Hokkaido Mathematical Journal Vol. 17 (1988), p. 289～332

The F. and M. Riesz theorem on certain transformation groups

Hiroshi YAMAGUCHI

(Received January 27, 1987, Revised January 28, 1988)

§ 0. Introduction.

Let $\boldsymbol{T}$ be the circle group. Then the classical F. and M. Riesz theorem is stated as follows: Let μ be bounded regular (complex-valued) measure on $\boldsymbol{T}$. Suppose μ is of analytic type (i.e., $\hat{\mu}(n)=\int_{0}^{2\pi} e^{-inx}d\mu(e^{ix})=0$ for $n<0$). Then

(A) μ is absolutely continuous with respect to the Lebesgue measure on $\boldsymbol{T}$.

Moreover, it is well-known that a measure of analytic type has the following important property:

(B) μ is quasi-invariant (i.e., $|\mu|*\delta_x \ll |\mu|$ for all $x\in\boldsymbol{T}$).

Helson and Lowdenslager extended (A) as follows:

THEOREM 0.1 (cf. [20, 8.2.3. Theorem]).
Let G be a compact abelian group with ordered dual $\hat{G}$. Let μ be a bounded regular measure on G that is of analytic type (i.e., $\hat{\mu}(\gamma)=0$ for $\gamma<0$). Then

(i) *μ_a and μ_s are of analytic type;*
(ii) *$\hat{\mu}_s(0)=0$,*

where μ_a and μ_s are the absolutely continuous part of μ and th singular part of μ respectively.

On the other hand, as for (A) and (B), deLeeuw and Glicksberg ([2]) extended the classical F. and M. Riesz theorem to compact abelian groups with certain ordered duals. Moreover, as an extension of the result of deLeeuw and Glicksberg, Forelli ([7]) extended the F. and M. Riesz theorem to a (topological) transformation group such that the reals $\boldsymbol{R}$ acts on a locally compact Hausdorff space. In fact, he proved the following theorems.

THEOREM 0.2 ([7, Theorem 3]). *Let $(\boldsymbol{R}, S)$ be a transformation group such that the reals $\boldsymbol{R}$ acts on a locally compact Hausdorff space S. Let*

μ be a bounded regular complex Baire measure on S. Suppose μ is an analytic measure. Then μ is quasi-invariant.

THEOREM 0.3 ([7, Theorem 4]). *Let $(\boldsymbol{R}, S)$ and μ be as in Theorem* 0.2. *Suppose $(\boldsymbol{R}, S)$ is equipped with the one-paramater group $\{T_t\}_{t\in\boldsymbol{R}}$ of homemorphisms on S. Suppose μ is an analytic measure. Then $T_t\mu$ moves continuously in $M(S)$.*

THEOREM 0.4 ([7, Theorem 5]). *Let $(\boldsymbol{R}, S)$ and μ be as in Theorem* 0.2. *Let σ be a positive Radon measure on S that is quasi-invariant. Suppose μ is an analytic measure. Then both $sp(\mu_a)$ and $sp(\mu_s)$ are contained in sp (μ), where μ_a is the absolutely continuous part of μ with respect to σ and μ_s is the singular part of μ with respect to σ respectively. In particular, if μ is an analytic measure, then μ_a and μ_s are also analytic measures.*

In this paper we give results corresponding to Theorems 0.2-0.4 on a transformation group with certain conditions (conditions (C. I) and (C. II)). We also extend Theorem 0.1 to such a transformation group. In section 1 we state definitions and our theorems of this paper. In section 2 we state several lemmas concerning properties of measures on certan transformation groups. In sections 3-5, we give the proofs of our theorems. In section 6, we shall state that if (G, X) is a transformation group such that a compact abelian group G acts freely on a locally compact Hausdorff space X or a transformation group such that a compact abelian group G acts on a locally compact metric space X, then (G, X) satisfies the conditions ((C. I) and (C. II)).

§ 1. Notations and results.

Let X be a locally compact Hausdorff space. Let $C_0(X)$ be the Banach space of continuous functions on X which vanish at infinity, and let $M(X)$ be the Banach space of complex-valued bounded regular Borel measures on X with the total variation norm. Let $M^+(X)$ be the set of nonnegative measures in $M(X)$. For $\mu\in M(X)$ and $f\in L^1(|\mu|)$, we often write $\mu(f)=\int_X f(x)d\mu(x)$. Let X' be another locally compact Hausdorff space, and let $S: X\to X'$ be a continuous map. For $\mu\in M(X)$, let $S(\mu)\in M(X')$ be the continuous image of μ under S. We denote by $\mathscr{B}(X)$ the σ-algebra of Borel sets in X. $\mathscr{B}_0(X)$ means the σ-algebra of Baire sets in X. In this paper we employ the definition of Baire sets in [6] or [19]. That is, $\mathscr{B}_0(X)$ is the σ-algebra generated by compact G_δ-sets in X.

Let G be a compact group and X a locally compact Hausdorff space. Suppose there exists a continuous map $(g, x)\to g\cdot x$ from $G\times X$ onto X with the following properties:

(1.1) $x \to g \cdot x$ is a homeomorphism on X for each $g \in G$ and $e \cdot x = x$, where e is the identity element in G,

(1.2) $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ for $g_1, g_2 \in G$ and $x \in X$.

Then a pair (G, X) is called a (topological) transformation group such that G acts on X. We say G acts freely on X if for any $x \in X$, $g \to g \cdot x$ is a one-to-one mapping. When G is commutative, we write customarily 0, $g_1 + g_2$ and $-g$ instead of e, $g_1 g_2$ and g^{-1} respectively. For a closed normal subgroup H of G and $x \in X$, the set $H(x) = \{h \cdot x : h \in H\}$ is called an orbit of x under H. Then $X/H = \{H(x) : x \in X\}$ is also a locally compact Hausdorff space with respect to the quotient topology. Define an action of G/H on X/H by $gH \cdot H(x) = H(g \cdot x)$. Then, by this action, $(G/H, X/H)$ becomes a transformation group (cf. [14, Theorem 2.9, p. 61]). Moreover, if G acts freely, then G/H also acts freely.

Let $Y = X/G$ be the quotient space, and let $\pi : X \to Y$ be the canonical map. Then, since G is compact, Y is a locally compact Hausdorff space and π is an open continuous map. A (Borel) measure σ on X is called quasi-invariant if $|\sigma|(F) = 0$ implies $|\sigma|(g \cdot F) = 0$ for all $g \in G$. $M(G)$ and $L^1(G)$ denote the measure algebra and the group algebra respectively. m_G means the Haar measure of G. By $M_a(G)$ we denote the set of measures in $M(G)$ which are absolutely continuous with respect to m_G. Then by the Radon-Nikodym theorem we can identify $M_a(G)$ with $L^1(G)$. When G is commutative, for a subset E of $\hat{G}$, $M_E(G)$ denotes the space of measures in $M(G)$ whose Fourier-Stieltjes transforms vanish off E. Put $L^1_E(G) = M_E(G) \cap L^1(G)$. For $\mu \in M(G)$, $\hat{\mu}$ denotes the Fourier-Stieltjes transform of μ. For a closed subgroup H of G, $H^\perp$ means the annihilator of H.

Let f be a Baire measurable function on X. Then

(1.3) $(g, x) \to f(g \cdot x)$ is a Baire function on $G \times X$.

In fact, let $L : G \times X \to G \times X$ be the map defined by $L(g, x) = (g, g \cdot x)$, and let $\pi_X : G \times X \to X$ be the projection. Then L is a homeomorphism. Set $\mathscr{F} = \{F \subset X : \pi_X^{-1}(F) \in \mathscr{B}_0(G \times X)\}$. Then $\mathscr{F}$ is a σ-algebra containing all compact G_δ-sets in X. Hence $\pi_X^{-1}(F)$ belongs to $\mathscr{B}_0(G \times X)$ for every $F \in \mathscr{B}_0(X)$. Thus, since $f(g \cdot x) = f \circ \pi_X \circ L(g, x)$, (1.3) follows easily.

For $\mu \in M(X)$, $\lambda \in M(G)$ and a bounded Baire function f on X, we define convolutions $\lambda * \mu$ and $\lambda * f$ as follows:

(1.4) $$\lambda * f(x) = \int_G f(g^{-1} \cdot x)\, d\lambda(g),$$

$$(1.5) \qquad \lambda*\mu(h)=\int_X\int_G h(g\cdot x)\,d\lambda(g)\,d\mu(x)=\int_G\int_X h(g\cdot x)\,d\mu(x)\,d\lambda(g)$$

for $h\in C_0(X)$. Then $\lambda*\mu\in M(X)$ and $\lambda*f$ is a bounded Baire function on X (see Lemma 2.1 later). We note that (1.5) holds for all bounded Baire functions h on X. Moreover we have

$$(1.6) \qquad \|\lambda*\mu\|\leqq\|\lambda\|\ \|\mu\| \text{ and } \|\lambda*f\|_\infty\leqq\|\lambda\|\ \|f\|_\infty.$$

For $\xi\in M(G)$, we also have

$$(1.7) \qquad \xi*(\lambda*\mu)=(\xi*\lambda)*\mu.$$

In fact, for $h\in C_0(X)$, we have

$$\begin{aligned}\xi*(\lambda*\mu)(h)&=\int_G\int_X h(g\cdot x)\,d(\lambda*\mu)(x)\,d\xi(g)\\&=\int_G\int_G\int_X h(g\cdot(s\cdot x))\,d\mu(x)\,d\lambda(s)\,d\xi(g)\\&=\int_X\int_G\int_G h((gs)\cdot x)\,d\lambda(s)\,d\xi(g)\,d\mu(x)\\&=(\xi*\lambda)*\mu(h).\end{aligned}$$

DEFINITION 1.1. Suppose G is a compact abelian group and $\mu\in M(X)$. Let $J(\mu)$ be the collection of all $f\in L^1(G)$ with $f*\mu=0$. We define the spectrum of μ, which is denoted by $\mathrm{sp}(\mu)$, as follows:

$$\mathrm{sp}(\mu)=\bigcap_{f\in J(\mu)}\hat{f}^{-1}(0).$$

REMARK 1.1. (I) By (1.6) and (1.7), $J(\mu)$ becomes a closed ideal in $L^1(G)$. Hence, since G is a compact abelian group, $J(\mu)$ coincides with $L^1_{E^c}(G)$, where $E=\mathrm{sp}(\mu)$ (cf. [20, p. 158-159]).

(II) By (I) we have

(II. 1) $\gamma\in\mathrm{sp}(\mu)$ if and only if $\gamma*\mu\neq0$.(1)

In particular, for μ, $\nu\in M(X)$, we have

(II. 2) $\mathrm{sp}(\mu+\nu)\subset\mathrm{sp}(\mu)\cup\mathrm{sp}(\nu)$.

In the sequel, (G, X) will denote a transformation group in which G is a compact abelian group and X is a locally compact Hausdorff space except in § 6. Before stating our results, we introduce two conditions (C. I) and (C. II).

(C. I) For any closed subgroup H of G with $H^\perp$ countable and any $\mu\in M^+(X/H)$, put $\eta=\pi(\mu)$, where π: $X/H\to Y=X/H/G/H(\cong X/G)$ is the canonical map. Then there exists a family $\{\lambda_y\}_{y\in Y}$ of measures in $M^+(X/H)$ with the following properties:

(1) $y\to\lambda_y(f)$ is η-measurable for each bounded Baire function f on X/H,

(2) $\|\lambda_y\|=1$,

(3) $\mathrm{supp}(\lambda_y)\subset\pi^{-1}(y)$,

(4) $\mu(f)=\int_Y\lambda_y(f)\,d\eta(y)$ for each bounded Baire function f on X/H.

(C. II) Let H be any closed subgroup of G with $H^\perp$ countable. Let Y and π be as in (C. I). Let $\eta\in M^+(Y)$, and let $\{\lambda_y^1\}_{y\in Y}$ and $\{\lambda_y^2\}_{y\in Y}$ be families of measures in $M(X/H)$ satisfying the following properties:

(1) $y\to\lambda_y^i(f)$ is η-integrable for each bounded Baire function f on X/H $(i=1,2)$,

(2) $\mathrm{supp}(\lambda_y^i)\subset\pi^{-1}(y)$ $(i=1,2)$,

(3) $\int_Y\lambda_y^1(f)\,d\eta(y)=\int_Y\lambda_y^2(f)\,d\eta(y)$ for all bounded Baire functions f on X/H.

Then $\lambda_y^1=\lambda_y^2$ η-a. a. $y\in Y$.

Now we state our results. We take the definition of Radon measure from [6].

THEOREM 1.1. *Assume that (G,X) satisfies conditions (C. I) and (C. II). Let P be a subsemigroup of $\hat{G}$ such that $P\cup(-P)=\hat{G}$. Let σ be a positive Radon measure on X that is quasi-invariant. Let $\mu\in M(X)$, and let $\mu=\mu_a+\mu_s$ be the Lebesgue decomposition of μ with respect to σ. Suppose $sp(\mu)\subset P$. Then both $sp(\mu_a)$ and $sp(\mu_s)$ are also contained in P. If, addition, $P\cap(-P)=\{0\}$ and $\pi(|\mu|)\ll\pi(\sigma)$, then $sp(\mu_s)\subset P\backslash\{0\}$, where $\pi: X\to X/G$ is the canonical map.*

THEOREM 1.2. *Let (G,X) be as in Theorem 1.1. Let E be a subset of G satisfying the following:*

(*)(2) *for any $0\neq\lambda\in M_E(G)$, $|\lambda|$ and m_G are mutually absolutely continuous.*

Let μ be a measure in $M(X)$ with $sp(\mu)\subset E$. Then μ is quasi-invariant.

DEFINITION 1.2. Let G be a LCA group, and let E be a closed subset

(1) On the left hand side, γ means a character of G. And we consider γ as an element in $L^1(G)$ on the right hand side. More exactly, on the right hand side, $\gamma*\mu$ means $(\gamma m_G)*\mu$.

of $\hat{G}$. E is called a Riesz set if $M_E(G)\subset L^1(G)$.

THEOREM 1.3. *Let (G, X) be as in Theorem 1.1. Suppose $E\subset\hat{G}$ is a Riesz set. Let μ be a measure in $M(X)$ with $sp(\mu)\subset E$. Then*

$$\lim_{g\to 0}\|\mu-\delta_g*\mu\|=0,$$

where δ_g denotes the point mass at $g\in G$.

