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Decomposition of convolution semigroups on Polish groups and zero-one laws

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Zero-one laws for infinitely divisible probability measures on a topological group G have quite a long history. (For a recent survey see the article [9] of A. Janssen.) Given a continuous convolution semigroup $(\mu_t)_{t\geq 0}$ of probability measures on G and a measurable subgroup H of G, one looks for conditions on $(\mu_t)_{t\geq 0}$ which yield $\mu_t(H)=0$ for all $t>0$ or $\mu_t(H)=1$ for all $t>0$. There are two classes of groups to which special attention has been given in this context: Locally compact groups; and topological vector spaces, in particular Banach spaces. But for technical reasons, on non-commutative groups mainly normal subgroups and normal convolution semi-groups have been considered (for example see [8, 9, 12]).

In 1983 a new idea was introduced in this field by T. Byczkowski and A. Hulanicki [4]. In order to obtain a zero-one law for Gaussian semigroups $(\mu_t)_{t\geq 0}$ on a Polish group G, they defined the resolvent measure $\mu=\int_0^\infty e^{-t}\mu_t\,dt$ and dealt with the space $L^1(\mu)$ (instead of a space of continuous functions on G). But this is quite natural since the indicator function of H is μ-integrable but not continuous (unless H is open). A further step along these lines was taken by T. Byczkowski and T. Żak [5]. If $\mu_t(H)>0$ for all $t>0$, then there exist a continuous convolution semigroup $(\lambda_t)_{t\geq 0}$ on G supported by H and a bounded measure ρ on G supported by $\complement H$ such that the infinitesimal generator of $(\mu_t)_{t\geq 0}$ is the sum of the infinitesimal generators of $(\lambda_t)_{t\geq 0}$ and of the Poisson semigroup $(e(t\rho))_{t\geq 0}$ with exponent ρ (decomposition theorem). An unsatisfactory aspect of [5] is that (for technical reasons) only Polish groups of the type $G=F^\infty$ are admitted (where F is a second countable locally compact group).

Although only normal subgroups H have been considered in [4] and [5], it is possible to get rid of this restriction by application of the following results: 1. An estimation of the growth of $\mu_t(H)$ as t tends to 0 ([10], cf. Lemma 1.6. below). 2. Every continuous convolution semigroup $(\mu_t)_{t\geq 0}$ admits a Lévy measure ([13], cf. 1.3. below). Indeed, the rôle played by the Lévy measure in the context of zero-one laws, is well known (cf. [8, 9]).

3. The expansion of a convolution semigroup into a perturbation series (cf. Section 2). This powerful technique already has been applied in various investigations (cf. [6, 7, 8, 10, 13]). Combining now the method of resolvent measures with 1., 2., 3. we obtain the decomposition theorem of Byczkowski and Żak in full generality (Theorem 4.6) i. e. for all Polish groups and for arbitrary measurable subgroups. Moreover, the uniqueness of the decomposition is established. As corollaries we obtain new zero-one laws, in particular for Gaussian semigroups and for semistable convolution semigroups; and a characterization of convolution semigroups with discrete part (Section 5).

Preliminaries

Let $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ denote the sets of integers, rational numbers, and real numbers, respectively. Moreover, let $\boldsymbol{Z}_+ :=\{n\in\boldsymbol{Z} : n\geq 0\}$, $\boldsymbol{N} :=\{n\in\boldsymbol{Z} : n>0\}$, $\boldsymbol{R}_+ :=\{r\in\mathbf{R} : r\geq 0\}$, $\boldsymbol{R}_+^* :=\{r\in\boldsymbol{R} : r>0\}$. If E is a vector lattice then E_+ denotes its positive cone.

By G we always denote a Polish group i. e. G is a topological group that admits a countable basis for its system $\mathscr{O}(G)$ of open subsets and a complete metric inducing $\mathscr{O}(G)$. By d we denote a left invariant metric on G inducing $\mathscr{O}(G)$. (Observe that d need not necessarily be complete; cf. Remark 1.3. below.) If B is a subset of G then ∂B denotes its boundary, $\bar{B}$ its closure, and 1_B its indicator function. A one-parameter group in G is a family $(x_t)_{t\in\boldsymbol{R}}$ in G such that $x_s x_t = x_{s+t}$ for all s, $t\in\boldsymbol{R}$ and such that $\lim_{t\to 0} x_t = e$.

By $\mathscr{B}(G)$ we denote the σ-field of Borel subsets of G. Moreover, $\mathscr{V}(e)$ denotes the system of neighbourhoods of the identity e of G which are in $\mathscr{B}(G)$. We put $G^* := G\backslash\{e\}$. By $\mathscr{C}(G)$ we denote the vector space of real valued bounded continuous functions on G furnished with the supremum norm $\|.\|$. Moreover, let $\mathscr{U}(G)$ denote the subspace of left uniformly (or d-uniformly) continuous functions in $\mathscr{C}(G)$. For every real valued function f on G we denote by $\mathrm{supp}(f)$ its support, and by f_x its right translate defined by $f_x(y) := f(yx)$ (all x, $y\in G$).

$\mathscr{M}(G)$ denotes the vector space of real valued (signed) measures on $\mathscr{B}(G)$. As is well known $\mathscr{M}(G)$ is a Banach algebra with repect to convolution $*$ and the norm $\|.\|$ of total variation. If μ, $\nu\in\mathscr{M}_+(G)$ have the same zero sets we write $\mu\approx\nu$. The weak topology on $\mathscr{M}_+(G)$ is generated by the seminorms $\nu\longrightarrow\int f\,d\nu$, where f runs through $\mathscr{C}_+(G)$ or $\mathscr{U}_+(G)$ respectively. As usual $\mathscr{M}^1(G) :=\{\nu\in\mathscr{M}_+(G) : \nu(G)=1\}$ is the set of probability measures on $\mathscr{B}(G)$. For every $x\in G$ the unit mass ε_x in $x\in G$ belongs to $\mathscr{M}^1(G)$. Moreover, if $\gamma\in\mathscr{M}_+(G)$ the Poisson measure with exponent γ is defined by $e(\gamma)=e^{-\gamma(G)}\sum_{n\geq 0}\frac{1}{n!}\gamma^n$ (where $\gamma^0:=\varepsilon_e$ and $\gamma^n:=\gamma*\gamma^{n-1}$ for all $n\in\boldsymbol{N}$). Obviously $e(\gamma)\in\mathcal{M}^1(G)$.

Let λ be a σ-finite positive measure on $\mathscr{B}(G)$. Moreover, let $f: G\longrightarrow\boldsymbol{R}_+$ be a Borel measurable and bounded function, and $\delta: G\longrightarrow G$ a Borel measurable mapping. Then by $\tilde{\lambda}(B):=\lambda(B^{-1})$, $(f.\lambda)(B):=\int 1_B.f\,d\lambda$ and $\delta(\lambda)(B):=\lambda(\delta^{-1}(B))$ (all $B\in\mathscr{B}(G)$) there are defined σ-finite positive measures $\tilde{\lambda}$, $f.\lambda$ and $\delta(\lambda)$ respectively. λ is said to be symmetric if $\lambda=\tilde{\lambda}$, and normal if $\lambda*\tilde{\lambda}=\tilde{\lambda}*\lambda$.

Let J be an interval on the real line and $t\longrightarrow\sigma_t$ a mapping of J into $\mathcal{M}(G)$ such that $t\longrightarrow\sigma_t^+(B)$ and $t\longrightarrow\sigma_t^-(B)$ are Lebesgue integrable over J for every $B\in\mathscr{B}(G)$ (where σ_t^+ and σ_t^- denote the positive and negative part of σ_t respectively). Then by $\sigma(B):=\int_J\sigma_t(B)\,dt$, $B\in\mathscr{B}(G)$, there is defined a measure σ in $\mathcal{M}(G)$. We write $\int_J\sigma_t\,dt:=\sigma$.

1. Preparations on convolution semigroups

1.1. By $x\longrightarrow f_x$ there is given a continuous representation of G by isometries on $\mathscr{U}(G)$. By $T_\nu f:=\int f_x\nu(dx)$ (Bochner integral) this extends to an isometric representation $\nu\longrightarrow T_\nu$ of the convolution algebra $\mathcal{M}(G)$ by bounded operators on $\mathscr{U}(G)$.

A convolution semigroup is a family $(\mu_t)_{t\geq 0}$ in $\mathcal{M}(G)$ such that $\mu_0=\varepsilon_e$ and $\mu_s*\mu_t=\mu_{s+t}$ for all $s, t\geq 0$. We speak of a convolution semigroup in $\mathcal{M}^1(G)$ if $\mu_t\in\mathcal{M}^1(G)$ for all $t\geq 0$. If all the measures μ_t have a certain property then we shall briefly say that $(\mu_t)_{t\geq 0}$ has this property.

1.2. With a convolution semigroup $(\mu_t)_{t\geq 0}$ in $\mathcal{M}(G)$ there is associated the operator semigroup $(T_{\mu_t})_{t\geq 0}$ on $\mathscr{U}(G)$; and T_{μ_0} is the identity operator I on $\mathscr{U}(G)$. Then $(\mu_t)_{t\geq 0}$ is said to be continuous if $(T_{\mu_t})_{t\geq 0}$ is (strongly) continuous (i. e. if $\lim_{t\downarrow 0}\|T_{\mu_t}f-f\|=0$ for all $f\in\mathscr{U}(G)$). If $(\mu_t)_{t\geq 0}$ is positive this is equivalent with the weak continuity of the mapping $t\longrightarrow\mu_t$.

