgroup of G. Then $[P, \sigma]\neq 1$.*

PROOF. If $[P, \sigma]=1$, then G has a normal p-complement by Lemma 3.4. This contradicts Lemma 4.1.

LEMMA 4.3. *Let P be the A-invariant Sylow p-subgroup of G, and set $N=N_G(P)$. Then the following conditions hold.*

(i) $N=N_G(Z(J(P)))$.

(ii) $N'\supseteq P$.

(iii) *N is a maximal A-invariant subgroup of G.*

PROOF. By the focal subgroup theorem (see Theorem 7.3.4 of [3]), $P\cap G'=\langle xy^{-1}\mid x, y\in P, x \text{ conjugate to } y \text{ in } G\rangle$. By Lemma 2.7, $y=x^n$ for some $n\in N_G(Z(J(P)))$. Hence $P\cap G'=P\cap N_G(Z(J(P)))'$. Since G is simple, $P=P\cap G'=P\cap N_G(Z(J(P)))'$ and so $P\subseteq N_G(Z(J(P)))'$. Moreover, $P\lhd N_G(Z(J(P)))$ by Lemma 3.8. Thus $N_G(Z(J(P)))=N_G(P)=N$ and $P\subseteq N'$.

Let M be a maximal A-invariant subgroup of G containing N. Then $M'\supseteq N'\supseteq P$. By Lemma 3.8, $P\lhd M$, this implies $M=N$.

LEMMA 4.4. *Let P be the A-invariant Sylow p-subgroup of G. Then $Z(P)$ is weakly closed in P.*

PROOF. If $Z(P)^g\subseteq P$, $Z(P)^g=Z(P)^n$ for some $n\in N_G(Z(J(P)))$ by Lemma 2.7. Since $N_G(Z(J(P)))=N_G(P)$ by Lemma 4.3, $Z(P)^n=Z(P)$. Thus $Z(P)$ is weakly closed in P.

LEMMA 4.5. *Let P be the A-invariant Sylow p-subgroup of G. If $N_G(P)/C_G(P)$ is an r'-group for some prime $r\neq p$, then for any p-subgroup P_0 of G, $N_G(P_0)/C_G(P_0)$ is an r'-group.*

PROOF. We may assume that $P_0\subseteq P$. Let x be an r-element of $N_G(P_0)$. By Lemma 2.7 and 4.3, $x=cn$ for some $c\in C_G(P_0)$ and $n\in N_G(P)$. Then $\bar{x}=\bar{n}$ in $N_G(P_0)/C_G(P_0)$. Since $N_G(P)/C_G(P)$ is an r'-group, $n^k\in C_G(P)$ for some integer k such that $(k, r)=1$, and so $\bar{n}^k=1$. Hence $\bar{x}^k=1$, this implies $\bar{x}=1$. Thus $N_G(P_0)/C_G(P_0)$ is an r'-group.

LEMMA 4.6. *Suppose p and r are distinct primes. For any p-subgroup P_0 of G, $N_G(P_0)/C_G(P_0)$ possesses an abelian Sylow r-subgroup.*

PROOF. Let P be the A-invariant Sylow p-subgroup of G. We may assume that $P_0\subseteq P$. Set $N=N_G(P_0)$. Since $Z(P)\subseteq C_G(P_0)$, $N=C_G(P_0)N_N(P)$ by the Frattini argument and Lemma 4.3 and 4.4. Let R_0 be a Sylow r-subgroup of N such that $N_{R_0}(P)$ is a Sylow r-subgroup of $N_N(P)$. Then

$R_0=(R_0\cap C_G(P_0))N_{R_0}(P)$. By Lemma 3.8, $N_{R_0}(P)'\subseteq C_G(P)\subseteq C_G(P_0)$. Hence $R_0'\subseteq(R_0\cap C_G(R_0))N_{R_0}(P)'\subseteq C_G(P_0)$. So $N_G(R_0)/C_G(P_0)$ possesses an abelian Sylow r-subgroup.

LEMMA 4.7. *Let M be a maximal A-invariant subgroup of G and P be the A-invariant Sylow p-subgroup of G. If $P\cap M$ is non-abelian, then $M=N_G(P)$.*

PROOF. By Lemma 3.8, $M=O_{p'}(M)N_M(P\cap M)$ and $1\neq(P\cap M)'\subseteq O_p(M)$. Hence $[O_{p'}(M), (P\cap M)']=1$ and $N_P(P\cap M)\rhd(P\cap M)'$, and so $(P\cap M)'\lhd M$. Thus $M=N_G((P\cap M)')$ by maximality of M. Hence $N_P(P\cap M)\subseteq M$ and so $N_P(P\cap M)=P\cap M$, this implies $P\cap M=P$. Hence $M=N_G(P')=N_G(P)$ by Lemma 4.3.

LEMMA 4.8. *Let P be the A-invariant Sylow p-subgroup of G and set $P_1=C_P(\tau)$. If x is a p'-element of $N_G(P)$ and $[x, P_1]=1$, then $[x, P]=1$. Furthermore, $C_G(P_1)$ has a normal p-complement.*

PROOF. Let H be the A-invariant Hall p'-subgroup of $N_G(P)$ and set $C=C_G(P_1)$. Then $x=hy$ for some $h\in H$ and some $y\in P$. Set $\bar{P}=P/\Phi(P)$. Then h acts on $\bar{P}$ and $[h, \bar{P}_1]=1$ since $[x, P_1]=1$. By Lemma 3.9, $[h, \bar{P}]=1$ and so $[x, \bar{P}]=1$. Hence $[x, P]=1$.

Let P^* be a Sylow p-subgroup of C containing $Z(P)$. By Sylow's theorem, $Z(P)^g\subseteq P^{*g}\subseteq P$ for some $g\in G$. Since $Z(P)$ is weakly closed in P by Lemma 4.4, $g\in N_G(Z(P))=N_G(P)$. Hence $P^*\subseteq P^{g^{-1}}=P$. Then, since $Z(J(P^*))\supseteq Z(P)$, $N_C(Z(J(P^*)))\subseteq N_C(Z(P))=N_C(P)$. By the argument of the preceeding paragraph, $N_C(P)$ has a normal p-complement, and so has $N_C(Z(J(P^*)))$. Since p is odd, the Glauberman-Thompson normal p-complement theorem (see Theorem 8.3.1 of [3]) now yields that C has a normal p-complement.

LEMMA 4.9. *Let P be the A-invariant Sylow p-subgroup of G and set $\bar{P}=P/P'$. Assume $C_{\bar{P}}(\sigma)=1$. If $x, y\in C_P(\tau)$ with x conjugate to y in G, then $\bar{x}=\bar{y}$ in $\bar{P}$. Moreover, there exists a normal subgroup K of $C_G(\tau)$ such that $K\cap C_P(\tau)\subseteq P'$. In particular, if $C_P(\sigma)=1$, then $C_G(\tau)$ has a normal p-complement.*

PROOF. Set $P_1=C_P(\tau)$ and $C=C_G(\tau)$, then P_1 is a Sylow p-subgroup of C by Lemma 3.3. Suppose $x, x^u\in P_1$ for some $u\in G$. By Lemmas 2.7 and 4.3, $u=cn$ for some $c\in C_G(x)$ and some $n\in N_G(P)$. Let H be the A-invariant Hall p'-subgroup of $N_G(P)$. Since $C_{\bar{P}}(\sigma)=1$, $[H, \sigma]\subseteq C_H(\bar{P})$

and so $[H, \sigma]\subseteq C_H(P)$. Since $H=C_H(\sigma)[H, \sigma]$ by Lemma 2.1(ii), $N_G(P)=C_H(\sigma)PC_G(P)$. Hence $n=ghk$ for some $g\in C_G(P)$, $h\in P$ and $k\in C_H(\sigma)$. Then $x^u=x^n=x^{ghk}=x^{hk}$. Set $\bar{N}=N_G(P)/P'$, then $\overline{x^u}=\overline{x^{hk}}=\bar{x}^{\bar{h}\bar{k}}=\bar{x}^{\bar{k}}$. Now A induces a group of automorphisms of $\bar{N}$. Then $(\overline{x^u})^\tau=\overline{x^u}=\bar{x}^{\bar{k}}$ and $(\overline{x^u})^\tau=(\bar{x}^{\bar{k}})^\tau=(\bar{k}^{-1}\bar{x}\bar{k})^\tau=\bar{k}\bar{x}\bar{k}^{-1}$. Hence $[\bar{k}^2, \bar{x}]=1$, it follows that $[\bar{k}, \bar{x}]=1$ since $|\bar{k}|$ is odd. Thus $\overline{x^u}=\bar{x}^{\bar{k}}=\bar{x}$, and so $x^{-1}x^u\in P'$.

By the focal subgroup theorem, $C'\cap P_1=<x^{-1}x^v|\, x, x^v\in P_1, v\in C>$. Hence $C'\cap P_1\subseteq P'$. Then there exists a normal subgroup K of C such that C/K is isomorphic to $P_1/P_1\cap C'$ by Theorem 7.3.1 of [3]. Then $P_1\cap C'$ is a Sylow p-subgroup of K, and hence $P_1\cap K=P_1\cap C'\subseteq P'$.