THEOREM 1.4. *Let (G, X) be as in Theorem 1.1. Let σ be a positive Radon measure on X that is quasi-invariant, and let E be a Riesz set in $\hat{G}$. Let μ be a measure in $M(X)$ with $sp(\mu)\subset E$. Then both $sp(\mu_a)$ and $sp(\mu_s)$ are contained in $sp(\mu)$, where $\mu=\mu_a+\mu_s$ is the Lebesgue decomposition of μ with respect to σ.*

Theorem 1.1 may be considered an extension of Theorem 0.1. And by the classical F. and M. Riesz theorem, we can consider that Theorems 1.2-1.4 are compact analogues of Theorems 0.2-0.4. If we regard $\boldsymbol{R}^+$ (non-negative real numbers) as a semigroup with $\boldsymbol{R}^+\cup(-\boldsymbol{R}^+)=\boldsymbol{R}$, Theorem 1.1 is also considered as one corresponding to Theorem 0.4.

If (G, X) is a transformation group such that a compact abelian group G acts freely on a locally compact Hausdorff space X or a transformation group such that a compact abelian group G acts on a locally compact metric space X, then (G, X) satisfies conditions (C. I) and (C. II) (see Theorem 6.4 and Remark 6.1). Hence the following corollary is obtained from Theorems 1.1-1.4.

COROLLARY 1.1. *Let (G, X) be a transformation group such that a compact abelian group G acts freely on a locally compact Hausdorff space X or a transformation group such that a compact abelian group G acts on a locally compact metric space X. Then the conclusions of Theorems 1.1-1.4 hold.*

Next we give several examples of transformation groups that satisfy the conditions in Corollary 1.1.

EXAMPLE 1.1. Let X be a locally compact group and G a compact abelian subgroup of X. Then $(g, x)\to gx$ is a continuous map from $G\times X$ onto X satisfying (1.1) and (1.2). Thus G and X form a transformation group. Exidently G acts freely on X because X is a group.

Let $\boldsymbol{C}$ be the complex plane. A subset $F\subset\boldsymbol{C}$ is said to be circular if

(2) If $E\subset\hat{G}$ satisfies condtion (*), then E is a Riesz set. However the converse is false in general (see Remark 5.1).

$e^{i\theta}z \in F$ for all $z \in F$ and $\theta \in \boldsymbol{R}$.

EXAMPLE 1.2. Let $\boldsymbol{T}$ be the circle group (i.e., $\boldsymbol{T}=\{e^{i\theta}: \theta \in [0, 2\pi)\}$). Let $X \subset \boldsymbol{C}$ be a locally compact set that is cirular. Then $(e^{i\theta}, z) \to e^{i\theta}z$ is a continuous map from $\boldsymbol{T} \times X$ onto X satisfying (1.1) and (1.2). Thus $\boldsymbol{T}$ and X form a transformation group. Evidently X is a locally compact metric space. If the origin is not contained in X, $\boldsymbol{T}$ acts freely on X.

EXAMPLE 1.3. Let (G_i, X_i) be a transformation group such that a compact abelian group G_1 acts freely on a locally compact Hausdorff space $X_i (i=1, 2)$. Then $((g_1, g_2), (x_1, x_2)) \to (g_1 \cdot x_1, g_2 \cdot x_2)$ is a continuous map from $G_1 \oplus G_2 \times X_1 \times X_2$ onto $X_1 \times X_2$ satisfying (1.1) and (1.2). Thus $G_1 \oplus G_2$ and $X_1 \times X_2$ form a transformation group. It is easy to see that $G_1 \oplus G_2$ acts freely on $X_1 \times X_2$.

EXAMPLE 1.4. Let (G_i, X_i) be a transformation group such that a compact abelian group G_i acts on a locally compact metric space X_i $(i=1, 2)$. Then $G_1 \oplus G_2$ and $X_1 \times X_2$ form a transformation group as in the previous example.

EXAMPLE 1.5. For each $i \in \boldsymbol{N}$ (the natural numbers), let (G_i, X_i) be a transformation group such that a compact abelian group G_i acts on a compact metric space X_i. Then $\prod_{i \in \boldsymbol{N}} G_i$ and $\prod_{i \in \boldsymbol{N}} X_i$ form a transformation group by the action $\langle g_i \rangle \cdot \langle x_i \rangle = \langle g_i \cdot x_i \rangle$, where $\langle g_i \rangle \in \prod_{i \in \boldsymbol{N}} G_i$ and $\langle x_i \rangle \in \prod_{i \in \boldsymbol{N}} X_i$.

EXAMPLE 1.6. Let Λ be an index set. For each $\alpha \in \Lambda$, let (G_α, X_α) be a transformation group such that a compact abelian group G_α acts freely on a compact Hausdorff space X_α. Then $\prod_{\alpha \in \Lambda} G_\alpha$ and $\prod_{\alpha \in \Lambda} X_\alpha$ form a transformation group by the action $\langle g_\alpha \rangle \cdot \langle x_\alpha \rangle = \langle g_\alpha \cdot x_\alpha \rangle$, where $\langle g_\alpha \rangle \in \prod_{\alpha \in \Lambda} G_\alpha$ and $\langle x_\alpha \rangle \in \prod_{\alpha \in \Lambda} X_\alpha$. Evidently $\prod_{\alpha \in \Lambda} G_\alpha$ acts freely on $\prod_{\alpha \in \Lambda} X_\alpha$.

Combining Corollary 1.1 with Proposition 4.3, which will be stated in section 4, we obtain the following corollaries.

COROLLARY 1.2. *Let Λ be an index set. For each $\alpha \in \Lambda$, let X_α be a compact circular set in $\boldsymbol{C}$. Let $X = \prod_{\alpha \in \Lambda} X_\alpha$ and $G = \prod_{\alpha \in \Lambda} \boldsymbol{T}_\alpha$, where $\boldsymbol{T}_\alpha = \boldsymbol{T}$ for all $\alpha \in \Lambda$. Let (G, X) be the transformation group defined by $\langle e^{it_\alpha} \rangle \cdot \langle x_\alpha \rangle = \langle e^{it_\alpha} x_\alpha \rangle$ for $\langle e^{it_\alpha} \rangle \in G$ and $\langle x_\alpha \rangle \in X$. Put $E = \{\langle m_\alpha \rangle \in \hat{G}: m_\alpha \geq 0$ for all $\alpha \in \Lambda\}$. Suppose that the origin is not contained in X_α for every $\alpha \in \Lambda$ or Λ is a countable set. Let μ be a measure in $M(X)$ with $sp(\mu) \subset E$.*

Then μ is quasi-invariant.

PROOF. Since (G, X) satisfies the conditions in Corollary 1.1, the corollary follows from Corollary 1.1 (cf. Theorem 1.2) and Proposition 4.3.

COROLLARY 1.3. *Let (G, X) and E be as in Corollary 1.2. Let S be a Sidon set in $\hat{G}$. Let μ be a measure in $M(X)$ with $sp(\mu)\subset E\cup S$, and let σ be a positive Radon measure on X that is quasi-invariant. Then the following hold.*

(i) $\lim_{g\to 0}\|\mu-\delta_g*\mu\|=0$;

(ii) *let $\mu=\mu_a+\mu_s$ be the Lebesgue decomposition of μ with respect to σ. Then both $sp(\mu_a)$ and $sp(\mu_s)$ are contained in $E\cup S$.*

PROOF. By Proposition 4.3 and [16, Corollary 4], $E\cup S$ is a Riesz set. Hence the corollary follows from Corollary 1.1 (cf. Theorems 1.3 and 1.4).

For a locally compact Hausdorff space X, $C_c(X)$ denotes the space of all complex-valued continuous functions on X with compact supports. Let $C_c^{\boldsymbol{R}}(X)$ be the set of all real-valued functions in $C_c(X)$. Before we close this section, we state several lemmas and propositions.

Let (G, X) be a transformation group such that a compact abelian group G acts on a locally compact Hausdorff space X. Let $\pi: X\to X/G$ be the canonical map. For $x\in X$, we define a map $B_x: G\to G\cdot x$ $(\subset X)$ by

$$B_x(g)=g\cdot x. \tag{1.8}$$

Then B_x is a continuous map. Put $G_x=\{g\in G: g\cdot x=x\}$. Then G_x is a closed subgroup of G. Let $\Pi_x: G\to G/G_x$ be the canonical map. We define a map $\tilde{B}_x: G/G_x\to G\cdot x$ by

$$\tilde{B}(g+G_x)=g\cdot x. \tag{1.9}$$

Then we have

$$B_x=\tilde{B}_x\circ\Pi_x. \tag{1.10}$$

$$\begin{tikzcd} G \arrow[r, "B_x"] \arrow[d, "\Pi_x"'] & G\cdot x \\ G/G_x \arrow[ur, "\tilde{B}_x"'] & \end{tikzcd}$$

Fig. I

PROPOSITION 1.1. *For each $x\in X$, $\tilde{B}_x$: $G/G_x\to G\cdot x$ is a homeomor-*

phism.

PROOF. It is easy to see that $\tilde{B}_x$ is bijective. Since B_x is continuous, $\tilde{B}_x$ is also continuous. Moreover, since G/G_x is compact, $\tilde{B}_x^{-1}$ is continuous. This completes the proof.

PROPOSITION 1.2. *For $x\in X$, $\lambda\in M(G/G_x)$ and $E\subset\hat{G}$, the following are equivalent.*

(I) $\lambda\in M_{E\cap G_x^{\perp}}(G/G_x)$;

(II) $sp(\tilde{B}_x(\lambda))\subset E.$

PROOF. For $g\in G$, let $\dot{g}$ denote the coset $g+G_x$. Then

$$
\begin{aligned}
&\mathrm{sp}(\tilde{B}_x(\lambda))\subset E\\
&\Longleftrightarrow \int_G\int_{G\cdot x} h(g\cdot y)\,d\tilde{B}_x(\lambda)(y)f(g)\,dm_G(g)=f*\tilde{B}_x(\lambda)(h)=0\\
&\qquad\qquad \text{for all } f\in L^1_{E^c}(G) \text{ and } h\in C_0(X)\\
&\Longleftrightarrow \int_G\int_{G/G_x} h(g\cdot\tilde{B}_x(\dot{s}))\,d\lambda(\dot{s})f(g)\,dm_G(g)=0\\
&\qquad\qquad \text{for all } f\in L^1_{E^c}(G) \text{ and } h\in C_0(X)\\
&\Longleftrightarrow \int_G\int_{G/G_x} h(\tilde{B}_x(\dot{g}+\dot{s}))\,d\lambda(\dot{s})f(g)\,dm_G(g)=0\\
&\qquad\qquad \text{for all } f\in L^1_{E^c}(G) \text{ and } h\in C_0(X).
\end{aligned}
$$

By Proposition 1.1, we note that $C_0(X)|_{G\cdot x}=C(G\cdot x)\cong C(G/G_x)$. Hence

$$
\begin{aligned}
&\Longleftrightarrow \int_G\int_{G/G_x} F(\dot{g}+\dot{s})\,d\lambda(\dot{s})f(g)\,dm_G(g)=0\\
&\qquad\qquad \text{for all } f\in L^1_{E^c}(G) \text{ and } F\in C(G/G_x)\\
&\Longleftrightarrow \int_{G/G_x}\int_{G/G_x}\int_{G_x} F(\dot{g}+\dot{s})f(g+u)\,dm_{G_x}(u)\,dm_{G/G_x}(\dot{g})\,d\lambda(\dot{s})=0\\
&\qquad\qquad \text{for all } f\in L^1_{E^c}(G) \text{ and } F\in C(G/G_x)\\
&\Longleftrightarrow \lambda*\Pi_x(f)(F)=\int_{G/G_x}\int_{G/G_x} F(\dot{g}+\dot{s})\Pi_x(f)(\dot{g})\,dm_{G/G_x}(\dot{g})\,d\lambda(\dot{s})=0\\
&\qquad\qquad \text{for all } f\in L^1_{E^c}(G) \text{ and } F\in C(G/G_x)\\
&\Longleftrightarrow \lambda\in M_{E\cap G_x^{\perp}}(G/G_x).
\end{aligned}
$$

This completes the proof.

The following two propositions are well-known.

PROPOSITION 1.3. *Let X be a locally compact Hausdorff space. For μ, $\nu\in M^+(X)$, let $\mu|_{\mathscr{B}_0(X)}$ and $\nu|_{\mathscr{B}_0(X)}$ be the restrictions of μ and ν to $\mathscr{B}_0(X)$ respectively. Then the following hold.*

(I) *The following are equivalent.*

(I.1) $\nu\ll\mu$;

(I.2) $\nu|_{\mathscr{B}_0(X)} \ll \mu|_{\mathscr{B}_0(X)}$.

(II) *The following are equivalent.*

(II.1) $\mu \perp \nu$;

(II.2) $\mu|_{\mathscr{B}_0(X)} \perp \nu|_{\mathscr{B}_0(X)}$.

PROPOSITION 1.4. *Let X be a locally compact Hausdorff space and μ a measure in $M(X)$. Then there exists a unimodular Baire function h on X such that $\mu = h|\mu|$.*

LEMMA 1.1. *Suppose $\sigma \in M^+(X)$ is quasi-invariant. Then σ and $m_G * \sigma$ are mutually absolutely continuous.*

PROOF. Let F be a Baire set in X with $m_G*\sigma(F)=0$. Then

$$0=\int_G\int_X \chi_F(g\cdot x)\,d\sigma(x)\,dm_G(g)$$
$$=\int_G \sigma((-g)\cdot F)\,dm_G(g),$$

which yields $\sigma((-g)\cdot F)=0$ for some $g \in G$. Hence $\sigma(F)=0$. Hence, by Proposition 1.3, we have $\sigma \ll m_G*\sigma$. Next suppose $\sigma(F)=0$ for a Baire set F in X. Then $\sigma((-g)\cdot F)=0$ for all $g \in G$, and so $m_G*\sigma(F)=\int_G \sigma((-g)\cdot F)\,dm_G(g)=0$. It follows from Proposition 1.3 that $m_G*\sigma \ll \sigma$, and the proof is complete.

LEMMA 1.2. *Let $x \in X$ and $g \in G$. Then the following hold.*

(I) *For μ, $\lambda \in M(G)$, we have $B_x(\mu*\lambda)=\mu*B_x(\lambda)$.*

(II) *For $\lambda \in M(G)$, we have*

$$B_{g\cdot x}(\lambda)=B_x(\delta_g*\lambda)=\delta_g*B_x(\lambda).$$

In particular, $B_{g\cdot x}(m_G)=B_x(m_G)$.

