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On almost Blaschke manifolds II

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§ 0. Introduction

In a previous paper [5] the auther studied the topology of a compact riemannian manifold (M, g) whose injectivity radius $i(M)$ is close to its diameter $d(M)$ and got the following results.

THEOREM A. *Let (M, g) be a 2-dimensional riemannian manifold and K denote its Gaussian curvature. Assume that one of the following holds;*

(i) *There is a positive number δ such that*

$$K \geq -\delta^2 \text{ and } \sinh(\delta i(M)) > (\sqrt{3}/2)\sinh(\delta d(M)).$$

(ii) *$K \geq 0$ and $i(M) > (\sqrt{3}/2) d(M)$ hold.*

Then M is diffeomorphic to the sphere S^2 or projective plane P^2.

THEOREM B. *Let (M, g) be a 3-dimensional riemannian manifold and K denote its sectional curvature. If there is a positive number δ such that*

$$K \geq \delta^2 \text{ and } \sin(\delta i(M)) > (\sqrt{3}/2)\sin(\delta d(M)),$$

then M is diffeomorphic to the sphere S^3 or projective space P^3.

These results are best possible.

On the other hand recently O. Durumeric has shown in [4] that in arbitrary dimension any manifold whose injectivity radius is sufficiently close to its diameter has either the trivial fundamental group or the homotopy type of the real projective space.

In this paper we prove the following theorem.

THEOREM. *Let (M, g) be a 3-dimensional compact riemannian manifold and K denote its sectional curvature. Assume that one of the following holds;*

(i) *There is a positive number δ such that we have*

$$K \geq -\delta^2 \text{ and } \sinh(\delta i(M)) > a(\delta d(M)) \cdot \sinh(\delta d(M))$$

where

$$a(\delta d(M)) := \sin\left[\frac{\pi}{2} - \frac{1}{10}\sin^{-1}\left\{\frac{\sin h(\frac{1}{10}\delta d(M))}{\sin h(2\delta d(M))}\right\}\right].$$

(ii) $K \geq 0$ *and* $i(M) > a \cdot d(M)$ *hold*
where

$$a := \sin\left(\frac{\pi}{2} - \frac{1}{10}\sin^{-1}\frac{1}{20}\right) = 0.99998\cdots\cdots.$$

Then M *is diffeomorphic to* 3*-dimensional sphere* S^3 *or real projective space* P^3.

REMARK 1. The inequalities of Theorem are invariant under homotheties.

To prove Theorem we use the stable cut locus whose local structure is well understood by M. Buchner in [2]. In a previous paper [5] the auther showed that the stable cut locus collapses on one point or a subcomplex which consists of some S^2 's, P^2 's and trees. We will show that the subcomplex is PL-homeomorphic to P^2 by Toponogov's comparison theorem.

The auther would like to express my sincere thanks to Prof. T. Sakai for his kind advices.

§ 1. Preliminaries

To begin with we state the following theorems which play important roles in our arguments. We denote the distance from p to q by $d(p, q)$.

TOPONOGOV'S COMPARISON THEOREM (T. C. T.) [3] *Let* M *be a complete manifold with sectional curvature* $K \geq c$. *Let* γ_1, γ_2 *be geodesic segments of length* l_1, l_2 *in* M *such that* $\gamma_1(l_1) = \gamma_2(0)$ *and* $\measuredangle(-\dot{\gamma}_1(l_1), \dot{\gamma}_2(0)) = \theta$. *We call such a configuration a hinge and denote it by* $(\gamma_1, \gamma_2, \theta)$. *Assume that* γ_1 *is minimal and* $l_2 \leq \pi/\sqrt{c}$, *if* $c > 0$. *Let* γ_1^*, γ_2^* *be geodesic segments of length* l_1, l_2 *in the simply connected* 2*-dimensional space of constant curvature* c *such that* $\gamma_1(l_1) = \gamma_2(0)$ *and* $\measuredangle(-\dot{\gamma}_1^*(l_1), \dot{\gamma}_2^*(0)) = \theta$. *Then* $d(\gamma_1(0), \gamma_2(l_2)) \leq d(\gamma_1^*(0), \gamma_2^*(l_2))$.

REMARK 2. By checking the proof of T. C. T. carefully, in the above we may choose a geodesic segment γ_3 from $\gamma_1(0)$ to $\gamma_2(l_2)$ in M which is homotopic to $\gamma_2 \circ \gamma_1$ and whose length is less than or equal to $d(\gamma_1^*(0), \gamma_2^*(l_2))$.

THEOREM (BUCHNER). *If dem* $M = 3$ *and* $p \in M$ *then the picture near a point* q *on stable cut locus* $C(p)$ *of* p *is* (i) *a plane through* q *or* (ii) *three planes meeting along a line through* q, *any two of the planes having regular intersection or* (iii) *the picture of* 6 *planes meeting along* 4 *lines all meeting* *at q obtained by viewing q as the barycenter of a tetrahedron and joining it to the* 4 *vertices or* (iv) *a half plane with q in the boundary or* (v) *a quarter plane glued onto a surface. See Figure* 1.

REMARK 3. By checking the proof of the above theorem, we have the following. There are n minimal geodesic segments from p to $q \in C(p)$ where $n=2$ in the case (i) of the above theorem, $n=3$ in the case (ii), $n=4$ in the case (iii) and $n=2$ is the case (v). There is one geodesic segment from p to q in the case (iv). Namely for any $q \in C(p)$ there are at most 4 minimal geodesic segments from p to q.