Suppose next that $C_P(\sigma)=1$. Then P has class at most 2 by Lemma 2.5, and so $P_1\cap K\subseteq P'\subseteq Z(P)$. We shall argue that K has a normal p-complement. Set $P_0=K\cap P_1$. Then $K'\cap P_0=<y^{-1}y^w|\, y, y^w\in P_0, w\in K>$. By Lemmas 2.7 and 4.3, $w=dm$ for some $d\in C_G(y)$ and some $m\in N_G(P)$. Moreover, since $m=rst$ for some $r\in C_G(P)$, $s\in P$ and $t\in C_H(\sigma)$. Then $y^w=y^m=y^{rst}=y^{st}=y^t$ since $y\in Z(P)$. Then a similar argument of the preceding paragraph gives $y^w=y^t=y$. Hence $K'\cap P_0=1$, it follows that K has a normal p-complement. Thus C has a normal p-complement.

For the remainder of this section, let Q and R be the A-invariant Sylow q- and r-subgroups of G, where q and r are distinct primes in $\pi(G)$.

LEMMA 4.10. *If $C_R(\sigma)\subseteq N_G(Q)$ and N be an A-invariant $\{q, r\}$-subgroup of G, then $[N\cap Q, \sigma]\subseteq O_q(N)$.*

PROOF. Set $\bar{N}=N/O_q(N)$ and $Q_0=[N\cap Q, \sigma]$. Then $[\overline{C_{N\cap R}(\sigma)}, \bar{Q}_0]\subseteq\overline{N\cap R}\cap\overline{N\cap Q}=1$. Hence $\bar{Q}_0$ stabilizes $\overline{N\cap R}\supseteq\overline{C_{N\cap R}(\sigma)\Phi(N\cap R)}\supseteq\overline{\Phi(N\cap R)}$ by Lemma 2.4, and so $\bar{Q}_0$ centralizes $\overline{N\cap R}$. This implies $\bar{Q}_0=1$ by Lemma 2.6. Thus $Q_0=[N\cap Q, \sigma]\subseteq O_q(N)$.

For the remainder of this paper, if L is a solvable A-invariant subgroup of G and π is a set of primes, let L_π denote the A-invariant Hall π-subgroup of L.

LEMMA 4.11. *Let M be a maximal A-invariant $\{q, r\}$-subgroup of G such that $O_r(M)=1$. Then $M\subseteq N_G(Q)$.*

PROOF. By Lemma 3.8, $M=O_{r,q,r}(M)$. Hence M is q-closed. Since $(N_G(O_q(M)))_{q,r}=M$ by maximality of M, $N_Q(O_q(M))=O_q(M)$. This implies $Q=O_q(M)$ and so $M\subseteq N_G(Q)$.

LEMMA 4.12. *Let R^* be an A-invariant r-subgroup of G such that $R^*=$*

$[R^*, \sigma] \subseteq N_G(Q)$. *If* N *is an* A*-invariant subgroup of* G *containing* $<R^*, C_Q(\sigma)>$, *then* $Q \triangleright [R^*, Q] = [R^*, O_q(N)]$.

PROOF. Let $1 \triangleleft O_q(N) \triangleleft Q \cap N = Q_1 \triangleleft Q_2 \triangleleft \ldots \triangleleft Q_n = Q$ be a normal series of Q, where $Q_{i+1} = N_Q(Q_i)$ for $i = 1, 2, \ldots, n-1$. Then each Q_i is R^*A-invariant. Since $C_Q(\sigma) \subseteq Q \cap N$, σ acts fixed-point-freely on Q/Q_{n-1}, and hence $[R^*, Q/Q_{n-1}] = 1$ by Lemma 2.4. This implies $[R^*, Q] \subseteq Q_{n-1}$. Hence $[Q, R^*] = [Q, R^*, R^*] = [Q_{n-1}, R^*]$. Repeating this argument, we have $[Q, R^*] = [Q_{n-1}, R^*] = \ldots = [Q_1, R^*]$. By Lemma 3.8, $[Q_1, R^*] \subseteq Q_1 \cap F(N) \subseteq O_q(N)$. Hence $[Q_1, R^*] = [Q_1, R^*, R^*] = [O_q(N), R^*]$. Thus $Q \triangleright [Q, R^*] = [O_q(N), R^*]$ by Lemma 2.1(ii).

LEMMA 4.13. *Set* $R_0 = C_R(\sigma) \cap N_G(Q)$ *and* $\bar{Q} = Q/Q'$. *If* $R_0 \neq 1$ *and* $C_{\bar{Q}}(\sigma) \neq 1$, *then* $C_Q(R_0) \supsetneq C_Q(\sigma)$.

PROOF. Let H be the A-invarianr Hall q'-subgroup of $N_G(Q)$. Suppose that $C_Q(R_0) = C_Q(\sigma)$. By Lemma 3.8, $H/C_H(Q)$ is abelian. Hence, if $h \in H$ and $a \in R_0$, then $h^{-1}ah = ab$ for some $b \in C_H(Q)$. Then $C_Q(a) = C_Q(a^h)$, this implies $C_Q(R_0) = C_Q(R_0)^h$ for each $h \in H$. Hence H acts on $\bar{Q}/\overline{C_Q(\sigma)}$ and $\overline{C_Q(\sigma)}$. Then, by Lemmas 2.4 and 3.4, $[H, \sigma]$ stabilizes $\bar{Q} \supseteq \overline{C_Q(\sigma)} \supseteq 1$, and so $[H, \sigma] \subseteq C_G(Q)$. Then $N_G(Q) = C_G(Q)QC_H(\sigma)$. Set $\bar{N} = N_G(Q)/Q$, then $\bar{N}' \cap \bar{Q} = [\overline{C_H(\sigma)}, \bar{Q}]$ and $\bar{Q} = [\overline{C_H(\sigma)}, \bar{Q}] \times C_{\bar{Q}}(\overline{C_H(\sigma)})$. Since $1 \neq \overline{C_Q(\sigma)} \subseteq \overline{C_Q(C_H(\sigma))}$ by Lemma 3.1, $\bar{N}' \cap \bar{Q} \subsetneq \bar{Q}$. Thus $N' \cap Q \subsetneq Q$, this contradicts Lemma 4.3.

LEMMA 4.14. *Let* Q^* *be an* A*-invariant* q*-subgroup of* G *and let* R_1 *and* R_2 *be* A*-invariant* r*-subgroups of* G *such that* $R_1 = [R, \sigma]$ *and* $[R_2, \sigma] = 1$. *If* $R_1 \times R_2 \subseteq N_G(Q^*)$, *then* $Q^* = <C_{Q^*}(R_1), C_{Q^*}(R_2)>$. *Furthermore, if* $R_1 = Z(R)$, *then* $[R_2, Q^*] = 1$.

PROOF. Set $\overline{Q^*} = Q^*/\Phi(Q^*)$. Then $\overline{C_{Q^*}(\sigma)} \subseteq C_{\overline{Q^*}}(R_2)$ by Lemma 3.1. Since R_1 acts on $\overline{Q^*}/C_{\overline{Q^*}}(R_2)$ and σ acts fixed-point-freely on $\overline{Q^*}/C_{\overline{Q^*}}(R_2)$, R_1 acts trivially on $\overline{Q^*}/C_{\overline{Q^*}}(R_2)$ by Lemma 2.4, and so $\overline{Q^*} = C_{\overline{Q^*}}(R_1)C_{\overline{Q^*}}(R_2)$. Thus $Q^* = <C_{Q^*}(R_1), C_{Q^*}(R_2)>$.

Now suppose that $R_1 = Z(R)$. Since $C_{Q^*}(Z(R)) \subseteq N_G(Z(R)) = N_G(R)$ by Lemma 4.3, $[C_{Q^*}(Z(R)), R_2] \subseteq R \cap Q^* = 1$. Hence $[R_2, Q^*] = 1$.

LEMMA 4.15. *Set* $Q_1 = C_Q(\tau)$ *and* $\bar{Q} = Q/Q'$. *If* $C_{\bar{Q}}(\sigma) = 1$, *then there exists a Sylow* r*-subgroup* R_0 *of* $C_G(\tau)$ *such that* $\overline{N_{Q_1}(R_0)} = \bar{Q}_1$ *in* $\bar{Q}$.

PROOF. Set $C = C_G(\tau)$. By Lemma 4.9, there exists a normal subgroup K of C such that $K \cap Q_1 \subseteq Q'$. By the the Frattini argument, $C = KN_C(R^*)$ for a Sylow r-subgroup R^* of C. Let Q_0 be a Sylow q-subgroup of C such that $N_{Q_0}(R^*)$ is a Sylow q-subgroup of $N_C(R^*)$. Since $Q_0^x = Q_1$ for some $x \in C$ by Lemma 3.3, $Q_1 = (Q_1 \cap K)N_{Q_1}(R^{*x})$. Setting $R_0 = R^{*x}$, $\overline{N_{Q_1}(R_0)} = \bar{Q}_1$ in $\bar{Q}$ since $Q_1 \cap K \subseteq Q'$.

LEMMA 4.16. *Set $\bar{Q} = Q/Q'$, $Q_1 = C_Q(\tau)$, and $C = C_G(\tau)$. Let Q_0 be a q-subgroup of C and let N be an A-invariant subgroup of G. Assume that the following conditions hold:*
(i) $C_{\bar{Q}}(\sigma) = 1$.
(ii) $\bar{Q}_0 = \bar{Q}_1$ *in* $\bar{Q}$.
(iii) $Q_0^z \subseteq N$ *for some* $z \in C$.
Then $Q \subseteq N$.