PROOF. (I): For any $f \in C_0(X)$, we have

$$\mu*B_x(\lambda)(f)=\int_G\int_X f(u\cdot y)\,dB_x(\lambda)(y)\,d\mu(u)$$
$$=\int_G\int_G f(u\cdot(s\cdot x))\,d\lambda(s)\,d\mu(u)$$
$$=\int_G f(u\cdot x)\,d\mu*\lambda(u)$$
$$=B_x(\mu*\lambda)(f).$$

Hence we have $\mu*B_x(\lambda)=B_x(\mu*\lambda)$.

(II): For any $f \in C_0(X)$, we have

$$
\begin{aligned}
B_{g\cdot x}(\lambda)(f) &= \int_G f(B_{g\cdot x}(u))\,d\lambda(u) \\
&= \int_G f((u+g)\cdot x)\,d\lambda(u) \\
&= \int_G f(u\cdot x)\,d\delta_g * \lambda(u) \\
&= B_x(\delta_g * \lambda)(f),
\end{aligned}
$$

which together with (I) yields $B_{g\cdot x}(\lambda)=B_x(\delta_g*\lambda)=\delta_g*B_x(\lambda)$. This completes the proof.

DEFINITION 1.3. For $x\in X$, put $\dot{x}=\pi(x)$. And define $m_{\dot{x}}\in M^+(X)$ by $m_{\dot{x}}=B_x(m_G)$.

REMARK 1.2. By Lemma 1.2 (II), $m_{\dot{x}}$ is well-defined. That is, $B_y(m_G)=B_x(m_G)$ for every $y\in\pi^{-1}(\dot{x})$.

LEMMA 1.3. *Let μ be a measure in $M^+(X)$ such that $\|\mu\|=1$ and supp $(\mu)\subset\pi^{-1}(\dot{x})$ for some $\dot{x}\in X/G$. Then $m_G*\mu=m_{\dot{x}}$.*

PROOF. Let $x\in\pi^{-1}(\dot{x})$, *and let* $\Pi_x: G\to G/G_x$ be the canonical map. Since $\mathrm{supp}(\mu)\subset\pi^{-1}(\dot{x})$, it follows from Proposition 1.1 that there exists a probability measure $\lambda\in M^+(G/G_x)$ such that $\mu=\tilde{B}_x(\lambda)$. Let ξ be a probability measure in $M^+(G)$ such that $\lambda=\Pi_x(\xi)$. Then by (1.10) we have $\mu=\tilde{B}_x(\Pi_x(\xi))=B_x(\xi)$. Hence, by Lemma 1.2 (I), we have

$$
\begin{aligned}
m_G*\mu &= m_G*B_x(\xi) \\
&= B_x(m_G*\xi) \\
&= B_x(m_G) \\
&= m_{\dot{x}}.
\end{aligned}
$$

This completes the proof.

PROPOSITION 1.5. *For $x\in\pi^{-1}(\dot{x})$, $\tilde{B}_x(m_{G/G_x})=m_{\dot{x}}$.*

PROOF. For a Borel set $B\subset G\cdot x$, put $\tilde{F}=\tilde{B}_x^{-1}(B)$ and $F=B_x^{-1}(B)$. It follows from (1.10) that $F=\Pi_x^{-1}(\tilde{F})$. Hence we have

$$
\begin{aligned}
m_{\dot{x}}(B) &= B_x(m_G)(B) \\
&= m_G(F) \\
&= \int_{G/G_x}\int_{G_x}\chi_F(s+t)\,dm_{G_x}(t)\,dm_{G/G_x}(\dot{s}) \\
&= \int_{G/G_x}\chi_{\tilde{F}}(\dot{s})\,dm_{G/G_x}(\dot{s}) \\
&= m_{G/G_x}(\tilde{F})
\end{aligned}
$$

$$=\tilde{B}_x(m_{G/G_x})(B).$$

This completes the proof.

For a locally compact Hausdorff space X, $\mathscr{B}_0(X)$ is the smallest σ-algebra with respect to which every function in $C_c(X)$ is measurable. Hence, by [4, 21 Theorem, 41-I], we get the following proposition.

PROPOSITION 1.6. *Let X be a locally compact Hausdorff space. Let $\mathscr{H}$ be a vector space of bounded real-valued functions on X, which contains the constants, is closed under uniform convergence and has the following property: for every uniformly bounded increasing sequence of positive functions $f_n \in \mathscr{H}$, the function $f=\lim_{n\to\infty} f_n$ belongs to $\mathscr{H}$. Suppose $\mathscr{H}\supset C_c^{\boldsymbol{R}}(X)$. Then $\mathscr{H}$ contains all bounded real-valued Baire measurable functions on X.*

§ 2. Several lemmas.

In this section we give several lemmas, which are used for proving our theorems later on. For locally compact Hausdorff spaces X_1 and X_2, $\mathscr{B}_0(X_1\times X_2)$ in general does not coincide with $\mathscr{B}_0(X_1)\times\mathscr{B}_0(X_2)$ (cf. [6, ch. 7, Exercise 31, p. 224]). However the following lemma holds. We give its proof for completeness.

LEMMA 2.1. *Let X_1 and X_2 be locally compact Hausdorrff spaces, and let $\mu\in M^+(X_1)$ and $\nu\in M^+(X_2)$. Then, for each bounded Baire function f on $X_1\times X_2$, we have*

(i) $x_1\to\int_{X_2} f(x_1, x_2)\,d\nu(x_2)$ *is Baire measurable on X_1, and*

(ii) $x_2\to\int_{X_1} f(x_1, x_2)\,d\mu(x_1)$ *is Baire measurable on X_2.*

PROOF. Let $\mathscr{H}=\{f(x_1, x_2)$: bounded real-valued functions on $X_1\times X_2$ satisfying (i) and (ii)$\}$. Then $\mathscr{H}$ is a vector space, which is closed under uniform convergence and contains the constants and $C_c^{\boldsymbol{R}}(X_1\times X_2)$. Moreover, for every uniformly bounded increasing sequence of positive functions $f_n\in\mathscr{H}$, the function $f=\lim_{n\to\infty} f_n$ belongs to $\mathscr{H}$. Hence, by Proposition 1.6, $\mathscr{H}$ contains all bounded real-valued Baire functions on $X_1\times X_2$. Thus, for every bounded Baire function f on $X_1\times X_2$, (i) and (ii) hold. This completes the proof.

LEMMA 2.2. *Let $\eta\in M^+(X/G)$ and $\lambda\in M(G)$. Let $\{\mu_{\dot{x}}\}_{\dot{x}\in X/G}$ be a family of measures in $M(X)$ such that $\dot{x}\to\mu_{\dot{x}}(f)$ is η-measurable for each bounded Baire function f on X. Then $\dot{x}\to\lambda*\mu_{\dot{x}}(f)$ is η-measurable for each* *bounded Baire function f on X.*

PROOF. Since $(g, x)\to f(g, x)$ is a Baire measurable function on $G\times X$, it follows from Lemma 2.1 that $x\to\int_G f(g\cdot x)d\lambda(g)$ is a bounded Baire function on X. Hence $\dot{x}\to\lambda*\mu_{\dot{x}}(f)=\int_X\int_G f(g\cdot x)d\lambda(g)d\mu_{\dot{x}}(x)$ is η-measurable. This completes the proof.

LEMMA 2.3. *Let $\mu\in M(X)$ and $\eta\in M^+(X/G)$. Let $\{\mu_{\dot{x}}\}_{\dot{x}\in X/G}$ be a family in $M(X)$ with the following properties:*

(1) $\dot{x}\to\mu_{\dot{x}}(f)$ *is η-integrable for each bounded Baire function f on X,*

(2) $\mu(f)=\int_{X/G}\mu_{\dot{x}}(f)d\eta(\dot{x})$ *for each bounded Baire function f on X.*

Then, for $\lambda\in M(G)$, the following hold.

(I) $\dot{x}\to\lambda*\mu_{\dot{x}}(f)$ *is an η-integrable function for each bounded Baire function f on X;*

(II) $\lambda*\mu(f)=\int_{X/G}\lambda*\mu_{\dot{x}}(f)d\eta(\dot{x})$ *for each bounded Baire function f on X.*

PROOF. (I): Since $(g, x)\to f(g\cdot x)$ is a bounded Baire function on $G\times X$, it follows from Lemma 2.1 that $x\to\int_G f(g\cdot x)d\lambda(g)$ is a bounded Baire function on X. Hence (1) implies that $\dot{x}\to\lambda*\mu_{\dot{x}}(f)=\int_X\int_G f(g\cdot x)d\lambda(g)d\mu_{\dot{x}}(x)$ is η-integrable.

(II): Since $x\to\int_G f(g\cdot x)d\lambda(g)$ is a bounded Baire function on X, it follows from (2) that

$$\begin{aligned}\lambda*\mu(f)&=\int_X\int_G f(g\cdot x)d\lambda(g)d\mu(x)\\&=\int_{X/G}\int_X\int_G f(g\cdot x)d\lambda(g)d\mu_{\dot{x}}(x)d\eta(\dot{x})\\&=\int_{X/G}\lambda*\mu_{\dot{x}}(f)d\eta(\dot{x}).\end{aligned}$$

This completes the proof.

From Lemma 2.4 through Lemma 2.8, we assume that (G, X) is a transformation group such that a metrizable compact abelian group G acts on a locally compact Hausdorff space X. For $\dot{x}\in X/G$, let $m_{\dot{x}}$ be the measure defined in Definition 1.3, and let $\pi: X\to X/G$ be the canonical map.

Moreover, from Lemma 2.4 through Lemma 2.8, we assume that (G, X) satisfies the following conditions (D. I) and (D. II).

(D. I) For any $\mu \in M^+(X)$, put $\eta = \pi(\mu)$. Then there exists a family $\{\lambda_{\dot{x}}\}_{\dot{x} \in X/G}$ of measures in $M^+(X)$ with the following:

(1) $\dot{x} \to \lambda_{\dot{x}}(f)$ is η-measurable for each bounded Baire function f on X,

(2) $\|\lambda_{\dot{x}}\| = 1$,

(3) $\mathrm{supp}(\lambda_{\dot{x}}) \subset \pi^{-1}(\dot{x})$,

(4) $\mu(f) = \int_{X/G} \lambda_{\dot{x}}(f) d\eta(\dot{x})$ for each bounded Baire function f on X.

(D. II) Let $\nu \in M^+(X/G)$. Suppose $\{\lambda^1_{\dot{x}}\}_{\dot{x} \in X/G}$ and $\{\lambda^2_{\dot{x}}\}_{\dot{x} \in X/G}$ are families of measures in $M(X)$ with the following properties:

(1) $\dot{x} \to \lambda^i_{\dot{x}}(f)$ is a ν-integrable function for each bounded Baire function f on X $(i=1, 2)$,

(2) $\mathrm{supp}(\lambda^i_{\dot{x}}) \subset \pi^{-1}(\dot{x})$ $(i=1, 2)$,

(3) $\int_{X/G} \lambda^1_{\dot{x}}(f) d\nu(\dot{x}) = \int_{X/G} \lambda^2_{\dot{x}}(f) d\nu(\dot{x})$ for all bounded Baire functions f on X.

Then we have $\lambda^1_{\dot{x}} = \lambda^2_{\dot{x}}$ ν-a.a. $\dot{x} \in X/G$.

Let $\mu \in M(X)$ and $\eta \in M^+(X/G)$. By an η-disintegration of μ, we mean a family $\{\lambda_{\dot{x}}\}_{\dot{x} \in X/G}$ of measures in $M(X)$ satisfying (1)′ $\dot{x} \to \lambda_{\dot{x}}(f)$ is η-integrable for each bounded Baire function f on X and (3)-(4) in (D. I). If, in addition, $\eta = \pi(|\mu|)$ and $\|\lambda_{\dot{x}}\| = 1$ for all $\dot{x} \in X/G$, then we call $\{\lambda_{\dot{x}}\}_{\dot{x} \in X/G}$ a canonical disintegration of μ. Thus condition (D. I) says that each $\mu \in M^+(X)$ has a canonical disintegration $\{\mu_{\dot{x}}\}_{\dot{x} \in X/G}$ with $\mu_{\dot{x}} \in M^+(X)$.

REMARK 2.1. For $\mu \in M^+(X)$, let $\{\lambda_{\dot{x}}\}_{\dot{x} \in X/G}$ be a canonical disintegration of μ. Then $\lambda_{\dot{x}} \in M^+(X)$ η-a.a. $\dot{x} \in X/G$.

REMARK 2.2. (i) For $\mu \in M(X)$, it follows from Proposition 1.4 that there exists a unimodular Baire function h on X such that $\mu = h|\mu|$. Suppose $\{\lambda_{\dot{x}}\}_{\dot{x} \in X/G}$ is a canonical disintegration of $|\mu|$. Then $\{h\lambda_{\dot{x}}\}_{\dot{x} \in X/G}$ is a canonical disintgration of μ. Hence the following are equivalent.

(1) every $\mu \in M^+(X)$ has a canonical disintegration;

(2) every $\mu \in M(X)$ has a canonical disintegration.

(ii) If (G, X) satisfiese (D. I), then every $\mu \in M(X)$ has a canonical disintegration.