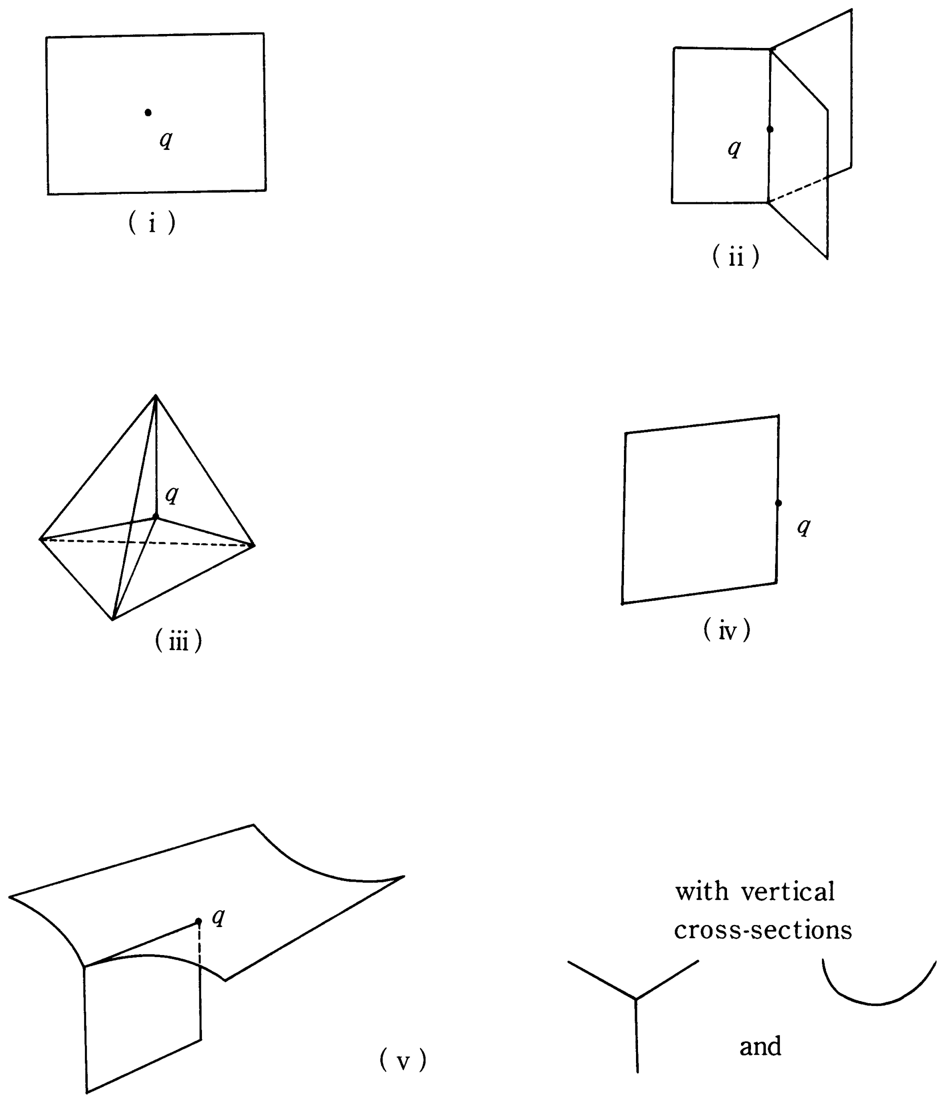

Figure 1.

We can take a cut stable metric g_0 which satisfies the hypotheses of Theorem by an approximation ([1]). Moreover we can take some triangulation of M which is compatible with the local structure of stable cut locus $C(p)$. Take a subcomplex $\tilde{C}(p)$ on which $C(p)$ collapses and which can not collapse further. Let $\{F_i\}$ be the family of connected components of the union of all 2-simplexes in $\tilde{C}(p)$ and let $\{E_j\}$ be the family of connected components of $(\tilde{C}(p)\backslash(\bigcup_i F_i))$, when $\tilde{C}(p)$ is not one point. Each F_i is a 2-complex and the closure of E_j is a 1-complex. Now we state the results which are shown in [5].

(1) Under the hypotheses of Theorem each F_i is a 2-dimensional PL-manifold (without boundary), which follows from Lemma 1 in [5]. In fact the hypotheses of the lemma is weaker than those of Theorem.

(2) F_i is PL-homeomorphic to S^2 or P^2 and the closure of E_j is a tree (Lemma 3 in [5]), Moreover if F_i is homeomorphic to P^2, then for ecch $\bar{E}_j$ there is at most one end point which is contained in F_i (See the proof of Lemma 3 in [5]).

(3) If $\tilde{C}(p)$ is PL-homeomorphic to one point or P^2, then Theorem follows immediately. See § 2 and § 3 in [5]. In the next section we will show that the hypothesis of the above result (3) holds.

We now define some function $l(b, \theta)$ as follows; Take a geodesic right triangle $(\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3)$ in the simply connected 2-dimensional space of constant curvature $-\delta^2$ such that $\tilde{\phi}_1=\pi/2$, $\tilde{\phi}_3=\theta$, $\tilde{l}_1=b$ where $\tilde{l}_i$ is the length of $\tilde{\gamma}_i$ and

$$\tilde{\phi}_i=\sphericalangle(-\dot{\tilde{\gamma}}_{i+1}(\tilde{l}_{i+1}),\ \dot{\tilde{\gamma}}_{i+2}(0)).$$

Put $l(b, \theta):=\tilde{l}_2$. We can represent $l(b,\theta)$ explicitly from sine and cosine rules.

§ 2. The proof of Theorem

In this section under the hypotheses of Theorem we show that $\tilde{C}(p)$ is homeomorphic to one point or P^2. Now it is known that $\tilde{C}(p)$ consists of $\{F_i\}$ and $\{E_j\}$ where $F_i\simeq S^2$ or P^2 and $\bar{E}_j$ is a tree, if $\tilde{C}(p)$ is not one point ([5]). To begin with we prepare the following Lemmas 1 and 2. Put

$$\Theta:=\frac{\pi}{2}-\frac{1}{10}\sin^{-1}\{\sinh(\frac{1}{10}\cdot\delta d(M))/\sinh(2\cdot\delta d(M))\},$$

then $a(\delta d(M))=\sin\Theta$.