PROOF. Now $N \cap Q$ is the A-invariant Sylow q-subgroup of N, in particular $N \cap Q$ is τ-invariant. By Lemma 2.1(iv), $Q_0^{zy} \subseteq N \cap Q$ for some $y \in C_N(\tau)$. Setting $zy = x$, $x \in C$ Since $\overline{Q_0^x} = \bar{Q}_0 = \bar{Q}_1$ by Lemma 4.9 and (ii), $Q_1 \subseteq Q_0^x Q'$. Hence $Q_1^\sigma \subseteq (Q_0^x)^\sigma Q'$. Since $C_{\bar{Q}}(\sigma) = 1$, $\bar{Q} = C_{\bar{Q}}(\tau) C_{\bar{Q}}(\sigma\tau)$ by Lemma 3.2 and so $Q = \langle Q_1, Q_1^\sigma \rangle$. Hence $Q = \langle Q_0^x, (Q_0^x)^\sigma, Q' \rangle = \langle Q_0^x, (Q_0^x)^\sigma \rangle \subseteq N$.

LEMMA 4.17. *Set $\bar{Q} = Q/Q'$. If $N_G(Q)/C_G(Q)$ is an r'-group and $C_{\bar{Q}}(\sigma) = 1$, then $Q \subseteq N_G(R)$.*

PROOF. Setting $R_1 = C_R(\tau)$, $C_G(R_1)$ has a normal complement by Lemma 4.8. Hence $Z(R)$ normalizes a Sylow q-subgroup Q_0 of $C_G(R_1)$. Since $N_G(Q)/C_G(Q)$ is an r'-group, $[Z(R), Q_0] = 1$ by Lemma 4.5, and so $Q_0 \subseteq N_G(Z(R)) = N_G(R)$. By the Frattini argument, $N_G(R_1) = C_G(R_1)(N_G(R_1) \cap N_G(R))$ by Lemmas 4.3 and 4.4.
Since $Q_0 \subseteq C_G(R_1) \cap N_G(R)$, $|C_G(R_1)|_q = |C_G(R_1) \cap N_G(R)|_q$. Hence $|N_G(R_1)|_q = |C_G(R_1)|_q |N_G(R_1) \cap N_G(R)|_q / |C_G(R_1) \cap N_G(R)|_q = |N_G(R_1) \cap N_G(R)|_q$.

Now, by Lemma 4.15, there exists a Sylow r-subgroup R_0 of $C_G(\tau)$ such that $\overline{N_{Q_1}(R_0)} = \bar{Q}_1$ in $\bar{Q}$, where $Q_1 = C_Q(\tau)$. By Lemma 3.3, $R_0^x = R_1$ for some $x \in C_G(\tau)$. Then $N_{Q_1}(R_0)^x \subseteq N_G(R_0)^x \subseteq N_G(R_1)$. Since $|N_G(R_1)|_q = |N_G(R_1) \cap N_G(R)|_q$ and $N_{Q_1}(R_0)^x$ is τ-invariant q-subgroup of $N_G(R_1)$, $N_{Q_1}(R_1)^{xy} \subseteq N_G(R_1) \cap N_G(R)$ for some $y \in C_G(\tau) \cap N_G(R_1)$. Setting $z = xy$, $z \in C_G(\tau)$ and $N_{Q_1}(R_0)^z \subseteq N_G(R)$. Then $Q \subseteq N_G(R)$ by Lemma 4.16.

LEMMA 4.18. *Set $\bar{Q} = Q/Q'$ and $\bar{R} = R/R'$. Assume that $C_{\bar{Q}}(\sigma) = 1 =$*

$C_{\bar{R}}(\sigma)$. *Then if* $C_R(\sigma)\cap N_G(Q)\neq 1$, $C_{\bar{Q}}(a)\neq 1$ *for some non-trivial element* $a\in C_R(\sigma)\cap N_G(Q)$.

PROOF. Suppose false and the proof will be by contradiction. Set $C=C_G(\tau)$ and $Q_1=C_Q(\tau)$. Then we break the proof of Lemma 4.8 into five steps.

STEP 1. $Q\nsubseteq N_G(R)$.

PROOF. Suppose $Q\subseteq N_G(R)$. Then $[C_R(\sigma)\cap N_G(Q), Q]\subseteq Q\cap R=1$, a contradiction.

STEP 2. $N_{Q_1}(R_0)\subseteq C_G(R_0)$ *and* $\bar{Q}_1=\overline{N_{Q_1}(R_0)}$ *in* $\bar{Q}$ *for some Sylow* r*-subgroup* R_0 *of* C.

PROOF. By Lemma 4.15, $\overline{N_{Q_1}(R_0)}=\bar{Q}_1$ in $\bar{Q}$ for some Sylow r-subgroup of C. Set $R_1=C_R(\sigma)$ and $N=N_G(R_1)$. By the Frattini argument, $N=C_G(R_1)N_N(R)$ by Lemmas 4.3 and 4.4. Since $C_{\bar{R}}(\sigma)=1$, $N_G(R)=C_G(R)R(C_G(\sigma)\cap N_G(R))$. Then $N_N(R)=C_G(R)R(C_G(\sigma)\cap N_G(R))\cap N=C_G(R)(R(C_G(\sigma)\cap N_G(R))\cap N)$ and so $N=C_G(R_1)(R(C_G(\sigma)\cap N_G(R))\cap N)$. Now $R_0=R_1^y$ for some $y\in C$ by Lemma 3.3. So $N_{Q_1}(R_0)\subseteq N_G(R_0)=N^y=C_G(R_0)(R(C_G(\sigma)\cap N_G(R))\cap N)^y$. Then, since τ acts trivially on $N_{Q_1}(R_0)$, $N_{Q_1}(R_0)\subseteq C_G(R_0)$ by Lemma 3.1.

STEP 3. *If x be an r-element of* $N_G(Q)$, *then* $C_Q(x)=Q$ *or* $C_{\bar{Q}}(x)=1$.

PROOF. Since $N_G(Q)=C_G(Q)Q(C_G(\sigma)\cap N_G(Q))$, $x\in C_G(Q)Q(C_R(\sigma)\cap N_G(Q))$, and so $x=ghk$ for some $g\in C_G(Q)$, $h\in Q$ and $k\in C_R(\sigma)\cap N_G(Q)$. Then $C_{\bar{Q}}(x)=C_{\bar{Q}}(k)$. If $k\neq 1$, then $C_{\bar{Q}}(k)=1$. If $k=1$, then $C_{\bar{Q}}(k)=\bar{Q}$ and hence $C_Q(x)=Q$.

STEP 4. $N_{Q_1}(R_0)^z\subseteq N_G(R)$ *for some* $z\in C$.

PROOF. Lemma 4.8, $C_G(R_0)$ has a normal r-complement. Let Q^* be a τ-invariant Sylow q-subgroup of $C_G(R_0)$ containing $N_{Q_1}(R_0)$. Then Q^* is normalizes by R^*, where R^* is a τ-invariant Sylow r-subgroup of $C_G(R_0)$.

Now we shall prove $[R^*, Q^*]=1$. Suppose false. By Lemma 2.1(iv), $Q^{*u}\subseteq Q$ for some $u\in C$. Setting $Q_0=Q^{*u}$, there exists an element $y\in R^{*u}$ such that $[y, Q_0]\neq 1$. By Lemmas 2.7 and 4.3, $y=cn$ for some $c\in C_G(Q_0)$ and some $n\in N_G(Q)$. Let H be the A-invariant Hall q'-subgroup of $N_G(Q)$. Then, since $C_Q(\sigma)=1$, $[H, \sigma]$ centralizes Q by Lemma 2.4, and so $[H, \sigma]\subseteq C_G(Q)$. Thus $N_G(Q)=C_G(Q)QC_H(\sigma)$. Hence $n=ghk$ for some $g\in C_G(Q)$,

$h\in Q$ and $k\in C_H(\sigma)$. Moreover, since $n\in N_G(Q_0)$, $hk\in N_G(Q_0)$, and so k normalizes $\bar{Q}_0$ in $\bar{Q}$. Since $N_{Q_1}(R_0)^u\subseteq Q^{*u}=Q_0$ for $u\in C$, $\bar{Q}_0\supseteq\overline{N_{Q_1}(R_0)^u}=\overline{N_{Q_1}(R_0)}=\bar{Q}_1$ by Lemma 4.9 and Step 2, and so $Q_1\subseteq Q_0Q'$. Now $<k>Q$ and Q_0Q' is τ-invariant. Since $<k>Q\rhd Q_0Q'$, $<k>Q/Q_0Q'$ is τ-invariant. Moreover, since $Q_1\subseteq Q_0Q'$, τ inverts $<k>Q/Q_0Q'$, and so $[k, Q]\subseteq Q_0Q'$ by Lemma 2.2(iv). Hence $Q=C_Q(k)Q_0Q'$.

If $Q_0Q'\subsetneq Q$, then $C_{\bar{Q}}(k)\neq 1$. Now k can be written uniquely in the form $k=k_1k_2$, where k_1 is an r-element and k_2 is an r'-element and $[k_1, k_2]=1$. Then, since $1\neq C_{\bar{Q}}(k)\subseteq C_{\bar{Q}}(k_1)$, $C_{\bar{Q}}(k_1)=\bar{Q}$ by Step 3, and so $[Q, k_1]=1$. Since $y=cghk_1k_2$, $\bar{y}=\overline{hk_2}$ in $N_G(Q_0)/C_G(Q_0)$. Moreover since $hk_2\in Q<k_2>$, hk_2 is an r'-element. Hence $\bar{y}$ is an r-element and $\overline{hk_2}$ is an r'-element, a contradiction.