LEMMA 2.4. *Let $\mu \in M(X)$ and $\eta \in M^+(X/G)$. Let $\{\lambda_{\dot{x}}\}_{\dot{x} \in X/G}$ be an η-disintegration of μ. Then $\{|\lambda_{\dot{x}}|\}_{\dot{x} \in X/G}$ is an η-disintegration of $|\mu|$. In* *particular,*

(1) $$\|\mu\|=\int_{X/G}\|\lambda_{\dot{x}}\|d\eta(\dot{x}).$$

PROOF. Let h be a unimodular Baire function on X such that $\mu=h|\mu|$. Then $\{\bar{h}\lambda_{\dot{x}}\}_{\dot{x}\in X/G}$ is an η-disintegration of $|\mu|$. Therefore, to establish the desired result, it will suffice to prove that $\lambda_{\dot{x}}\geq 0$ for η-a.a. $\dot{x}\in X/G$ assuming that $\mu\geq 0$. So suppose $\mu\geq 0$. By (D. I), μ has a canonical disintegration $\{\mu_{\dot{x}}\}_{\dot{x}\in X/G}$. By Remark 2. 1, $\mu_{\dot{x}}$ is a probability measure for $\pi(\mu)$-a.a. $\dot{x}\in X/G$. Moreover,

(2) $$\int_{X/G}\lambda_{\dot{x}}(f)\,d\eta(\dot{x})=\mu(f)=\int_{X/G}\mu_{\dot{x}}(f)\,d\pi(\mu)(\dot{x})$$

for each bounded Baire function f on X. We claim that $\pi(\mu)\ll\eta$. To see this, pick any Baire set $B\subset X/G$ with $\eta(B)=0$. Since $\mathrm{supp}(\lambda_{\dot{x}})\subset\pi^{-1}(\dot{x})$, $\lambda_{\dot{x}}(\chi_B\circ\pi)=0$ for $\dot{x}\notin B$. Hence we have, by (2) with $f=\chi_B\circ\pi$,

$$\begin{aligned}\pi(\mu)(B)&=\mu(\chi_B\circ\pi)\\&=\int_{X/G}\lambda_{\dot{x}}(\chi_B\circ\pi)\,d\eta(\dot{x})\\&=\int_B\lambda_{\dot{x}}(\chi_B\circ\pi)\,d\eta(\dot{x})+\int_{B^c}\lambda_{\dot{x}}(\chi_B\circ\pi)\,d\eta(\dot{x})\\&=0,\end{aligned}$$

which establishes our claim. Finally let ϕ be the nonnegative Radon-Nikodym derivative of $\pi(\mu)$ with respect to η. Then $\pi(\mu)=\phi\eta$ by our claim. So (2) ensures that

$$\int_{X/G}\lambda_{\dot{x}}(f)\,d\eta(\dot{x})=\int_{X/G}[(\phi\circ\pi)\mu_{\dot{x}}](f)\,d\eta(\dot{x})$$

for each bounded Baire function f on X. Therefore we have

$$\lambda_{\dot{x}}=(\phi\circ\pi)\mu_{\dot{x}}\geq 0 \text{ for } \eta\text{-a.a. } \dot{x}\in X/G$$

by condition (D. II). This completes the proof.

LEMMA 2. 5. *Let $\sigma\in M^+(X)$ be a quasi-invariant measure. Let $\mu\in M^+(X)$ and $\eta\in M^+(X/G)$. Let $\{\mu_{\dot{x}}\}_{\dot{x}\in X/G}$ be an η-disintegration of μ with $\mu_{\dot{x}}\in M^+(X)$. Then the following hold.*

(I) *If $\eta\ll\pi(\sigma)$, then the following are equivalent.*

(I .1) $\mu\ll\sigma$;

(I .2) $\mu_{\dot{x}}\ll m_{\dot{x}}$ *η-a.a. $\dot{x}\in X/G$.*

(II) *If $\eta\ll\pi(\sigma)$, then the following are equivalent.*

(II.1) $\mu \perp \sigma$;

(II.2) $\mu_{\dot{x}} \perp m_{\dot{x}}$ η-a.a. $\dot{x} \in X/G$.

PROOF. We first prove (I). (I.2)$\Longrightarrow$(I.1): Let E be a Baire set in X with $\sigma(E)=0$. Let $\{\sigma_{\dot{x}}\}_{\dot{x}\in X/G}$ be a canonical disintegration of σ. Then by Lemmas 1.1 and 2.3 we have

$$\begin{aligned} 0 &= m_G * \sigma(E) \\ &= \int_{X/G} m_G * \sigma_{\dot{x}}(E)\, d\pi(\sigma)(\dot{x}); \end{aligned}$$

hence

$$m_G * \sigma_{\dot{x}}(E) = 0 \ \pi(\sigma)\text{-a.a. } \dot{x} \in X/G.$$

Thus Lemma 1.3 and the hypothesis yield

$$\mu(E) = \int_{X/G} \mu_{\dot{x}}(E)\, d\eta(\dot{x}) = 0,$$

which together with Proposition 1.3 shows $\mu \ll \sigma$.

(I.1)$\Longrightarrow$(I.2): By Lemmas 1.1, 1.3 and 3.3, we have

(1) $\mu \ll m_G * \sigma$, and

(2) $$m_G * \sigma(f) = \int_{X/G} m_{\dot{x}}(f)\, d\pi(\sigma)(\dot{x})$$

for each bounded Baire function f on X. Since $\eta \ll \pi(\sigma)$, there is a nonnegative real-valued Baire function F on X/G such that $\eta = F\pi(\sigma)$. Then

(3) $\dot{x} \to F(\dot{x})\mu_{\dot{x}}(f)$ is a $\pi(\sigma)$-integrable function

and

(4) $$\mu(f) = \int_{X/G} F(\dot{x})\mu_{\dot{x}}(f)\, d\pi(\sigma)(\dot{x})$$

for each bounded Baire function f on X. Moreover,

(5) $F(\dot{x})\mu_{\dot{x}} \ll m_{\dot{x}}$ $\pi(\sigma)$-a.a. $\dot{x} \in X/G$.

In fact, since $\mu \ll m_G * \sigma$, there exists a nonnegative real-valued Baire function K on X such that $\mu = K m_G * \sigma$. It follows from Lemma 2.2 and (2) that $\dot{x} \to m_{\dot{x}}(K)$ is $\pi(\sigma)$-integrable. Hence there exists a $\pi(\sigma)$-null set $\tilde{E}$ in X/G such that $\|K m_{\dot{x}}\| < \infty$ for $\dot{x} \notin \tilde{E}$. We define a family $\{V_{\dot{x}}\}_{\dot{x}\in X/G}$ of measures in $M^+(X)$ by

(6) $$V_{\dot{x}} = \begin{cases} K m_{\dot{x}} & \text{for } \dot{x} \notin \tilde{E} \\ 0 & \text{for } \dot{x} \in \tilde{E} \end{cases}.$$

Then $\{V_{\dot{x}}\}_{\dot{x}\in X/G}$ is a $\pi(\sigma)$-disintegration of μ. Hence, by (3)-(4) and the hypothesis (D. II), we get

$$V_{\dot{x}}=F(\dot{x})\mu_{\dot{x}} \quad \pi(\sigma)\text{-a.a. } \dot{x}\in X/G,$$

which shows that (5) holds. By (5), we have $\mu_{\dot{x}}\ll m_{\dot{x}}$ η-a.a. $\dot{x}\in X/G$.

Next we prove (II). By (2) and (4), $\{m_{\dot{x}}-F(\dot{x})\mu_{\dot{x}}\}_{\dot{x}\in X/G}$ is a $\pi(\sigma)$-disintegration of $m_G*\sigma-\mu$. It follows from Lemma 2. 4 that

$$(7) \qquad \|m_G*\sigma-\mu\|=\int_{X/G}\|m_{\dot{x}}-F(\dot{x})\mu_{\dot{x}}\|d\pi(\sigma)(\dot{x}).$$

Notice that $m_G*\sigma\perp\mu$ if and only if $\|m_G*\sigma-\mu\|=\|m_G*\sigma\|+\|\mu\|$ since all of the measures σ, m_G and μ are nonnegative. It follows from (7) that $m_G*\sigma\perp\mu$ if and only if

$$(8) \qquad \|m_{\dot{x}}-F(\dot{x})\mu_{\dot{x}}\|=\|m_{\dot{x}}\|+\|F(\dot{x})\mu_{\dot{x}}\|$$

for $\pi(\sigma)$-a.a. $\dot{x}\in X/G$. But (8) is obvious for all $\dot{x}$ with $F(\dot{x})=0$. Therefore, since $\eta=F\pi(\sigma)$, (8) holds if and only if $m_{\dot{x}}\perp F(\dot{x})\mu_{\dot{x}}$ for η-a.a. $\dot{x}\in X/G$. Hence (II) follows from Lemma 1. 1. This completes the proof.

LEMMA 2. 6. *Let $\mu\in M(X)$ and $\eta\in M^+(X/G)$. Let E be a subset of $\hat{G}$. Let $\{\mu_{\dot{x}}\}_{\dot{x}\in X/G}$ be an η-disintegration of μ. If $sp(\mu)\subset E$, then*

$$(1) \qquad sp(\mu_{\dot{x}})\subset E \quad \eta\text{-a.a. } \dot{x}\in X/G.$$

PROOF. Let $\mathscr{A}\subset C(G)$ be a countable dense set in $L^1_{E^c}(G)$. For $f_n\in\mathscr{A}$, it follows from Lemma 2. 3 that $\dot{x}\to f_n*\mu_{\dot{x}}(h)$ is η-integrable and $0=f_n*\mu(h)=\int_{X/G}f_n*\mu_{\dot{x}}(h)d\eta(\dot{x})$ for each bounded Baire function h on X. Moreover, we have $\mathrm{supp}(f_n*\mu_{\dot{x}})\subset\pi^{-1}(\dot{x})$. Hence the hypothesis (D. II) yields $f_n*\mu_{\dot{x}}=0$ η-a.a. $\dot{x}\in X/G$, and so

$$f*\mu_{\dot{x}}=0 \quad \eta\text{-a.a. } \dot{x}\in X/G$$

for all $f\in L^1_{E^c}(G)$. Thus we get (1), and the proof is complete.

LEMMA 2. 7. *Let $\sigma\in M^+(X)$ be a quasi-invariant measure and E a subset of $\hat{G}$. Then the following hold.*

(I) *Each $\mu\in M(X)$ can be uniquely represented as $\mu=\mu_1+\mu_2$, where μ_1 and μ_2 are measures in $M(X)$ such that $\pi(|\mu_1|)\ll\pi(\sigma)$ and $\pi(|\mu_2|)\perp\pi(\sigma)$;*

(II) *Let μ and μ_1 be as in (I). If $sp(\mu)\subset E$, then $sp(\mu_1)\subset E$.*

PROOF. (I): Let $\eta=\pi(|\mu|)$. Choose disjoint Baire sets A and B so that $A\cup B=X/G$, $\eta|_A\ll\pi(\sigma)$ and $\eta|_B\perp\pi(\sigma)$. Define $\mu_1=\mu|_{\pi^{-1}(A)}$ and $\mu_2=$

$\mu|_{\pi^{-1}(B)}$. Plainly μ_1 and μ_2 have the desired properties. The uniqueness is obvious.

(II): Let $\{\mu_{\dot{x}}\}_{\dot{x}\in X/G}$ be a canonical disintegration of μ. Define measures ω_1, $\omega_2 \in M(X)$ by

$$(1) \qquad \omega_1(f)=\int_{X/G}\mu_{\dot{x}}(f)\,d\pi(|\mu|)_a(\dot{x}),$$
$$\omega_2(f)=\int_{X/G}\mu_{\dot{x}}(f)\,d\pi(|\mu|)_s(\dot{x})$$

for $f\in C_0(X)$, where $\pi(|\mu|)=\pi(|\mu|)_a+\pi(|\mu|)_s$ is the Lebesgue decomposition of $\pi(|\mu|)$ with respect to $\pi(\sigma)$. Then $\mu=\omega_1+\omega_2$, and we can verify that $\pi(|\omega_1|)\ll\pi(\sigma)$ and $\pi(|\omega_2|)\perp\pi(\sigma)$. Hence we have

$$(2) \qquad \mu_1=\omega_1 \text{ and } \mu_2=\omega_2.$$

On the other hand, it follows from Lemma 2.6 that

$$\mathrm{sp}(\mu_{\dot{x}})\subset E \quad \pi(|\mu|)\text{-a.a. } \dot{x}\in X/G;$$

hence

$$\mathrm{sp}(\mu_{\dot{x}})\subset E \quad \pi(|\mu|)_a\text{-a.a. } \dot{x}\in X/G.$$

Hence

$$f*\mu_{\dot{x}}=0 \quad \pi(|\mu|)_a\text{-a.a. } \dot{x}\in X/G$$

for all $f\in L^1_{E^c}(G)$. Thus we have, by (1)-(2) and Lemma 2.3,

$$f*\mu_1=0 \text{ for all } f\in L^1_{E^c}(G),$$

which yields $\mathrm{sp}(\mu_1)\subset E$. This completes the proof.

LEMMA 2.8. *Let $\eta\in M^+(X/G)$ and $K>0$, and let $\{\mu_{\dot{x}}\}_{\dot{x}\in X/G}$ be a family of measures in $M^+(X)$ with the following properties*:

(1) $\dot{x}\to\mu_{\dot{x}}(f)$ is η-integrable for each bounded Baire function f on X,

(2) $\mathrm{supp}(\mu_{\dot{x}})\subset\pi^{-1}(\dot{x})$,

(3) $\sup\{|\int_{X/G}\mu_{\dot{x}}(f)\,d\eta(\dot{x})|: f\in C_0(X),\ \|f\|_\infty\le 1\}\le K.$

Let $\mu_{\dot{x}}=\mu^a_{\dot{x}}+\mu^s_{\dot{x}}$ be the Lebesgue decomposition of $\mu_{\dot{x}}$ with respect to $m_{\dot{x}}$. Then

(4) *$\dot{x}\to\mu^a_{\dot{x}}(f)$ and $\dot{x}\to\mu^s_{\dot{x}}(f)$ are η-integrable functions for each bounded Baire function f on X.*

PROOF. We may assume that $\eta\neq 0$. For any $f\in C_0(X)$, we note that

$x \to \int_G f(g \cdot x) dm_G(g)$ is a function on X which belongs to $C_0(X)$ and is constant on each orbit under G. For each $f \in C_0(X)$, define a function $[f] \in C_0(X/G)$ by $[f](\dot{x}) = \int_G f(g \cdot x) dm_G(g)$, where $\dot{x} = \pi(x)$. We define a measure $\sigma \in M^+(X)$ by

$$\sigma(f) = \int_{X/G} [f](\dot{x}) d\eta(\dot{x})$$

for $f \in C_0(X)$.

Claim 1. $\delta_g * \sigma = \sigma$ for all $g \in G$.

For $f \in C_0(X)$, define $f_g \in C_0(X)$ by $f_g(x) = f(g \cdot x)$. Then $[f_g] = [f]$. Hence we have

$$\begin{aligned} \delta_g * \sigma(f) &= \int_G f_g(x) d\sigma(x) \\ &= \int_{X/G} [f_g](\dot{x}) d\eta(\dot{x}) \\ &= \int_{X/G} [f](\dot{x}) d\eta(\dot{x}) \\ &= \sigma(f) \end{aligned}$$

for all $f \in C_0(X)$, and Claim 1 follows.