LEMMA 1. *Under the hypotheses of Theorem let q be a cut point of p and γ_v, $\gamma_{v'}$ be geodesic segments from p to q with initial vectors v, v' such* *that* $\gamma_v \circ (\gamma_{v'})^{-1}$ *is a homotopically non trivial closed curve. Then* $\sphericalangle(v, v') > 2\Theta$.

PROOF. Let b be the distance from p to q and γ_2 be the geodesic from p with the unit initial direction $(v+v')/|v+v'|$. In this section we simply denote $l(b,\Theta)$ by l_b, Put $x := \gamma_2(l_b)$. Assume that $\sphericalangle(v, v') \leq 2\Theta$ (i.e. $\sphericalangle(\dot{\gamma}_v(0), \dot{\gamma}_2(0)) \leq \Theta$, $\sphericalangle(\dot{\gamma}_{v'}(0), \dot{\gamma}_2(0)) \leq \Theta$). We apply Remark 2 after T. C. T. in § 1 to two hinges $((\gamma_v)^{-1}, \gamma_2|[0, l_b], \sphericalangle(\dot{\gamma}_v(0), \dot{\gamma}_2(0)))$ and $((\gamma_{v'})^{-1}, \gamma_2|[0, l_b], \sphericalangle(\dot{\gamma}_{v'}(0), \dot{\gamma}_2(0)))$. Then there are geodesic segments γ_3^1, γ_3^2 from q to x whose lengths are less than $i(M)$ such that γ_3^1 is homotopic to $(\gamma_2|[0, l_b]) \circ (\gamma_v)^{-1}$ and γ_3^2 is homotopic to $(\gamma_2|[0, l_b]) \circ (\gamma_{v'})^{-1}$. Hence $(\gamma_3^1)^{-1} \circ \gamma_3^2$ is a homotopically non trivial closed curve whose length is less than $2i(M)$. This is a contradiction. Q. E. D.

Put $\Psi := \pi - 2\Theta$, then

$$\Psi = \frac{1}{5}\sin^{-1}\{\sinh(\frac{1}{10}\cdot\delta d(M))/\sinh(2\delta d(M))\}.$$

From

$$\sin 5\psi = \{\sinh(\frac{1}{10}\cdot\delta d(M))/\sinh(2\delta d(M))\}$$

it follows that

$$\sin(\frac{1}{2}\psi) = \sin(\frac{1}{2}(\pi - 2\Theta)) < \{\sinh(\frac{1}{10}\cdot\delta d(M))/\sinh(\delta d(M))\}.$$

From $\sinh(\delta i(M)) > a(\delta d(M)) \cdot \sinh(\delta d(M))$ and the definition of $a(\delta d(M))$, we have $d(M) - i(M) < (1/100)d(M)$.

LEMMA 2. *Under the hypotheses of Theorem if there are* q_i $(i=1,2)$ *which are cut points of* p *and minimal geodesic segments* γ_{v_i}, $\gamma_{v'_i}$ $(i=1, 2)$ *from* p *to* q_i *with initial vectors* v_i, v'_i *such that* $\gamma_{v_i} \circ (\gamma_{v'_i})^{-1}$ *is homotopically non trivial and* $\gamma_{v_1} \circ (\gamma_{v'_1})^{-1}$ *is not homotopic to* $\gamma_{v_2} \circ (\gamma_{v'_2})^{-1}$, *then* $\sphericalangle(v_1, v_2) > 10 \cdot \psi = 10(\pi - 2\Theta)$.

PROOF. Let $\tilde{M}$ be the universal covering space of M, Π be its covering map. Take a point $\tilde{p}_0 \in \Pi^{-1}(p)$. Denote by $\tilde{v}_i$, the lift of v_i to $T_{\tilde{p}_0}\tilde{M}$ and by $\tilde{\gamma}_{\tilde{v}_i}$, the geodesic from $\tilde{p}_0$ with initial direction $\tilde{v}_i$. Let $\tilde{q}_i$ be the first point along $\tilde{\gamma}_{\tilde{v}_i}$ such that $\Pi(\tilde{q}_i) = q_i$. We denote the distance from p to q_i by b_i. Take a geodesic α_i from $\tilde{q}_i$ with $\Pi(\alpha_i|[0, b_i]) = (\gamma_{v'_i})^{-1}$. Put $\tilde{p}_i := \alpha_i(b_i)$, then $\Pi(\tilde{p}_i) = p$. Thus we have $i(M) < d(\tilde{q}_i, \tilde{p}_i) = d(\tilde{q}_i, \tilde{p}_0) = d(q_i, p) = b_i < d(M)$.

Define $y_i := \tilde{\gamma}_{\tilde{v}_i}(2d(M))$. Now we will show that $d(y_i, \tilde{p}_i) < \frac{3}{10} d(M)\ (i=1, 2)$. Put $w_{0,i} := -\dot{\tilde{\gamma}}_{\tilde{v}_i}|_{\tilde{q}_i}$, $w_i := \dot{\alpha}_i|_{\tilde{q}_i}$. From Lemma 1 it follows that $\measuredangle(w_{0,i}, w_i) > 2\Theta$. Put $w_{y_i} := \dot{\tilde{\gamma}}_{\tilde{v}_i}|_{\tilde{q}_i} (= -w_{0,i})$. We take the geodesic segment β_i from $\tilde{q}_i$ to y_i with initial vector w_{y_i} $(\beta_i \subset \tilde{\gamma}_{\tilde{v}_i})$. Put $y'_i := \beta_i(d(M)) (= \tilde{\gamma}_{\tilde{v}_i}(d(M) + b_i))$ and $\tilde{p}'_i := \alpha_i(d(M))$. Then $d(y'_i, 7_i) = d(\tilde{p}'_i, \tilde{q}_i) = d(M)$ and $\measuredangle(w_i, w_{y_i}) < \psi = \pi - 2\Theta$ hold. From T. C. T. it follows that $d(y'_i, \tilde{p}'_i) < (2/10) d(M)$, $d(y_i, y'_i) < (d(M) - i(M)) < (1/100) d(M)$ and $d(\tilde{p}_i, \tilde{p}'_i) < (d(M) - i(M)) < (1/100)\ d(M)$ (See Figure 2). Hence $d(y_i, \tilde{p}_i) < (3/10) d(M)$.