By the infinitesimal generator of a continuous convolution semigroup $(\mu_t)_{t\geq 0}$ we understand the infinitesimal generator $(N, D(N))$ of the operator semigroup $(T_{\mu_t})_{t\geq 0}$.

1.3. A continuous convolution semigroup $(\mu_t)_{t\geq 0}$ in $\mathcal{M}^1(G)$ admits a Lévy measure η i. e. η is a σ-finite positive measure on $\mathscr{B}(G)$ such that $\eta(\{e\})=0$ and such that $\int f\,d\eta=\lim_{t\downarrow 0}\frac{1}{t}\int f\,d\mu_t$ for all $f\in\mathscr{C}(G)$ with $e\notin\mathrm{supp}(f)$ (cf.

[13]).

If $\gamma \in \mathcal{M}_+(G)$ then by $(e(t\gamma))_{t\geq 0}$ there is given a continuous convolution semigroup in $\mathcal{M}^1(G)$ with infinitesimal generator $(T_\gamma - \gamma(G).I, \mathcal{U}(G))$ and Lévy measure $1_{G^*}.\ \gamma$; the Poisson semigroup with exponent γ.

REMARK: In [13] we have considered a slightly smaller class of Polish groups (though still including second countable locally compact groups and separable Frechét spaces): There should exist a complete left invariant metric d on G inducing the topology. In fact, if G is a Polish group as defined in the Preliminaries, there need not exist a left invariant metric that is complete. Nevertheless the results of [13] remain true in this more general situation: The stochastic process $(X_t)_{t\geq 0}$ with independent left increments associated with the continuous convolution semigroup $(\mu_t)_{t\geq 0}$ in $\mathcal{M}^1(G)$ can always be chosen to have right continuous paths that admit left-hand limits ([3], Theorem 3). In fact this is the only place in our paper where we need that G is completely metrizable. Otherwise it always suffices that G is metrizable and separable (i. e. that $\mathcal{O}(G)$ admits a countable basis).

1.4. A crucial part in our investigations is played by the following idea due to T. Byczkowski and A. Hulanicki [4]: If $(\mu_t)_{t\geq 0}$ is a continuous convolution semigroup in $\mathcal{M}^1(G)$ then by $\mu := \int_0^\infty e^{-t}\,\mu_t\,dt$ there is defined a measure in $\mathcal{M}^1(G)$. By $\mathcal{L}^1(\mu)$ we denote the vector space of all Borel measurable real valued functions on G that are μ-integrable. Convergence in $\mathcal{L}^1(\mu)$ means convergence in the mean. Thus $\mathcal{U}(G) \subset \mathcal{L}^1(\mu)$, norm convergence implies convergence in the mean, und $\mathcal{U}(G)$ is dense in $\mathcal{L}^1(\mu)$. The equivalence class $\{g \in \mathcal{L}^1(\mu) : g = f\ \mu\text{-almost everywhere}\}$ of $f \in \mathcal{L}^1(\mu)$ is denoted by $[f]$. The space $L^1(\mu)$ of all $[f]$, $f \in \mathcal{L}^1(\mu)$, furnished with the norm $\|[f]\|_1 := \int |f|\,d\mu$, becomes a separable Banach space. If $f \in \mathcal{U}(G)$ then $\|[f]\|_1 \leq \|f\|$.

Let $\nu \in \mathcal{M}_+(G)$. If there exists some $c \in \boldsymbol{R}_+$ such that $\int (\int f(xy)\,\nu(dy))\,\mu(dx) \leq c \int f\,d\mu$ for all $f \in \mathcal{L}^1_+(\mu)$ then T_ν extends uniquely to a bounded operator $\bar{T}_\nu$ on $L^1(\mu)$ with norm $\|\bar{T}_\nu\|_1 \leq c$; in fact, $\bar{T}_\nu[f] := [\int f_y\,\nu(dy)]$ for all $f \in \mathcal{L}^1(\mu)$.

1.5. Now let $(\lambda_t)_{t\geq 0}$ be a continuous convolution semigroup in $\mathcal{M}^1(G)$ with infinitesimal generator $(L, D(L))$. We assume that every T_{λ_t} admits a bounded extension $\bar{T}_{\lambda_t}$ onto $L^1(\mu)$ (in the sense of 1.4.) such that sup $\{\|\bar{T}_{\lambda_t}\|_1 :$

$0 \leq t \leq 1\}$ is finite. Then $(\bar{T}_{\lambda_t})_{t \geq 0}$ is a (strongly) continuous operator semigroup on $L^1(\mu)$ (with $\bar{T}_{\lambda_0} = \bar{I}$) whose infinitesimal generator $(\bar{L}, D(\bar{L}))$ is an extension of $(L, D(L))$ i. e. $[f] \in D(\bar{L})$ and $\bar{L}[f] = [Lf]$ for all $f \in D(L)$. In particular, $(T_{\mu_t})_{t \geq 0}$ admits an extension $(\bar{T}_{\mu_t})_{t \geq 0}$ onto $L^1(\mu)$ such that $\|\bar{T}_{\mu_t}\|_1 = e^t$ for all $t \in \boldsymbol{R}_+$ (cf. [4], proof of Proposition 1).

1.6. A further decisive result for our investigations is prepared by the following general facts:

LEMMA. *Let f be a mapping of $\boldsymbol{R}_+^*$ into $[0, 1]$ such that $0 \leq f(s+t) - f(s)f(t) \leq (1-f(s))(1-f(t))$ for all $s, t \in \boldsymbol{R}_+^*$ and such that $\lim_{t \downarrow 0} f(t) = 1$. Then the following assertions are valid:*

(i) *f is uniformly continuous.*

(ii) *$b := -\inf_{t>0} \frac{1}{t} \ln f(t)$ is finite.*

(iii) *$f(t) \geq e^{-bt}$ for all $t > 0$.*

(iv) *b is the least real number c such that $f(t) \geq e^{-ct}$ for all $t > 0$.*

(v) *$b = -\lim_{t \downarrow 0} \frac{1}{t} \ln f(t)$.*

(vi) *$b = \lim_{t \downarrow 0} \frac{1}{t}(1 - f(t))$.*

PROOF. First of all (i) follows from $|f(t+s) - f(t)| \leq 2(1 - f(|s|))$ for all $t \in \boldsymbol{R}_+^*$ and $s \in \boldsymbol{R}$ such that $0 < |s| < t$ and from $\lim_{t \downarrow 0} f(t) = 1$. Moreover there exists some $c \in \boldsymbol{R}_+$ such that $f(t) \geq e^{-ct}$ for all $t > 0$ ([10], Lemma 3 and its proof). Hence $\frac{1}{t} \ln f(t) \geq -c$ for all $t > 0$. This shows $c \geq b \geq 0$; hence (ii), (iii) and (iv).

For the proof of (v) one may proceed similarly as in [14], p. 232 (proof of the Proposition).

Finally the series expansion of the logarithm yields $\ln f(t) = -(1-f(t)) + (1-f(t))^2 r(t)$ with $|r(t)| \leq 1/f(t)$ (all $t > 0$). But $0 \leq 1 - f(t) \leq bt$ in view of (iii). Hence (vi) follows from (v).

1.7. PROPOSITION. *Let $(\mu_t)_{t \geq 0}$ be a continuous convolution semigroup in $\mathscr{M}^1(G)$, and let H be a measurable subgroup of G. Then the following assertions are equivalent:*

(i) *The subset $\{t \in \boldsymbol{R}_+ : \mu_t(H) > 0\}$ of $\boldsymbol{R}$ has positive Lebesgue measure.*

(ii) $\lim_{t \downarrow 0} \mu_t(H) = 1$.

(iii) $b:=\lim_{t\downarrow 0}\frac{1}{t}\mu_t(\complement H)$ *exists and is finite and* $\mu_t(H)\geq e^{-bt}$ *for all* $t \in \boldsymbol{R}_+$.

PROOF. "(iii)$\Longrightarrow$(i)" is obvious; "(i)$\Longrightarrow$(ii)" cf. [4], Proposition 1.

"(ii)$\Longrightarrow$(iii)" We put $f(t):=\mu_t(H)$ for all $t>0$. Then $f(s+t)=\mu_{s+t}(H)=\mu_s*\mu_t(HH)\geq\mu_s(H)\mu_t(H)=f(s)f(t)$; and $f(s+t)-f(s)f(t)=\int\mu_t(x^{-1}H)\mu_s(dx)-\int 1_H(x)\mu_t(x^{-1}H)\mu_s(dx)=\int 1_{\complement H}(x)\mu_t(x^{-1}H)\mu_s(dx)\leq \int 1_{\complement H}(x)\mu_t(\complement H)\mu_s(dx)=\mu_s(\complement H)\mu_t(\complement H)=(1-\mu_s(H))(1-\mu_t(H))=(1-f(s))(1-f(t))$ (all s, $t>0$). Thus f fulfills the assumptions of Lemma 1.6. Hence (iii).

1.8. REMARK. Let $G:=\boldsymbol{R}$, $H:=\boldsymbol{Q}$, and $\mu_t:=\varepsilon_t$ for all $t\in\boldsymbol{R}_+$. Then $\mu_t(H)=1$ if t is rational and $\mu_t(H)=0$ if t is irrational. Hence the density of $\{t\in\boldsymbol{R}_+: \mu_t(H)>0\}$ in $\boldsymbol{R}_+$ is not sufficient for Proposition 1.7. to hold.

