Hokkaido Mathematical Journal Vol. 16 (1987), p. 145~150

A note on a theorem of Wada

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(Received March 10, 1986, Revised November 10, 1986)

In his papers [6], [7], [8] T. Wada introduced and studied two new invariants, which may be associated with a p-block B of a finite group G. If B is considered as an ideal in the group algebra FG, where F is an algebraically closed field of characteristic $p>0$ and P is a p-Sylow subgroup of G, then $m(B)$ (or $n(B)$) is the number of indecomposable summands of B when restricted to the diagonal group $\Delta(P)$ (or $P\times P$). A main result of [8] was that if $D=\delta(B)$ is a defect group of B and if $D\leq Z(P)$, then

(*) $|P:D|k(B)\leq m(B)$,

where $k(B)$ is the number of ordinary characters in B. Some examples were given to show that (*) does not hold in general. In this note we prove a result which has as a consequence the following obvious generalization of Wada's theorem:

Let B be a p-block with $\delta(B)=D$. If P_1 is a p-Sylow subgroup of $DC_G(D)$ then

$$|P_1 : D|k(B)\leq m(B).$$

By Brauer's second main theorem on blocks $k(B)$ may be decomposed according to the subsections of B (See [4], IV, § 6). To prove our main result we decompose $m(B)$ in a similar way and compare the corresponding summands of $k(B)$ and $m(B)$.

Let P be a p-Sylow subgroup of G and B a block with $\delta(B)=D$. Then there is an integer $v(B)$ such that

(1) $\dim_F B=p^{2a-d}v(B)$

(see [2] or [8]). A subsection for G is a pair (π, b_π) where π is a p-element in G and b_π a block of $C_G(\pi)$. If $b_\pi^G=B$, we call (π, b_π) a B-subsection.

If (π, b_π) is a subsection, define

(2) $m(\pi, b_\pi) := \dfrac{1}{|P|}|\pi^G\cap P| \dim b_\pi.$

(here π^G is the G-conjugacy class containing π).

Of course $m(\pi, b_\pi)$ is invariant uder conjugation. In particular, using (1) and (2)

(3) $m(1, B)=p^{a-d}v(B)$

LEMMA 1 *In the above notation* $m(\pi, b_\pi)\in\boldsymbol{N}$.

PROOF Write the nonempty set $\pi^G\cap P$ as a disjoint union of P-conjugacy classes

$$\pi^G\cap P=x_1^P\cup x_2^P\cup \cdots \cup x_t^P$$

so that

$$|\pi^G\cap P|=\sum_{i=1}^{t}|P:C_P(x_i)|.$$

Applying (1) to b_π we get from the definition

$$m(\pi, b_\pi)=\sum_{i=1}^{t}\frac{|C_G(\pi)|_p|C_G(\pi)|_p}{|C_P(x_i)||\delta(b_\pi)|}v(b_\pi),$$

where generally n_p is the p-part of the integer n. Since $x_i\sim_G\pi$ $|C_P(x_i)|$ divides $|C_G(\pi)|_p=|C_G(x_i)|_p$. Moreover $|\delta(b_\pi)|$ divides $|C_G(\pi)|_p$, since $\delta(b_\pi)$ is a p-subgroup of $C_G(\pi)$. This proves the lemma. We note

COROLLARY 2 *In the above notation*

$$\frac{|C_G(\pi)|_p}{|\delta(b_\pi)|} \text{ divides } m(\pi, b_\pi).$$

Moreover

$$m(\pi, b_\pi)\geq\frac{|C_G(\pi)|_p}{|\delta(b_\pi)|}v(b_\pi).$$

PROPOSITION 3 *Let B be a p-block. We have*

$$m(B)=\sum_{(\pi,b_\pi)_G}m(\pi, b_\pi),$$

where the sum is on a full set of representatives for the G-conjugacy classes of B-subsections.

PROOF If σ is a p-element in G, let $d_B(\sigma)$ be defined by

$$d_B(\sigma)=\sum_{b_\sigma}\dim_F b_\sigma$$

where $b_\sigma \in Bl(C_G(\sigma), B)$. Note that $d_B(\sigma) \neq 0$, if and only if $\sigma \in D^G$, where $D = \delta(B)$. By [8] (2.3) we have

$$(4) \quad m(B) = \frac{1}{|P|} \sum_{\sigma} d_B(\sigma)$$

where $\sigma \in D^G \cap P$. Let $\pi_1, \pi_2, \dots, \pi_t$ be a set of representatives of the G-conjugacy classes, whose intersection with D is nonempty. Then (4) implies that

$$m(B) = \frac{1}{|P|} \sum_{i=1}^{t} |\pi_i^G \cap P| d_B(\pi_i).$$

Now the proposition follows from the definition of $m(\pi, b_\pi)$.
We need another lemma.

LEMMA 4 *Let (π, b_π) be a B-subsection and let $D = \delta(B)$, $D_\pi = \delta(b_\pi)$. Then*

$$\frac{|C_G(\pi)|_p}{|D_\pi|} \geq \min_{\pi' \in \pi^G \cap D} \frac{|C_G(\pi')|_p}{|C_G(\pi') \cap D|}$$

PROOF Let $X = \{x \in G \mid \pi^x \in D\}$. By a remark of Brauer (see [1] p. 901) there exists an $x \in X$, such that

$$\pi^x \in D_\pi^x \leq D \cap C_G(\pi^x)$$

Thus, since for $x \in X$ $\pi^x \in \pi^G \cap D$, we have

$$\begin{aligned} |D_\pi| &\leq \max_{x \in X} |D \cap C_G(\pi^x)| \\ &= \max_{\pi' \in \pi^G \cap D} |D \cap C_G(\pi')| \end{aligned}$$

From this the lemma follows.