Put $\xi := \measuredangle(\tilde{v}_i, \tilde{v}_2)$. We consider the hinge $((\tilde{\gamma}_{\tilde{v}_1}|[0, 2d(M)])^{-1}, \tilde{\gamma}_{\tilde{v}_2}|[0, 2d(M)], \xi)$ at p_0. From T. C. T. it follows that

$$d(y_1, y_2) < \frac{2}{\delta} \sinh^{-1}\{\sin\left(\frac{\xi}{2}\right) \cdot \sinh(2\delta d(M))\}.$$

We will show that $\xi > 10 \cdot \psi = 10(\pi - 2\Theta)$. If $\xi \leq 10 \cdot \psi$ holds, we have $d(y_1, y_2) < (2/10) d(M)$. Then

$$\begin{aligned} d(p_1, p_2) &\leq d(p_1, y_1) + d(y_1, y_2) + d(y_2, p_2) \\ &< (8/10) d(M). \end{aligned}$$

On the other hand for $\Pi(\tilde{p}_1) = \Pi(\tilde{p}_2)$ and $\tilde{p}_1 \neq \tilde{p}_2$, $d(\tilde{p}_1, \tilde{p}_2) < 2i(M)$ holds. Hence an inequality $2i(M) < (8/10) d(M)$ contradicts the hypotheses.

Q. E. D.

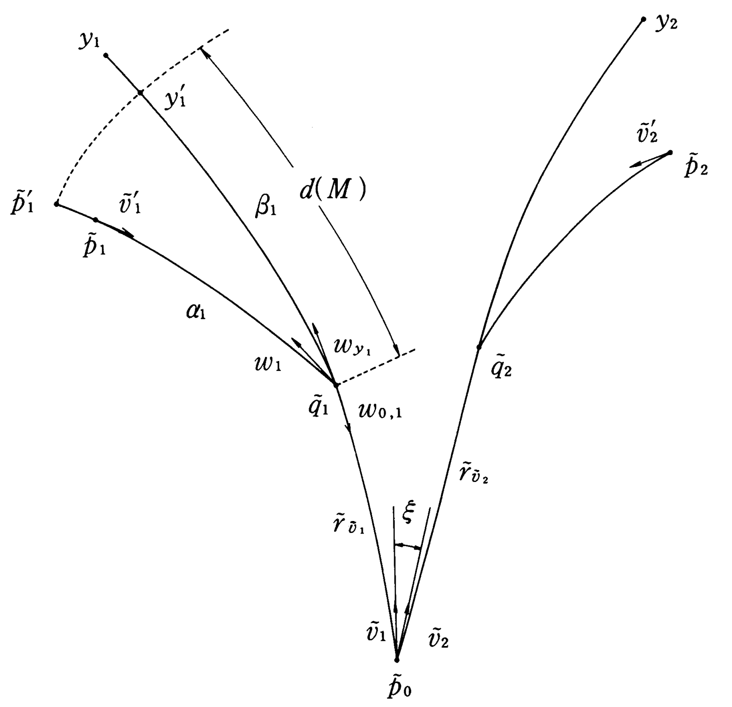

Figure 2.

We show that $\tilde{C}(p)$ consists of just one F_i which is homeomorphic to P^2, if $\tilde{C}(p)$ is not one point. From (2) in § 1 if there is only one F_i which is homeomorphic to P^2 or $\{F_i\}$ is empty, then $\{E_j\}$ is empty. From now on we will derive a contradiction under one of the following assumptions; (i) There are at least two 2-complexes F_i and F_j which are homeomorphic to P^2. (ii) There is at least one 2-complex F_i which is homeomorphic to S^2. Firstly we consider the case (i). There are homotopically non trivial closed curves c_1 on F_i and c_2 on F_j which are transverse to all 1-simplexes and do not contain any 0-simplex of F_i, F_j. Let Φ be a map from U_pM to $C(p)$ such that for any $x\in U_pM$ $\Phi(x)$ is the cut point of gecodesic from p with initial direction x. For any $q\in C(p)$ we have $\#\{\Phi^{-1}(q)\}\leq 4$ from Remark 3. Moreover for any $q\in c_1$ or c_2 we have $\#\{\Phi^{-1}(q)\}\leq 3$. In the following a minimal geodesic segment on U_pM, which is S^2 with the cannonical metric, is called a great arc.

LEMMA 3. *Under the above situation* $\sphericalangle(\Phi^{-1}(c_1),\ \Phi^{-1}(c_2)):=\min\{\sphericalangle(x_1,x_2)\,|\,x_1\in\Phi^{-1}(c_1),\ x_2\in\Phi^{-1}(c_2)\}<4\psi$.

