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# On a certain property of closed hypersurfaces with constant mean curvature in a Riemannian manifold

Dedicated to the late Professor Yoshie Katsurada

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## § 1. Introduction.

H. Liebmann [7] proved

THEOREM A. *The only ovaloids with constant mean curvature $H$ in Euclidean space $E^3$ are the spheres.*

W. Süss [11] generalized this result for a closed convex hypersurface in an $n$-dimensional Euclidean space $E^n$. To prove this Theorem we need integral formulas of Minkowski type. Y. Katsurada ([4], [6]) derived integral formulas of Minkowski type which are valid in an Einstein space and proved the following:

THEOREM B. *Let $R^{n+1}$ be an Einstein space which admits a vector field $\xi^i$ generating a continuous one-parameter group of conformal transformations in $R^{n+1}$ and $V^n$ a closed orientable hypersurface in $R^{n+1}$ such that*

(i) $H_1=const.$,

(ii) $C^i\xi_i$ *has fixed sign on $V^n$. Then every point of $V^n$ is umbilic, where $H_1$ and $C^i$ denote the first mean curvature of $V^n$ and the unit normal vector to $V^n$ respectively.*

It is one of our interesting problems to find a certain condition for a closed orientable hypersurface in a Riemannian manifold to be isometric to a sphere. On this subject, she [5] also proved the following two Theorems:

THEOREM C. *Let $\xi^i$ be a proper conformal Killing vector field such that $\nabla_j\xi_i+\nabla_i\xi_j=2\varphi G_{ji}$ in an Einstein space $R^{n+1}$ and $V^n$ a closed orientable hypersurface such that*

(i) $H_1=const.$,

(ii) $C^i\nabla_i\varphi$ *has fixed sign on $V^n$ and is not constant along $V^n$. Then $V^n$ is isometric to a sphere, where $G_{ji}$ and $\nabla_i$ denote the positive definite fundamental tensor of $R^{n+1}$ and the operator of covariant differentiation with respect to Christoffel symbols $\left\{ {k \atop ji} \right\}$ formed with $G_{ji}$ respectively.*

THEOREM D. *Let $\xi^i$ be a proper conformal Killing vector field in an Einstein space $R^{n+1}$ and $V^n$ a closed orientable hypersurface such that*

(i) *$H_1$=const.,*

(ii) *$C^i\xi_i$ has fixed sign on $V^n$,*

(iii) *$\varphi$ is not constant along $V^n$. Then $V^n$ is isometric to a sphere.*

It is known that if an Einstein space $R^{n+1}$ of dimension $n+1$ admits a proper conformal Killing vector field $\xi^i$, then it admits a non-constant scalar function $v$ which satisfies the partial differential equation given by

$$\nabla_j\nabla_i v=\lambda v G_{ji}\ \ (\lambda=-R/n(n+1))([13], [15]),$$

where $R$ denotes the scalar curvature of $R^{n+1}$. Such being the case, we assume in this paper the existence of a non-constant scalar function $\Phi$ which satisfies the partial differential equation defined by

$$(1.1)\qquad \nabla_j\nabla_i\Phi=\rho\Phi G_{ji}\ \ (\rho=\text{non-zero const.}).$$

The purpose of the present paper is to prove some analogous Theorems to Theorem B, C and D, replacing an Einstein manifold by a more general one. In § 3, we derive some integral formulas which are valid for a closed orientable hypersurface in a Riemannian manifold $R^{n+1}$. In § 4, we discuss properties of $R^{n+1}$ admitting the scalar field $\Phi$ defined by (1.1). In § 5, we apply the integral formulas obtained in § 3 to a closed orientable hypersurface whose first mean curvature $H_1$ is non zero constant. And, in the last section 6, we give a certain condition for a closed orientable hypersurface to be isometric to a sphere.

## § 2. Notation and general formulas.

Let $R^{n+1}$ be an $(n+1)$-dimensional orientable Riemannian manifold with local coordinates $x^i$, and $G_{ji}$ the positive definite fundamental tensor of $R^{n+1}$.

We now consider an orientable hypersurface $V^n$ imbedded in $R^{n+1}$ and locally given by

$$x^i=x^j(u^\alpha)\qquad i=1,\ 2,\dots,n+1\ ;\ \alpha=1,\ 2,\dots,n,$$

where $u^\alpha$ are local coordinates of $V^n$. Throughout the present paper Latin indices $i,\ j,\ k,\dots$ run from 1 to $n+1$ and Greek indices $\alpha,\ \beta,\ \gamma,\dots$ from 1 to $n$.

If we put

$$B^i_\alpha=\partial x^i/\partial u^\alpha,$$

then $B^i_\alpha$ are $n$ linearly independent vectors tangent to $V^n$ and the first fundamental tensor $g_{\beta\alpha}$ of $V^n$ is given by

$$(2.1)\qquad g_{\beta\alpha}=G_{ji}B^j_\beta B^i_\alpha.$$

We assume that $n$ vectors $B^i_1, B^i_2, \dots, B^i_n$ give the positive orientation on $V^n$, and we denote by $C^i$ the unit normal vector to $V^n$ such that

$$B^i_1,\ B^i_2, \dots, B^i_n,\ C^i$$

give the positive orientation in $R^{n+1}$.