1.9. COROLLARY. *Let* $\mu_{t_0}(H)>0$ *for a certain* $t_0>0$. *Then the assertions of Proposition* 1.7. *are valid in each of the following three situations*:

(i) *The semigroup* $(\mu_t)_{t\geq 0}$ *is symmetric.*

(ii) $\lim_{t\downarrow t_1}\|\mu_t-\mu_{t_1}\|=0$ *for a certain* $t_1>0$.

(iii) μ_{t_1} *is (left) absolutely continuous for a certain* $t_1>0$ *(i. e.* $x\longrightarrow\mu_{t_1}(xB)$ *is continuous at* e *for every* $B\in\mathscr{B}(G)$*).*

PROOF. For (i) cf. [4], Proposition 2. For (ii) we first observe that $\|\mu_{t_1+t}-\mu_{t_1+s}\|\leq\|\mu_{t_1+|t-s|}-\mu_{t_1}\|$ for all s, t$\in\boldsymbol{R}_+$; hence the mapping $t\longrightarrow\mu_t$ is norm continuous on $[t_1,\infty[$. Now we can choose some $n\in\boldsymbol{N}$ such that $nt_0>t_1$. Then $\mu_{nt_0}(H)\geq(\mu_{t_0}(H))^n>0$ and hence $\mu_t(H)>0$ for all $t>0$ in some neighbourhood of nt_0. Finally, (iii) is a special case of (ii).

2. Perturbations of convolution semigroups

2.1. LEMMA. *Let* $(\alpha_t)_{t\geq 0}$ *be a continuous convolution semigroup in* $\mathscr{M}(G)$ *with infinitesimal generator* $(A, D(A))$; *and let* $\|\alpha_t\|\leq c\, e^{at}$ *for certain* a, $c\in\boldsymbol{R}_+$ *and for all* $t\in\boldsymbol{R}_+$. *Moreover, let* $\pi\in\mathscr{M}(G)$. *We define* $Bf:=Af+(T_\pi f-\pi(G)f)$ *for all* $f\in D(A)$. *Then the following assertions are valid*:

(i) *There exists a (unique) continuous convolution semigroup* $(\beta_t)_{t\geq 0}$ *in* $\mathscr{M}(G)$ *with infinitesimal generator* $(B, D(A))$; *and* $\|\beta_t\|\leq c\, e^{(a+c\|\pi\|-\pi(G))t}$ *for all* $t\in\boldsymbol{R}_+$.

(ii) *For every* $t\geq 0$ *we define by induction*: $\pi_0(t):=\alpha_t$, *and*

$\pi_k(t):=\int_0^t \alpha_r * \pi * \pi_{k-1}(t-r)\, dr$ *for all* $k \in \boldsymbol{N}$. *Then* $\pi_k(t) \in \mathscr{M}(G)$ *such that* $\|\pi_k(t)\| \leq c\, e^{at}(c\|\pi\| t)^k / k\,!$ *(all* $t \in \mathbf{R}_+$, $k \in \boldsymbol{Z}_+$*).*

(iii) *The series* $e^{-t\pi(G)} \sum_{k \geq 0} \pi_k(t)$ *converges to* β_t *with respect to the norm of* $\mathscr{M}(G)$ *(all* $t \in \boldsymbol{R}_+$*).*

Proof. First of all T_π is a bounded operator on $\mathscr{U}(G)$. Now for every $t \in \boldsymbol{R}_+$ we define by induction bounded operators on $\mathscr{U}(G)$: $S_0(t):=T_{\alpha_t}$, and $S_k(t):=\int_0^t S(r) T_\pi S_{k-1}(t-r)\, dr$ for all $k \in \boldsymbol{N}$ (where the integrals are taken with respect to the strong operator topology). Moreover let $Cf:=Af+T_\pi f$ for all $f \in D(A)$. Then $(C, D(A))$ is the infinitesimal generator of a continuous operator semigroup $(R_t)_{t \geq 0}$ on $\mathscr{U}(G)$; and the series $\sum_{k \geq 0} S_k(t)$ converges to R_t with respect to the operator norm ([6], p. 11, Hilfssatz 11).

By induction we conclude $\|\pi_k(t)\| \leq c\, e^{at}(c\|\pi\| t)^k / k\,!$ and $S_k(t)=T_{\pi_k(t)}$ for all $t \in \boldsymbol{R}_+$ and $k \in \boldsymbol{Z}_+$. Consequently, the series $\sum_{k \geq 0} \pi_k(t)$ converges to some measure $\gamma_t \in \mathscr{M}(G)$ with respect to the norm; hence $R_t=T_{\gamma_t}$ (all $t \in \boldsymbol{R}_+$). Thus $(\gamma_t)_{t \geq 0}$ is a continuous convolution semigroup in $\mathscr{M}(G)$ with infinitesimal generator $(C, D(A))$. Finally, we define $\beta_t:=e^{-t\pi(G)}\gamma_t$ for all $t \in \boldsymbol{R}_+$. Since $B=C-\pi(G) I$, the assertions now follow.

2.2. Corollary. *In the situation of Lemma* 2.1. *the following assertions are valid*:

(i) $\lim_{t \downarrow 0} \frac{1}{t} \int f\, d\pi_1(t)=\int f\, d\pi$ *for all* $f \in \mathscr{U}(G)$.

(ii) $\|\sum_{k \geq 2} \pi_k(t)\| \leq (c\|\pi\| t)^2 c\, e^{(a+c\|\pi\|)t}$ *for all* $t \in \boldsymbol{R}_+$.

Proof. The continuity of the operator semigroup $(T_{\alpha_t})_{t \geq 0}$ yields (i); and (ii) is obvious.

2.3. Definition. *The convolution semigroup* $(\beta_t)_{t \geq 0}$ *obtained in Lemma* 2.1. *is said to be a* <u>*perturbation*</u> *of the convolution semigroup* $(\alpha_t)_{t \geq 0}$ *by means of the measure* π. *We apply the notation* $(\beta_t)_{t \geq 0}=p((\alpha_t)_{t \geq 0};\pi)$.

2.4. Corollary. *Let* $(\alpha_t)_{t \geq 0}$ *be a continuous convolution semigroup as in Lemma* 2.1. *and let* $\pi, \rho \in \mathscr{M}(G)$. *Then we have* $p(p((\alpha_t)_{t \geq 0};\pi);\rho)=p((\alpha_t)_{t \geq 0};\pi+\rho)$; *and in particular* $p(p((\alpha_t)_{t \geq 0};\pi);-\pi)=(\alpha_t)_{t \geq 0}$.

Proof. Follows immediately from Lemma 2.1.

2.5. Proposition. *Let* $(\alpha_t)_{t \geq 0}$ *be a continuous convolution semigroup in* $\mathscr{M}^1(G)$ *with infinitesimal generator* $(A, D(A))$ *and Lévy measure* ω.

Moreover, for some measure $\pi\in\mathscr{M}(G)$ *we define* $(\beta_t)_{t\geq 0}=p((\alpha_t)_{t\geq 0};\ \pi)$. *Then the following assertions are equivalent*:

(i) $\omega+\pi\geq 0$.

(ii) $\beta_t\in\mathscr{M}^1(G)$ *for all* $t\in\boldsymbol{R}_+$.

In the affirmative case, $\omega+\pi$ *is the Lévy measure of* $(\beta_t)_{t\geq 0}$.

PROOF. We keep the notations of Lemma 2. 1.

"(i)$\Longrightarrow$(ii)" In view of $B1_G=0$ it suffices to show: $T_{\beta_t}f\geq 0$ for all $f\in\mathscr{U}_+(G)$ and $t\in\boldsymbol{R}_+$. But this is equivalent with the following condition ([1], Corollary 5. 2): (P) For every positive linear functional ϕ on $\mathscr{U}(G)$ and for every $f\in D(B)$ such that $f\geq 0$ and $\phi(f)=0$ it holds $\phi(Bf)\geq 0$.

Hence let ϕ and f as in (P). We put $g(y):=\phi(f_y)$ for all $y\in G$. Then $g\in\mathscr{C}_+(G)$ such that $g(e)=0$. Since

$$Bf=\lim_{t\downarrow 0}\frac{1}{t}\int[f_y-f]\ \alpha_t(dy)+\int[f_y-f]\ \pi(dy)$$

and since ϕ is continuous we conclude:

$$\phi(Bf)=\lim_{t\downarrow 0}\frac{1}{t}\int g\ d\alpha_t+\int g\ d\pi.$$

(Observe [14], V. 5; in particular Corollary V. 5. 2.) There exists a sequence $(V_n)_{n\geq 1}$ in $\mathscr{V}(e)$ descending to $\{e\}$ such that $\omega(\partial V_n)=0$ for all $n\in\boldsymbol{N}$. Thus

$$\begin{aligned}\phi(Bf)&\geq\lim_{t\downarrow 0}\frac{1}{t}\int 1_{\complement V_n}\, g\ d\alpha_t+\int g\ d\pi\\&=\int 1_{\complement V_n}\, g\ d\omega+\int g\ d\pi.\end{aligned}$$

Since the sequence $(1_{\complement V_n}\, g)_{n\geq 1}$ ascends to g we conclude $\phi(Bf)\geq\int g\ d\omega+\int g\ d\pi\geq 0$. Thus (P) is fulfilled.

"(ii)$\Longrightarrow$(i)" In view of Lemma 2. 1. (ii) we have $\beta_t=e^{-t\pi(G)}(\alpha_t+\pi_1(t)+\sum_{k\geq 2}\pi_k(t))$ for all $t\geq 0$. Let ξ denote the Lévy measure of $(\beta_t)_{t\geq 0}$. If $f\in\mathscr{U}(G)$ such that $e\notin\operatorname{supp}(f)$ then Corollary 2. 2. yields $\int fd\xi=\int fd\omega+\int fd\pi$. Hence $\xi=\omega+\pi$. Since ξ is positive this proves (i) and the supplement.