PROOF. We note that for any $x\in\Phi^{-1}(c_k)$ there is $x^*\in\Phi^{-1}(c_k)$ such that $\Phi(x)=\Phi(x^*)$ and $\gamma_x\circ(\gamma_{x^*})^{-1}$ is htmotopically non trivial closed curve where $k=1$ or 2 and γ_x, γ_{x^*} are geodesic segments from p to their cut points with initial directions x, x^*. Moreover it follows that $\sphericalangle(x, x^*)>2\Theta$ from Lemma 1. We denote by $\{c_k^l\}$, the family of connected components of $\Phi^{-1}(c_k)$. Note that c_k^l is a curve on U_pM. For any end point x of c_k^l there is only one end point y of $c_k^{l'}$ such that $\Phi(x)=\Phi(y)$ and $\gamma_x\circ(\gamma_y)^{-1}$ is homotopically trivial closed curve. For any above pair (x, y) we have $\sphericalangle(x, y)<2\psi$. In fact since we have $\sphericalangle(x, x^*)>2\Theta$ and $\sphericalangle(y, x^*)>2\Theta$, it follows that $\sphericalangle(x, -x^*)<\pi-2\Theta=\psi$ and $\sphericalangle(y, -x^*)<\psi$. Take the closed curve s_k $(k=1, 2)$ which consists of $\{c_k^l\}$ and great arcs connecting all of the above pairs of end points of c_k^l's. Fix two points $x_1\in\Phi^{-1}(c_1)$ and $x_2\in\Phi^{-1}(c_2)$. Let s_k' be a curve which is a part of s_k connecting x_k and x_k^*. We denote by $\tilde{c}_k$, the closed curve on U_pM which consists of s_k', the antipodal curve s_k'' of s_k', and two great arcs connecting end points of s_k', s_k'' whose lengths are less than 2ψ. For any $z\in\tilde{c}_k$ it holds that $\sphericalangle(z, \Phi^{-1}(c_k))<2\psi$ from the construction. Moreover for any $z\in\tilde{c}_k$ the antipodal point of z is contained in $\tilde{c}_k$. Thus evidently $\tilde{c}_1$ and $\tilde{c}_2$ have intersections. Then $\sphericalangle(\Phi^{-1}(c_1),\ \Phi^{-1}(c_2))<4\psi$. Q. E. D.

Now we can take vectors v_1, $v_2\in U_pM$ with $\sphericalangle(v_1, v_2)<4\psi$ such that the cut point of p along γ_{v_1} (resp. γ_{v_2}) is contained in $F_i\simeq P^2$ (resp. $F_j\simeq P^2$). This contradicts the conclusion of Lemma 2. Thus in $\tilde{C}(p)$ there is at most one surface F_i which is homeomorphic to P^2.

Next we consider the case (ii). In this case $F_i(\simeq S^2)$ is two-sided in M. We simply denote F_i by F. Put $D:=\Phi^{-1}(F)(\subset U_pM)$. The set D coincides with the union of of D_+ and D_- such that all geodesics from p with initial vectors in D_+ (resp. D_-) strike on F $(\simeq S^2)$ at their cut points from the fixed one (resp. the other) side of F $(\simeq S^2)$ and $(D_+\cap D_-)=\Phi$.

LEMMA 4. *Under the above situation we have* $\measuredangle(v_1, v_2)<\pi-9\psi$ *for any* v_1, $v_2\in D_+$ *(resp.* D_-*).*

PROOF. For any $v_1\in D_+$ there is $v_1'\in D_-$ such that $\measuredangle(v_1, v_1')>2\Theta$ from Lemma 1. Then $\measuredangle(-v_1, v_1')<\psi$ for $\psi=\pi-2\Theta$. Since $v_1'\in D_-$ we have $\measuredangle(v_1', v_2)>10\cdot\psi$ for any $v_2\in D_+$ from Lemma 2. Hence $\measuredangle(-v_1, v_2)>9\psi$, namely $\measuredangle(v_1, v_2)<\pi-9\psi$. Q. E. D.

Let $\{D_{+,i}\}$ (resp. $\{D_{-,i}\}$) be the family of conncted components of D_+ (resp. D_-). Any 1-simplex σ_k^i in $\partial D_{+,i}$ is identified by Φ with the other 1-simplex $\sigma_l^{i'}$ in $\partial D_{+,i'}$ where D has a triangulation induced by Φ^{-1} from F. Let $\Xi_{j,k;j',l}^+$ be the union of all great arcs on U_pM connecting points of σ_k^j and those of $\sigma_l^{j'}$ which are identified by Φ. We note that the lengths of these great arcs are less than $2\psi=2(\pi-2\Theta)$. In fact, when v_1 and v_2 are the end points of a great arc, there is $v_1'\in D_-$ such that $\Phi(v'_1)=\Phi(v_1)=\Phi(v_2)$. From Lemma 1 it follows that $\measuredangle(v_1, v_1')>2\Theta$ and $\measuredangle(v_2, v_1')>2\Theta$. Then $\measuredangle(v_1, v_2)<2\psi$. We denote by D'_+, the union of D_+ and all $\Xi_{j,k;j',l}^+$'s. Define $\Xi_{j,k;j',l}^-$ and D'_- from D_- in the same way as above. It is easy to show that D'_+ and D'_- do not intersect. In fact suppose $x\in(D'_+\cap D'_-)$, then since D_+ and D_- do not intersect there are two great arcs each of which contains x and whose lengths are less than 2ψ. Hence there are $y\in D_+$, $z\in D_-$ such that $\measuredangle(x, y)<\psi$ and $\measuredangle(x, z)<\psi$, namely $\measuredangle(y, z)<2\psi$ holds. This contradicts Lemma 2. Moreover D'_+ and D'_- are connected from the construction. We denote by D''_+ (resp. D''_-), the simply connected domain on $U_pM(\simeq S^2)$ which contains D'_+ (resp. D'_-), does not intersect D'_- (resp. D'_+) and whose boundary is contained in $\partial D'_+$(resp. $\partial D'_-$). The domains D''_+ and D''_- do not intersect. From now on we only consider D_+, $D_{+,i}$, D'_+, D''_+, $\Xi_{j,k;j',l}^+$ and simply denote them by D, D_i, D', D'', $\Xi_{j,k;j',l}$ respectively. Take the disjoint union of $\{D_i\}$ and $\{\Xi_{j,k;j',l}\}$. We attach all $\Xi_{j,k;j',l}$'s to $\{D_i\}$ by the identity maps on σ_k^j, $\sigma_l^{j'}$ and get a set D'''. For any $y\in\partial D$ we have $\sharp\{(\Phi|_D)^{-1}(\Phi(y))\}\leq 3$, because we put $D=D_+$. Then each connected component e_m of $\partial D'''$ consists of three great arcs in $\bigcup_{j,j',k,l}\Xi_{j,k;j',l}$ whose lengths are less than 2ψ. On U_pM we denote by E_m, the domain bounded by e_m which is contained in $(3/2)\psi$-disk. We attach