Denoting by $\nabla_\alpha$ the van der Wärden-Bortolotti covariant differentiation along $V^n$ [10], we can write the equations of Gauss and Weingarten in the form

$$(2.2)\qquad \nabla_\beta B^i_\alpha = b_{\beta\alpha} C^i,$$
$$(2.3)\qquad \nabla_\beta C^i = -b_\beta{}^\alpha B^i_\alpha$$

respectively, where $b_{\beta\alpha}$ is the second fundamental tensor of $V^n$ and $b_\gamma{}^\alpha = g^{\beta\alpha} b_{\gamma\beta}$. Also, the equations of Codazzi are written as follows:

$$(2.4)\qquad \nabla_\gamma b_{\beta\alpha} - \nabla_\beta b_{\gamma\alpha} = R_{kjih} B^k_\gamma B^j_\beta B^i_\alpha C^h,$$

where $R_{kjih}$ is the curvature tensor of $R^{n+1}$. Transvecting $g^{\beta\alpha}$ to (2. 4) and remembering $g^{\beta\alpha} B^j{}_\beta B^i{}_\alpha = G^{ji} - C^j C^i$, we find that

$$(2.5)\qquad \nabla_\gamma b_\beta{}^\beta - \nabla_\beta b_\gamma{}^\beta = R_{kh} B^k_\gamma C^h,$$

where $b_\beta{}^\beta = g^{\beta\alpha} b_{\beta\alpha}$ and $R_{kh} = G^{ji} R_{kjih}$.

Now, if we denote by $k_1, k_2, \dots, k_n$ the principal curvatures of $V^n$, that is, the roots of the characteristic equation

$$\det(b_{\beta\alpha} - k g_{\beta\alpha}) = 0,$$

then the first mean curvature $H_1$ and the second mean curvature $H_2$ of $V^n$ are respectively given by

$$(2.6)\qquad nH_1 = \sum_\alpha k_\alpha = b_\alpha{}^\alpha$$

and

$$(2.7)\qquad \binom{n}{2} H_2 = \sum_{\beta<\alpha} k_\beta k_\alpha = \frac{1}{2}\{(b_\beta{}^\beta)^2 - b_\beta{}^\alpha b_\alpha{}^\beta\}.$$

## § 3. Integral formulas in a Riemannian manifold $R^{n+1}$ admitting a special concircular scalar field $\Phi$.

As mentioned in § 1, we assume the existence of a non-constant scalar field $\Phi$ which satisfies the partial differential equations defined by

$$(3.1)\qquad \nabla_j \Phi_i = \rho \Phi G_{ji} \qquad (\rho = \text{non-zero const.}),$$

where $\Phi_i = \nabla_i \Phi$, and hereafter we shall call this scalar field $\Phi$ a *special* *concircular* scalar field.

And, if $\Phi=0$ on $V^n$, since the second covariant derivative of $\Phi=0$ along $V^n$ is given by $\nabla_j\Phi_i B^j_\beta B^i_\alpha+\Phi_i\nabla_\beta B^i_\alpha=0$, substituting (2.2) and (3.1) into this equation and transvecting $g^{\beta\alpha}$ to the resulting equation, we see that $H_1\Theta=0$ on $V^n$, where $\Theta=C^i\Phi_i$. Hence we have the following:

LEMMA 3.1 *Let $R^{n+1}$ be an $(n+1)$-dimensional Riemannian manifold which admits a special concircular scalar field $\Phi$. If $V^n$ is a hypersurface in $R^{n+1}$ such that $H_1\Theta\neq 0$ there, then we have $\Phi\neq 0$ on $V^n$.*

On the hypersurface $V^n$, we can put

$$\Phi^j=B^j_\beta\phi^\beta+\Theta C^j, \tag{3.2}$$

where $\Phi^j=G^{ji}\Phi_i$. Transvecting $G_{ji}B^i_\alpha$ to this equation and making use of (2.1), we get $\phi_\alpha=B^i_\alpha\Phi_i$, from which, by covariant differentiation along $V^n$ and by virtue of (2.2), (3.1) and (2.1), we obtain

$$\nabla_\beta\phi_\alpha=\Theta b_{\beta\alpha}+\rho\Phi g_{\beta\alpha}.$$

Transvecting $g^{\beta\alpha}$ to this equation and making use of (2.6), we get

$$\nabla_\beta\phi^\beta=nH_1\Theta+n\rho\Phi, \tag{3.3}$$

where $\nabla_\beta\phi^\beta=g^{\beta\alpha}\nabla_\beta\phi_\alpha$.

We now put

$$\eta_\beta=b_\beta{}^\alpha B^i_\alpha\Phi_i,$$

from which, by covariant differentiation along $V^n$, we obtain, by virtue of (2.2), (3.1), (2.1) and $C^i\Phi_i=\Theta$.

$$\nabla_\gamma\eta_\beta=\nabla_\gamma b_\beta{}^\alpha B^i_\alpha\Phi_i+b_\beta{}^\alpha b_{\gamma\alpha}\Theta+\rho\Phi b_{\beta\gamma}.$$

Transvecting $g^{\gamma\beta}$ to this equation, we get

$$\nabla_\gamma\eta^\gamma=\nabla_\gamma b_\beta{}^\gamma\phi^\beta+b_\beta{}^\gamma b_\gamma{}^\beta\Theta+\rho\Phi b_\gamma{}^\gamma \tag{3.4}$$

by virtue of (3.2).

On the other hand, we have, from (2.6) and (2.7),

$$b_\gamma{}^\gamma=nH_1,\qquad b_\beta{}^\gamma b_\gamma{}^\beta=n^2H_1^2-n(n-1)H_2,$$

and consequently, we have, from (3.4),

$$\nabla_\gamma\eta^\gamma=\nabla_\gamma b_\beta{}^\gamma\phi^\beta+n\{nH_1^2-(n-1)H_2\}\Theta+n\rho\Phi H_1. \tag{3.5}$$

We now assume that the hypersurface $V^n$ is closed, and apply Green's formula [12] to (3.3) and (3.5). Then we obtain

$$(3.6)\qquad \int_{V^n} H_1 \Theta dA + \int_{V^n} \rho\Phi dA = 0$$

and

$$(3.7)\qquad \frac{1}{n}\int_{V^n} \nabla_\gamma b_\beta{}^\gamma \phi^\beta dA + \int_{V^n}\{nH_1^2 - (n-1)H_2\}\Theta dA + \int_{V^n} \rho\Phi H_1 dA = 0$$

respectively [4], where $dA$ denotes the area element of $V^n$.