Claim 2. $\pi(\sigma) = \eta$.

For $F \in C_0(X/G)$, we note that $[F \circ \pi] = F$. Hence

$$\begin{aligned} \pi(\sigma)(F) &= \int_X F \circ \pi(x) d\sigma(x) \\ &= \int_{X/G} [F \circ \pi](\dot{x}) d\eta(\dot{x}) \\ &= \int_{X/G} F(\dot{x}) d\eta(\dot{x}) \end{aligned}$$

for all $F \in C_0(X/G)$. Thus Claim 2 follows.

By Claims 1 and 2, we have

(5) σ is quasi-invariant and $\pi(\sigma) = \eta$.

On account of (1) and (3), we can define a measure $\mu \in M^+(X)$ by

(6) $\mu(f) = \int_{X/G} \mu_{\dot{x}}(f) d\eta(\dot{x})$ for $f \in C_0(X)$.

We note that (6) holds for all bounded Baire functions f on X. Let $\mu = \mu_a +$

μ_s be the Lebesgue decomposition of μ with respect to σ. Since $\pi(\mu_a)\ll\eta$, it follows from (D. I) and Remark 2.1 that μ_a has an η-disintegration $\{\xi_{\dot{x}}^1\}_{\dot{x}\in X/G}$ with $\xi_{\dot{x}}^1\in M^+(X)$. Similary μ_s has an η-disintegration $\{\xi_{\dot{x}}^2\}_{\dot{x}\in X/G}$ with $\xi_{\dot{x}}^2\in M^+(X)$. Then

$$\mu_{\dot{x}}=\xi_{\dot{x}}^1+\xi_{\dot{x}}^2 \quad \eta\text{-a.a. } \dot{x}\in X/G \tag{7}$$

by the uniqueness assumption (D. II). Moreover, Lemma 2.5 ensures that

$$\xi_{\dot{x}}^1\ll m_{\dot{x}} \text{ and } \xi_{\dot{x}}^2\perp m_{\dot{x}}$$

for η-a.a. $\dot{x}\in X/G$. From this and (7), we have

$$\mu_{\dot{x}}^a=\xi_{\dot{x}}^1 \text{ and } \mu_{\dot{x}}^s=\xi_{\dot{x}}^2$$

for η-a.a. $\dot{x}\in X/G$. Thus (4) holds, and the proof is complete.

From Lemma 2.9 through Lemma 2.13, let (G, X) be a transformation group such that a compact abelian group G acts on a locally compact Hausdorff space X. For a closed subgroup H of G, let $q_H: G\to G/H$ and $\pi_H: X\to X/H$ be the canonical maps respectively.

LEMMA 2.9. *For $\mu\in M(X)$ and $\lambda\in M(G)$, we have*

$$\pi_H(\lambda*\mu)=q_H(\lambda)*\pi_H(\mu).$$

PROOF. For $F\in C_0(X/H)$, we have

$$\begin{aligned}
\pi_H(\lambda*\mu)(F)&=\int_X F(\pi_H(x))\,d\lambda*\mu(x)\\
&=\int_X\int_G F(\pi_H(g\cdot x))\,d\lambda(g)\,d\mu(x)\\
&=\int_X\int_G F(q_H(g)\cdot\pi_H(x))\,d\lambda(g)\,d\mu(x)\\
&=\int_X\int_{G/H} F(\dot{g}\cdot\pi_H(x))\,dq_H(\lambda)(\dot{g})\,d\mu(x)\\
&=\int_{G/H}\int_X F(\dot{g}\cdot\pi_H(x))\,d\mu(x)\,dq_H(\lambda)(\dot{g})\\
&=\int_{G/H}\int_{X/H} F(\dot{g}\cdot\dot{x})\,d\pi_H(\mu)(\dot{x})\,dq_H(\lambda)(\dot{g})\\
&=q_H(\lambda)*\pi_H(\mu)(F).
\end{aligned}$$

Hence we have $\pi_H(\lambda*\mu)=q_H(\lambda)*\pi_H(\mu)$, and the proof is complete.

LEMMA 2.10. *Put $\Gamma=H^\perp$. Let μ be a measure in $M(X)$ with $sp(\mu)=E$. Then $sp(\pi_H(\mu))\subset E\cap\Gamma$.*

PROOF. By Lemma 2.9, we have $\{q_H(f): f\in J(\mu)\}\subset J(\pi_H(\mu))$.

Hence, noting $q_H(f)^\wedge = \hat{f}|_\Gamma$, we get $\mathrm{sp}(\pi_H(\mu)) \subset \bigcap_{f \in J(\mu)} (q_H(f)^\wedge)^{-1}(0) = E \cap \Gamma$. This completes the proof.

LEMMA 2.11. *Let μ be a nonzero measure in $M(X)$. Then there exists a countable subgroup Γ_0 of $\hat{G}$ such that $\pi_{\Gamma^\perp}(\mu) \neq 0$ for all subgroups Γ of $\hat{G}$ with $\Gamma \supset \Gamma_0$.*

PROOF. Since $\mu \neq 0$, there exists a compact set K is X such that $\mu(K) \neq 0$. Put $|\mu(K)| = 2\delta > 0$. Then there exists an open set $U \supset K$ in X such that $|\mu|(U \backslash K) < \delta$. Let V be an open neighborhood of 0 in G such that

(1) $\qquad V \cdot K \subset U.$

Then there exists a countable subgroup Γ_0 of $\hat{G}$ such that $\Gamma_0^\perp \subset V$. Let Γ be a subgroup of $\hat{G}$ such that $\Gamma \supset \Gamma_0$. Then (1) yields

$$\begin{aligned} |\pi_{\Gamma^\perp}(\mu)(\pi_{\Gamma^\perp}(K))| &= |\mu(\Gamma^\perp \cdot K)| \\ &\geq |\mu|(K) - |\mu|(U \backslash K) \\ &> \delta. \end{aligned}$$

Thus $\pi_{\Gamma^\perp}(\mu) \neq 0$, and the proof is complete.

The following lemma follows easily from the definition of transformation group.

LEMMA 2.12. *Let K and F be disjoint compact sets in X. Then there exists an open neighborhood V of 0 in G such that $V \cdot K \cap V \cdot F = \phi$.*

On account of Lemma 2.12, the following lemma can be obtained as in [18, Lemma 4.1].

LEMMA 2.13 (cf. [18, Lemma 4.1]).
Let μ and ξ be measures in $M^+(X)$ with $\mu \perp \xi$. Then there exists a countable subgroup Γ_0 of $\hat{G}$ such that

(1) $\qquad \pi_{\Gamma^\perp}(\mu) \perp \pi_{\Gamma^\perp}(\xi)$

for all subgroups Γ of $\hat{G}$ with $\Gamma \supset \Gamma_0$.

§ 3 Proof of Theorem 1.1.

In this section we prove Theorem 1.1.

LEMMA 3.1. *Let σ be a measure in $M^+(X)$ that is quasi-invariant. If G is a metrizable compact abelian group and (G, X) satisfies conditions (D. I) and (D. II), then the conclusion of Theorem 1.1. holds.*

PROOF. As for the first assertion, it is sufficient to prove that $\mathrm{sp}(\mu_a)\subset P$ because of Remark 1.1 (II). Moreover, by Lemma 2.7, we may assume that $\pi(|\mu|)\ll\pi(\sigma)$. Let $\{\lambda_{\dot{x}}\}_{\dot{x}\in X/G}$ be a canonical disintegration of $|\mu|$. Let h be a unimodular Baire function on X with $\mu=h|\mu|$. We define measures $\mu_{\dot{x}}\in M(X)$ by $\mu_{\dot{x}}=h\lambda_{\dot{x}}$. Then $\{\mu_{\dot{x}}\}_{\dot{x}\in X/G}$ is a canonical disintegration of μ. For each $\dot{x}\in X/G$, let $\lambda_{\dot{x}}=\lambda_{\dot{x}}^a+\lambda_{\dot{x}}^s$ and $\mu_{\dot{x}}=\mu_{\dot{x}}^a+\mu_{\dot{x}}^s$ be the Lebesgue decompositions of $\lambda_{\dot{x}}$ and $\mu_{\dot{x}}$ with respect to $m_{\dot{x}}$ respectively. Then

(1) $\mu_{\dot{x}}^a=h\lambda_{\dot{x}}^a$ and $\mu_{\dot{x}}^s=h\lambda_{\dot{x}}^s$.

Since $\mathrm{sp}(\mu)\subset P$, it follows from Lemma 2.6 that

(2) $\mathrm{sp}(\mu_{\dot{x}})\subset P$ η-a.a. $\dot{x}\in X/G$,

where $\eta=\pi(|\mu|)$. Let $x\in\pi^{-1}(\dot{x})$, and let $\xi_{\dot{x}}$ be the measure in $M(G/G_x)$ such that $\tilde{B}_x(\xi_{\dot{x}})=\mu_{\dot{x}}$, where $\tilde{B}_x: G/G_x\to G\cdot x$ is the homeomorphism defined in (1.9). Then, by (2) and Proposition 1.2, we have

(3) $\xi_{\dot{x}}\in M_{P\cap G_x^\perp}(G/G_x)$ η-a.a. $\dot{x}\in X/G$,

which together with [24, Corollary] yields

(4) $\xi_{\dot{x}}^a,\ \xi_{\dot{x}}^s\in M_{P\cap G_x^\perp}(G/G_x)$ η-a.a. $\dot{x}\in X/G$,

where $\xi_{\dot{x}}=\xi_{\dot{x}}^a+\xi_{\dot{x}}^s$ is the Lebesgue decomposition of $\xi_{\dot{x}}$ *whith respect to* m_{G/G_x}. By Proposition 1.5, we note that $\tilde{B}_x(\xi_{\dot{x}}^a)=\mu_{\dot{x}}^a$ and $\tilde{B}_x(\xi_{\dot{x}}^s)=\mu_{\dot{x}}^s$. If follows from (4) and Proposition 1.2 that

(5) $\mathrm{sp}(\mu_{\dot{x}}^a)$, $\mathrm{sp}(\mu_{\dot{x}}^s)\subset P$ η-a.a. $\dot{x}\in X/G$.

By Lemma 2.8, we have

(6) $\dot{x}\to\lambda_{\dot{x}}^a(f)$ and $\dot{x}\to\lambda_{\dot{x}}^s(f)$ are η-measurable for each bounded Baire function f on X;

hence (1) yields

(7) $\dot{x}\to\mu_{\dot{x}}^a(f)$ and $\dot{x}\to\mu_{\dot{x}}^s(f)$ are η-measurable for each bounded Baire function f on X.

By (6) and (7), we can define measures $\omega_1, \omega_2\in M^+(X)$ and $\mu_1, \mu_2\in M(X)$ as follows:

$$\omega_1(f)=\int_{X/G}\lambda_{\dot{x}}^a(f)\,d\eta(\dot{x}),\quad \omega_2(f)=\int_{X/G}\lambda_{\dot{x}}^s(f)\,d\eta(\dot{x});$$
$$\mu_1(f)=\int_{X/G}\mu_{\dot{x}}^a(f)\,d\eta(\dot{x}),\quad \mu_2(f)=\int_{X/G}\mu_{\dot{x}}^s(f)\,d\eta(\dot{x})$$

for $f\in C_0(X)$. Then, by (1), we have $\mu_1\ll\omega_1$ and $\mu_2\ll\omega_2$. Since $\eta\ll\pi(\sigma)$, it follows from Lemma 2.5 that

$$\omega_1\ll\sigma \text{ and } \omega_2\perp\sigma;$$

hence

$$\mu_1\ll\sigma \text{ and } \mu_2\perp\sigma. \tag{8}$$

Since $\mu=\mu_1+\mu_2$, (8) yields $\mu_1=\mu_a$. For any $\gamma\notin P$ and $f\in C_0(X)$, we have

$$\begin{aligned}\gamma*\mu_a(f)&=\gamma*\mu_1(f)\\ &=\int_{X/G}\gamma*\mu_{\dot{x}}^a(f)\,d\eta(\dot{x}) && \text{(by Lemma 2.3)}\\ &=0. && \text{(by (5) and Remark 1.1 (II))}\end{aligned}$$

Hence, by Remark 1.1 (II), we get $\mathrm{sp}(\mu_a)\subset P$.

Next we prove the latter half. If $P\cap(-P)=\{0\}$, then by (4) and Theorem 0.1 (ii) we have

$$\hat{\xi}_{\dot{x}}^s(0)=0 \quad \eta\text{-a.a. } \dot{x}\in X/G;$$

hence Proposition 1.2 yields

$$0\notin\mathrm{sp}(\mu_{\dot{x}}^s) \quad \eta\text{-a.a. } \dot{x}\in X/G.$$

Thus, by Remark 1.1 (II), we have

$$1*\mu_{\dot{x}}^s=0 \quad \eta\text{-a.a. } \dot{x}\in X/G,$$

where 1 is the constant function on G with value one. On the other hand, (8) yields $\mu_s=\mu_2$. Hence, by Lemma 2.3 and the construction of μ_2, we have

$$1*\mu_s=0,$$

which together with Remark 1.1 (II. 1) yields $0\notin\mathrm{sp}(\mu_s)$. Thus $\mathrm{sp}(\mu_s)\subset P\setminus\{0\}$, and the proof is complete

Now we prove Theorem 1.1. Since μ is bounded regular, there exists a σ-compact open set X_0 in X with $G\cdot X_0=X_0$ and a quasi-invariant measure $\sigma'\in M^+(X)$ satisfying the following:

(3.1) μ is concentrated on X_0,

(3.2) $\sigma'|_{X_0}\ll\sigma|_{X_0}$ and $\sigma|_{X_0}\ll\sigma'|_{X_0}$.