E_m to D''' by the identity map on e_m for all m and get $\boldsymbol{D}$. We note that $\boldsymbol{D}$ is homeomorphic to a sphere from the construction. Put $\tilde{D}:=D'\cup(\bigcup_m E_m)$ on U_pM. Let $\Psi:\boldsymbol{D}\to\tilde{D}\subset U_pM$ be a natural projection, namely $\Psi|_{D_i}$, $\Psi|_{\Xi_{j,k;j',l}}$ and $\Psi|_{E_m}$ are the identity maps.

Now we define some property of subsets of U_pM $(\simeq S^2)$ as follows; A subset $s\subset U_pM$ has the property $(*)$ with respect to $x\in U_pM$ if and only if for any great circle c through x on U_pM the subset s is not contained in each open hemi-sphere with boundary c. Put $B_x^\varepsilon:=\{v\in U_pM \mid \measuredangle(v,x)\le\Theta-\varepsilon\}$ $(\varepsilon>0)$.

Lemma 5. *Under the above situation for (PL)-simple closed curve* $A:=\partial\tilde{D}$ on U_pM either (I) or (II) of the following holds.

(I) There are $\varepsilon>0$ *and* $x\in U_pM$ *such that* $B_x^\varepsilon\supset A$.

(II) *For any* $\varepsilon>0$ *there are* $x\in U_pM$ *and a connected component* $\mathscr{A}$ *of* $(B_x^\varepsilon\cap A)$ *such that* $(\partial B_x^\varepsilon\cap\mathscr{A})$ *on* U_pM *has the property* $(*)$ *with respect to* x.

Proof. For any $z\in U_pM$ and for any $y\in A$ with $\measuredangle(z,y)=\Theta-\varepsilon$, we denote by $\mathscr{A}_{z,y}$, the connected component of $(B_z^\varepsilon\cap A)$ containing y. Under the negotiation of (II), we will derive (I). If for some $\varepsilon>0$, $z\in U_pM$ and $y\in A$ with $\measuredangle(z,y)=\Theta-\varepsilon$ a subset $(\partial B_z^\varepsilon\cap\mathscr{A}_{z,y})$ does not have the property $(*)$ with respect to z, then there are a great circle c through z and hemi-sphere O with boundary c which contains $(\partial B_z^\varepsilon\cap\mathscr{A}_{z,y})$. Denote by $\{z_t\}$, the great arc from $z_0=z$ to the pole of O. For sufficiently small $t>0$ there is $y_t\in A$ such that $\measuredangle(z_t,y_t)=\Theta-\varepsilon$ and $\mathscr{A}_{z_t,y_t}\supset\mathscr{A}_{z,y}$. Denote the maximum of such number by t_0. We exchange $z^{(1)}$, $y^{(1)}$ for z_{t_0}, y_{t_0} and repeat the same procedure successively. At last there is n such that $\mathscr{A}_{z^{(n)},y^{(n)}}$ coincides with A, then (I) holds. Q. E. D.

Put

$$l:=l(d(M),\ \Theta-\varepsilon)$$
$$i_\varepsilon:=\frac{1}{\delta}\sinh^{-1}(\sin(\Theta-\varepsilon)\cdot\sinh(\delta d(M)))<i(M).$$

Let $\mathscr{B}_z^\varepsilon$ be a closed ball in M centered at $\mathrm{Exp}_p lz$ with radius i_ε. Under the above preparations we will show that the existence $F(\simeq S^2)$ derives a contradiction. If there are $\varepsilon>0$ and $x\in U_pM$ such that $B_x^\varepsilon\supset D$, then we get $\mathscr{B}_x^\varepsilon\supset F$. In fact from T. C. T. $\mathscr{B}_x^\varepsilon$ contains the subset consisting of all geodesic segments from p whose lengths are equal to $d(M)$ and whose initial vectors are contained in $B_x^\varepsilon(D\subset\tilde{D}\subset B_x^\varepsilon)$. Then the homotopically non trivial sphere F is contained in contractible ball $\mathscr{B}_x^\varepsilon$. This is a contradiction.