If $Q_0Q'=Q$, then $Q=Q_0$. So we have $R_0^u\subseteq C_G(Q)$ for $u\in G$. Setting $R_1=C_R(\tau)$, $R_1^v=R_0$ for some $v\in C$ by Lemma 3.3. Then $R_1^{vu}\subseteq C_G(Q)$ and $vu\in C$. By Lemma 4.16, $R\subseteq C_G(Q)$. This contradicts Step 1. Hence $[Q^*, R^*]=1$, in particular $[N_{Q_1}(R_0), R^*]=1$.

Let R_2 be a τ-invariant Sylow r-subgroup of G containing R^*. Then $Z(R_2)\subseteq R^*$ since $R^*=C_{R_2}(R_0)$. Thus $[N_{Q_1}(R_0), Z(R_2)]=1$. By Lemma 2.1(iv), $R_2^z=R$ for some $z\in C$, and so $[N_{Q_1}(R_0)^z, Z(R)]=1$. Hence $N_{Q_1}(R_0)^z\subseteq C_G(Z(R))\subseteq N_G(Z(R))=N_G(R)$ by Lemma 4.3.

STEP 5. *We have a contradiction.*

PROOF. By Steps 2 and 4, $\overline{N_{Q_1}(R_0)}=\bar{Q}_1$ in $\bar{Q}$ and $N_{Q_1}(R_0)^z\subseteq N_G(R)$ for some $z\in C$. By Lemma 4.16, $Q\subseteq N_G(R)$, This contradicts Step 1.

LEMMA 4.19. *Set $R_0=C_R(\sigma)\cap N_G(Q)$ and $Q_0=C_Q(\sigma)\cap N_G(R)$. If $R_0\neq 1\neq Q_0$, then one of the following holds:*

(i) *There exists a non-trivial element $a\in R_0$ such that $C_Q(a)\supsetneq C_Q(\sigma)$, or*

(ii) *there exists a non-trivial element $b\in Q_0$ such that $C_R(b)\supsetneq C_R(\sigma)$.*

PROOF. Suppose false. By Lemma 4.13, $C_{\bar{Q}}(\sigma)=1=C_{\bar{R}}(\sigma)$, where $\bar{Q}=Q/Q'$ and $\bar{R}=R/R'$. By Lemma 4.18, there exists a non-trivial element $a\in R_0$ such that $C_{\bar{Q}}(a)\neq 1$. If $C_Q(a)=C_Q(\sigma)$, then $C_{\bar{Q}}(a)=\overline{C_Q(a)}=\overline{C_Q(\sigma)}=C_{\bar{Q}}(\sigma)=1$, a contradiction. Hence $C_Q(a)\supsetneq C_Q(\sigma)$, a contradiction.

LEMMA 4.20. *Assume that $QR\neq RQ$ and $C_Q(\sigma)\neq 1\neq C_R(\sigma)$. Then there exists a maximal A-invariant $\{q, r\}$-subgroup H of G such that $O_q(H)\neq 1\neq O_r(H)$ and $<C_Q(\sigma), C_R(\sigma)>\subseteq H$.*

PROOF. Suppose $C_{Z(Q)}(\sigma)\neq 1$. Let $1\neq a\in C_{Z(Q)}(\sigma)$, and let H be a maximal A-invariant $\{q, r\}$-subgroup containing $(C_G(a))_{q,r}$. Then $<C_R(\sigma), Q>\subseteq (C_G(a))_{q,r}\subseteq H$. By Lemma 3.8, $H=O_{q,r,q}(H)$. Hence, if $O_q(H)=1$, then H is r-closed and so $R\subseteq H$ by maximality of H. Hence $QR=H=RQ$, a contradiction. Hence $O_q(H)\neq 1$. If $O_r(H)\neq 1$, H satisfies the required conditions. So we may assume that $O_r(H)=1$. Then the argument of the preceding paragraph gives $H\triangleright Q$, and so $C_R(\sigma)\subseteq N_G(Q)$.

Suppose next that $C_{Z(Q)}(\sigma)=1$. Let $1\neq b\in C_Q(\sigma)$, and let M be a maximal A-invariant $\{q, r\}$-subgroup containing $(C_G(b))_{q,r}$. Then $<C_R(\sigma), Z(Q), C_Q(\sigma)>\subseteq (C_G(b))_{q,r}\subseteq M$. If $O_q(M)=1$, then $M\triangleright R$ and so $<Z(Q), C_Q(\sigma)>\subseteq N_G(R)$. By Lemma 4.14, $[R, C_Q(\sigma)]=1$. Hence $1\neq C_Q(\sigma)\subseteq C_M(O_r(M))\subseteq O_r(M)$ by Lemma 2.6, a contradiction. Hence $O_q(M)\neq 1$. If $O_r(M)\neq 1$, M satisfies the required conditions. So we may assume that $O_r(M)=1$ and so $C_R(\sigma)\subseteq M=N_G(Q)$. Interchanging Q and R and applying the argument of the preceding paragraph gives $C_Q(\sigma)\subseteq N_G(R)$.

By Lemma 4.19, one of the following holds:

(i) there exists a non-trivial element $c\in R_0$ such that $C_Q(c)\supsetneq C_Q(\sigma)$, or

(ii) there exists a non-trivial element $d\in R_0$ such that $C_R(d)\supsetneq C_R(\sigma)$.

Suppose first that (i) holds. Setting $Q^*=[C_Q(c), \sigma]$, $Q^*\neq 1$. Let N be a maximal A-invariant $\{q, r\}$-subgroup of G containing $(C_G(c))_{q,r}$. Then $<C_Q(\sigma), C_R(\sigma), Q^*>\subseteq (C_G(c))_{q,r}\subseteq N$. By Lemma 4.10, $[N\cap Q, \sigma]\subseteq O_q(N)$ and $[N\cap R, \sigma]\subseteq O_r(N)$. Then we have $1\neq Q^*\subseteq [N\cap Q, \sigma]\subseteq O_q(N)$. By Lemmas 3.4 and 4.2, $[C_R(C_R(\sigma)), \sigma]\neq 1$ and so $1\neq [C_R(c), \sigma]\subseteq [N\cap R, \sigma]\subseteq O_r(N)$. Thus N satisfies the required conditions. Suppose next that (ii) holds. Then, similarly, we can show the existence of the subgroup of G which satisfies the required conditions.

For the remainder of this section, H be a maximal A-invariant $\{q, r\}$-subgroup of G with $O_q(H)\neq 1\neq O_r(H)$.

LEMMA 4.21. *If K is an A-invariant subgroup of $F(H)$ with $O_q(K)\neq 1\neq O_r(K)$, then H is the only maximal $\{q, r\}$-subgroup of G to contain K.*

PROOF. See Lemma 4 of [4].

LEMMA 4.22. *$R\subseteq H$ or $Z(R)\subseteq N_G(Q)$.*

PROOF. Since $O_r(H)\neq 1$, $H=(N_G(O_r(H)))_{q,r}\supseteq Z(R)$. Similarly $Z(Q)\subseteq H$. Then $[Z(R), Z(Q)]\subseteq O_q(H)Z(R)\cap F(H)$ by Lemmas 3.8

and 4.4. If $O_q(H)Z(R)\cap F(H)=O_q(H)$, then $Z(R)\subseteq N_G(Z(Q)O_q(H))\subseteq N_G(Z(Q))=N_G(Q)$ by Lemmas 4.3 and 4.4. If $O_q(H)Z(R)\cap F(H)\supsetneqq O_q(H)$, then $Z(R)\cap O_r(H)\neq 1$. Let $K=(Z(R)\cap O_r(H)\times O_q(H)$. Then $K\subseteq F(H)$ and $O_q(K)\neq 1\neq O_r(K)$. Since $K\subseteq(C_G(Z(R))\cap O_r(H))_{q,r}$, $(C_G\ (Z(R)\cap O_r(H)))_{q,r}\subseteq H$ by Lemma 4.21, and so $R\subseteq(C_G(Z(R)\cap O_r(H)))_{q,r}\subseteq H$.

LEMMA 4.23. *If* $Q\nsubseteq H$, *then* $[N_{R\cap H}(Q),\sigma]\subseteq C_G(Q)$, *furthermore,* $[R\cap H,\sigma]\subseteq O_r(H)$.