2. 6. REMARKS. 1. If there exists a Lévy-Khintchine formula for the continuous convolution semigroups in $\mathscr{M}^1(G)$ (e. g. if G is a second countable

locally compact group or a separable Frechēt space) then Proposition 2. 5. can also be derived from this very formula.

2. Let $(\lambda_t)_{t\geq 0}$ be a continuous convolution semigroup in $\mathscr{M}^1(G)$ with Lēvy measure ξ. Moreover, let $V\in\mathscr{V}(e)$. Then there exists a continuous convolution semigroup $(\lambda_t^V)_{t\geq 0}$ in $\mathscr{M}^1(G)$ with Lēvy measure $1_V.\ \xi$ such that $(\lambda_t)_{t\geq 0}=p((\lambda_t^V)_{t\geq 0};\ 1_{\complement V}.\ \xi)$.

[Define $(\lambda_t^V)_{t\geq 0}: =p((\lambda_t)_{t\geq 0};\ -1_{\complement V}.\ \xi)$. Since $1_{\complement V}.\ \xi\leq\xi$, we have $\lambda_t^V\in\mathscr{M}^1(G)$ for all $t\in\boldsymbol{R}_+$, and $\xi-1_{\complement V}.\ \xi=1_V.\ \xi$ is the Lēvy measure of $(\lambda_t^V)_{t\geq 0}$ (Proposition 2. 5.). The last assertion follows from Corollary 2. 4.] For locally compact groups this result has been observed by W. Hazod ([6], p. 34, Korollar zu Satz 2. 2.).

3. Let $\varphi: G\longrightarrow\boldsymbol{R}_+^*$ be a continuous function that is submultiplicative i. e. $\varphi(xy)\leq\varphi(x)\varphi(y)$ for all $x, y\in G$. Then for every continuous convolution semigroup $(\lambda_t)_{t\geq 0}$ in $\mathscr{M}^1(G)$, with Lēvy measure ξ, are equivalent: (i) $\int\varphi\, d\lambda_{t_0}<\infty$ for a certain $t_0\in\boldsymbol{R}_+$; (ii) $\int 1_{\complement V}.\ \varphi\, d\xi<\infty$ for all $V\in\mathscr{V}(e)$

[For second countable locally compact groups and separable Frechēt spaces this is Theorem 5 in [13]. The crucial step in its proof has been the inequality $\lambda_t\geq e^{-t\xi(\complement V)}\int_0^t\lambda_r^V*(1_{\complement V}.\ \xi)*\lambda_{t-r}^V\, dr$. But in view of Remark 2 this inequality is now available for every Polish group.]

3. Construction of a measure

3. 1. At first let us recall that a vector sublattice $\mathscr{F}$ of $\mathscr{C}(G)$ is said to be <u>Stonian</u> if $\min(f, 1_G)\in\mathscr{F}$ for every $f\in\mathscr{F}$. Moreover, a subset O of G is said to be $\mathscr{F}$-open if there exists a sequence $(f_n)_{n\geq 1}$ in $\mathscr{F}_+$ ascending to 1_O. By $\mathscr{O}(\mathscr{F})$ we denote the system of all $\mathscr{F}$-open subsets of G. Obviously we have $\mathscr{O}(\mathscr{F})\subset\mathscr{O}(G)$.

Finally, for every $n\in\boldsymbol{N}$, we put $V_n: =\{x\in G: d(x, e)<\frac{1}{n}\}$; hence $(V_n)_{n\geq 1}$ is a countable basis for $\mathscr{V}(e)$.

3. 2. Lemma. *There exists a separable closed subalgebra $\mathscr{F}$ of $\mathscr{U}(G)$ containing 1_G which enjoys the following additional properties:*

a) *There exists a sequence $(f_n)_{n\geq 1}$ in $\mathscr{F}$ such that $1_{\complement V_n}\leq f_n\leq 1_{\complement V_{n+1}}$ for all $n\in\boldsymbol{N}$.*

b) *$\mathscr{F}_{00}: =\{f\in\mathscr{F}: e\notin\operatorname{supp}(f)\}$ is a Stonian vector lattice.*

c) *$\mathscr{F}_{00}$ generates $\mathscr{B}(G)$.*

d) *$\mathscr{F}_{00}$ is dense in $\mathscr{F}_0: =\{f\in\mathscr{F}: f(e)=0\}$.*

e) *$\mathscr{O}(\mathscr{F})=\mathscr{O}(G)$.*

f) $\mathscr{F}$ *is dense in* $\mathscr{L}^1(\nu)$ *for every* $\nu \in \mathscr{M}^1(G)$.

PROOF. There exists a countable basis $(O_n)_{n\geq 1}$ for $\mathscr{O}(G)$, and for every $n \in \boldsymbol{N}$ there exists a sequence $(f_{nk})_{k\geq 1}$ in $\mathscr{U}_+(G)$ ascending to 1_{O_n}. Moreover there exists a sequence $(f_n)_{n\geq 1}$ in $\mathscr{U}_+(G)$ such that $1_{\complement V_n} \leq f_n \leq 1_{\complement V_{n+1}}$ for all $n \in \boldsymbol{N}$. Now we successively define:

$$\mathscr{F}_1: =\{f_{nk}: n, k \in \boldsymbol{N}\} \cup \{f_n: n \in \boldsymbol{N}\} \cup \{1_G\};$$
$$\mathscr{F}_2: =\{rf: f \in \mathscr{F}_1;\ r \in \boldsymbol{Q}\};$$
$$\mathscr{F}_3: =\{g_1 \dots g_n: g_1, \dots, g_n \in \mathscr{F}_2;\ n \in \boldsymbol{N}\};$$
$$\mathscr{F}_4: =\{h_1 + \dots + h_n: h_1, \dots, h_n \in \mathscr{F}_3;\ n \in \boldsymbol{N}\}.$$

Then $\mathscr{F}_4$ is a countable algebra over $\boldsymbol{Q}$ such that $1_G \in \mathscr{F}_4$. Let $\mathscr{F}$ denote the closure of $\mathscr{F}_4$ in $\mathscr{U}(G)$. Then $\mathscr{F}$ is a separable closed subalgebra of $\mathscr{U}(G)$ such that $1_G \in \mathscr{F}$. Hence $\mathscr{F}$ is a vector lattice too. This yields property b); whereas properties a) and c) follow by the choice of $\mathscr{F}_1$.

Let $f \in \mathscr{F}_0$. Then property a) yields $ff_n \in \mathscr{F}_{00}$ for all $n \in \boldsymbol{N}$ and $\lim_{n\geq 1} \|ff_n - f\| = 0$. This proves property d). By definition we have $O_n \in \mathscr{O}(\mathscr{F})$ for all $n \in \boldsymbol{N}$; moreover $\mathscr{O}(\mathscr{F})$ is closed with respect to countable unions. Hence property e). Finally, property f) follows by the Daniell-Stone theorem ([2], Korollar 39.5), since $\mathscr{F}$ is a Stonian lattice that generates $\mathscr{B}(G)$.

3.3. COROLLARY. *For every* $f \in \mathscr{C}(G)$ *such that* $f(e)=0$ *there exists some* $g \in \mathscr{F}_0$ *such that* $|f| \leq g$ *and* $g(x) > 0$ *for all* $x \in G^*$.

PROOF. By induction we obtain a strictly increasing sequence $(k_n)_{n\geq 1}$ of positive integers such that $|f(x)| < 2^{-(n+1)}$ for all $x \in V_{k_n}$. We define $h: = \sum_{n\geq 1} 2^{-n} f_{k_n}$; obviously, $h \in \mathscr{F}_0$.

Let $x \in G^*$. If $x \in \complement V_{k_1}$ then $f_{k_n}(x) = 1$ for all $n \in \boldsymbol{N}$, hence $h(x)=1$. Otherwise, there exists a certain $m \in \boldsymbol{N} \setminus \{1\}$ such that $x \in V_{k_{m-1}} \setminus V_{k_m}$. Consequently, $|f(x)| < 2^{-m}$ and $f_{k_m}(x) = 1$. Thus $|f(x)| < 2^{-m} f_{k_m}(x) < h(x)$. Hence $g: =(1+\|f\|)h$ has the desired properties.

3.4. CONVENTIONS. For the remainder of this section and for Section 4 let $(\mu_t)_{t\geq 0}$ denote a fixed continuous convolution semigroup in $\mathscr{M}^1(G)$ with infinitesimal generator $(N, D(N))$ and Lévy measure η. Moreover, let H denote a Borel measurable subgroup of G such that $\mu_t(H) > 0$ for all $t > 0$. Let $b: = \lim_{t\downarrow 0} \frac{1}{t}\mu_t(\complement H)$ so that $\mu_t(H) \geq e^{-bt}$ for all $t \in \boldsymbol{R}_+$ (cf. Proposition 1.7.).

Finally, we fix a subalgebra $\mathscr{F}$ of $\mathscr{U}(G)$ together with its subspaces

$\mathscr{F}_0$ and $\mathscr{F}_{00}$ as in Lemma 3.2.

3.5. LEMMA. *There exist a sequence $(t(n))_{n\geq 1}$ in $\boldsymbol{R}_+$ descending to* 0 *and a measure $\sigma\in\mathscr{M}_+(G)$ with the following properties:*

a) $\int f\ d\sigma=\lim_{n\geq 1}\frac{1}{t(n)}\int 1_{\complement H}f\ d\mu_{t(n)}$ *for all* $f\in\mathscr{F}_0$.

b) $\sigma\leq\eta$ *and* $\sigma(G)\leq b$.