If we assume, moreover, that the first mean curvature of $V^n$ is non zero constant: $H_1 = \text{const.} \neq 0$, then we obtain, from (2.5),

$$\nabla_\gamma b_\beta{}^\gamma = -R_{ji}B_\beta^j C^i,$$

and consequently, we have, from (3.7),

$$(3.8)\qquad -\frac{1}{n}\int_{V^n} R_{ji}B_\beta^j \phi^\beta C^i dA + \int_{V^n}\{nH_1^2 - (n-1)H_2\}\Theta dA + H_1\int_{V^n} \rho\Phi dA = 0.$$

Eliminating $\int_{V^n} \rho\Phi dA$ from (3.6) and (3.8), we find that

$$(3.9)\qquad -\frac{1}{n}\int_{V^n} R_{ji}B_\beta^j \phi^\beta C^i dA + (n-1)\int_{V^n}\{H_1^2 - H_2\}\Theta dA = 0.$$

## § 4. Some properties of a Riemannian manifold admitting a special concircular scalar field $\Phi$.

Let $R^{n+1}$ be a Riemannian manifold of dimension $n+1$ which admits a special concircular scalar field $\Phi$ defined by (3.1). Substituting (3.1) into the Ricci identity

$$\nabla_k \nabla_j \Phi_i - \nabla_j \nabla_k \Phi_i = -R_{kji}{}^l \Phi_l,$$

we find that

$$R_{kji}{}^l \Phi_l = \rho(\Phi_j G_{ki} - \Phi_k G_{ji}),$$

from which, by covariant differentiation, we obtain

$$(4.1)\qquad \nabla_h R_{kji}{}^l \Phi_l = -\rho\Phi\{R_{kjih} - \rho(G_{ki}G_{jh} - G_{kh}G_{ji})\}.$$

This shows that the tensor $\nabla_h R_{kji}{}^l \Phi_l$ is skew-symmetric in $h$ and $i$, that is,

$$\nabla_h R_{kji}{}^l \Phi_l + \nabla_i R_{kjh}{}^l \Phi_l = 0,$$

and consequently, transvecting $G^{hi}$ to this equation, we get

$$(4.2)\qquad \nabla_h R_{kjl}{}^h \Phi^l = 0.$$

Also, transvecting $G^{ji}$ to (4.1), we obtain

$$(4.3)\qquad \nabla_h R_{kl}\Phi^l = -\rho\Phi(R_{hk} + n\rho G_{hk}),$$

and if we put

$$(4.4)\qquad S_{hk} = R_{hk} + n\rho G_{hk},$$

then (4.3) is rewritten as follows:

$$(4.5)\qquad \nabla_h R_{kl}\Phi^l = -\rho\Phi S_{hk}.$$

Moreover, transvecting $G^{hk}$ to this equation and making use of $2\nabla_k R_l^k = \nabla_l R$, we get

$$(4.6)\qquad \nabla_l R\Phi^l = -2\rho\Phi S,$$

where $S = G^{hk}S_{hk}$. Next, transvecting $G^{hk}$ to (4.4), we obtain

$$(4.7)\qquad S = R + n(n+1)\rho.$$

Thus from (4.6), we have

THEOREM 4.1 *Let $R^{n+1}$ be a Riemannian manifold which admits a special concircular scalar field $\Phi$ such that $\nabla_j\Phi_i = \rho\Phi G_{ji}$ ($\rho$ = non-zero const.). If its scalar curvature $R$ is constant, then we have*

$$\rho = -R/n(n+1).$$

Now, transvecting $G^{hj}$ to (4.1), we get, from $R_{kjil} = R_{ilkj}$ and (4.4),

$$(4.8)\qquad \nabla_h R_{lik}{}^h\Phi^l = -\rho\Phi S_{ik}.$$

On the other hand, transvecting $G^{lh}$ to the Bianchi's identity: $\nabla_l R_{kjih} + \nabla_k R_{jlih} + \nabla_j R_{lkih} = 0$, we find that

$$(4.9)\qquad \nabla_h R_{kji}{}^h = \nabla_k R_{ji} - \nabla_j R_{ki},$$

and consequently, transvecting $\Phi^k$ to this equation, we get, from (4.8) and (4.5),

$$(4.10)\qquad \nabla_k R_{ji}\Phi^k = -2\rho\Phi S_{ji}.$$

Also, transvecting $\Phi^i$ to (4.9) and making use of (4.2), we obtain

$$(4.11)\qquad \nabla_k R_{ji}\Phi^i = \nabla_j R_{ki}\Phi^i \qquad \text{(that is, symmetric in } k \text{ and } j).$$

LEMMA 4.2 *Let $R^{n+1}$ be a Riemannian manifold which admits a special concircular scalar field $\Phi$. If the scalar field $\Phi$ satisfies the following equation:*

$$(4.12)\qquad \Phi S_{kj} = 0,$$

*then we have* $(\Phi_l\Phi^l)S_{kj}=0$ *in* $R^{n+1}$.

PROOF. Covariantly differentiating (4.12), we get, from (4.4),

$$\Phi_l S_{kj}+\Phi\nabla_l R_{kj}=0.$$

Transvecting $\Phi^l$ to this equation and making use of (4.10), we obtain

$$(\Phi_l\Phi^l)S_{kj}-2\rho\Phi^2 S_{kj}=0,$$

from which, taking account of the assumption (4.12), we conclude that Lemma 4.2 holds.

## § 5. Closed orientable hypersurfaces with $H_1=$const.$\neq 0$.

First, we shall prove the following Theorem:

THEOREM 5.1 *Let* $R^{n+1}$ *be an* $(n+1)$*-dimensional orientable Riemannian manifold with* $\nabla_k R_{ji}=\nabla_j R_{ki}$ *which admits a special concircular scalar field* $\Phi$ *such that*

$$\nabla_j\Phi_i=\rho\Phi G_{ji} \qquad (\rho=\textit{non-zero const.}),$$

*and* $V^n$ *a closed orientable hypersurface in* $R^{n+1}$ *such that*

(i) $H_1=const.\neq 0$,

(ii) $C^i\Phi_i$ $(=\Theta)$ *has fixed sign on* $V^n$.