THEOREM 5 *Let B be a p-block, $D = \delta(B)$. Let $\alpha \geq 0$ such that*

$$p^\alpha = \min_{\pi \in D} |C_G(\pi) : C_G(\pi) \cap D|_p.$$

Then

$$p^\alpha k(B) \leq m(B).$$

PROOF If b is a block, let $l(b)$ be the number of modular irreducible characters in b. It was shown in [6] and [7] that generally $l(b) \leq v(b)$. We have then (summing as in Proposition 3)

$$
\begin{aligned}
m(B) &= \sum_{(\pi, b_\pi)_G} m(\pi, b_\pi) && \text{(by Proposition 3)}\\
&\geq \sum_{(\pi, b_\pi)_G} \frac{|C_G(\pi)|_p}{|\delta(b_\pi)|} v(b_\pi) && \text{(by Corollary 2)}\\
&\geq \sum_{(\pi, b_\pi)_G} \frac{|C_G(\pi)|_p}{|\delta(b_\pi)|} l(b_\pi) && \\
&\geq p^\alpha \sum_{(\pi, b_\pi)_G} l(b_\pi) && \text{(by Lemma 4)}\\
&= p^\alpha k(B) && \text{(by [4], IV, 6. 5)}
\end{aligned}
$$

Corollary 6 *In the notation of Theorem* 5 *we have*

$$|DC_G(D) : D|_p\, k(B) \leq m(B)$$

Proof Let $\pi \in D$. Consider the following diagram of subgroups

$$
\begin{tikzcd}
& C_G(\pi) \arrow[d, dash] & \\
& C_G(D)\ (D\cap C_G(\pi)) \arrow[dl, dash] \arrow[dr, dash] & \\
C_G(D) \arrow[dr, dash] & & D\cap C_G(\pi) \arrow[dl, dash] \\
& Z(D) &
\end{tikzcd}
$$

We see that

$$
\begin{aligned}
&|DC_G(D) : D|_p = |C_G(D) : Z(D)|_p\\
&= |C_G(D)\ (D\cap C_G(\pi)) : D\cap C_G(\pi)|_p\\
&\leq |C_G(\pi) : D\cap C_G(\pi)|_p
\end{aligned}
$$

Thus $|DC_G(D) : D|_p \leq p^\alpha$, so Corollary 6 follows from Theorem 5.

Corollary 7 *If* $DC_P(D) = P$ *(e. g. if* $D \leq Z(P)$*) then*

$$|P : D|\, k(B) \leq m(B).$$

Additional remarks

1. A similar argument as above may be applied to $n(B)$; put

$$n(\pi, b_\pi) := \frac{1}{|P|^2}\, |\pi^G \cap P|^2 \dim b_\pi .$$

Then $n(1, B)=\frac{1}{|D|} v(B)$.

The result thus obtained is that $k(B) \leq |D| n(B)$, which can also be deduced from the facts that $k(B) \leq |D| l(B)$ (wellknown) and that $l(B) \leq n(B)$ ([7]).

2. From the definition it is clear that

$$\sum_{B} m(B) = |\{g_P \mid g \in G\}|,$$

the number of P-orbits of G. Similary we have

$$\sum_{B} m(1, B) = |G : P|.$$

3. One may consider the possibility of decomposing $m(B)$ according to the vertices of the indecomposable summands of $B_{\Delta(P)}$. This would correspond roughly to the decomposition of $k(B)$ according to the multiplicities of lower defect groups.

4. Finally we mention a formal analogy between Brauer's " matrices of contribution " and some other matrices associated to the integers $m(\pi, b_\pi)$. This may be of interest because the best known general inequalities for the invariants $k(B)$ and $l(B)$ (see [3] and [5]) were proved using the " contributions ". If $s=(\pi, b_\pi)$ is a B-subsection and χ_i an irreducible character in B, then in [1] Brauer defined a class function $\chi_i^{(s)}$, such that

$$\chi_i = \sum_{(s)_G} \chi_i^{(s)}.$$

Then he called the inner product

$$(\chi_i^{(s)}, \chi_j^{(s)})_G = m_{ij}^{(s)}$$

the contribution of s to the inner product $(\chi_i, \chi_j)_G$. The $k(B) \times k(B)$ matrix $M^{(s)}$

$$M^{(s)} = (m_{ij}^{(s)})$$

has the properties that $M^{(s)}M^{(s)} = M^{(s)}$ and $\mathrm{Tr}(M^{(s)}) = l(b_\pi)$ (where Tr denotes trace). If

$$A^{(s)} = (a_{ij}^{(s)})$$

where

$$a_{ij}^{(s)}=(\chi_{iP}^{(s)}, \chi_{jP}^{(s)})_P$$

then $A^{(s)}A^{(s)}=m(\pi, b_\pi)A^{(s)}$ and $\mathrm{Tr}A^{(s)}=m(\pi, b_\pi)$ Moreover

$$A^{(s)}M^{(s)}=A^{(s)}.$$

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