Next we consider the case that for any $\varepsilon>0$ and $z\in U_pM$, F is not contained in $\mathscr{B}_z^\varepsilon$. Namely for a closed curve $A=\partial\tilde{D}$, (II) of Lemma 5 holds. Take a point $z\in U_pM$, sufficiently small number $\varepsilon>0$ ($\varepsilon\ll\psi$) and a connected component $\boldsymbol{G}_0$ of $(\boldsymbol{D}\cap\Psi^{-1}(B_z^\varepsilon))$ such that $(\partial B_z^\varepsilon\cap\partial\Psi(\boldsymbol{G}_0))$ has the property $(*)$ with respect to z and fix them. Put $G_0:=\Psi(\boldsymbol{G}_0)$. Denote by $\{\boldsymbol{G}_k\}$, the family of closure of each connected component of $(\boldsymbol{D}\backslash\boldsymbol{G}_0)$. Put $G_k:=\Psi(\boldsymbol{G}_k)$. We simply denote B_z^ε, $\mathscr{B}_z^\varepsilon$ by B_z, $\mathscr{B}_z$. We can deform $\mathscr{B}_z$ to sufficiently near $\mathscr{B}'_z$ so that $\mathscr{B}'_z$ is transverse to F and $(\partial\mathscr{B}'_z\cap F)$ consists of closed curves. Let R_0 be a connected component of $(F\cap\mathscr{B}'_z)$ such that $\Phi(G_0\cap D)\subset R_0$. Each connected component of $(F\backslash R_0)$ is homeomorphic to a disk. We denote by $\{R_j\}$, the family of closure of each connected component of $(F\backslash R_0)$. For any j there is $k(j)$ such that $(\Phi|_D)^{-1}(R_j)$ intersects $G_{k(j)}$. Put $c_j:=R_j\cap R_0$. Now for each connected component G_k take a point $y_k\in\partial B_z$ such that

$$\sphericalangle(y_k,\ \partial B_z\cap G_k)=\max_{y\in\partial B_z}\sphericalangle(y,\ \partial B_z\cap G_k).$$

The 4ψ-neighborhood of y_k does not intersect $\tilde{D}$ from Lemma 4. For any G_k there are $n,m\in\{k\}$ such that $(G_k\cap\partial B_z)$ is contained in the inferior arc $y_n\cdot y_m$ on ∂B_z. Define a great arc γ_k by $\gamma_k:=\{$the great arc connecting y_n and $y_m\}\backslash\{4\psi$-neighborhood of y_n and $y_m\}$. Let N_k be a 2ψ-neighborhood of γ_k in $B_z^{2\varepsilon}$ We can take a homotopy H_t: $S^1\times[0,1]\to F$ and a simple closed curve c_j^1 on F such that $H_0=c_j$, $H_1=c_j^1$, $(\Phi|_D)^{-1}(c_j^1)\subset N_{k(j)}$, $(\Phi|_D)^{-1}(H_t(S^1\times[0,1]))\subset(B_z\cup G_{k(j)})$, c_j^1 is transverse to all 1-simplexes of F and c_j^1 does not contain any 0-simplex of F. This is possible from the construction of $\tilde{D}$. For any circle c_j^1 on $F\cap\mathscr{B}_z$ we denote by $\mathscr{D}_z(c_j^1)$, the union of geodesic segments connecting $\mathrm{Exp}_p lz$ (the center of $\mathscr{B}_z$) and all points of c_j^1. Note that there is $\tilde{\varepsilon}>0$, which depends on ε, such that for any z' with $\sphericalangle(z,z')\leq\tilde{\varepsilon}$ and for any j it holds that $\mathscr{D}_z(c_j^1)\subset\mathscr{B}_{z'}$. We denote by F_j^1, the domain on F bounded by c_j^1 such that F_j^1 and $\{\bigcup_{l\neq j}c_l^1\}$ are disjoint. Then $(F_j^1\cup\mathscr{D}_z(c_j^1))$ is homeomorphic to a sphere and moreover there is at least one $J\in\{j\}$ such that $(F_J^1\cup\mathscr{D}_z(c_J^1))$ is homotopically non trivial, for F is homotopically non trivial.

From now on we simply denote F_J^1 by F^1. The set $(\Phi|_D)^{-1}(c_J^1)$ is the union of curves. We can get the closed curve $\tilde{c}_J^1$ in $N_{k(J)}$ by connecting the corresponding end points of $(\Phi|_D)^{-1}(c_J^1)$ with great arcs. Let $\boldsymbol{D}^1$ be the disk in $\boldsymbol{D}$ such that $\Phi(\Psi(\boldsymbol{D}^1)\cap D)\supset F^1$ and $\partial\boldsymbol{D}^1\subset(\Psi^{-1}(\tilde{c}_j^1))$. Put $\tilde{D}^1:=\Psi(\boldsymbol{D}^1)$. If there is $x\in U_pM$ such that $\measuredangle(x, z)\leq\tilde{\varepsilon}$ and $B_x\supset\tilde{D}^1$, then a homotopically non trivial sphere $(F^1\cup\mathscr{D}_z(c_j^1))$ is contained in the contractible ball $\mathscr{B}_x$, which is a contradiction. Then for any $x\in U_pM$ with $\measuredangle(x, z)\leq\tilde{\varepsilon}$, F^1 is not contained in $\mathscr{B}_x$. We will repeat the similar procedure as above. Let U be the $\tilde{\varepsilon}$-neighborhood of z in U_pM. For any $z'\in U$ we denote by $\boldsymbol{G}_0^{1\prime}(z')$, the connected component of $(\boldsymbol{D}\cap\Psi^{-1}(B_{z'}))$ intersecting $\boldsymbol{G}_0$. Put $G_0^{1\prime}(z'):=\Psi(\boldsymbol{G}_0^{1\prime}(z'))$ and $G_0^1(z'):=\tilde{D}^1\cap G_0^{1\prime}(z')$. We note that for any $z'\in u$ $(\partial G_0^{1\prime}(z')\cap\partial B_{z'})$ has the property $(*)$ with respect to z' from Lemma 4. Take a point $z^1\in U$ $(\subset U_pM)$ such that