PROOF. We have $\frac{1}{t}\mu_t(\complement H)\leq\frac{1}{t}(1-e^{-bt})\leq b$ for all $t>0$. Consequently, $|\frac{1}{t}\int 1_{\complement H}fd\mu_t|\leq b\|f\|$ for all $f\in\mathscr{C}(G)$ and $t>0$. Together with $\mathscr{F}$ also $\mathscr{F}_{00}$ is separable. Applying a diagonal procedure, we obtain a sequence $(t(n))_{n\geq 1}$ in $\boldsymbol{R}_+$ descending to 0 such that for every $f\in\mathscr{F}_{00}$ the following limit exists:

$$\phi(f):=\lim_{n\geq 1}\frac{1}{t(n)}\int 1_{\complement H}fd\mu_{t(n)}.$$

Obviously, ϕ is a positive linear functional on $\mathscr{F}_{00}$ such that $|\phi(f)|\leq b\|f\|$ for all $f\in\mathscr{F}_{00}$. Moreover, for all $f\in(\mathscr{F}_{00})_+$ we have (cf. 1.3.):

$$(*)\qquad \phi(f)=\lim_{n\geq 1}\frac{1}{t(n)}\int 1_{\complement H}f\ d\mu_{t(n)}$$
$$\leq\lim_{n\geq 1}\frac{1}{t(n)}\int fd\mu_{t(n)}=\int fd\eta<\infty.$$

Hence ϕ is a σ-smooth (positive linear) functional on the Stonian vector lattice $\mathscr{F}_{00}$. By the Daniel-Stone theorem ([2], Satz 39.4) there exists a positive measure σ' on $\mathscr{B}(G)$ such that $\phi(f)=\int f\ d\sigma'$ for all $f\in\mathscr{F}_{00}$. Let $\sigma:=1_{G^*}.\ \sigma'$. Then we also have $\phi(f)=\int f\ d\sigma$ for every $f\in\mathscr{F}_{00}$ (since $f(e)=0$).

From $(*)$ and $\sigma(\{e\})=0$ we conclude $\sigma\leq\eta$. Moreover, there exists a sequence $(f_n)_{n\geq 1}$ in $(\mathscr{F}_{00})_+$ ascending to 1_{G^*} (cf. Lemma 3.2.). Since $\int f_n\ d\sigma=\phi(f_n)\leq b\|f_n\|\leq b$ for all $n\in\boldsymbol{N}$, we conclude $\sigma(G)\leq b$ (observe $\sigma(\{e\})=0$). Hence b) is proved.

Now let $f\in\mathscr{F}_0$. Given $\varepsilon>0$ there exists some $g\in\mathscr{F}_{00}$ such that $\|f-g\|<\varepsilon$ (Lemma 3.2.). Since $\lim_{n\geq 1}\frac{1}{t(n)}\int 1_{\complement H}g\ d\mu_{t(n)}=\int g\ d\sigma$ and since $\frac{1}{t(n)}\mu_{t(n)}(\complement H)\leq b$ for all $n\in\boldsymbol{N}$, it follows that $\lim_{n\geq 1}\frac{1}{t(n)}\int 1_{\complement H}fd\mu_{t(n)}=\int fd\sigma$.

Hence a) is also proved.

3.6. Corollary. *In the situation of Lemma* 3.5. *the following assertions are valid*:

(i) *For every* $g\in(\mathscr{F}_0)_+$ *the sequence* $(\frac{1}{t(n)}(1_{\complement H}g)\cdot\mu_{t(n)})_{n\geq 1}$ *converges weakly to* $g.\sigma$ (*in* $\mathscr{M}_+(G)$).

(ii) $\int f d\sigma=\lim_{n\geq 1}\frac{1}{t(n)}\int 1_{\complement H} f d\mu_{t(n)}$ *for all* $f\in\mathscr{C}(G)$ *such that* $f(e)=0$.

Proof. (i) For some fixed $g\in(\mathscr{F}_0)_+$ we put $\lambda_n:=\frac{1}{t(n)}(1_{\complement H}g)\cdot\mu_{t(n)}$, $n\in N$, and $\lambda:=g\cdot\sigma$. For every $f\in\mathscr{F}$ we have $fg\in\mathscr{F}_0$; hence we conclude from property a) of Lemma 3.5. that $\int fd\lambda=\lim_{n\geq 1}\int fd\lambda_n$. Thus $\lambda(G)=\underline{\lim}_{n\geq 1}\lambda_n(G)$, and in view of property e) of Lemma 3.2., also $\lambda(O)\leq\lim_{n\geq 1}\lambda_n(O)$ for all $O\in\mathscr{O}(G)$. This proves (i).

(ii) Now let $f\in\mathscr{C}(G)$ such that $f(e)=0$. We choose some $g\in\mathscr{F}_0$ according to Corollary 3.3. and define $h(x):=f(x)/g(x)$ if $x\in G^*$ and $h(e):=0$. Then h is a bounded function on G that is continuous on G^*, hence is continuous σ-almost everywhere. Taking into account (i) we conclude:

$$\begin{aligned}\int f d\sigma=\int hg\, d\sigma&=\lim_{n\geq 1}\frac{1}{t(n)}\int 1_{\complement H}hg\, d\mu_{t(n)}\\&=\lim_{n\geq 1}\frac{1}{t(n)}\int 1_{\complement H}fd\mu_{t(n)};\end{aligned}$$

hence (ii).

4. Decomposition of a convolution semigroup

4.1. We recall that the conventions 3.4. are still in force. We put $\mu:=\int_0^\infty e^{-t}\mu_t\, dt$ (cf. 1.4.). Moreover, let σ be a measure with the properties of Lemma 3.5. and Corollary 3.6.

We define $\mu_t^H:=\frac{1}{\mu_t(H)}(1_H\cdot\mu_t)$; thus $\mu_t^H\in\mathscr{M}^1(G)$ and $\mu_t^H(H)=1$ (all $t>0$). Finally, let $N^Hf:=Nf-(T_\sigma-\sigma(G)I)f$ for all $f\in D(N)$.

4.2. Lemma. *For every* $f\in D(N)$ *the following assertions are valid*:

(i) $N^Hf(x)=\lim_{n\geq 1}\frac{1}{t(n)}\int[f(xy)-f(x)]\mu_{t(n)}^H(dy)$ (*all* $x\in G$).

(ii) $N^H f = \lim_{n\geq 1} \frac{1}{t(n)}[T_{\mu^H_{t(n)}} f - f]$ *in* $\mathscr{L}^1(\mu)$.

PROOF. (i) For all $n\in \boldsymbol{N}$ we have:

$$(*)\qquad \frac{1}{t(n)}\int [f(xy)-f(x)]\mu_{t(n)}(dy)$$
$$=\mu_{t(n)}(H)\frac{1}{t(n)}\int [f(xy)-f(x)]\mu^H_{t(n)}(dy)$$
$$+\frac{1}{t(n)}\int 1_{\complement H}(y)[f(xy)-f(x)]\mu_{t(n)}(dy).$$

In view of Corollary 3.6. (ii) we have:

$$(**)\qquad \lim_{n\geq 1}\frac{1}{t(n)}\int 1_{\complement H}(y)[f(xy)-f(x)]\mu_{t(n)}(dy)$$
$$=\int [f(xy)-f(x)]\sigma(dy) = T_\sigma f(x) - \sigma(G)f(x).$$

Hence, letting n tend to infinity in (*), we obtain assertion (i) (observe Proposition 1.7.).

(ii) First of all we have $Nf = \lim_{n\geq 1}\frac{1}{t(n)}\int [f_y - f]\mu_{t(n)}(dy)$ in $\mathscr{U}(G)$and hence also in $\mathscr{L}^1(\mu)$. Moreover, for all $x\in G$ and $n\in \boldsymbol{N}$, we have:

$$\left|\frac{1}{t(n)}\int 1_{\complement H}(y)[f(xy)-f(x)]\mu_{t(n)}(dy)\right| \leq 2\|f\|\frac{1}{t(n)}\mu_{t(n)}(\complement H)\leq 2\|f\|b.$$

Hence, taking into account (**), the theorem of majorized convergence yields:

$$\lim_{n\geq 1}\frac{1}{t(n)}\int 1_{\complement H}(y)[f_y - f]\mu_{t(n)}(dy) = (T_\sigma - \sigma(G)I)f$$

in $\mathscr{L}^1(\mu)$. Together with (*), this yields assertion (ii).

4.3. LEMMA. *The convolution semigroup* $(\nu_t)_{t\geq 0} := p((\mu_t)_{t\geq 0}; -\sigma)$ *possesses the following properties*:

a) $\nu_t \in \mathscr{M}^1(G)$ *for all* $t\in \boldsymbol{R}_+$.

b) $(\nu_t)_{t\geq 0}$ *has the Lévy measure* $\eta - \sigma$ *and the infinitesimal generator* $(N^H, D(N))$.

c) $\nu_t \leq e^{bt}\mu_t$ *for all* $t\in \boldsymbol{R}_+$.

d) $\nu_t(H)=1$ *for all* $t\in \boldsymbol{R}_+$.