*Then every point of* $V^n$ *is umbilic.*

PROOF. Transvecting $\Phi^k$ to the assumption $\nabla_k R_{ji}=\nabla_j R_{ki}$, we get, from (4.10) and (4.5), $\Phi S_{ji}=0$. Thus, using Lemma 4.2, we have $(\Phi_k\Phi^k)\cdot S_{ji}=0$ in $R^{n+1}$. Moreover, by the assumption that $C^i\Phi_i$ has fixed sign on $V^n$, we find that $S_{ji}=0$ on $V^n$, that is, $R_{ji}=-n\rho G_{ji}$ on $V^n$. Consequently, from (3.9), we obtain

$$\int_{V^n}(H_1^2-H_2)\Theta dA=0. \tag{5.1}$$

On the other hand, since

$$H_1^2-H_2=\frac{1}{n^2(n-1)}\sum_{\alpha<\beta}(k_\alpha-k_\beta)^2, \tag{5.2}$$

we see that $H_1^2-H_2\geqq 0$. Thus, from (5.1) and the assumption that $\Theta$ has fixed sign on $V^n$, we conclude that $H_1^2-H_2=0$, and consequently, because of (5.2), that $k_1=k_2=\cdots=k_n$ at each point of $V^n$. This means that every point of $V^n$ is umbilic.

COROLLARY 5.2 *Let* $R^{n+1}$ *be an orientable Riemannian manifold with*

$\nabla_k R_{ji}=0$ *which admits a special concircular scalar field* $\Phi$, *and* $V^n$ *a closed orientable hypersurface in* $R^{n+1}$ *such that*

(i) $H_1=const.\neq 0$,

(ii) $C^i\Phi_i$ *has fixed sign on* $V^n$.

*Then every point of* $V^n$ *is umbilic.*

COROLLARY 5.3 *Let* $R^{n+1}$ *be an orientable Riemannian manifold with* $\nabla_l R_{kjih}=0$ *which admits a special concircular scalar field* $\Phi$, *and* $V^n$ *a closed orientable hypersurface in* $R^{n+1}$ *such that*

(i) $H_1=const.\neq 0$,

(ii) $C^i\Phi_i$ *has fixed sign on* $V^n$.

*Then every point of* $V^n$ *is umbilic.*

We next assume that $R^{n+1}$ is a conformally flat Riemannian manifold:

$$R_{kji}{}^h=-\frac{1}{n-1}(R_{ki}\delta_j^h-R_{ji}\delta_k^h+G_{ki}R_j{}^h-G_{ji}R_k{}^h)$$
$$+\frac{R}{n(n-1)}(G_{ki}\delta_j^h-G_{ji}\delta_k^h).$$

By covariant differentiation, we have

$$\nabla_l R_{kji}{}^h=-\frac{1}{n-1}(\nabla_l R_{ki}\delta_j^h-\nabla_l R_{ji}\delta_k^h+G_{ki}\nabla_l R_j{}^h-G_{ji}\nabla_l R_k{}^h)$$
$$+\frac{\nabla_l R}{n(n-1)}(G_{ki}\delta_j^h-G_{ji}\delta_k^h),$$

from which, replacing $l$ by $h$ and summing for $h$, we get

$$\nabla_h R_{kji}{}^h=-\frac{1}{n-1}(\nabla_j R_{ki}-\nabla_k R_{ji}+G_{ki}\nabla_h R_j{}^h-G_{ji}\nabla_h R_k{}^h)$$
$$+\frac{1}{n(n-1)}(G_{ki}\nabla_j R-G_{ji}\nabla_k R).$$

And, making use of (4.9) and $2\nabla_h R_j{}^h=\nabla_j R$, we find that

$$\nabla_j R_{ki}-\nabla_k R_{ji}-\frac{1}{2n}(G_{ki}\nabla_j R-G_{ji}\nabla_k R)=0\quad (n>2). \tag{5.3}$$

Also, in case $n=2$, a conformally flat Riemannian manifold is defined by

$$\nabla_j R_{ki}-\nabla_k R_{ji}-\frac{1}{4}(G_{ki}\nabla_j R-G_{ji}\nabla_k R)=0. \tag{5.4}$$

Therefore, assuming the scalar curvature $R$ to be constant, we see easily, from (5.3) and (5.4), that $\nabla_j R_{ki}-\nabla_k R_{ji}=0$. Consequently, using Theorem 5.1, we have the following:

COROLLARY 5.4 *Let $R^{n+1}$ be an orientable conformally flat Riemannian manifold with $R=$const. which admits a special concircular scalar field $\Phi$, and $V^n$ a closed orientable hypersurface in $R^{n+1}$ such that*

(i) $H_1=$ *const.* $\neq 0$,

(ii) $C^i\Phi_i$ *has fixed sign on* $V^n$.

*Then every point of $V^n$ is umbilic.*

Moreover, making use of (4.11), (4.5) and (4.10), we can prove the following Theorem by an argument similar to that used in the proof of Theorem 5.1:

THEOREM 5.5 *Let $R^{n+1}$ be an orientable Riemannian manifold with $\nabla_k R_{ji}+\nabla_j R_{ik}+\nabla_i R_{kj}=0$*[1] *which admits a special concircular scalar field $\Phi$, and $V^n$ a closed orientable hypersurface in $R^{n+1}$ such that*

(i) $H_1=$ *const.* $\neq 0$,

(ii) $C^i\Phi_i$ *has fixed sign on* $V^n$.