PROOF. Setting $R^*=[N_{R\cap H}(Q),\sigma]$, $Q\triangleright[Q,R^*]=[O_q(H),R^*]$ by Lemma 4.12. If $Q\cap H$ is non-abelian, $Q\subseteq H$ by Lemma 4.7, a contradiction. Hence $Q\cap H$ is abelian. Moreover, $(R\cap H)'\subseteq O_r(H)$ by Lemma 3.8, and so $(R\cap H)'\subseteq C_H(O_q(H))$. Thus $H/C_H(O_q(H))$ is an abelian r-group. Let $x\in O_q(H)$, $y\in R^*$ and $h\in H$. Then $y^h=ay$ for some $a\in C_H(O_q(H))$. Hence $[x,y]^h=(x^h)^{-1}(y^h)^{-1}x^hy^h=(x^h)^{-1}y^{-1}a^{-1}x^hay=(x^h)^{-1}y^{-1}x^hy=[x^h,y]\in[O_q(H),R^*]$. Thus we have $Q\triangleright[O_q(H),R^*]\triangleleft H$. Suppose $[O_q(H),R^*]\neq 1$. Then, since $H=(N_G([O_q(H),R^*]))_{q,r}$ by maximality of H, $Q\subseteq H$, a contradiction. Hence $[Q^*,R]=[O_q(H),R^*]=1$. Thus $[N_{R\cap H}(Q),\sigma]\subseteq C_G(Q)$. Now $Z(Q)\subseteq H$ since $H=(N_G(O_q(H))_{q,r}$. By Lemma 3.8, $H=O_r(H)N_H(H\cap Q)$. Since $Z(Q)\subseteq H\cap Q$, $N_H(H\cap Q)=N_H(Z(Q))=N_H(Q)$ by Lemmas 4.3 and 4.4. Set $\bar{H}=H/O_r(H)$. Then $[\overline{R\cap H,\sigma}]=[\overline{N_{R\cap H}(Q),\sigma}]=\overline{R^*}$. Since $[\overline{R^*},\ \overline{Q\cap H}]=1$, $\overline{R^*}=1$ by Lemma 2.6. Hence $[R\cap H,\sigma]\subseteq O_r(H)$.

Suppose p and q are distinct primes. Let P and Q be the A-invariant Sylow p- and q-subgroups of G. Then we shall show that $PQ=QP$.

Now we can divide the A-invariant Sylow p-subgroups for $p\in\pi(G)$ into three disjoint sets,

$$\pi_1=\{P^A=P\in Syl_p(G)\mid C_P(\sigma)=1\},$$
$$\pi_2=\{Q^A=Q\in Syl_q(G)\mid C_Q(\sigma)\neq 1 \text{ and } C_{Z(Q)}(\sigma)=1\},$$
$$\pi_3=\{R^A=R\in Syl_r(G)\mid C_{Z(R)}(\sigma)\neq 1\}.$$

5. The case $P\in\pi_1$.

In this section, P, Q be the A-invariant Sylow p- and q-subgroups of G (where p, q are distinct primes) such that $P\in\pi_1$, ie., $C_P(\sigma)=1$.

LEMMA 5.1. *If* $N_G(Q)/C_G(Q)$ *is a* p'-*group, then* $PQ=QP$.

PROOF. Setting $P_1=C_P(\tau)$, P_1 is a Sylow p-subgroup of $C_G(\tau)$ by

Lemma 3.3. By Lemma 4.9, P_1 normalizes some Sylow q-subgroup Q_1 of $C_G(\tau)$. By hypothesis and Lemma 4.5, $Q_1 \subseteq C_G(P_1) \supseteq Z(P)$. Setting $L = C_G(P_1)$, L has a normal p-complement by Lemma 4.8. Hence $Z(P)$ normalizes some τ-invariant Sylow q-subgroup Q^* of L. By Lemma 2.1(iv), $Q_1{}^x \subseteq Q^*$ for some $x \in C_L(\tau)$. By hypothesis and Lemma 4.5, $Q^* \subseteq C_G(Z(P)) \subseteq N_G(Z(P)) = N_G(P)$. Moreover, by Lemma 2.1(iv), $Q^{*y} \subseteq N_Q(P)$ for some $y \in C_G(\tau) \cap N_G(P)$, in particular, $Q_1{}^{xy} \subseteq C_G(\tau) \cap N_Q(P)$. Since Q_1 is a Sylow q-subgroup of $C_G(\tau)$, $Q_1{}^{xy} = C_Q(\tau) \subseteq N_G(P)$. Hence $< C_Q(\tau), C_Q(\sigma\tau) > \subseteq N_G(P)$. Now, Since $Q \triangleright [Q, \sigma]$ and τ, $\sigma\tau$ invert every element of $Q/[Q, \sigma]$, $< C_Q(\tau), C_Q(\sigma\tau) > \subseteq [Q, \sigma]$. Setting $\overline{[Q, \sigma]} = [Q, \sigma]/\Phi([Q, \sigma])$, $\overline{[Q, \sigma]} = \overline{C_Q(\tau)}\ \overline{C_Q(\sigma\tau)}$ by Lemma 3.2. This implies that $< C_Q(\tau), C_Q(\sigma\tau) > = < C_Q(\tau), C_Q(\sigma\tau), \Phi([Q, \sigma]) > = [Q, \sigma]$. So $[Q, \sigma] \subseteq N_G(P)$. Since $C_P(\sigma) = 1$, $P \subseteq C_G([Q, \sigma])$ by Lemma 2.4. Since $Q \triangleright [Q, \sigma]$ by Lemma 2.1(ii), $< P, Q > \subseteq N_G([Q, \sigma])$. This implies that $PQ = QP$.

LEMMA 5.2. $PQ = QP$.

PROOF. Suppose false and the proof will be by contradiction. By Lemmas 4.17 and 5.1, we may assume that $p \,\|\, |N_G(Q)/C_G(Q)|$ and $q \,\|\, |N_G(P)/C_G(P)|$. Setting $Q_0 = C_Q(\sigma) \cap N_G(P)$ and $P_0 = P \cap N_G(Q)$, $P_0 \neq 1$. Since $N_G(P) = C_G(P)P(C_G(\sigma) \cap N_G(P))$, $Q_0 \neq 1$. Furthermore, we set $P_1 = C_P(\tau)$ and $L = C_G(\sigma) \cap N_G(P)$. Now we divide the proof of Lemma 5.2 into seven steps.

STEP 1. *There exists a maximal A-invariant subgroup of G containing $< C_G(\sigma), P_0, Q >$.*

PROOF. $[P_0, Q_0] \subseteq P \cap Q = 1$ Let M be a maximal A-invariant subgroup of G containing $C_G(Q_0)$. Then $< C_G(\sigma), P_0, Z(Q) > \subseteq C_G(Q_0) \subseteq M$. we subdivide the proof according to whether $Z(Q) \cap O_q(H) \neq 1$ or $Z(Q) \cap O_q(H) = 1$.

CASE I. $Z(Q) \cap O_q(M) \neq 1$

Since $Z(Q) \subseteq M$, $M = O_q(M)N_M(Z(Q))$ by Lemmas 3.8 and 4.4. Furthermore, since $[O_q(M), Z(Q) \cap O_q(M)] = 1$, $M = O_q(M)N_M(Z(Q)) = N_G(Z(Q) \cap O_q(M))$ by maximality of M. Hence $Q \subseteq M$. Then M satisfies the required conditions.

CASE II. $Z(Q) \cap O_q(M) = 1$

Then $[P_0, Z(Q)] \subseteq Z(Q) \cap F(M) = Z(Q) \cap O_q(M) = 1$. Set $H = (N_G(P_0))_{p,q}$. Then $Z(Q) \subseteq H$ and $H = O_{q,p,q}(H)$.

Suppose $C_{Z(Q)}(\sigma) = 1$. Setting $\bar{H} = H/O_q(H)$, $[\overline{Z(Q)}, O_p(\bar{H})] = 1$ by Lemma 2.4. By Lemma 2.6, $\overline{Z(Q)} = 1$ and so $Z(Q) \subseteq O_q(H)$. By Lemmas 4.3 and 4.4, $H \subseteq N_G(Z(Q)) = N_G(Q)$. Hence $N_P(P_0) = P_0$. This implies $P = P_0$. Thus $P \subseteq N_G(Q)$ and so $PQ = QP$, a contradiction. Hence $C_{Z(Q)}(\sigma) \neq 1$.

Setting $Z = C_{Z(Q)}(\sigma)$, $\langle P_0, C_G(\sigma), Q \rangle \subseteq C_G(Z)$. Let T be a maximal A-invariant subgroup of G containing $C_G(Z)$. Then T satisfies the required conditions.

STEP 2. $C_G(\sigma) \subseteq N_G(Q)$.

PROOF. If Q is non-abelian, then $M = N_G(Q)$ by Lemma 4.7. Thus $C_G(\sigma) \subseteq N_G(Q)$. Hence we may assume that Q is abelian.

Set $X = [C_Q(P_0), \sigma]$. Suppose $X \neq 1$. Let K be a maximal A-invariant $\{p, q\}$-subgroup which contains $(C_G(P_0))_{p,q}$. Then $\langle Z(P), X \rangle \subseteq (C_G(P_0))_{p,q} \subseteq K$. Setting $\bar{K} = K/O_q(K)$, $\bar{X} \subseteq C_{\bar{K}}(O_p(\bar{K})) \subseteq O_p(\bar{K})$ by Lemmas 2.4 and 2.6, and hence $X \subseteq O_q(K)$. Since Q is abelian, $Q \subseteq (N_G(O_q(K)))_{p,q} = K$ by maximality of K. Suppose $O_p(K) \cap Z(P) \neq 1$. Since $O_q(K) \times (O_p(K) \cap Z(P)) \subseteq (C_G(O_p(K) \cap Z(P)))_{p,q}$ and $O_q(K) \neq 1 \neq O_p(K) \cap Z(P)$, $P \subseteq (C_G(O_p(K) \cap Z(P)))_{p,q} \subseteq K$ by Lemma 4.21. Hence $\langle P, Q \rangle \subseteq K$ and so $PQ = QP$, a contradiction. Hence $O_p(K) \cap Z(P) = 1$. Then $[Q_0, Z(P)] \subseteq Z(P) \cap F(K) = 1$. By Lemmas 4.3 and 4.4, $K = O_q(K) N_K(Z(P)) = O_q(K) N_K(P)$. Since $Q \subseteq K$ and $[N_Q(P), \sigma] \subseteq O_q(K)$ by Lemmas 2.4 and 2.6, $Q = Q_0 O_q(K)$. Hence $Z(P) \subseteq N_G(Q)$. Thus $Z(P) \subseteq P_0$. Let U be a maximal A-invariant $\{p, q\}$-subgroup which contains $(C_G(Z(P)))_{p,q}$. Then $\langle X, P \rangle \subseteq (C_G(Z(P)))_{p,q} \subseteq U$. Since $1 \neq X \subseteq O_q(U)$ by Lemmas 2.4 and 2.6, $Q \subseteq (N_G(O_q(N)))_{p,q} = U$ by maximality of N. Hence $\langle P, Q \rangle \subseteq U$ and so $PQ = QP$, a contradiction. Hence $X = [C_Q(P_0), \sigma] = 1$.