PROOF. 1. In view of Lemma 3.5. we have $\sigma \leq \eta$. Hence property a) and the first part of b) by taking into account Proposition 2.5. The second part of b) follows from Lemma 2.1. (i). Since $(\mu_t)_{t\geq 0} = p((\nu_t)_{t\geq 0}; \sigma)$

(Corollary 2.4.) and since σ is positive, property c) follows from Lemma 3.5. and Lemma 2.1. (iii).

2. For the proof of d) we define first of all $\pi_t^{(n)} := e(\frac{t}{t(n)}\mu_{t(n)}^H)$ for all $t \geq 0$ and $n \in \boldsymbol{N}$. Now simple calculations show that for every $f \in \mathscr{L}_+^1(\mu)$ we have:

$$\iint f(xy)\mu_t^H(dy)\mu(dx) \leq e^{(b+1)t}\int f\, d\mu \quad \text{and}$$
$$\iint f(xy)\pi_t^{(n)}(dy)\mu(dx) \leq \exp\{(b+1)t\ e^{(b+1)t(n)}\}\int fd\mu.$$

Hence for every $n \in \boldsymbol{N}$ we may define $T_t^{(n)} := \bar{T}_{\pi_t^{(n)}}$, $t \in \boldsymbol{R}_+$, and $N^{(n)} := \frac{1}{t(n)}[\bar{T}_{\mu_{t(n)}^H} - \bar{I}]$; and $(T_t^{(n)})_{t\geq 0}$ is a continuous operator semigroup on $L^1(\mu)$ with infinitesimal generator $(N^{(n)}, L^1(\mu))$ (cf. 1.4. and 1.5.).

Moreover, taking into account property c), we obtain for all $f \in \mathscr{L}_+^1(\mu)$:

$$\iint f(xy)\nu_t(dy)\mu(dx) \leq e^{(b+1)t}\int fd\mu.$$

Hence we may define $T_t := \bar{T}_{\nu_t}$, $t \in \boldsymbol{R}_+$; and $(T_t)_{t\geq 0}$ is a continuous operator semigroup on $L^1(\mu)$ with infinitesimal generator $(\overline{N^H}, D(\overline{N^H}))$ (observe property b) and 1.5.). Furthermore, by the Hille-Yosida theory, $D(N)$ and $(N^H - cI)D(N)$ are dense in $\mathscr{C}(G)$ and hence also in $\mathscr{L}^1(\mu)$ (all $c > 0$). Finally, by Lemma 4.2. (ii) we have for all $f \in D(N)$:

$$\overline{N^H}[f] = [N^H f] = \lim_{n\geq 1} N^{(n)}[f] \quad \text{in } L^1(\mu).$$

Hence the Trotter approximation theorem ([6], p. 5, Hilfssatz 1.1.5) yields for all $f \in \mathscr{L}^1(\mu)$ and $t \in \boldsymbol{R}_+$:

$$(*) \qquad T_t[f] = \lim_{n\geq 1} T_t^{(n)}[f] \quad \text{in } L^1(\mu).$$

Let us fix $t \in \boldsymbol{R}_+$. Since $\pi_t^{(n)}$ is concentrated on H, we have

$$T_t^{(n)}[1_H] = [\int (1_H)_y\ \pi_t^{(n)}(dy)] = [1_H] \quad (\text{all } n \in \boldsymbol{N}).$$

Thus $(*)$ yields $[1_H] = T_t[1_H] = [\int (1_H)_y\ \nu_t(dy)]$. But $\mu(H) > 0$ assures the existence of some $x \in H$ such that $1 = 1_H(x) = \int 1_H(xy)\nu_t(dy) = \nu_t(H)$. This proves property d).

4.4. PROPOSITION. *There exist a continuous convolution semigroup* $(\lambda_t)_{t\geq 0}$ *in* $\mathscr{M}^1(G)$ *and a measure* $\rho \in \mathscr{M}_+(G)$ *with the following properties:*

a) $\rho\leq\eta$ *and* $\rho(H)=0$.

b) $\lambda_t(H)=1$ *for all* $t\in\boldsymbol{R}_+$.

c) $(\mu_t)_{t\geq 0}=p((\lambda_t)_{t\geq 0};\ \rho)$.

PROOF. We define $\rho:=1_{\complement H}\cdot\sigma$, $\tau:=1_H\cdot\sigma$, and $(\lambda_t)_{t\geq 0}:=p((\nu_t)_{t\geq 0};\ \tau)$. First of all $\rho\leq\sigma\leq\eta$ (Lemma 3.5.), and $\rho(H)=0$, hence a). Since $(\eta-\sigma)+\tau=\eta-\rho\geq 0$, we have $\lambda_t\in\mathscr{M}^1(G)$ for all $t>0$ taking into account Lemma 4.3. and Proposition 2.5.

Since $p((\mu_t)_{t\geq 0};\ -\sigma)=(\nu_t)_{t\geq 0}=p((\lambda_t)_{t\geq 0};\ -\tau)$ (Lemma 4.3. and Corollary 2.4.) and since $\sigma-\tau=\rho$, we conclude $(\mu_t)_{t\geq 0}=p(p((\lambda_t)_{t\geq 0};\ -\tau);\ \sigma)$ $=p((\lambda_t)_{t\geq 0};\ \sigma-\tau)=p((\lambda_t)_{t\geq 0};\ \rho)$ (Corollary 2.4.). This proves c).

Moreover, by Lemma 2.1. we have $\lambda_t=e^{-t\tau(H)}\ \sum_{k\geq 0}\tau_k(t)$, where $\tau_0(t):=\nu_t$ and $\tau_k(t):=\int_0^t\nu_r*\tau*\tau_{k-1}(t-r)\ dr$ for all $k\in\boldsymbol{N}$ (and $t\in\boldsymbol{R}_+$). But $\nu_t(H)=1=\nu_t(G)$ (Lemma 4.3.) and $\tau(H)=\tau(G)$; hence $\tau_k(t)(H)=\tau_k(t)(G)$ for all $k\in\boldsymbol{Z}_+$ by induction, and thus $\lambda_t(H)=1$ (all $t\in\boldsymbol{R}_+$). This proves b).

4.5. COROLLARY. *The following assertions are equivalent*:

(i) $\eta(\complement H)=0$.

(ii) $\mu_t(H)=1$ *for all* $t\in\boldsymbol{R}_+$.

PROOF. "(i)$\Longrightarrow$(ii)" In view of a) of Proposition 4.4. we have $\rho=0$. Thus c) of Proposition 4.4. yields $\mu_t=\lambda_t$ for all $t>0$. Hence the assertion by b) of Proposition 4.4.

"(ii)$\Longrightarrow$(i)" Let $V\in\mathscr{V}(e)$. Then $\eta(\complement V)<\infty$; hence $\kappa:=1_{\complement V}\cdot\eta\in\mathscr{M}_+(G)$ and $\kappa\leq\eta$. Let $(\nu_t)_{t\geq 0}:=p((\mu_t)_{t\geq 0};\ -\kappa)$. By Proposition 2.5. we have $\nu_t\in\mathscr{M}^1(G)$ for all $t>0$. Moreover, $(\mu_t)_{t\geq 0}=p((\nu_t)_{t\geq 0};\ \kappa)$ (Corollary 2.4.) and $\mu_t\geq e^{-t\kappa(G)}\{\nu_t+\int_0^t\nu_r*\kappa*\nu_{t-r}\ dr\}$ (Lemma 2.1. (iii)). This yields first of all $\nu_t(\complement H)\leq e^{t\kappa(G)}\mu_t(\complement H)=0$ and hence $\nu_t(H)=1$ (all $t>0$). Consequently, $\nu_r*\kappa*\nu_{t-r}(\complement H)=\kappa(\complement H)$ if $0\leq r\leq t$. Hence $\kappa(\complement H)=(\int_0^1\nu_r*\kappa*\nu_{1-r}\ dr)(\complement H)\leq e^{\kappa(G)}\mu_1(\complement H)=0$ and thus $\eta(\complement H\cap\complement V)=\kappa(\complement H)=0$. Since $V\in\mathscr{V}(e)$ was arbitrary and since $\mathscr{V}(e)$ admits a countable basis this yields $\eta(\complement H)=0$.

4.6. THEOREM. *Let* $(\mu_t)_{t\geq 0}$ *be a continuous convolution semigroup in* $\mathscr{M}^1(G)$ *with Lévy measure* η. *Moreover, let H be a measurable subgroup of G such that* $\mu_t(H)>0$ *for all* $t\in\boldsymbol{R}_+$. *Then the following assertions are valid*:

(i) *There exist a unique continuous convolution semigroup* $(\lambda_t)_{t\geq 0}$ *in*

$\mathcal{M}^1(G)$ and a unique measure $\rho\in\mathcal{M}_+(G)$ with the following properties:

a) $\lambda_t(H)=1$ *for all* $t\in\boldsymbol{R}_+$ *and* $\rho(H)=0$.

b) $(\mu_t)_{t\geq 0}=p((\lambda_t)_{t\geq 0};\ \rho)$.

(ii) $\rho=1_{\complement H}.\ \eta$; *and* $(\lambda_t)_{t\geq 0}=p((\mu_t)_{t\geq 0};\ -1_{\complement H}.\ \eta)$.

(iii) $1_H.\ \eta$ *is the Lévy measure of* $(\lambda_t)_{t\geq 0}$; *and* $\eta(\complement H)<\infty$.

PROOF. The existence part of (i) follows readily from Proposition 4.4. The uniqueness part of (i) obviously follows from assertion (ii). Thus let ξ denote the Lévy measure of $(\lambda_t)_{t\geq 0}$. By Corollary 4.5. we have $\xi(\complement H)=0$. By Proposition 2.5. we have $\eta=\xi+\rho$. Since $\rho(H)=0$ this yields $1_{\complement H}.\ \eta=\rho$ and $1_H.\ \eta=\xi$; hence (iii) and the first part of (ii) are proved. The second part of (ii) now follows from property b) in (i) taking into account Corollary 2.4.