*Then every point of $V^n$ is umbilic.*

We now assume that the length of the Ricci tensor is constant in $R^{n+1}$, that is,

$$R^{ji}R_{ji}=\alpha \qquad (\alpha=\text{const.}). \tag{5.5}$$

By covariant differentiation, we have

$$\nabla_k R^{ji}R_{ji}=0. \tag{5.6}$$

On the other hand, from (4.4), we see that $R^{ji}R_{ji}=S^{ji}S_{ji}-2n\rho S+n^2(n+1)\rho^2$. Thus, we have, from (5.5),

$$S^{ji}S_{ji}=2n\rho S-n^2(n+1)\rho^2+\alpha. \tag{5.7}$$

Also, transvecting $\nabla_k R^{ji}$ to (4.4), that is, $S_{ji}=R_{ji}+n\rho G_{ji}$ and using (5.6), we have $\nabla_k R^{ji}S_{ji}=n\rho\nabla_k R$, and, moreover, transvecting $\Phi^k$ to this equation, from (4.10) and (4.6), we find that

$$\Phi S^{ji}S_{ji}=n\rho\Phi S. \tag{5.8}$$

Consequently, substituting (5.7) into (5.8), we can see easily that

$$\Phi\{n\rho S-n^2(n+1)\rho^2+\alpha\}=0. \tag{5.9}$$

So, covariantly differentiating and taking account of the fact that $\nabla_i S$ is equal to $\nabla_i R$, we have

$$\Phi_i\{n\rho S-n^2(n+1)\rho^2+\alpha\}+n\rho\Phi\nabla_i R=0,$$

1) See [2].

and consequently, multiplying by $\Phi$ and using (5.9), we obtain

(5.10) $\quad \Phi^2\nabla_i R=0.$

LEMMA 5.6 *Let $R^{n+1}$ be a Riemannian manifold with $R^{ji}R_{ji}=const.$ which admits a special concircular scalar field $\Phi$, $V^n$ a hypersurface in $R^{n+1}$, and $\Phi_k\Phi^k>0$ on $V^n$. If $\beta\neq 0$, then we have $\nabla_i R=0$, that is, $R=const.$ in $R^{n+1}$, and if $\beta=0$, then we have $\nabla_i R=0$ on $V^n$, where $\beta$ is a constant number defined by $\beta=\rho\Phi^2-\Phi_j\Phi^j$.*

PROOF. Since $\nabla_i(\rho\Phi^2-\Phi_j\Phi^j)=0$, we see that $\beta$ is constant in $R^{n+1}$. Now, by covariant differentiation of (5.10), we get

$$2\Phi\Phi_j\nabla_i R+\Phi^2\nabla_j\nabla_i R=0,$$

from which, because of $\nabla_j\nabla_i R=\nabla_i\nabla_j R$, we obtain $\Phi(\Phi_j\nabla_i R-\Phi_i\nabla_j R)=0$. Moreover, covariantly differentiating and making use of (3.1), we have

$$\Phi_k(\Phi_j\nabla_i R-\Phi_i\nabla_j R)+\rho\Phi^2(G_{kj}\nabla_i R-G_{ki}\nabla_j R)+\Phi(\Phi_j\nabla_k\nabla_i R-\Phi_i\nabla_k\nabla_j R)=0,$$

and consequently, taking account of (5.10), we obtain

$$\Phi_k(\Phi_j\nabla_i R-\Phi_i\nabla_j R)+\Phi(\Phi_j\nabla_k\nabla_i R-\Phi_i\nabla_k\nabla_j R)=0.$$

And, transvecting $G^{kj}$ to this equation, we have, from (5.10) and (4.6),

(5.11) $\quad -\beta\nabla_i R+\Phi\{\Phi^j\nabla_j\nabla_i R+\Phi_i(2\rho S-\nabla^j\nabla_j R)\}=0,$

where $\nabla^j\nabla_j R=G^{kj}\nabla_k\nabla_j R$. Thus, if $\beta\neq 0$, then we have, from (5.10) or (5.11), $\nabla_i R=0$. And if $\beta=0$, then we have $\rho\Phi^2=\Phi_i\Phi^i$, which is non zero on $V^n$ and hence $\nabla_i R=0$ there because of (5.10).

LEMMA 5.7 *Let $R^{n+1}$ be a Riemannian manifold which admits a special concircular scalar field $\Phi$, and $V^n$ a hypersurface in $R^{n+1}$ such that*
*(i) $H_1=const.\neq 0$,*
*(ii) $C^i\Phi_i$ $(=\Theta)$ has fixed sign on $V^n$.*
*If $\nabla_i R=0$ on $V^n$, then we have $S=0$ on $V^n$.*

PROOF. Transvecting $B^i_a$ to the both sides of $\nabla_i R=0$, we get $\nabla_a R=0$, that is, $R=$const. on $V^n$. Consequently, because of (4.7), we have

(5.12) $\quad S=\text{const.} \qquad \text{on } V^n.$

Moreover, making use of the assumption $\nabla_i R=0$ on $V^n$, we get, from (4.6),

$$\Phi S=0 \qquad \text{on } V^n.$$

Here $\Phi\neq 0$ on $V^n$. Because, from the assumption (i) and (ii), we have $H_1\Theta\neq 0$ on $V^n$, which shows that this holds because of Lemma 3.1. Thus, by virtue of (5.12), we can prove that $S=0$ on $V^n$.

Next, making use of these Lemmas, we shall prove the following Theorem:

THEOREM 5.8 *Let $R^{n+1}$ be an orientable Riemannian manifold with $R^{ji}R_{ji}=$ const. which admits a special concircular scalar field $\Phi$, and $V^n$ a closed orientable hypersurface in $R^{n+1}$ such that*

(i) $H_1=$ *const.* $\neq 0$,

(ii) $C^i\Phi_i$ *has fixed sign on* $V^n$.