Since $[P_0, Q] \subseteq Q \cap F(M) = O_q(M)$, $Q = C_Q(P_0) O_q(M)$. Since $[C_Q(P_0), \sigma] = 1$, $[Q, \sigma] \subseteq O_q(M)$. Now $C_G(\sigma)$ is abelian by Lemma 3.1 and so $C_G(\sigma)$ normalizes $C_Q(\sigma) O_q(M) = C_Q(\sigma)[Q, \sigma] = Q$ by Lemma 2.1(ii). Thus $C_G(\sigma) \subseteq N_G(Q)$.

STEP 3. $Z(P) \cap F(N_G(Q)) = 1$.

PROOF. Set $P_2 = Z(P) \cap F(N_G(Q))$. If $P_2 \neq 1$, then $\langle P, Q \rangle \subseteq$

$C_G(P_2)$ and so $PQ=QP$, a contradiction. Hence $Z(P)\cap F(N_G(Q))=1$.

STEP 4. *$[L, P_0]\neq 1$ or $P_0\subseteq P'$.*

PROOF. Suppose $[L, P_0]\neq 1$ and $P_0\nsubseteq P'$. Set $N=N_G(P)$ and $\bar{N}=N/P'$. Since $N=C_G(P)PL$ by Lemma 2.4, $\bar{N}'\cap\bar{P}=[\bar{L}, \bar{P}]$. Now since $\bar{P}=[\bar{P}, \bar{L}]\times C_{\bar{P}}(\bar{L})$ and $1\neq\bar{P}_0\subseteq C_{\bar{P}}(\bar{L})$, $\bar{P}\supsetneq[\bar{P}, \bar{L}]=\bar{N}'\cap\bar{P}\supseteq\overline{N'\cap P}$. Hence $N'\cap P\subsetneq P$. This contradicts Lemma 4.3.

STEP 5. *P is non-abelian. In particular, $P_1'\neq 1$.*

PROOF. Suppose P is abelian. Since $\langle P_0, C_G(\sigma)\rangle\subseteq N_G(Q)$, $[L, P_0]\subseteq P\cap F(N_G(Q))=1$ by Step 3. This contradicts Step 4. Hence P is non-abelian. By Lemma 3.7, $P_1'\neq 1$.

STEP 6. *$P_0\cap P_1'=1$.*

PROOF. Set $P_3=P_0\cap P_1'$ and assume that $P_3\neq 1$. Since class $P\leq 2$ by Lemma 2.5, $P_3\subseteq P_1'\subseteq Z(P)$. Then $[L, P_3]\subseteq Z(P)\cap F(N_G(Q))=1$ by Steps 2 and 3, where $L=C_G(\sigma)\cap N_G(P)$. Hence $P_3\subseteq Z(N_G(P))$. By Lemmas 2.7 and 4.3, every element of P_3 is weakly closed in P with respect to G. By Lemma 4.9, $P_1\subseteq N_G(Q_1)$ for some Sylow q-subgroup Q_1 of $C_G(\tau)$. By Lemma 4.6, $P_3\subseteq P_1'\subseteq C_G(Q_1)$. Let Q^* be a τ-invariant Sylow q-subgroup of G containing Q_1. Then $Z(Q^*)Q_1$ normalizes a τ-invariant Sylow p-subgroup P^* by Lemma 4.8. By Lemma 2.1(iv), $Q^{*x}=Q$ for some $x\in C_G(\tau)$. Then $Z(Q^*)^xQ_1^x=Z(Q)C_Q(\tau)$ normalizes P^{*x}. Now since $P_3\subseteq C_G(Q_1)$, $P_3^y\subseteq P^*$ for some $y\in C_G(Q_1)$ by Sylow's theorem. Since $P_3^{yx}\subseteq P^{*x}$ and every element of P_3 is weakly closed, $P_3^{yx}\subseteq C_G(Z(Q)C_Q(\tau))=C_G(C_Q(\tau))\cap C_G(Z(Q))\subseteq C_G(C_Q(\tau))\cap N_G(Q)$ by Lemma 4.3. By Lemma 4.8, $P_3^{yx}\subseteq C_G(Q)$. Since $C_P(Q)$ is a Sylow p-subgroup of $C_G(Q)$, $P_3^{yxz}\subseteq C_P(Q)$ for some $z\in C_G(Q)$ by Sylow's theorem. Since P_3 is weakly closed in P, $P_3=P_3^{yxz}\subseteq C_P(Q)$. Thus $1\neq P_3\subseteq C_{Z(P)}(Q)$. Setting $Z=C_{Z(P)}(Q)$, $\langle P, Q\rangle\subseteq C_G(Z)$ and so $PQ=QP$, a contradiction. Hence $1=P_3=P_0\cap P_1'$.

STEP 7. *We have a contradiction.*

PROOF. By Step 2, $[P_0, L]\subseteq P\cap F(N_G(Q))$. Set $P_4=P\cap F(N_G(Q))$. If $P_4\neq 1$, then $Z(P)\subseteq N_G(P_4)=N_G(Q)$. Hence $P_1'\subseteq Z(P)\subseteq N_P(Q)=P_0$. Thus $1\neq P_1'=P_0\cap P_1'$. This contradicts Step 6. Hence $1=P_4=[P_0, L]$. By Step 4, $P_0\subseteq P'$. By Lemma 3.7, $P'=C_P(\tau)'C_P(\sigma\tau)'$ and so $C_{P'}(\tau)=$

$C_P(\tau)'=P_1'$. Since $C_{P_0}(\sigma)=1$, $P_0=C_{P_0}(\tau)C_{P_0}(\sigma\tau)$ by Lemma 3.2 and so $1\neq C_{P_0}(\tau)\subseteq C_{P'}(\tau)=P_1'$. Thus $P_0\cap P_1'\neq 1$. This contradicts Step 6.

6. The case Q, R$\in\pi_2$

LEMMA 6.1. *Suppose q and r are distinct primes. Let Q and R be the A-invariant Sylow q- and r-subgroups of G such that $Q, R\in\pi_2$, ie., $C_Q(\sigma)\neq 1\neq C_R(\sigma)$ and $C_{Z(Q)}(\sigma)=1=C_{Z(R)}(\sigma)$. Then $QR=RQ$.*

PROOF. Suppose false. By Lemma 4.20, there exists a maximal A-invariant $\{q, r\}$-subgroup H of G such that $<C_Q(\sigma), C_R(\sigma)>\subseteq H$ and $O_q(H)\neq 1\neq O_r(H)$. If $Q\nsubseteq H$, $Z(Q)\subseteq N_G(R)$ by Lemma 4.22. Since $C_{Z(Q)}(\sigma)=1=C_{Z(R)}(\sigma)$, $[Z(Q), Z(R)]=1$. Now $Z(R)\subseteq(N_G(O_r(H)))_{q,r}=H$ by maximality of H. By Lemma 4.23, $Z(R)=[Z(R), \sigma]\subseteq[R\cap H, \sigma]\subseteq O_r(H)$. Since $Z(R)\times O_q(H)\subseteq(C_G(Z(Q)))_{q,r}$ and $Z(R)\times O_q(H)\subseteq F(H)$, $(C_G(Z(Q)))_{q,r}\subseteq H$ by Lemma 4.21. Thus $Q\subseteq(C_G(Z(Q)))_{q,r}\subseteq H$. By symmetry between Q and R, we also have $R\subseteq H$ and so $QR=RQ$, a contradiction.

7. The case Q, R$\in\pi_3$

LEMMA 7.1. *Suppose q and r are distinct primes. Let Q and R be the A-invariant Sylow q- and r-subgroups of G such that $Q, R\in\pi_3$, ie., $C_{Z(Q)}(\sigma)\neq 1\neq C_{Z(R)}(\sigma)$. Then $QR=RQ$.*

PROOF. Suppose false and the proof will be by contradiction. Now we divide the proof of Lemma 7 1 into two steps.

STEP 1. *There exists a maximal A-invariant $\{q, r\}$-subgroup H of G such that $O_q(H)\neq 1\neq O_r(H)$, and $<Q, C_R(\sigma)>\subseteq H$ or $<R, C_Q(\sigma)>\subseteq H$.*

PROOF. Suppose false. Let $1\neq a\in C_{Z(Q)}(\sigma)$. Let H be a maximal A-invariant $\{q, r\}$-subgroup of G containing $(C_G(a))_{q,r}$. Then $<C_R(\sigma), Q>\subseteq(C_G(a))_{q,r}\subseteq H$. If $O_q(H)=1$, then $H\subseteq N_G(R)$ by Lemma 4.11. Thus $Q\subseteq N_G(R)$ and so $QR=RQ$, a contradiction. Hence $O_q(H)\neq 1$. If $O_r(H)\neq 1$, then H satisfies the required conditions, a contradiction. Hence $O_r(H)=1$. Then $H\subseteq N_G(Q)$ by Lemma 4.11. Thus $C_R(\sigma)\subseteq H\subseteq N_G(Q)$. By symmetry between Q and R, we also have $C_Q(\sigma)\subseteq N_G(R)$.