4.7. COROLLARY. *In the situation of Theorem* 4.6. *the following assertions are valid*:

(i) $\displaystyle\int 1_{\complement H}f\ d\eta=\lim_{t\downarrow 0}\frac{1}{t}\int 1_{\complement H}f\ d\mu_t$ *for all* $f\in\mathscr{C}(G)$; *in particular* $\displaystyle\eta(\complement H)=\lim_{t\downarrow 0}\frac{1}{t}\mu_t(\complement \mathrm{H})$.

(ii) $\displaystyle\int 1_H f d\eta=\lim_{t\downarrow 0}\frac{1}{t}\int 1_H f d\mu_t$ *for all* $f\in\mathscr{C}(G)$ *such that* $e\notin\mathrm{supp}(f)$.

(iii) $\eta(\complement H)$ *is the least real number* c *such that* $\mu_t(H)\geq e^{-ct}$ *for all* $t\in\mathbf{R}_+$.

PROOF. In view of (i) b) of Theorem 4.6. and of Lemma 2.1. we have for all $t>0$:

$$(*)\qquad \mu_t=e^{-t\rho(G)}\{\lambda_t+\rho_1(t)+\rho(t)\}$$

where $\rho_1(t):=\int_0^t\lambda_r*\rho*\lambda_{t-r}\ dr$ and where $\rho(t)\in\mathcal{M}_+(G)$ such that $\|\rho(t)\|\leq t^2\rho(G)^2e^{t\rho(G)}$ (cf. Corollary 2.2. (ii)). In view of (i) and (ii) of Theorem 4.6. we have $\lambda_r*\rho*\lambda_{t-r}(\complement H)=\rho(\complement H)=\eta(\complement H)$ if $0\leq r\leq t$; hence $\rho_1(t)(\complement H)=t\eta(\complement H)=\rho_1(t)(G)$. Thus $\rho_1(t)$ is supported by $\complement H$ (all $t>0$).

Now let $f\in\mathscr{C}(G)$. Then $(*)$ (together with (i) a) of Theorem 4.6.) yields:

$$\int 1_{\complement H}fd\mu_t=e^{-t\rho(G)}\Big\{\int fd\rho_1(t)+\int 1_{\complement H}fd\rho(t)\Big\}.$$

Taking into account Corollary 2.2. (i) we conclude (i) (since in $\mathcal{M}_+(G)$ pointwise convergence on $\mathscr{U}(G)$ and on $\mathscr{C}(G)$ respectively coincide).

Moreover, in view of (*), we have

$$\int 1_H f d\mu_t = e^{-t\rho(G)}\left\{\int f d\lambda_t + \int 1_H f d\rho(t)\right\}.$$

Since $1_H.\eta$ is the Lévy measure of $(\lambda_t)_{t\geq 0}$ (by (iii) of Theorem 4.6.) this yields (ii).

Finally, (iii) follows from (i) together with Lemma 1.6. (iv) and (vi).

4.8. REMARK. Corollary 4.7. (i) shows that the auxiliary measure σ constructed in Section 3 already equals $1_{\complement H}.\eta$.

5. Applications of the decomposition theorem

5.1. PROPOSITION. *Let G be a Polish group and H a measurable subgroup of G. Moreover let $(\mu_t)_{t\geq 0}$ be a normal continuous convolution semigroup in $\mathscr{M}^1(G)$ with Lévy measure η. Then the following assertions are valid:*

(i) If $\eta(\complement H)=\infty$ then $\mu_t(xH)=0$ and $\mu_t(Hx)=0$ for all $x\in G$ and $t\in \boldsymbol{R}_+^*$.

(ii) If $\eta(\complement H)=0$ then either

a) $\mu_t(xH)=0$ and $\mu_t(Hx)=0$ for all $x\in G$ and $t\in\boldsymbol{R}_+^*$; or

b) $\mu_t(x_tH)=1$ and $\mu_t(Hy_t)=1$ for appropriate $x_t\in G$ and $y_t\in G$ and for all $t\in\boldsymbol{R}_+^*$.

PROOF. Let $\nu_t:=\tilde{\mu}_t*\mu_t$ for all $t\in\boldsymbol{R}_+$. Then $(\nu_t)_{t\geq 0}$ is a symmetric continuous convolution semigroup in $\mathscr{M}^1(G)$ with Lévy measure $\xi:=\eta+\tilde{\eta}$. Hence $\xi(\complement H)=2\eta(\complement H)$.

Now we assume that $\mu_{t_0}(x_0H)>0$ *for certain* $x_0\in G$ and $t_0>0$. Then $\nu_{t_0}(H)=\tilde{\mu}_{t_0}*\mu_{t_0}((x_0H)^{-1}(x_0H))\geq\mu_{t_0}(x_0H)^2>0$.

[Moreover $0<\nu_{t_0}(H)=\mu_{t_0}*\tilde{\mu}_{t_0}(H)=\int\mu_{t_0}(Hy)\mu_{t_0}(dy)$ yields $\mu_{t_0}(Hy_0)>0$ for some $y_0\in G$. Hence it suffices to consider left H-cosets only.]

Consequently, in view of Corollary 1.9. (i), we have $\nu_t(H)>0$ for all $t>0$. Taking into account Theorem 4.6. (iii) we conclude $\xi(\complement H)<\infty$ and hence $\eta(\complement H)<\infty$. This proves (i).

Let us assume in addition that $\eta(\complement H)=0$ and hence $\xi(\complement H)=0$. Then Corollary 4.5. yields $\nu_t(H)=1$ for all $t>0$. Consequently, in view of $1=\nu_t(H)=\int\mu_t(xH)\mu_t(dx)$, there exists some $x_t\in G$ such that $\mu_t(x_tH)=1$. This proves (ii).

5.2. REMARKS. 1. If G is a locally compact group (not necessarily with a countable basis) and if H is a normal subgroup then Proposition 5.1. is due to A. Janssen ([8], Corollary 7).

2. Let G be a separable Banach space. Then Proposition 5. 1. is again due to A. Janssen ([8], Theorem 10 and Corollary 11).

5. 3. Let $(\mu_t)_{t\geq 0}$ be a Gaussian semigroup in $\mathscr{M}^1(G)$ i. e. $(\mu_t)_{t\geq 0}$ is a non-degenerate continuous convolution semigroup in $\mathscr{M}^1(G)$ with Lévy measure $\eta=0$, or equivalently, with $\lim_{t\downarrow 0}\frac{1}{t}\mu_t(\complement V)=0$ for all $V\in\mathscr{V}(e)$. Moreover, let H be a measurable subgroup of G. Then the following assertions are valid:

(i) If $\mu_t(H)>0$ for all $t>0$ then $\mu_t(H)=1$ for all $t=0$.
[This follows immediately from Corollary 4. 5.]

(ii) Let $t\longrightarrow\mu_t$ be norm continuous at some $t_0>0$. Then either $\mu_t(H)=0$ for all $t>0$ or $\mu_t(H)=1$ for all $t>0$.
[This follows from (i) taking into account Corollary 1. 9. (ii).]

(iii) Let $(\mu_t)_{t>0}$ be symmetric. Then either $\mu_t(xH)=0$ and $\mu_t(Hx)=0$ for all $x\in G$ and $t>0$ or $\mu_t(H)=1$ for all $t>0$.
[This follows from Proposition 5. 1. (ii) since $\mu_t(x_tH)=1$ implies $\mu_{2t}(H)\geq(\mu_t(x_tH))^2=1$ for all $t>0$.]

5. 4. Remarks. 1. If H is a normal subgroup then assertion (i) is the result of T. Byczkowski and A. Hulanicki ([4], Theorem).

2. If G is a Lie group then assertion (iii) has been proved by completely different methods in [11] (Theorem 5 (i)).

5. 5. For symmetric Gaussian semigroups we can obtain further information by applying the following concept: If $\mu\in\mathscr{M}^1(G)$ then $E(\mu):=\{x\in G: \varepsilon_x*\mu\approx\mu\}$ is called the set of equivalent (left) translates of μ. Then $E(\mu)$ is a measurable subgroup of G that is contained in every measurable subgroup H of G such that $\mu(H)=1$. Moreover, if $\nu\in\mathscr{M}^1(G)$, then $E(\mu)\subset E(\mu*\nu)$; and $\nu(E(\mu))=1$ implies $\nu*\mu\approx\mu$, hence $E(\nu*\mu)=E(\mu)$ (cf. [7], Chapter II).

Now let $(\mu_t)_{t\geq 0}$ be a symmetric Gaussian semigroup in $\mathscr{M}^1(G)$. Then, in view of assertion 5. 3. (iii), we have for every $t>0$ either $\mu_t(E(\mu_t))=0$ or $\mu_t(E(\mu_t))=1$; and in the second case we have $\mu_s(E(\mu_t))=1$ and thus $\mu_t\approx\mu_{t+s}$ for all $s\geq 0$.

Consequently, if we define $t_0:=\inf\{t>0: \mu_t(E(\mu_t))=1\}$, then $\mu_t(E(\mu_t))=1$ for all $t>t_0$ and $\mu_t(E(\mu_t))=0$ for all $t<t_0$. Moreover, if $H_m:=E(\mu_t)$ for a certain $t>t_0$, then $H_m=E(\mu_t)$ for all $t>t_0$; and H_m is the least measurable subgroup H of G such that $\mu_t(H)=1$ for some and hence for all $t>0$ (cf. [11], Theorem 5 (ii)).