*Then every point of $V^n$ is umbilic.*

PROOF. Transvecting $\Phi^k$ to (5.6), and substituting (4.10) and (4.4) into this equation, we obtain

$$\Phi(S^{ji}S_{ji}-n\rho S)=0.$$

Now, by covariant differentiation, we get, because of (4.4) and (4.7),

$$\Phi_k(S^{ji}S_{ji}-n\rho S)+\Phi(2\nabla_k R^{ji}S_{ji}-n\rho\nabla_k R)=0,$$

and, substituting (4.4) into the second term of the left-hand side of this equation, and using (5.6), we find that

$$\Phi_k(S^{ji}S_{ji}-n\rho S)+n\rho\Phi\nabla_k R=0.$$

Thus, making use of Lemma 5.6 and Lemma 5.7 and transvecting $\Phi^k$ to its equation, it follows that

$$(\Phi_k\Phi^k)(S^{ji}S_{ji})=0 \qquad \text{on } V^n,$$

and, using the assumption that $C^i\Phi_i$ has fixed sign on $V^n$, we conclude that

$$S_{ji}=0, \text{ that is, } R_{ji}=-n\rho G_{ji} \text{ on } V^n.$$

Therefore, from (3.9), we obtain

$$\int_{V^n}(H_1^2-H_2)\Theta dA=0,$$

and consequently, by the argument similar to that used in the proof of Theorem 5.1, it follows that Theorem 5.8 holds.

REMARK 1. We can see that Theorem 5.8 is a generalization of Corollary 5.2.

REMARK 2. We need to keep in mind the fact that Theorems and Corollaries mentioned in this section afford examples of the following Theorem, because it can be proved that $S_{ji}=0$ on $V^n$, that is, $R_{ji}=-n\rho G_{ji}$

on $V^n$ in all cases:

THEOREM E. (T. Ôtsuki, [9]) *Let $R^{n+1}$ be an orientable Riemannian manifold which admits a conformal Killing vector field $\xi^i$, and $V^n$ a closed orientable hypersurface in $R^{n+1}$ such that*

(i) $H_1 = const.$,

(ii) $C^i\xi_i$ *has fixed sign on* $V^n$,

(iii) $R_{ji}B^j_\beta\bar{\xi}^\beta C^i = 0$ *on* $V^n$.

*Then every point of $V^n$ is umbilic, where $\xi^j = B^j_\beta\bar{\xi}^\beta + \omega C^j$ on $V^n$.*

## § 6. Some characterizations of a hypersurface in $R^{n+1}$ to be isometric to a sphere.

To prove that the hypersurface under consideration is isometric to a sphere, we use the following Theorem due to M. Obata [8]:

THEOREM F. *Let $V^n$ $(n \geqq 2)$ be a complete Riemannian manifold which admits a non-null function $\Psi$ such that $\nabla_\beta\nabla_\alpha\Psi = -C^2\Psi g_{\beta\alpha}$ $(C = const.)$. Then $V^n$ is isometric to a sphere of radius $1/C$.*

Making use of Theorem 5.1, we obtain the following:

THEOREM 6.1 *Let $R^{n+1}$ be an orientable Riemannian manifold with $\nabla_k R_{ji} = \nabla_j R_{ki}$ which admits a special concircular scalar field $\Phi$, and $V^n$ a closed orientable hypersurface in $R^{n+1}$ such that*

(i) $H_1 = const. \neq 0$,

(ii) $C^i\Phi_i$ *has fixed sign on $V^n$ and is not constant along $V^n$.*

*Then $V^n$ is isometric to a sphere.*

PROOF. If we put $\Phi^i = B^i_\alpha\phi^\alpha + \Theta C^i$ on $V^n$, then we obtain

$$(6.1) \qquad \Theta = C^i\Phi_i.$$

By covariant differentiation of (6.1) along $V^n$ and by virtue of (3.1) and (2.3), we have

$$(6.2) \qquad \nabla_\beta\Theta = -b_\beta{}^\alpha B^i_\alpha\Phi_i.$$

Furthermore, by virtue of Theorem 5.1, every point of $V^n$ is umbilic, that is,

$$(6.3) \qquad b_{\beta\gamma} = H_1 g_{\beta\gamma}.$$

So, transvecting $g^{\gamma\alpha}$ to this equation, we see that $b_\beta{}^\alpha = H_1\delta^\alpha_\beta$. Thus, substituting this equation into (6.2), we have

$$(6.4) \qquad \nabla_\beta\Theta = -H_1 B^i_\beta\Phi_i, \text{ that is, } \nabla_\beta\Theta + H_1\nabla_\beta\Phi = 0.$$

Accordingly, under the assumption, that is, $H_1 = \text{const.}$, we get

(6.5) $\qquad \Theta+H_1\Phi=c \ (c=\text{const.}).$

Moreover, by covariant differentiation of (6. 4) along $V^n$ and by virtue of (3. 1), (2. 1), (2. 2) and (6. 1), we obtain

$$\nabla_\gamma\nabla_\beta\Theta=-H_1(\rho\Phi g_{\gamma\beta}+\Theta b_{\gamma\beta}).$$

Thus, from (6. 3) and (6. 5), it follows that

(6.6) $\qquad \nabla_\gamma\nabla_\beta\Theta=-\{(H_1^2-\rho)\Theta+\rho c\}g_{\gamma\beta}.$

If $H_1^2-\rho=0$, then (6. 6) becomes $\nabla_\gamma\nabla_\beta\Theta=-\rho c g_{\gamma\beta}$, from which $\Delta\Theta=-n\rho c$, that is, $\Delta\Theta=\text{const.}$, where $\Delta\Theta=g^{\gamma\beta}\nabla_\gamma\nabla_\beta\Theta$. However this is impossible, unless $\Theta=\text{const.}$ on $V^n$ ([3], [1], [12]). Thus, $H_1^2-\rho$ being different from zero, we have, from (6. 6),

(6.7) $\qquad \nabla_\gamma\nabla_\beta\left(\Theta+\dfrac{\rho c}{H_1^2-\rho}\right)=-(H_1^2-\rho)\left(\Theta+\dfrac{\rho c}{H_1^2-\rho}\right)g_{\gamma\beta},$

from which we get

$$\Delta\left(\Theta+\frac{\rho c}{H_1^2-\rho}\right)=-n(H_1^2-\rho)\left(\Theta+\frac{\rho c}{H_1^2-\rho}\right),$$

and consequently, it follows that $H_1^2-\rho>0$ [14]. Therefore, using Theorem F, the equation (6. 7) shows that the hypersurface $V^n$ under consideration is isometric to a sphere ([5], [6]).