Hence $\mu=\mu_a+\mu_s$ is the Lebesgue decomposition of μ with respect to σ'. Thus we may assume that σ is a measure in $M^+(X)$ that is quasi-invariant. As for the first assertion, it is sufficient to prove that $\mathrm{sp}(\mu_s)\subset P$ because of

Remark 1.1 (II). We may assume $\mu_s \neq 0$. Suppose there exists $\gamma_0 \notin P$ such that $\gamma_0 \in \mathrm{sp}(\mu_s)$. Then $\gamma_0 * \mu_s \neq 0$. It follows from Lemmas 2.11 and 2.13 that there exists a countable subgroup Γ of $\hat{G}$ with $\gamma_0 \in \Gamma$ such that

$$(3.3) \qquad \pi_H(\gamma_0 * \mu_s) \neq 0$$

and

$$(3.4) \qquad \pi_H(|\mu_s|) \perp \pi_H(\sigma),$$

where $H = \Gamma^{\perp}$ and $\pi_H : X \to X/H$ is the canonical map. By (3.4), $\pi_H(\mu_s)$ is the singular part of $\pi_H(\mu)$ with respect to $\pi_H(\sigma)$. Since σ is quasi-invariant, $\pi_H(\sigma)$ is also quasi-invariant. $\Gamma = H^{\perp}$ is countable, and $(G/H, X/H)$ satisfies conditions (D.I) and (D.II). Hence we have

$$(3.5) \qquad \mathrm{sp}(\pi_H(\mu_s)) \subset P \cap \Gamma$$

by Lemmas 2.10 and 3.1. It follows from (3.3) and Lemma 2.9 that $q_H(\gamma_0) * \pi_H(\mu_s) \neq 0$. On the other hand, since $\gamma_0 \in \Gamma$, we get $q_H(\gamma_0) = \gamma_0$. Hence $\gamma_0 \in \mathrm{sp}(\pi_H(\mu_s))$. Thus, by (3.5), we have $\gamma_0 \in P \cap \Gamma$, which contradicts the choice of γ_0.

As to the latter half, we may repeat a similar argument for $\gamma_0 = 0$ (i.e., $\gamma_0(x) = 1$ for $x \in G$). This completes the proof of Theorem 1.1.

§ 4. Proof of Theorem 1.2.

In this section, we first prove Theorem 1.2. The latter half of this section is devoted to the consideration of the sets satisfying condition (*) in Theorem 1.2. (Such sets shall be defined in LCA groups.). For a LCA group G, let $M(G)$ and $L^1(G)$ be the measure algebra and the group algebra respectively. For a subset E of $\hat{G}$, $M_E(G)$ denotes the space of measures in $M(G)$ whose Fourier-Stieltjes transforms vanish off E. m_G means the Haar measure on G. For a closed subgroup H of G, $H^{\perp}$ denotes the annihilator of H.

Definition 4.1. Let G be a LCA group, and let μ be a measure in $M(G)$. μ is said to be quasi-invariant if $|\mu| * \delta_x \ll |\mu|$ for all $x \in G$.

Remark 4.1. Suppose there exists a nonzero measure $\mu \in M(G)$ that is quasi-invariant. Then, by regularity of μ, G must be σ-compact. That is, if G is not σ-compact, $M(G)$ has no nonzero quasi-invariant measures.

The following proposition is well-known.

Proposition 4.1. *Let G be a LCA group, and let μ be a nonzero measure in $M(G)$. Then the following are equivalent.*

(i) *μ is quasi-invariant;*

(ii) *$|\mu|$ and m_G are mutually absolutely continuous.*

DEFINITION 4.2. Let G be a LCA group, and let E be a closed subset of $\hat{G}$. We say that E satisfies condition (*) if the following holds.

(*) For $\mu \in M_E(G)$, μ is quasi-invariant.

REMARK 4.2. When $G=\boldsymbol{T}$, $\boldsymbol{Z}^+$ satisfies condition (*). When $G=\boldsymbol{R}$, $\boldsymbol{R}^+$ also satisfies condition (*).

LEMMA 4.1. *Let G be a LCA group, and let E be a closed subset of $\hat{G}$ satisfying condition (*) in Definition 4.2. Then, for any open subgroup Γ of $\hat{G}$, the following $(*)_\Gamma$ holds.*

$(*)_\Gamma$ *For any nonzero measure $\xi \in M_{E\cap\Gamma}(G/H)$, $|\xi|$ and $m_{G/H}$ are mutually absolutely continuous, where $H=\Gamma^\perp$.*

PROOF. Let ξ be a nonzero measure in $M_{E\cap\Gamma}(G/H)$. Let $q_H: G\to G/H$ be the natural homomorphism. We note that H is compact.

Step 1. $|\xi| \ll m_{G/H}$.

In fact, there exists a nonzero measure $\lambda \in M_{E\cap\Gamma}(G)$ such that $\hat{\lambda}(\gamma)=\hat{\xi}(\gamma)$ for $\gamma\in\Gamma$ and $\hat{\lambda}(\gamma)=0$ for $\gamma\in\hat{G}\backslash\Gamma$. Hence, by the hypothesis and Proposition 4.1, we have $\lambda\in L^1(G)$, and so $\xi=q_H(\lambda)\in L^1(G/H)$. Thus Step 1 is obtained.

Step 2. $|\xi|$ and $m_{G/H}$ are mutually absolutely continuous.

By [10, (28.54) Theorem (iv) and (28.55) Theorem (iii), (Vol. 2)], we note that the following (1) holds.

$$(1) \qquad g\circ q_H \in L^1(G) \text{ and } \int_{G/H} g(\dot{x})\,dm_{G/H}(\dot{x}) = \int_G g(q_H(x))\,dm_G(x)$$

for all $g\in L^1(G/H)$. We define a map $J: L^1(G/H)\to L^1(G)$ by $J(g)=g\circ q_H$. Then we have $J(\xi)\in L^1(G)$ by Step 1. Moreover, for any $g\in L^1(G/H)$, we have

$$(2) \qquad J(g)^\wedge(\gamma)=\begin{cases}\hat{g}(\gamma) & \text{for } \gamma\in\Gamma \\ 0 & \text{for } \gamma\in\hat{G}\backslash\Gamma\end{cases}.$$

Hence, by the hypothesis, $|J(\xi)|$ and m_G are mutually absolutely continuous. Hence

(3) $\qquad q_H(|J(\xi)|)$ and $m_{G/H}$ are mutually absolutely continuous.

On the other hand, by the definition of the map, we have

$$J(|\xi|)=|J(\xi)|.$$

Moreover, by (2), we have

$$q_H(J(|\xi|))=|\xi|,$$

which together with (3) yields that $|\xi|$ and $m_{G/H}$ are mutually absolutely continuous. This completes the proof.

LEMMA 4.2. *If G is a metrizable compact abelian group and (G, X) satisfies conditions (D. I) and (D. II), then the conclusion of Theorem 1.2 holds.*

PROOF. Put $\eta=\pi(|\mu|)$. Let $\{\mu_{\dot{x}}\}_{\dot{x}\in X/G}$ be a canonical disintegration of μ. Then, by Lemma 2.6, we have

(1) $\quad \mathrm{sp}(\mu_{\dot{x}})\subset E \ \ \eta\text{-a.a. } \dot{x}\in X/G.$

Hence we have

(2) $\quad |\mu_{\dot{x}}|\ll m_{\dot{x}}$ and $m_{\dot{x}}\ll|\mu_{\dot{x}}| \ \ \eta$-a.a. $\dot{x}\in X/G.$

In fact, let $x\in\pi^{-1}(\dot{x})$, and let $\tilde{B}_x$: $G/G_x\to G\cdot x$ be the homeomorphism defined in (1.9). Let $\xi_{\dot{x}}$ be the measure in $M(G/G_x)$ such that $\tilde{B}_x(\xi_{\dot{x}})=\mu_{\dot{x}}$. Then, by (1) and Proposition 1.2, we have

$$\xi_{\dot{x}}\in M_{E\cap G_{\dot{x}}^{\perp}}(G/G_x) \ \ \eta\text{-a.a. } \dot{x}\in X/G\,;$$

hence Lemma 4.1 yields

(3) $\quad |\xi_{\dot{x}}|\ll m_{G/G_x}$ and $m_{G/G_x}\ll|\xi_{\dot{x}}| \ \ \eta$-a.a. $\dot{x}\in X/G.$

Thus, since $|\tilde{B}_x(\xi_{\dot{x}})|=\tilde{B}_x(|\xi_{\dot{x}}|)$, (2) follows from (3) and Proposition 1.5. Let F be a Baire set in X with $|\mu|(F)=0$. Then, by Lemma 2.4, we have

$$\int_{X/G}|\mu_{\dot{x}}|(F)\,d\eta(\dot{x})=0,$$

which yields

$$|\mu_{\dot{x}}(F)|=0 \ \ \eta\text{-a.a. } \dot{x}\in X/G.$$

For any $g\in G$, it follows from (2) that

$$|\mu_{\dot{x}}|(g\cdot F)=0 \ \ \eta\text{-a.a. } \dot{x}\in X/G,$$

which shows

$$\begin{aligned}|\mu|(g\cdot F)&=\int_{X/G}|\mu_{\dot{x}}|(g\cdot F)\,d\eta(\dot{x})\\&=0.\end{aligned}$$

This completes the proof.

Now we prove Theorem 1. 2. Suppose there exists a measure $\mu\in M(X)$ with $\mathrm{sp}(\mu)\subset E$ such that μ is not quasi-invariant. Then there exists $g_0\in G$ such that $|\mu|$ is not absolutely continuous with respect to $\delta_{g_0}*|\mu|$. Let $\mu=\mu_1+\mu_2$ be the Lebesgue decomposition of μ with respect to $\delta_{g_0}*|\mu|$, where $\mu_1\ll\delta_{g_0}*|\mu|$ and $\mu_2\perp\delta_{g_0}*|\mu|$. Then $\mu_2\neq 0$. By Lemmas 2. 11 and 2. 13, there exists a countable subgroup Γ of $\hat{G}$ such that

$$(4.1)\qquad \pi_H(\mu_2)\neq 0,$$
$$(4.2)\qquad \pi_H(|\mu_1|)\perp\pi_H(|\mu_2|),\ \text{and}$$
$$(4.3)\qquad \pi_H(|\mu_2|)\perp\pi_H(\delta_{g_0}*|\mu|),$$

where $H=\Gamma^{\perp}$ and $\pi_H: X\rightarrow X/H$ is the canonical map. We note $|\pi_H(\mu_2)|\ll\pi_H(|\mu_2|)$, $|\pi_H(\delta_{g_0}*\mu)|\ll\pi_H(\delta_{g_0}*|\mu|)$ and $|\pi_H(\delta_{g_0}*\mu)|=\delta_{q_H(g_0)}*|\pi_H(\mu)|$. It follows from (4. 3) that

$$(4.4)\qquad |\pi_H(\mu_2)|\perp\delta_{q_H(g_0)}*|\pi_H(\mu)|.$$

Let $(G/H, X/H)$ be the transformation group induced by (G, X). Then $(G/H, X/H)$ satisfies conditions (D. I) and (D. II). It follows from Lemma 2. 10 that $\mathrm{sp}(\pi_H(\mu))\subset\Gamma\cap E$. Hence, by Lemmas 4. 1 and 4. 2, we have

$$(4.5)\qquad |\pi_H(\mu)|\ll\delta_{q_H(g_0)}*|\pi_H(\mu)|.$$

On the other hand, $\pi_H(\mu)=\pi_H(\mu_1)+\pi_H(\mu_2)$, and (4. 2) implies that $|\pi_H(\mu_1)|\perp|\pi_H(\mu_2)|$. Hence $|\pi_H(\mu_2)|\ll|\pi_H(\mu)|$. Thus, by (4. 1) and (4. 5), we have $0\neq|\pi_H(\mu_2)|\ll\delta_{q_H(g_0)}*|\pi_H(\mu)|$, which contradicts (4. 4). This completes the proof.

Before we close this section, we give several examples of sets satisfying condition (*) in Definition 4. 2 together with propositions.

PROPOSITION 4. 2. *Let G be a LCA group, and let Γ be an open subgroup of $\hat{G}$. Let E be a closed subset of $\hat{G}$ contained in Γ. Suppose that E satisfies condition (*) in Γ. Then E also satisfies condition (*) in $\hat{G}$.*

PROOF. Put $H=\Gamma^{\perp}$. *For* $x\in G$, let $\dot{x}=q_H(x)$. Let μ be a nonzero measure in $M_E(G)$. Then $q_H(\mu)$ is a nonzero measure in $M_E(G/H)$. We note, by Remark 4. 1, that G is σ-compact. By Proposition 4. 1, E is a Riesz set in Γ. Hence E is a Riesz set in $\hat{G}$. Thus we have

$$(1)\qquad \mu\in L^1(G).$$

Let $J: L^1(G/H)\rightarrow L^1(G)$ be the map defined in the proof of Lemma 4. 1 (i.e., $J(g)=g\circ q_H$). Then, for $g\in L^1(G/H)$, $J(g)^{\wedge}(\gamma)=\hat{g}(\gamma)$ for $\gamma\in\Gamma$ and $J(g)^{\wedge}(\gamma)=0$ for $\gamma\in\hat{G}\backslash\Gamma$. Hence, since $\mathrm{supp}(\hat{\mu})\subset E$, we have

(2) $\qquad J(q_H(\mu))=\mu.$

Let K be a Borel set in G with $|\mu|(K)=0$. Let F be the Radon-Nikodym derivative of $q_H(\mu)$ with respect to $m_{G/H}$. Then, by (2), we have

$$\begin{aligned}0&=\int_G \chi_K(x)|F|\circ q_H(x)\,dm_G(x)\\&=\int_{G/H}\int_H \chi_K(x+y)|F|\circ q_H(x+y)\,dm_H(y)\,dm_{G/H}(\dot{x})^{(3)}\\&=\int_{G/H}|F|(\dot{x})\int_H \chi_K(x+y)\,dm_H(y)\,dm_{G/H}(\dot{x}),\end{aligned}$$

which shows

(3) $\qquad \int_H \chi_K(x+y)\,dm_H(y)=0 \ |F|dm_{G/H}\text{-a.a. } \dot{x}\in G/H.$

Since $q_H(\mu)\in M_E(G/H)$, $|q_H(\mu)|$ and $m_{G/H}$ are mutually absolutely continuous. Hence (3) yields

$$\int_H \chi_K(x+y)\,dm_H(y)=0 \ m_{G/H}\text{-a.a. } \dot{x}\in G/H.$$

Hence we have

$$\begin{aligned}m_G(K)&=\int_{G/H}\int_H \chi_K(x+y)\,dm_H(y)\,dm_{G/H}(\dot{x})\\&=0,\end{aligned}$$

which shows $m_G\ll|\mu|$. Thus, by (1), $|\mu|$ and m_G are mutually absolutely continuous. This completes the proof.