5.6. Let $(\mu_t)_{t\geq 0}$ be a δ-semistable continuous convolution semigroup in $\mathcal{M}^1(G)$ with coefficient c i. e. δ is a continuous endomorphism of G and $c\in]0, 1[\cup]1, \infty[$ such that $\delta(\mu_t)=\mu_{ct}$ for all $t>0$ (cf. [12]). Then the Lévy measure η of $(\mu_t)_{t\geq 0}$ obeys the relation $\delta(\eta)=c\eta$.

Now let H be a measurable subgroup of G such that $\delta^{-1}(H)\subset H$ if $c<1$ and $\delta^{-1}(H)\supset H$ if $c>1$. Then $\eta(\complement H)=0$ or $\eta(\complement H)=\infty$.
[Let $c<1$ and $\delta^{-1}(H)\subset H$. Then $c\eta(\complement H)=\delta(\eta)(\complement H)=\eta(\complement\delta^{-1}(H))\geq \eta(\complement H)$; hence the assertion. The second case follows analogously.]
Hence the following assertions are valid:

(i) If $\mu_t(H)>0$ for all $t>0$ then $\mu_t(H)=1$ for all $t>0$.
[By (iii) of Theorem 4.6. we have $\eta(\complement H)<\infty$ and hence $\eta(\complement H)=0$. Thus the assertion by Corollary 4.5.]

(ii) Let $t\longrightarrow\mu_t$ be norm continuous at some $t_0>0$. Then either $\mu_t(H)=0$ for all $t>0$ or $\mu_t(H)=1$ for all $t>0$.
[This follows from (i) taking into account Corollary 1.9. (ii).]

(iii) If $(\mu_t)_{t\geq 0}$ is normal then we have a complete alternative in view of Proposition 5.1.

5.7. PROPOSITION. *For a continuous convolution semigroup $(\mu_t)_{t\geq 0}$ in $\mathcal{M}^1(G)$ the following assertions are equivalent:*

(i) *The semigroup $(\mu_t)_{t\geq 0}$ has a discrete part.*

(ii) *There exist a one-parameter group $(x_t)_{t\in\boldsymbol{R}}$ in G and a measure $\kappa\in \mathcal{M}_+(G)$ such that $(\mu_t)_{t\geq 0}=p((\varepsilon_{x_t})_{t\geq 0};\ \kappa)$.*

PROOF. "(i)$\Longrightarrow$(ii)" There exists a one-parameter group $(y_t)_{t\in\boldsymbol{R}}$ in G such that $\mu_t(\{y_t\})>0$ for all $t\geq 0$ ([10], Lemma 2). Thus $H:=\{y_t: t\in\boldsymbol{R}\}$ is a measurable subgroup of G such that $\mu_t(H)>0$ for all $t\geq 0$. Taking into account Theorem 4.6., there exist a continuous convolution semigroup $(\lambda_t)_{t\geq 0}$ in $\mathcal{M}^1(G)$ and a measure $\rho\in\mathcal{M}_+(G)$ such that $\lambda_t(H)=1$ for all $t\geq 0$, $\rho(H)=0$, and $(\mu_t)_{t\geq 0}=p((\lambda_t)_{t\geq 0};\ \rho)$.

Since the diffuse measures form a closed ideal in the Banach algebra $\mathcal{M}(G)$ we derive from Lemma 2.1. (ii), (iii) that the convolution semigroup $(\lambda_t)_{t\geq 0}$ has a discrete part too. But $(\lambda_t)_{t\geq 0}$ can be considered as a continuous convolution semigroup in $\mathcal{M}^1(\bar{H})$; and $\bar{H}$ is a commutative Polish group. Hence there exist a one-parameter group $(x_t)_{t\in\boldsymbol{R}}$ in $\bar{H}$ and a measure $\gamma\in\mathcal{M}_+(\bar{H})$ such that $\lambda_t=\varepsilon_{x_t}*e(t\gamma)$ for all $t\geq 0$ ([10], Corollary 2 of Lemma 2). Consequently, $(\lambda_t)_{t\geq 0}=p((\varepsilon_{x_t})_{t\geq 0};\ \gamma)$ in $\mathcal{M}(\bar{H})$ and thus in $\mathcal{M}(G)$. Finally, Corollary 2.4. now yields $(\mu_t)_{t\geq 0}=p((\lambda_t)_{t\geq 0};\rho)=p((\varepsilon_{x_t})_{t\geq 0};\ \gamma+\rho)$.

"(ii)$\Longrightarrow$(i)" immediately follows from Lemma 2.1.

5.8. REMARK. If G is a locally compact group (not necessarily with a countable basis) then this result has already been known ([10], Theorem 2).

5.9. COROLLARY. *Every Gaussian semigroup* $(\mu_t)_{t\geq 0}$ *is diffuse.*

PROOF. If $(\mu_t)_{t\geq 0}$ would have a discrete part then it would be of the form described in Proposition 5.7. (ii). Since the Lévy measure of $(\varepsilon_{x_t})_{t\geq 0}$ is zero, $\varkappa$ would be the Lévy measure of $(\mu_t)_{t\geq 0}$ (Proposition 2.5.). Hence $\varkappa=0$ (cf. 5.3.) and thus $(\mu_t)_{t\geq 0}=(\varepsilon_{x_t})_{t\geq 0}$. But this is a contradiction since Gaussian semigroups by definition are non-degenerate.

5.10. COROLLARY. *For a continuous convolution semigroup* $(\mu_t)_{t\geq 0}$ *in* $\mathscr{M}^1(G)$ *the following assertions are equivalent*:

(i) *The semigroup* $(\mu_t)_{t\geq 0}$ *is discrete.*

(ii) *There exist a one-parameter group* $(x_t)_{t\in\mathbf{R}}$ *in* G *and a discrete measure* $\gamma\in\mathscr{M}_+(G)$ *such that* $\mu_t=\varepsilon_{x_t}*e(t\gamma)=e(t\gamma)*\varepsilon_{x_t}$ *for all* $t\in\mathbf{R}_+$.

(iii) *There exist a one-parameter group* $(x_t)_{t\in\mathbf{R}}$ *in* G *and a discrete measure* $\gamma\in\mathscr{M}_+(G)$ *such that* $(\mu_t)_{t\geq 0}=p((\varepsilon_{x_t})_{t\geq 0};\gamma)$ *and* $x_tx=xx_t$ *for all* $x\in G$ *with* $\gamma(\{x\})>0$ *and for all* $t\in\mathbf{R}_+$.

[Taking into account Proposition 5.7. the proof is literally the same as the proof of Theorem 3 in [10].]

Note Added in Proof.

There is a recent paper by H. Byczkowska and T. Byczkowski (Zero-one law for subgroups of paths of group valued stochastic processes. To appear in Studia Math.) that contains a result similar to Theorem 4.6; but for symmetric convolution semigroups only.

References

[1] ARENDT, W., Chernoff, P. R., Kato, T.: A generalization of dissipativity and positive semigroups. J. Operator Theory 8, 167-180 (1982).

[2] BAUER, H.: Wahrscheinlichkeitstheorie und Grundzüge der Maßtheorie. 3. Aufl. Berlin-New York: De Gruyter 1978.

[3] BAXENDALE, P.: Brownian motions in the diffeomorphism group I. Compositio Math. 53, 19-50 (1984).

[4] BYCZKOWSKI, T., Hulanicki, A.: Gaussian measure of normal subgroups. Ann. Probab. 11, 685-691 (1983).

[5] BYCZKOWSKI, T., Żak, T.: Decomposition of convolution semigroups of probability measures on groups. In: Probability Measures on Groups VII. Proceedings, Oberwolfach 1983, pp. 23-35. Lecture Notes in Math. Vol. 1064. Berlin-Heidelberg-New York-Tokyo: Springer 1984.

[6] HAZOD, W.: Stetige Faltungshalbgruppen und erzeugende Distributionen. Lecture Notes in Math. Vol. 595. Berlin-Heidelberg-New York: Springer 1977.

[7] JANSSEN, A.: Zulässige Translationen von Faltungshalbgruppen. Dissertation, Dortmund 1979.

[8] JANSSEN, A.: Zero-one laws for infinitely divisible probability measures on groups. Z. Wahrscheinlichkeitstheorie verw. Gebiete 60, 119-138 (1982).

[9] JANSSEN, A.: A survey about zero-one laws for probability measures on linear spaces and locally compact groups. In: Probability Measures on Groups VII. Proceedings, Oberwolfach 1983, pp. 551-563. Lecture Notes in Math. Vol. 1064. Berlin-Heidelberg-New York-Tokyo: Springer 1984.

[10] SIEBERT, E.: Diffuse and discrete convolution semigroups of probability measures on topological groups. Rendiconti di Matematica Roma (2) Ser. VII, 1, 219-236 (1981).

[11] SIEBERT, E.: Absolute continuity, singularity, and supports of Gauss semigroups on a Lie group. Monatsh. Math. 93, 239-253 (1982).

[12] SIEBERT, E.: Semistable convolution semigroups on measurable and topological groups. Ann. Inst. H. Poincaré 20, 147-164 (1984).

[13] SIEBERT, E.: Jumps of stochastic processes with values in a topological group. Probab. Math. Statist. 5, 197-209 (1985).

[14] YOSIDA, K.: Functional Analysis. 3rd ed. Berlin-Heidelberg-New York: Springer 1971.

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