COROLLARY 6. 2 *Let $R^{n+1}$ be an orientable Riemannian manifold with $\nabla_k R_{ji}=0$ which admits a special concircular scalar field $\Phi$, and $V^n$ a closed orientable hypersurface in $R^{n+1}$ such that*

(i) *$H_1=\text{const.}\neq 0$,*

(ii) *$C^i\Phi_i$ has fixed sign on $V^n$ and is not constant along $V^n$.*

*Then $V^n$ is isometric to a sphere.*

COROLLARY 6. 3 *Let $R^{n+1}$ be an orientable Riemannian manifold with $\nabla_l R_{kjih}=0$ which admits a special concircular scalar field $\Phi$, and $V^n$ a closed orientable hypersurface in $R^{n+1}$ such that*

(i) *$H_1=\text{const.}\neq 0$,*

(ii) *$C^i\Phi_i$ has fixed sign on $V^n$ and is not constant along $V^n$.*

*Then $V^n$ is isometric to a sphere.*

COROLLARY 6. 4 *Let $R^{n+1}$ be an orientable conformally flat Riemannian manifold with $R=\text{const.}$ which admits a special concircular scalar field $\Phi$, and $V^n$ a closed orientable hypersurface in $R^{n+1}$ such that*

(i) *$H_1=\text{const.}\neq 0$,*

(ii) *$C^i\Phi_i$ has fixed sign on $V^n$ and is not constant along $V^n$.*
*Then $V^n$ is isometric to a sphere.*

Moreover, making use of Theorem 5.5 and Theorem 5.8 respectively, we obtain the following two Theorems:

THEOREM 6.5 *Let $R^{n+1}$ be an orientable Riemannian manifold with $\nabla_k R_{ji}+\nabla_j R_{ik}+\nabla_i R_{kj}=0$ which admits a special concircular scalar field $\Phi$, and $V^n$ a closed orientable hypersurface in $R^{n+1}$ such that*

(i) $H_1=const.\neq 0$,

(ii) *$C^i\Phi_i$ has fixed sign on $V^n$ and is not constant along $V^n$.*

*Then $V^n$ is isometric to a sphere.*

THEOREM 6.6 *Let $R^{n+1}$ be an orientable Riemannian manifold with $R^{ji}R_{ji}=const.$ which admits a special concircular scalar field $\Phi$, and $V^n$ a closed orientable hypersurface in $R^{n+1}$ such that*

(i) $H_1=const.\neq 0$,

(ii) *$C^i\Phi_i$ has fixed sign on $V^n$ and is not constant along $V^n$.*

*Then $V^n$ is isometric to a sphere.*

Next, under the new assumption of $\Phi$, that is, $\Phi$ is not constant along $V^n$, we shall prove some Theorems in the same way. Making use of Theorem 5.1, we have

THEOREM 6.7 *Let $R^{n+1}$ be an orientable Riemannian manifold with $\nabla_k R_{ji}=\nabla_j R_{ki}$ which admits a special concircular scalar field $\Phi$, and $V^n$ a closed orientable hypersurface in $R^{n+1}$ such that*

(i) $H_1=const.\neq 0$,

(ii) *$C^i\Phi_i$ has fixed sign on $V^n$,*

(iii) *$\Phi$ is not constant along $V^n$.*

*Then $V^n$ is isometric to a sphere.*

PROOF. Since $\nabla_\beta(\Phi_i B^i_\alpha)=\nabla_j\Phi_i B^j_\beta B^i_\alpha+\Phi_i\nabla_\beta B^i_\alpha$, we see, from (3.1), (2.2) and $C^i\Phi_i=\Theta$, that

$$\nabla_\beta\nabla_\alpha\Phi=\rho\Phi g_{\beta\alpha}+\Theta b_{\beta\alpha}. \tag{6.8}$$

Also, making use of Theorem 5.1. every point of $V^n$ is umbilic, that is, $b_{\beta\alpha}=H_1 g_{\beta\alpha}$. Thus we have, from (6.8),

$$\nabla_\beta\nabla_\alpha\Phi=(\rho\Phi+H_1\Theta)g_{\beta\alpha}.$$

And, substituting (6.5) into this equation, we find that

$$\nabla_\beta\nabla_\alpha\Phi=\{-(H_1^2-\rho)\Phi+cH_1\}g_{\beta\alpha}. \tag{6.9}$$

If $H_1^2-\rho=0$, then (6.9) becomes $\nabla_\beta\nabla_\alpha\Phi=cH_1 g_{\beta\alpha}$, from which $\Delta\Phi=ncH_1$, that is, $\Delta\Phi=\text{const.}$ However this is impossible, unless $\Phi=\text{const.}$ on $V^n$. Thus, $H_1^2-\rho$ being different from zero, we have, from (6.9),

$$\nabla_\beta\nabla_\alpha\left(\Phi-\frac{cH_1}{H_1^2-\rho}\right)=-(H_1^2-\rho)\left(\Phi-\frac{cH_1}{H_1^2-\rho}\right)g_{\beta\alpha}, \tag{6.10}$$

from which we get

$$\Delta\left(\Phi-\frac{cH_1}{H_1^2-\rho}\right)=-n(H_1^2-\rho)\left(\Phi-\frac{cH_1}{H_1^2-\rho}\right),$$

and consequently, it follows that $H_1^2-\rho>0$. Therefore, using Theorem F, the equation (6.10) shows that the hypersurface $V^n$ under consideration is isometric to a sphere ([5], [6], [15]).