$$\text{area of } G_0^1(z^1)=\max_{z'\in U}\text{ area of } G_0^1(z').$$

Denote simply $G_0^1(z')$, $\boldsymbol{G}_0^{1\prime}(z^1)$ by G_0^1, $\boldsymbol{G}_0^{1\prime}$ respectively. Denote by $\{\boldsymbol{G}_{k^1}^1\}$, the family of closure of each connected component of $((\boldsymbol{D}^1\cup\boldsymbol{G}_0^{1\prime})\setminus\text{int}(\boldsymbol{G}_0^{1\prime}))$. Put $G_{k^1}^1:=\Psi(\boldsymbol{G}_{k^1}^1)$. We can deform $\mathscr{B}_{z^1}$ to sufficiently near $\mathscr{B}'_{z^1}$ so that $\mathscr{B}'_{z^1}$ is transverse to F^1 and $(\partial\mathscr{B}'_{z^1}\cap F^1)$ consists of closed curves. Let R_0^1 be a connected component of $(F^1\cap\mathscr{B}'_{z^1})$ such that $\Phi(G_0^1\cap D)\subset R_0^1$. If the set $(F^1\setminus R_0^1)$ is in empty, we are done. Otherwise each connected component of $(F^1\setminus R_0^1)$ is homeomorphic to a disk. Let $\{R_{j^1}^1\}$ be the family of closure of each connected component of $(F^1\setminus R_0^1)$. For any j^1 there is k^1 such that $(\Phi|_D)^{-1}(R_{j^1}^1)$ intersects $G_{k^1}^1$. Put $c_{j^1}^1:=R_{j^1}^1\cap R_0^1$ $(\subset F^1)$. For each connected component $G_{k^1}^1$, take a point $y_{k^1}^1\in\partial B_{z^1}$ such that

$$\measuredangle(y_k,\ \partial B_{z^1}\cap G_{k^1}^1)=\max_{y\in\partial B_{z^1}}\measuredangle(y,\ \partial B_{z^1}\cap G_{k^1}^1).$$

The 4ψ-neighborhood of each $y_{k^1}^1$ does not intersect $\tilde{D}$. For any $G_{k^1}^1$ with $G_{k^1}^1\subset\tilde{D}^1$ there are n, $m\in\{k^1\}$ such that $(G_{k^1}^1\cap\partial B_{z^1})$ is contained in the inferior arc $\widehat{y_n^1\cdot y_m^1}$ on ∂B_{z^1}. Next we define $\gamma_{k^1}^1$. Firstly we denote by V, the domain in B_{z^1} whose boundary consists of the inferior arc $\widehat{y_n^1\cdot y_m^1}$ and the great arc connecting y_n^1 and y_m^1. When $\gamma_{k(J)}\cap V=\phi$, we define $\gamma_{k^1}^1$ by $\gamma_{k^1}^1:=\{$the great arc connecting y_n^1 and $y_m^1\}\setminus\{4\psi$-neighborhood of y_n^1 and $y_m^1\}$. Note that $(\gamma_{k(J)}\cap V)$ is a great arc, if $\gamma_{k(J)}\cap V\neq\phi$. Take a piecewise great arc which consists of the three great arcs as follows ; The first great arc connects y_n^1 and one end point of $(\gamma_{k(J)}\cap V)$. The second one is $(\gamma_{k(J)}\cap V)$. The third one connects y_m^1 and the other end point of $(\gamma_{k(J)}\cap V)$. We define $\gamma_{k^1}^1$ by $\gamma_{k^1}^1:=\{$the above piecewise great arc$\}\setminus\{4\psi$-neighborhood of y_n^1 and $y_m^1\}$. Let $N_{k^1}^1$ be a 2ψ-neighborhood of $\gamma_{k^1}^1$ in $B_{z^1}^{2\varepsilon}$. We can take a homotopy H_t^1:

$S^1\times[0,1]\to F^1$ and a simple closed curve $c^2_{j^1}$ on F^1 such that $H^1_0=c^1_{j^1}$, $H^1_1=c^2_{j^1}$, $(\Phi\mid_D)^{-1}(c^2_{j^1})\subset N_{k^1(j^1)}$, $(\Phi\mid_D)^{-1}(H^1_t(S^1\times[0,1]))\subset(B_{z^1}\cup G^1_{k^1(j^1)})$, $c^2_{j^1}$ is transverse to all 1-simplexes of F and $c^2_{j^1}$ does not contain any 0-simplex of F. We denote by $F^2_{j^1}$, the domain on F^1 bounded by $c^2_{j^1}$ such that $F^2_{j^1}$ and $\{\bigcup_{l\neq j^1} c^2_l\}$ have no intersection. There is at least one $J^1\in\{j^1\}$ such that $(F^2_{J^1}\cup\mathscr{D}_{z^1}(c^2_{J^1}))$ is a homotopically non trivial sphere. Denote simply $F^2_{J^1}$ by F^2. Define a closed curve $\tilde{c}^2_{J^1}$ in $N^1_{k^1(J^1)}$ and a domain $\tilde{D}^2$ in $\tilde{D}^1$ by the same way as $\tilde{c}^1_J$ in $N_{k(J)}$ and $\tilde{D}^1$ in $\tilde{D}$. If there is $x\in U_pM$ such that $\sphericalangle(x,z^1)\leq\tilde{\varepsilon}$ and $B_x\supset\tilde{D}^2$, then we get a contradiction. Then any $x\in U_pM$ with $\sphericalangle(x,z^1)\leq\tilde{\varepsilon}$, F^2 is not contained in $\mathscr{B}_x$.

After repeating the same procedure finitely many times -say n times-, we have $z^n\in U_pM$ such that $\tilde{D}^n\subset B_{z^n}$ because $\tilde{D}$ is bounded. This contradicts the fact that $(F^n\cup\mathscr{D}_{z^n}(c^n_{J^{n-1}}))$ is a homotopically non trivial sphere. Hence $\tilde{C}(p)$ is homeomorphic to one point or P^2, which complete the proof of Theorem.

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