Now suppose that $C_R(\sigma)$ is non-cyclic. Then $Q=<C_Q(x)\mid 1\neq x\in C_R(\sigma)>=<C_Q(x)\mid C_Q(x)\supsetneqq C_Q(\sigma),\ 1\neq x\in C_R(\sigma)>$. Let $1\neq x\in C_R(\sigma)$ such that $C_Q(x)\supsetneqq C_Q(\sigma)$. Setting $Q^*=[C_Q(x), \sigma]$, $Q^*=[Q^*, \sigma]\neq 1$. Let K be a maximal A-invariant $\{q, r\}$-subgroup of G containing $(C_G(x))_{q,r}$. Then $<C_R(x), Q^*>\subseteq(C_G(x))_{q,r}\subseteq K$. By Lemma 4.10, $1\neq Q^*=[Q^*, \sigma]\subseteq O_q(K)$. By Lemmas 3.4 and 4.2, $1\neq[C_R(C_R(\sigma)), \sigma]$ and so $1\neq[C_R(x), \sigma]$. Then $[C_R(x), \sigma]\subseteq O_r(K)$ by Lemma 4.10. Thus $O_q(K)\neq 1\neq O_r(K)$. If $R\cap K$ is non-abelian, $R\subseteq K$ by Lemma 4.7. Then K satisfies the required conditions. Hence $R\cap K$ is abelian. Then $R\cap K=C_R(x)=C_R(C_R(\sigma))$ since $x\in C_R(\sigma)$ and $C_R(\sigma)\subseteq C_R(x)\subseteq R\cap K$. Setting $R^*=C_R(C_R(\sigma))$, $1\neq[R^*, \sigma]\subseteq O_r(K)$ by Lemma 4.10. Suppose $C_{R^*}(\sigma)\cap O_r(K)\neq 1$. Since $(C_{R^*}(\sigma)\cap O_r(K))\times O_q(K)\subseteq F(K)$ and $(C_{R^*}(\sigma)\cap O_r(K))\times O_q(K)\subseteq(C_G(C_{Z(Q)}(\sigma)))_{q,r}$, $Q\subseteq(C_G(C_{Z(Q)}(\sigma)))_{q,r}\subseteq K$ by Lemma 4.21. Thus K satisfies the required conditions, a contradiction. Hence $C_{R^*}(\sigma)\cap O_r(K)=1$ and so $O_r(K)=[R^*, \sigma]$. Hence $C_Q(x)\subseteq K\subseteq N_G(O_r(K))=N_G([R^*, \sigma])$. Since $Q=<C_Q(x)\mid 1\neq x\in C_R(\sigma),\ C_Q(x)\supsetneqq C_Q(\sigma)>$, $Q\subseteq N_G([R^*, \sigma])$. Hence $Q\subseteq(N_G([R^*, \sigma]))_{q,r}=(N_G(O_r(K)))_{q,r}=K$ by maximality of K, a contradiction. Hence $C_R(\sigma)$ is cyclic. By symmetry between Q and R, $C_Q(\sigma)$ is cyclic.

By Lemma 4.19, we may assume that $C_Q(a)\supsetneqq C_Q(\sigma)$ for some $1\neq a\in\Omega_1(C_R(\sigma))$. Since $C_R(\sigma)$ is cyclic and $C_{Z(R)}(\sigma)\neq 1$, $a\in Z(R)$. Setting $Q_0=[C_Q(a), \sigma]$, $Q_0\neq 1$. Let M be a maximal A-invariant $\{q, r\}$-subgroup which contains $(C_G(a))_{q,r}$. Then $<R, Q_0>\subseteq(C_G(a))_{q,r}\subseteq M$. By Lemma 4.10, $1\neq Q_0\subseteq O_q(M)$. If $O_r(M)=1$, then $Q\subseteq M$ by Lemma 4.11. Then $<Q, R>\subseteq M$ and so $QR=RQ$, a contradiction. Hence $O_r(M)\neq 1$. Then M satisfies the required conditions, a contradiction. This completes the proof.

STEP 2. *We have a contradiction.*

PROOF. By Step 1, we may assume that there exists a maximal A-invariant $\{q, r\}$-subgroup H of G such that $O_q(H)\neq 1\neq O_r(H)$ and $<Q, C_R(\sigma)>\subseteq H$. Setting $R_1=C_{Z(R)}(\sigma)$ and $R_2=[Z(R), \sigma]$, $R_1\neq 1$ and $Z(R)=R_1\times R_2$ by hypothesis and Lemma 2.1(ii). By Lemma 4.14, $1\neq O_q(H)=<C_{O_q(H)}(R_1), C_{O_q(H)}(R_2)>$, and hence $C_{O_q(H)}(R_1)\neq 1$ or $C_{O_q(H)}(R_2)\neq 1$. Suppose that $C_{O_q(H)}(R_1)\neq 1$. Since $C_{O_q(H)}(R_1)\times O_r(H)\subseteq F(H)$ and $C_{O_q(H)}(R_1)\neq 1\neq O_r(H)$, $C_{O_q(H)}(R_1)\times O_r(H)\subseteq(C_G(R_1))_{q,r}\subseteq H$ by Lemma 4.21. Thus $R\subseteq(C_G(R_1))_{q,r}\subseteq H$. Then since $<Q, R>\subseteq H$, $QR=RQ$, a contradiction. Hence $C_{O_q(H)}(R_2)\neq 1$. If $R_2\neq 1$, similarly, we have a contradiction. Hence $R_2=1$ and so $[Z(R), \sigma]=1$.

If Q is non-abelian, $Q\subseteq H$ by Lemma 4.7. Since $O_r(H)\times C_Q(\sigma)\subseteq$

$F(H)$ and $O_r(H)\neq 1\neq C_Q(\sigma)$, $O_r(H)\times C_Q(\sigma)\subseteq(C_G(Z(R)))_{q,r}\subseteq H$ by Lemma 4.21. Then $<Q, R>\subseteq H$ since $R\subseteq(C_G(Z(R)))_{q,r}\subseteq H$, and so $QR=RQ$, a contradiction. Hence Q is abelian.

By Lemma 4.22, $Z(R)\subseteq N_G(Q)$. By Lemma 4.13, $C_Q(Z(R))\supsetneqq C_Q(\sigma)$. Setting $Q_1=[C_Q(Z(R)), \sigma]$, $Q_1\neq 1$. Then $Q_1\subseteq C_G(Z(R))\subseteq N_G(R)$. By Lemma 4.23, $[N_{Q\cap H}(R), \sigma]\subseteq C_G(R)$ and so $Q_1\subseteq C_G(R)$. Since Q is abelian, $<Q, R>\subseteq C_G(Q_1)$ and so $QR=RQ$. Thus we have a contradiction and the lemma is proved.

8. The case $Q\in\pi_2$ and $R\in\pi_3$

LEMMA 8.1. *Suppose q and r are distinct primes. Let Q and R be the A-invariant Sylow q- and r-subgroups of G such that $Q\in\pi_2$ and $R\in\pi_3$, i.e., $C_Q(\sigma)\neq 1\neq C_{Z(R)}(\sigma)$ and $C_{Z(Q)}(\sigma)=1$. Then $QR=RQ$.*

PROOF. Suppose false and the proof will be by contradiction. Now we divide the proof of Lemma 8.1 into eleven steps.

STEP 1. *There exists a maximal A-invariant $\{q, r\}$-subgroup H of G such that $O_q(H)\neq 1\neq O_r(H)$ and $<C_Q(\sigma), C_R(\sigma)>\subseteq H$.*

PROOF. See Lemma 4.20.

STEP 2. $R\subseteq H$.

PROOF. Suppose $R\nsubseteq H$. Since $O_q(H)\neq 1$, $Z(Q)\subseteq(N_G(O_q(H)))_{q,r}=H$ by maximality of H. By Lemma 4.23, $Z(Q)\subseteq[Q\cap H, \sigma]\subseteq O_q(H)$. Hence $H\subseteq N_G(Z(Q))=N_G(Q)$ by Lemmas 4.3 and 4.4, and so $Q\lhd H$. Since $C_Q(\sigma)\times O_r(H)\subseteq F(H)$ and $C_Q(\sigma)\neq 1\neq O_r(H)$, $C_Q(\sigma)\times O_r(H)\subseteq(C_G(C_{Z(R)}(\sigma)))_{q,r}\subseteq H$ by Lemma 4.21. Hence $R\subseteq H$, a contradiction.

STEP 3. *The following conditions hold.*
(i) $C_Q(\sigma)\subseteq C_G(R)$.
(ii) *Q is non-abelian and $Q\cap H$ is abelian.*
(iii) $[R, \sigma]\subseteq O_r(H)$.