EXAMPLE 4.1. Let G be a connected compact abelian group, and let E be a finite subset of $\hat{G}$. Since $\hat{G}$ is ordered, it follows from [20, 8.4.1 Theorem, p. 206] that E satisfies condition (*).

EXAMPLE 4.2. Let G be a compact abelian group, and let γ_0 be an element of $\hat{G}$ with infinite order. Put $E=\{n\gamma_0 : n\in \boldsymbol{Z}^+\}$. Then, by Proposition 4.2, E satisfies condition (*).

EXAMPLE 4.3. Let $G=\boldsymbol{T}^n$, and let $E=\{m_1, m_2, \dots, m_n)\in \boldsymbol{Z}^n : m_i\geq 0$ $(i=1,2,3,\dots,n)\}$. Then, by [2, Main Theorem], we can verify that E satisfies condtion (*).

---

(3) We noralize the Haar measures on G/H and H so that

$$\int f(x)\,dm_G(x)=\int_{G/H}\int_H f(x+y)\,dm_H(y)\,dm_{G/H}(\dot{x})$$

for all $f\in L^1(G)$.

EXAMPLE 4.4. Let $G=\boldsymbol{T}^{\infty}$ (countable infinite dimensional torus), and let $E=\{(m_1, m_2, \ldots, m_n, \ldots)\in\hat{\boldsymbol{T}}^{\infty}: m_i\geq 0\ (i\in\boldsymbol{N})\}$. Then, by ([2, p. 191]), E satisfies condition (*).

Moreover, the following proposition holds.

PROPOSITION 4.3 (cf. [17, Corollary 2]).
For any index set Λ, let $G=\prod_{\alpha\in\Lambda}\boldsymbol{T}_{\alpha}$, where $\boldsymbol{T}_{\alpha}=\boldsymbol{T}$ for all $\alpha\in\Lambda$. Let $E=\{<m_{\alpha}>\in\hat{G}: m_{\alpha}\geq 0$ for all $\alpha\in\Lambda\}$. Then E satisfies condition (*).

PROOF. Let $\mu\in M_E(G)$. Then Examples 4.3 and 4.4 ensure that $\hat{\mu}|_{\Gamma}\in A(\Gamma)=\{\hat{f}: f\in L^1(G/\Gamma^{\perp})\}$ for each countable subgroup Γ of $\hat{G}$. It follows from [16, Lemma 4] that $\mu\in L^1(G)$. In particular, $\operatorname{supp}(\hat{\mu})$ is countable. Hence μ is quasi-invariant by Examples 4.3 and 4.4 combined with Proposition 4.2. This completes the proof.

In [23, Theorem 2], the author proved that the product set of two Riesz sets is also a Riesz set. As for the sets satisfying condition (*), we prove that an analogous result holds.

PROPOSITION 4.4. *Let G_1 and G_2 be LCA groups. Let E_i be a closed subset of $\hat{G}_i$ satisfying condition (*) $(i=1,2)$. Then $E_1\times E_2$ also satisfies condition (*).*

PROOF. Let $q_i: G_1\oplus G_2\to G_i$ be a projection $(i=1,2)$. Let μ be a nonzero measure in $M_{E_1\times E_2}(G_1\oplus G_2)$. Put $\eta_1=q_1(|\mu|)$ and $\eta_2=q_2(|\mu|)$. First we prove the proposition in case that G_1 and G_2 are metrizable LCA groups. By the theory of disintegration, there exists a family $\{\mu_y\}_{y\in G_2}$ of measures in $M(G_1\oplus G_2)$ with the following properties (cf. [18, p. 114, (6)-(9)]):

(1) $y\to\mu_y(f)$ is η_2-measurable for each bounded Borel function f on $G_1\oplus G_2$,

(2) $\|\mu_y\|=1$,

(3) $\operatorname{supp}(\mu_y)\subset G_1\times\{y\}$,

(4) $\mu(f)=\int_{G_2}\mu_y(f)\,d\eta_2(y)$ for each bounded Borel function f on $G_1\oplus G_2$.

By (3), there exists $\lambda_y\in M(G_1)$ such that $\mu_y=\lambda_y\times\delta_y$. Since $\operatorname{supp}(\hat{\mu})\subset E_1\times E_2\subset E_1\times\hat{G}_2$, the argument in [18, p. 115] implies that

(5) $\operatorname{supp}(\hat{\lambda}_y)\subset E_1$ η_2-a.a. $y\in G_2$.

Hence we have

(6) $|\lambda_y|*\delta_x\ll|\lambda_y|$ η_2-a.a. $y\in G_2$

for all $x \in G_1$. We note that $y \to |\mu_y|(f)$ is η_2-measurable and

(7) $$|\mu|(f) = \int_{G_2} |\mu_y|(f)\, d\eta_2(y)$$

for each bounded Borel function f on $G_1 \oplus G_2$. Let K be a Borel set in $G_1 \oplus G_2$ with $|\mu|(K)=0$. Then, by (7), we have

$$|\lambda_y|(K_y)=0 \ \eta_2\text{-a.a. } y \in G_2,$$

where $K_y = \{x \in G_1 : (x, y) \in K\}$. Hence, for all $x \in G_1$, (6) yields

$$|\lambda_y| * \delta_x(K_y)=0 \ \eta_2\text{-a.a. } y \in G_2.$$

Thus we have

$$\begin{aligned} |\mu| * \delta_{(x,0)}(K) &= |\mu|(K-(x, 0)) \\ &= \int_{G_2} |\mu_y|K-(x, 0))\, d\eta_2(y) \qquad \text{(by (7))} \\ &= \int_{G_2} |\lambda_y|(K_y - x)\, d\eta_2(y) \\ &= \int_{G_2} |\lambda_y| * \delta_x(K_y)\, d\eta_2(y) \\ &= 0, \end{aligned}$$

which yields

(8) $$|\mu| * \delta_{(x,0)} \ll |\mu| \text{ for all } x \in G_1.$$

On the other hand, we have $\operatorname{supp}(\mu \hat{*} \delta_{(x,0)}) \subset E_1 \times E_2 \subset \hat{G}_1 \times E_2$ for all $x \in G_1$. Hence, by a similar argument, we have

$$|\mu * \delta_{(x,0)}| * \delta_{(0,y)} \ll |\mu * \delta_{(x,0)}|$$

for all $x \in G_1$ and $y \in G_2$. Hence we have, by (8),

$$|\mu| * \delta_{(x,y)} = (|\mu| * \delta_{(x,0)}) * \delta_{(0,y)} \ll |\mu|$$

for all $(x, y) \in G_1 \oplus G_2$. Thus the proposition holds when G_1 and G_2 are metrizable LCA groups.

Next we prove the proposition in case that G_1 and G_2 are LCA groups. By [23, Theorem 2], $E_1 \times E_2$ is a Riesz set. So $\mu \in L^1(G_1 \oplus G_2)$; in particular, there exists an open σ-compact subgroup $\Gamma = \Gamma_1 \times \Gamma_2$ of $\widehat{G_1 \oplus G_2}$ such that $\operatorname{supp}(\hat{\mu}) \subset \Gamma$. Then $\operatorname{supp}(\hat{\mu}) \subset (\Gamma_1 \cap E_1) \times (\Gamma_2 \cap E_2)$ and $G_1 \oplus G_2 / \Gamma^{\perp}$ is metrizable. Therefore μ is quasi-invariant by the metrizable case combined with Lemma 4. 1 and Proposition 4. 2. This completes the proof.

§ 5. Proofs of Theorems 1.3 and 1.4.

In this section we prove Theorems 1.3 and 1.4. We first prove Theorem 1.3. Let (G, X) be a transformation group such that a compact abelian group G acts on a locally compact Hausdorff space X. Let $M_{aG}(X)$ be an L-subspace of $M(X)$ defined by

$$M_{aG}(X)=\left\{\mu\in M(X): \begin{array}{l}\mu\ll\rho*\nu \text{ for some } \rho\in L^1(G)\cap M^+(G)\\ \text{and } \nu\in M^+(X)\end{array}\right\}.$$

Put $M_{aG}(X)^{\perp}=\{\nu\in M(X): \nu\perp\mu$ for all $\mu\in M_{aG}(X)\}$. Then $M_{aG}(X)^{\perp}$ is also an L-subspace of $M(X)$, and $M(X)=M_{aG}(X)\oplus M_{aG}(X)^{\perp}$.

DEFINITION 5.1. We say that $\mu\in M(X)$ translates G-continuously if $\lim_{g\to 0}\|\mu-\delta_g*\mu\|=0$, where $g\in G$.

PROPOSITION 5.1. (cf. [18, Proposition 3.1]).
For $\mu\in M(X)$, the following are equivalent.

(I) $\mu\in M_{aG}(X)$,
(II) μ *translates G-continuously.*

PROOF. (I)$\Longrightarrow$(II): Since $\mu\in M_{aG}(X)$, there exist $\nu\in M^+(X)$ and $\rho\in L^1(G)\cap M^+(G)$ such that $\mu\ll\rho*\nu$. For $f\in C_c(X)$ and $g\in G$, we note that $\delta_g*f(x)=\int_G f((-u)\cdot x)\,d\delta_g(u)=f((-g)\cdot x)\in C_c(X)$ and

$$\lim_{g\to 0}\|f-\delta_g*f\|_{\infty}=0. \tag{1}$$

Claim. For $f\in C_c(X)$, $f(\rho*\nu)$ translates G-continuously.

In fact, we note $\delta_g*(f(\rho*\nu))=(\delta_g*f)(\rho*\delta_g*\nu)$. Then we have

$$\begin{aligned}&\|f(\rho*\nu)-\delta_g*(f(\rho*\nu))\|\\ &\quad\leq\|f(\rho*\nu)-(\delta_g*f)(\rho*\nu)\|\\ &\qquad+\|(\delta_g*f)(\rho*\nu)-(\delta_g*f)(\rho*\delta_g*\nu))\|\\ &\quad\leq\|f-\delta_g*f\|_{\infty}\|\rho*\nu\|+\|f\|_{\infty}\|\rho-\rho*\delta_g\|\|\nu\|.\end{aligned}$$

Hence, by (1) and the fact that $\lim_{g\to 0}\|\rho-\rho*\delta_g\|=0$, we have $\lim_{g\to 0}\|f(\rho*\nu)-\delta_g*(f(\rho*\nu))\|=0$, and the claim follows.

For $\varepsilon>0$, since $\mu\ll\rho*\nu$, there exists $f\in C_c(X)$ such that $\|\mu-f(\rho*\nu)\|<\varepsilon$. By Claim, there exists a neighborhood V of 0 in G such that

$$\|f(\rho*\nu)-\delta_g*(f(\rho*\nu))\|<\varepsilon \text{ for all } g\in V.$$

Hence, for any $g\in V$, we have

$$\begin{aligned}\|\mu-\delta_g*\mu\| &\leq \|\mu-f(\rho*\nu)\|+\|f(\rho*\nu)-\delta_g*(f(\rho*\nu))\| \\ &\quad+\|\delta_g*(f(\rho*\nu))-\delta_g*\mu\| \\ &<3\varepsilon,\end{aligned}$$

which shows that μ translates G-continuously.

(II)$\Longrightarrow$(I): By a similar method in [18, Proposition 3.1], we can prove that (II) implies (I). This completes the proof.

For $x\in X$, let G_x be the closed subgroup of G defined by $G_x=\{g\in G: g\cdot x=x\}$. Let $\Pi_x: G\to G/G_x$ be the natural homomorphism. We define a map $J_x: L^1(G/G_x)\to L^1(G)$ by $J_x(f)=f\circ\Pi_x$ (cf. [10, (28.54) Theorem (iv) and (28.55) Theorem (iii), (Vol. 2)]). Then, for $f\in L^1(G/G_x)$, we have $\|J_x(f)\|_1=\|f\|_1$. Moreover, $J_x(f)^\wedge(\gamma)=\hat{f}(\gamma)$ on $G_x^\perp$ and $J_x(f)^\wedge(\gamma)=0$ on $\hat{G}\backslash G_x^\perp$. Let B_x and $\tilde{B}_x$ be the maps defined in (1.8) and (1.9).

$$\begin{tikzcd} L^1(G) \arrow[r, "B_x"] & M(G\cdot x)\subset M(X) \\ L_1(G/G_x) \arrow[u, "J_x"] \arrow[ur, "\tilde{B}_x"'] & \end{tikzcd}$$

Fig. II

PROPOSITION 5.2. *Let $\xi\in L^1(G/G_x)$ and $g\in G$. Then we have*

(i) $\tilde{B}_x(\xi)=B_x(J_x(\xi))$, *and*

(ii) $J_x(\xi)*\delta_g=J_x(\xi*\delta_{\Pi_x(g)})$.

PROOF. We first prove (i). We note that $\Pi_x(J_x(\xi))=\xi$. Hence, for $F\in C(G\cdot x)$, we have

$$\begin{aligned}\tilde{B}_x(\xi)(F) &=\tilde{B}_x(\Pi_x(J_x(\xi)))(F) \\ &=\int_{G/G_x} F(\tilde{B}_x(\dot{u}))\,d\Pi_x(J_x(\xi))(\dot{u}) \\ &=\int_G F\circ\tilde{B}_x(\Pi_x(u))\,dJ_x(\xi)(u) \\ &=\int_G F\circ B_x(u)\,dJ_x(\xi)(u) \qquad \text{(by (1.10))} \\ &=B_x(J_x(\xi))(F).\end{aligned}$$

Thus we have $\tilde{B}_x(\xi)=B_x(J_x(\xi))$. Next we prove (ii). For $\gamma\in G_x^\perp$, we have

$$\begin{aligned}J_x(\xi*\delta_{\Pi_x(g)})^\wedge(\gamma) &=(\xi*\delta_{\Pi_x(g)})^\wedge(\gamma) \\ &=\xi(\gamma)(-g,\gamma) \\ &=J_x(\xi)^\wedge(\gamma)(-g,\gamma)\end{aligned}$$

$$=(J_x(\xi)*\delta_g)^\wedge(\gamma).$$

For $\gamma\in\hat{G}\backslash G_x^\perp$, we have

$$J_x(\xi*\delta_{\Pi_x(g)})^\wedge(\gamma)=0=(J_x(\xi)*\delta_g)^\wedge(\gamma).$$

Thus, by the uniqueness for Fourier-Stieltjes transforms, we have $J_x(\xi)*\delta_g=J_x(\xi*\delta_{\Pi_x(g)})$. This completes the proof.