COROLLARY 6.8 *Let $R^{n+1}$ be an orientable Riemannian manifold with $\nabla_k R_{ji}=0$ which admits a special concircular scalar field $\Phi$, and $V^n$ a closed orientable hypersurface in $R^{n+1}$ such that*

(i) $H_1=\text{const.}\neq 0$,

(ii) $C^i\Phi_i$ *has fixed sign on* $V^n$,

(iii) $\Phi$ *is not constant along* $V^n$.

*Then $V^n$ is isometric to a sphere.*

COROLLARY 6.9 *Let $R^{n+1}$ be an orientable Riemannian manifold with $\nabla_l R_{kjih}=0$ which admits a special concircular scalar field $\Phi$, and $V^n$ a closed orientable hypersurface in $R^{n+1}$ such that*

(i) $H_1=\text{const.}\neq 0$,

(ii) $C^i\Phi_i$ *has fixed sign on* $V^n$,

(iii) $\Phi$ *is not constant along* $V^n$.

*Then $V^n$ is isometric to a sphere.*

COROLLARY 6.10 *Let $R^{n+1}$ be an orientable conformally flat Riemannian manifold with $R=\text{const.}$ which admits a special concircular scalar field $\Phi$, and $V^n$ a closed orientable hypersurface in $R^{n+1}$ such that*

(i) $H_1=\text{const.}\neq 0$,

(ii) $C^i\Phi_i$ *has fixed sign on* $V^n$,

(iii) $\Phi$ *is not constant along* $V^n$.

*Then $V^n$ is isometric to a sphere.*

Similarly, making use of Theorem 5.5 and Theorem 5.8 respectively, we have the following two Theorems:

THEOREM 6.11 *Let $R^{n+1}$ be an orientable Riemannian manifold with $\nabla_k R_{ji}+\nabla_j R_{ik}+\nabla_i R_{kj}=0$ which admits a special concircular scalar field $\Phi$, and $V^n$ a closed orientable hypersurface in $R^{n+1}$ such that*

(i) $H_1 = const. \neq 0$,
(ii) $C^i \Phi_i$ *has fixed sign on* $V^n$,
(iii) $\Phi$ *is not constant along* $V^n$.
*Then* $V^n$ *is isometric to a sphere.*

THEOREM 6.12 *Let* $R^{n+1}$ *be an orientable Riemannian manifold with* $R^{ji}R_{ji} = const.$ *which admits a special concircular scalar field* $\Phi$, *and* $V^n$ *a closed orientable hypersurface in* $R^{n+1}$ *such that*
(i) $H_1 = const. \neq 0$,
(ii) $C^i \Phi_i$ *has fixed sign on* $V^n$,
(iii) $\Phi$ *is not constant along* $V^n$.
*Then* $V^n$ *is isometric to a sphere.*

REMARK 1. We can see easily that Theorem 6.6 and Theorem 6.12 is a generalization of Corollary 6.2 and of Corollary 6.8 respectively.

REMARK 2. For the same reason as mentioned in Remark 2 of § 5, Theorems and Corollaries from 6.7 to 6.12 afford examples of the following Theorem:

THEOREM G. (K. Yano, [15]) *Let* $R^{n+1}$ *be an orientable Riemannian manifold which admits a non-constant scalar field* $\Phi$ *such that*

$$\nabla_j \nabla_i \Phi = k \Phi G_{ji}, \quad k = const.,$$

*and* $V^n$ *a closed orientable hypersurface in* $R^{n+1}$ *such that*
(i) $H_1 = const.$,
(ii) $C^i \nabla_i \Phi$ *has fixed sign on* $V^n$,
(iii) $(R_{ji} + nkG_{ji}) C^j C^i \geqq 0$ *on* $V^n$.
*Then every point of* $V^n$ *is umbilic. If, moreover,*
(iv) $\Phi$ *is not constant on* $V^n$, *then* $V^n$ *is isometric to a sphere.*

**References**

[1] BOCHNER, S.: Vector fields and Ricci curvature, Bull. Amer. Math. Soc., 52 (1946), 776-797.
[2] GRAY, A.: Einstein-like manifold which are not Einstein, Geometriae Dedicata, 7 (1978), 259-280.
[3] HOPF, E.: Elementare Bemerkungen über die Lösungen partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus, Sitzungs ber. Preuss Akad. Wiss., 19 (1972), 147-152.
[4] KATSURADA, Y.: On a certain property of closed hypersurfaces in Einstein space, Comment. Math. Helv., 38 (1964), 165-171.
[5] KATSURADA, Y.: Some characterizations of a submanifold which is isometric to a sphere, Jour. Fac. Sci. Hokkaido Univ. (Ser. I), 21 (1970), 85-96.
[6] KATSURADA, Y., Nagai, T. and Kôjyô, H.: On submanifolds with constant mean curva-

ture in a Riemannian manifold, (Publ. study group of Geom. 9 (1975))

[7] LIEBMANN, H.: Über die Verbiegung der geschlossenen Flächen positiver Krümmung, Math. Ann., 53 (1900), 91-112.

[8] OBATA, M.: Certain conditions for a Riemannian manifold to be isometric with a sphere, Jour. Math. Soc. Japan, 14 (1962), 333-340.

[9] ÔTSUKI, T.: Integral formulas for hypersurfaces in a Riemannian manifold and their applications, Tôhoku Math. Jour., 17 (1965), 335-348.

[10] SCHOUTEN, J. A.: Ricci-Calculus, 2nd ed. (Springer, Berlin, 1954)

[11] SÜSS, W.: Zur relativen Differentialgeometrie V., Tôhoku Math. Jour., 30 (1929), 202-209.

[12] YANO, K. and BOCHNER, S.: Curvature and Betti numbers, (Princeton, Ann. Math. Studies 32 (1953))

[13] YANO, K. and NAGANO, T.: Einstein space admitting a one-parameter group of conformal transformations, Ann. Math., 69 (1959), 451-461.

[14] YANO, K.: Differential geometry on complex and almost complex space, (Pergamon, 49 (1965))

[15] YANO, K.: Note on hypersurfaces in a Riemannian manifold, Canadian Jour. Math., 19 (1967), 439-446.

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