PROOF. By Lemma 4.23 and Step 2, $[R, \sigma]\subseteq O_r(H)$ and so $R=O_r(H)C_R(\sigma)$. Since $O_q(H)\neq 1$, $Z(Q)\subseteq(N_G(O_q(H)))_{q,r}=H$ by maximality of H. Then, since $Z(Q)\times C_Q(\sigma)\subseteq N_G(O_r(H))$ and $Z(Q)=[Z(Q), \sigma]$, $[O_r(H), C_Q(\sigma)]=1$ by Lemma 4.14. Then

$[C_Q(\sigma), O_r(H)\ C_R(\sigma)]=1$ by Lemma 3.1, and hence $[C_Q(\sigma), R]=1$.

Now, if Q is abelian, then $Q=Z(Q)\subseteq H$. Hence $<Q, R>\subseteq H$ and so

$QR=RQ$, a contradiction. Hence Q is non-abelian. Next, if $Q\cap H$ is abelian, then $Q\subseteq H$ by Lemma 4.7, and so $<Q, R>\subseteq H$, a contradiction. Hence $Q\cap H$ is abelian.

STEP 4. $C_Q(Q\cap H)=Q\cap H$.

PROOF. Set $Q_0=C_Q(\sigma)$. Then $H=<R, Q\cap H>\subseteq(C_G(Q_0))_{q,r}=H$ by Step 3. Hence $C_Q(Q_0)=Q\cap H$, in particular, $C_Q(Q\cap H)=Q\cap H$.

STEP 5. $[Z(Q), R]=O_r(H)$.

PROOF. By Lemma 4.22, $Z(Q)\subseteq N_G(R)$. Suppose $C_{O_r(H)}(Z(Q))\neq 1$. Then, since $O_q(H)\times C_{O_r(H)}(Z(Q))\subseteq F(H)$ and $O_q(H)\neq 1\neq C_{O_r(H)}(Z(Q))$, $O_q(H)\times C_{O_r(H)}(Z(Q))\subseteq(C_G(Z(Q)))_{q,r}\subseteq H$ by Lemma 4.21. Thus $Q\subseteq H$ and so $<Q, R>\subseteq H$, a contradiction. Hence $C_{O_r(H)}(Z(Q))=1$ and so $[Z(Q), R]=[Z(Q), O_r(H)]=O_r(H)$ by Lemmas 2.1(ii) and 4.12.

STEP 6. $R\lhd H$.

PROOF. If R in non-abelian, then $R\lhd H$ by Lemma 4.7. Hence we may assume that R is abelian. Let L be the A-invariant Hall r'-subgroup of $N_G(R)$. Since $N_G(R)'$ is nilpotent by Lemma 3.8, $N_G(R)'\subseteq C_G(R)$ and so $N_G(R)/C_G(R)$ is abelian. Let $x\in R$, $y\in Z(Q)$ and $h\in N_G(R)$. Then $y^h=ay$ for some $a\in C_G(R)$. Hence $[x, y]^h=(x^h)^{-1}(y^h)^{-1}x^hy^h=(x^h)^{-1}y^{-1}x^hay=[x^h, y]\in[R, Z(Q)]$. Thus we have $O_r(H)=[Z(Q), R]\lhd N_G(R)$, and hence L normalizes $O_r(H)$. Since $[R, \sigma]\subseteq O_r(H)$ by Lemma 4.23, $[R/O_r(H), \sigma]=1$. Hence L centralizes $R/O_r(H)$ by Lemma 3.4. Then $R=R\cap N_G(R)'=[L, R]\subseteq O_r(H)$ by Lemma 4.3. Thus $R=O_r(H)\lhd H$.

STEP 7. $r\nmid|N_G(Q)/C_G(Q)|$.

PROOF. By Lemma 4.23, $[N_R(Q), \sigma]\subseteq C_R(Q)$. Hence $N_R(Q)=(C_G(\sigma)\cap N_R(Q))C_R(Q)$. Setting $R_0=C_G(\sigma)\cap N_G(Q)$, $[R_0, Q\cap H]\subseteq Q\cap R$ since $H\rhd R$. Since $(Q\cap H)\times R_0$ normalizes Q and $C_Q(Q\cap H)=Q\cap H$ by Step 4, $[R_0, Q\cap H]=1$ by $A\times B$-theorem (see [3], Theorem 5.3.4). Hence $N_R(Q)=C_R(Q)$.

STEP 8. *$C_{\bar{Q}}(\sigma)\neq 1$, where $\bar{Q}=Q/Q'$.*

PROOF. See Lemma 4.17.

STEP 9. *For some prime $p\in\pi(N_G(Q))-\{q, r\}$, there exists the A-invariant Sylow p-subgroup P_0 of $N_G(Q)$ such that $[P_0, \sigma]\not\subseteq C_G(Q)$.*

PROOF. Suppose false. Let U be the A-invariant Hall q'-subgroup of $N_G(Q)$. Then $N_G(Q)=C_G(Q)QC_U(\sigma)$ by Step 7. Set $N=N_G(Q)$ and $\bar{N}=N/Q'$. Then $\bar{N}'\cap\bar{Q}=[\overline{C_U(\sigma)}, \bar{Q}]$. Since $\bar{Q}=[\overline{C_U(\sigma)}, \bar{Q}]\times C_{\bar{Q}}(\overline{C_U(\sigma)})$ and $1\neq C_{\bar{Q}}(\sigma)\subseteq C_{\bar{Q}}(\overline{C_U(\sigma)})$, $\bar{Q}\supsetneqq[\overline{C_U(\sigma)}, \bar{Q}]=\bar{N}'\cap\bar{Q}$, and so $N'\cap Q\subsetneqq Q$. This contradicts Lemma 4. 3.

STEP 10. *Let P be the A-invariant Sylow p-subgroup of G. Then $PR\neq RP$.*

PROOF. Suppose false. Setting $T=PR=RP$ and $Q_0=C_Q(\sigma)$, $[Q_0, R]=1$ by Step 3. Hence $<C_G(\sigma), Z(Q), R>\subseteq C_G(Q_0)$. Setting $K=C_G(Q_0)$, $R=[Z(Q), R]\subseteq F(K)$ by Lemma 3. 8 and Steps 5 and 6. Hence $O_r(K)=R$ and so $C_G(\sigma)\subseteq K\subseteq N_G(R)$.

Now $1\neq[R, \sigma]\subseteq O_r(T)$ by Lemmas 4. 2 and 4. 10. Setting $M=N_G(O_r(T))$, $<P, Q_0>\subseteq M$. Moreover, setting $P_1=[P_0, \sigma]$, $1\neq[P_1, Q]=[P_1, O_q(M)]$ by Lemma 4. 12. Since $[P_1, Q]\lhd Q$, $Z(Q)\cap[P_1, O_q(M)]\neq 1$. Setting $Q_1=Z(Q)\cap[P_1, O_q(M)]$, $Q_1\subseteq O_q(M)$, and so $[Q_1, O_r(M)]=1$. Since $O_r(T)\subseteq O_r(M)$, $[Q_1, O_r(T)]=1$.
Now, since $O_q(H)\times O_r(T)\subseteq F(H)$ and $O_q(H)\neq 1\neq O_r(T)$, $O_q(H)\times O_r(T)\subseteq(C_G(Q_1))_{q,r}\subseteq H$ by Lemma 4. 21. Thus $Q\subseteq H$ and so $<Q, R>\subseteq H$, a contradiction.

STEP 11. *We have a contradiction.*

PROOF. By Step 10, $PR=RP$. By Lemmas 5. 2 and 7. 1 and Step 3, $C_P(\sigma)\neq 1=C_{Z(P)}(\sigma)$ and P is non-abelian. Then $PQ=QP$ by Lemma 6. 1. Since P and Q are non-abelian by Step 3, $PQ\rhd P$ and $PQ\rhd Q$ by Lemma 4. 7. Thus $[P, Q]=1$. Hence $p\nmid|N_G(Q)/C_G(Q)|$. This contradicts Step 9.

By Lemma 5. 2, 6. 1, and 7. 1, 8. 1, if P, Q are the A-invariant Sylow p- and q-subgroups of G for p, q in $\pi(G)$ $(p\neq q)$, then $PQ=QP$. By P. Hall's characterization of solvable groups, G is solvable. Thus we have a final contradiction. Hence no minimal counterexample to the theorem can exist, and the theorem is proved.

Additional Comment. After I submitted this paper for publication, I was informed that the same result was also obtained by B. Dolman in his unpublished Ph. D. Thesis at the University of Adelaide.

References

[1] G. GLAUBERMAN; Fixed points in groups with operator groups, Math. Z. 84 (1964), 120

-125.
[2] G. GLAUBERMAN; A sufficient condition for p-stability, Proc. London Math. Soc. (3) 25 (1972), 253-287.
[3] D. GORENSTEIN; Finite Groups, Harper & Row, New YorK, 1968.
[4] R. P. MARTINEAU; Elementary abelian fixed-point-free automorphism groups, Quart. J. Math. Oxford (2) 23 (1972), 205-212.
[5] B. H. NEUMANN; Groups with automorphisms that leave only the neutral element fixed, Archiv der Math. 7 (1956), 1-5.
[6] E. W. RALSTON; Solvability of finite groups admitting fixed-point-free automorphisms of order rs, J. Algebra. 23 (1972), 164-180.
[7] E. SHULT; On groups admitting fixed point free abelian operator groups, Illinois J. Math. 9 (1965), 701-720.
[8] H. WIELANDT; Beziehungen zwischen den Fixpunktzahlen von Automorphismengruppen einer endlichen Gruppe, Math. Z. 73 (1960), 146-158.

Department of Mathematics,
Faculty of General Study
Gunma University,
Japan