LEMMA 5.1 *If G is a metrizable compact abelian group and (G, X) satisfies conditions (D. I) and (D. II), then the conclusion of Theorem* 1.3 holds.

PROOF. Put $\eta=\pi(|\mu|)$, and let $\{\mu_{\dot{x}}\}_{\dot{x}\in X/G}$ be a canonical disintegration of μ. Then, by Lemma 2.6, we have

(1) $\qquad \mathrm{sp}(\mu_{\dot{x}})\subset E\ \ \eta\text{-a.a. } \dot{x}\in X/G.$

Let $g\in G$. Since $\{\mu_{\dot{x}}-\delta_g*\mu_{\dot{x}}\}_{\dot{x}\in X/G}$ is an η-disintegration of $\mu-\delta_g*\mu$, it follows from Lemma 2.4 that $\dot{x}\to\|\mu_{\dot{x}}-\delta_g*\mu_{\dot{x}}\|$ is η-integrable and

(2) $\qquad \|\mu-\delta_g*\mu\|=\int_{X/G}\|\mu_{\dot{x}}-\delta_g*\mu_{\dot{x}}\|d\eta(\dot{x}).$

For each $\dot{x}\in X/G$, choose an element $x\in\pi^{-1}(\dot{x})$ and fix it. Let $\xi_{\dot{x}}$ be the measure in $M(G/G_x)$ such that $\tilde{B}_x(\xi_{\dot{x}})=\mu_{\dot{x}}$. By (1) and Proposition 1.2, we have

(3) $\qquad \xi_{\dot{x}}\in M_{E\cap G_x^\perp}(G/G_x)\ \ \eta\text{-a.a. } \dot{x}\in X/G.$

Hence there exists a Borel set $\tilde{B}\subset X/G$ such that $\eta(\tilde{B}^c)=0$ and

(4) $\qquad \xi_{\dot{x}}\in L^1(G/G_x)$ for $\dot{x}\in\tilde{B}$

since E is a Riesz set. Define $\zeta_{\dot{x}}\in L^1(G)$ by

(5) $\qquad \zeta_{\dot{x}}=\begin{cases} J_x(\xi_{\dot{x}}) & \text{for } \dot{x}\in\tilde{B} \\ 0 & \text{for } \dot{x}\notin\tilde{B}\end{cases}.$

Then, by (4), Lemma 1.2 and Proposition 5.2, we have

$$\begin{aligned}\|\mu_{\dot{x}}-\delta_g*\mu_{\dot{x}}\|&=\|B_x(J_x(\xi_{\dot{x}}))-\delta_g*B_x(J_x(\xi_{\dot{x}}))\|\\&=\|B_x(J_x(\xi_{\dot{x}}-\xi_{\dot{x}}*\delta_{\Pi_x(g)}))\|\\&=\|\tilde{B}_x(\xi_{\dot{x}}-\xi_{\dot{x}}*\delta_{\Pi_x(g)})\|\\&=\|\xi_{\dot{x}}-\xi_{\dot{x}}*\delta_{\Pi_x(g)})\|\\&=\|J_x(\xi_{\dot{x}})-J_x(\xi_{\dot{x}})*\delta_g\|\\&=\|\zeta_{\dot{x}}-\zeta_{\dot{x}}*\delta_g\|\end{aligned}$$

for $\dot{x}\in\tilde{B}$. Hence $\dot{x}\to\|\xi_{\dot{x}}-\xi_{\dot{x}}*\delta_g\|$ is η-integrable, and (2) yields

$$(6)\qquad \|\mu-\delta_g*\mu\|=\int_{X/G}\|\xi_{\dot{x}}-\xi_{\dot{x}}*\delta_g\|d\eta(\dot{x})$$

On the other hand, since $\xi_{\dot{x}}\in L^1(G)$, we have

$$\lim_{g\to 0}\|\xi_{\dot{x}}-\xi_{\dot{x}}*\delta_g\|=0$$

for all $\dot{x}\in X/G$. Hence, by (6) and Lebesgue's dominated convergence theorem, we have $\lim_{g\to 0}\|\mu-\delta_g*\mu\|=0$. This completes the proof.

Now we prove Theorem 1. 3. Let μ be a measure in $M(X)$ such that sp $(\mu)\subset E$. Suppose μ does not translate G-continuously. Let

$$\mu=\mu_1+\mu_2,$$

where $\mu_1\in M_{aG}(X)$ and $\mu_2\in M_{aG}(X)^{\perp}$. Then, by Proposition 5. 1 and the hypothesis, we have $\mu_2\neq 0$. Since $m_G*|\mu_2|\in M_{aG}(X)$, $|\mu_2|\perp m_G*|\mu_2|$. Hence, by Lemmas 2. 11 and 2. 13, there exists a countable subgroup Γ of $\hat{G}$ such that

(5.1) $\pi_H(\mu_2)\neq 0$, and

(5.2) $\pi_H(|\mu_2|)\perp\pi_H(m_G*|\mu_2|)$,

where $H=\Gamma^{\perp}$ and $\pi_H: X\to X/H$ is the canonical map. Let $q_H: G\to G/H$ be the natural homomorphism. Since $q_H(L^1(G))=L^1(G/H)$, we have

$$(5.3)\qquad \pi_H(M_{aG}(X))\subset M_{aG/H}(X/H)$$

by Lemma 2. 9. Next we calim that

$$(5.4)\qquad \pi_H(\mu)\notin M_{aG/H}(X/H).$$

On account of (5. 3), it is sufficient to show that $\pi_H(\mu_2)\notin M_{aG/H}(X/H)$. Suppose $\pi_H(\mu_2)\in M_{aG/H}(X/H)$. Then it follows from Proposition 5. 1 that $|\pi_H(\mu_2)|$ translates G/H-continuously. We note that

$$(5.5)\qquad |\pi_H(\mu_2)|\ll q_H(m_G)*|\pi_H(\mu_2)|.$$

In fact, let F be a Baire set in X/H with $q_H(m_G)*|\pi_H(\mu_2)|(F)=0$. Then, since $m_{G/H}=q_H(m_G)$, we have

$$\int_{G/H}\int_{X/H}\chi_F(u\cdot z)d|\pi_H(\mu_2)|(z)dm_{G/H}(u)=0.$$

Hence, since G/H is metrizable, there exists a sequence $\{u_n\}$ in G/H such that $\lim_{n\to\infty} u_n=0$ and

$$\int_{X/H} \chi_F(u_n \cdot z)\, d|\pi_H(\mu_2)|(z) = 0 \qquad (n=1, 2, 3, \ldots).$$

Thus

$$\delta_{u_n} * |\pi_H(\mu_2)|(F) = 0 \qquad (n=1, 2, 3, \ldots);$$

hence

$$\begin{aligned} |\pi_H(\mu_2)|(F) &= \lim_{n\to\infty} \delta_{u_n} * |\pi_H(\mu_2)|(F) \\ &= 0, \end{aligned}$$

which shows that (5. 5) holds. Since $\pi_H(m_G * |\mu_2|) = q_H(m_G) * \pi_H(|\mu_2|)$, (5. 5) contradicts (5. 1) and (5. 2). Thus (5. 4) holds.

Let $(G/H, X/H)$ be the transformation group induced by (G, X). Then $(G/H, X/H)$ satisfies conditions (D. I) and (D. II). It follows from Lamma 2. 10 that $\mathrm{sp}(\pi_H(\mu)) \subset \Gamma \cap E$. Since $\Gamma \cap E$ is a Riesz set in Γ and G/H is a metrizable compact abelian group, it follows from Lemma 5. 1 that $\pi_H(\mu)$ translates G/H-continuously. Hence, by Proposition 5. 1, we have $\pi_H(\mu) \in M_{aG/H}(X/H)$, which contradicts (5. 4). This completes the proof.

LEMMA 5. 2. *Let σ be a measure in $M^+(X)$ that is quasi-invariant. If G is a metrizable compact abelian group and (G, X) satisfies conditions (D. I) and (D. II), then the conclusion of Theorem* 1. 4 holds.

PROOF Let $E_0 = \mathrm{sp}(\mu)$. Then $E_0 \subset E$ and E_0 is also a Riesz set. Put $\eta = \pi(|\mu|)$, and let $\{\mu_{\dot{x}}\}_{\dot{x}\in X/G}$ be a canonical disintegration of μ. Then, by Lemma 2. 6, we have

(1) $\qquad \mathrm{sp}(\mu_{\dot{x}}) \subset E_0 \ \eta\text{-a.a. } \dot{x} \in X/G.$

Since E_0 is a Riesz set, it follows from (1) and a similar argument below (2) of Lemma 4. 2 that

(2) $\qquad |\mu_{\dot{x}}| \ll m_{\dot{x}} \ \eta\text{-a.a. } \dot{x} \in X/G.$

Let $\eta = \eta_a + \eta_s$ be the Lebesgue decomposition of η with respect to $\pi(\sigma)$. Then, for each bounded Baire function f on X, $\dot{x} \to \mu_{\dot{x}}(f)$ and $\dot{x} \to |\mu_{\dot{x}}|(f)$ are both η_a-measurable and η_s-measurable. Hence we can define measures ω_1, $\omega_2 \in M^+(X)$ and ξ_1, $\xi_2 \in M(X)$ as follows:

(3) $$\omega_1(f) = \int_{X/G} |\mu_{\dot{x}}|(f)\, d\eta_a(\dot{x}), \quad \omega_2(f) = \int_{X/G} |\mu_{\dot{x}}|(f)\, d\eta_s(\dot{x});$$
$$\xi_1(f) = \int_{X/G} \mu_{\dot{x}}(f)\, d\eta_a(\dot{x}), \quad \xi_2(f) = \int_{X/G} \mu_{\dot{x}}(f)\, d\eta_s(\dot{x})$$

for $f \in C_0(X)$. By Lemma 2. 4, we note that $|\xi_1| = \omega_1$ and $|\xi_2| = \omega_2$. By (2) and Lemma 2. 5, we have $\omega_1 \ll \sigma$. It is easy to see that $\omega_2 \perp \sigma$. Hence we get $\mu_a = \xi_1$ and $\mu_s = \xi_2$ since $\mu = \xi_1 + \xi_2$. Note that (3) holds for all bounded Baire functions f on X. Let $\gamma_0 \notin E_0$. Then, by (1) and Remark 1. 1 (II), we have

$$\gamma_0 * \mu_{\dot{x}} = 0 \quad \eta\text{-a.a. } \dot{x} \in X/G,$$

which together with Lemma 2. 3 yields

$$\begin{aligned}\gamma_0 * \mu_a(h) &= \gamma_0 * \xi_1(h) \\ &= \int_{X/G} \gamma_0 * \mu_{\dot{x}}(h)\, d\eta_a(\dot{x}) \\ &= 0\end{aligned}$$

for all $h \in C_0(X)$. Hence $\gamma_0 * \mu_a = 0$. Thus, by Remark 1. 1 (II), we have $\gamma_0 \notin \mathrm{sp}(\mu_a)$, which shows $\mathrm{sp}(\mu_a) \subset E_0 = \mathrm{sp}(\mu)$. By Remark 1. 1 (II), we also have $\mathrm{sp}(\mu_s) = \mathrm{sp}(\mu - \mu_a) \subset \mathrm{sp}(\mu)$. This completes the proof.

Now we prove Theorem 1. 4. As seen in the proof of Theorem 1. 1, we may assume that σ is a measure in $M^+(X)$ that is quasi-invariant. Let μ be a measure in $M(X)$ such that $\mathrm{sp}(\mu) \subset E$. Let $E_0 = \mathrm{sp}(\mu)$. We may assume $\mu_s \neq 0$. Suppose there exists $\gamma_0 \in \hat{G} \backslash E_0$ with $\gamma_0 * \mu_s \neq 0$. Then there exists a countable subgroup Γ of $\hat{G}$ with $\gamma_0 \in \Gamma$ satisfying (3. 3) and (3. 4). Let $H = \Gamma^{\perp}$, and let $\pi_H : X \to X/H$ be the canonical map. Then $\pi_H(\mu_s)$ is the singular part of $\pi_H(\mu)$ with respect to $\pi_H(\sigma)$, and $\pi_H(\sigma)$ is also quasi-invariant. It follows from Lemma 2. 10 that $\mathrm{sp}(\pi_H(\mu)) \subset E_0 \cap \Gamma$. Since $E_0 \cap \Gamma$ is a Riesz set in Γ and G/H is a metrizable compact abelian group, it follows from Lemma 5. 2 that

$$\mathrm{sp}(\pi_H(\mu_s)) \subset E_0 \cap \Gamma. \tag{5.6}$$

On the other hand, as in the proof of Theorem 1. 1, we get $\gamma_0 \in \mathrm{sp}(\pi_H(\mu_s))$. Hence, by (5. 6), we have $\gamma_0 \in E_0 \cap \Gamma$, which contradicts the choice of γ_0. This completes the proof.

REMARK 5. 1. Let G be a compact abelian group and E a subset of $\hat{G}$ satisfying condition (*). Then E is a Riesz set. However the converse is false in general. In fact, suppose $\hat{G}$ has a nonzero element γ_0 of finite order. Let $E = \{0, \gamma_0\}$. Then E_0 is a Riesz set. But it does not satisfy condition (*).

REMARK 5. 2. Let G be a compact abelian group with dual $\hat{G}$. It is known that a Sidon set in $\hat{G}$ is a Riesz set. It is also known that the union of a Riesz set and a Sidon set is a Riesz set (cf. [16, Corollary 4]). Thus it seems that there exist many Riesz sets. We can find appropriate references in [12, 10.5, p. 162-163].

§ 6. Transformation groups that satisfy conditions (C. I) and (C. II).

In this section, we shall show that, if (G, X) is a transformation group such that a compact abelian group G acts freely on a locally compact Hausdorff space X, then (G, X) satisfies conditins (C. I) and (C. II).

Let X be a locally compact Hausdorff space and $C_c(X)$ the space of all continuous functions on X with compact supports, with the topology of uniform convergence on compact sets. Let $C^*(X)$ be the dual space of $C_c(X)$. Then $C^*(X)$ coincides with the space of Radon measures on X. We will assume $C^*(X)$ is given the vague topology.

DEFINITION 6.1. Let W be a locally compact Hausdorff space and η a positive measure on W. A map $\lambda: W \to C^*(X)$ is called η-Lusin measurable if, for each compact set $K \subset W$, there is a sequence $\{K_i\}$ of pairewise disjoint compact sets such that (i) $\eta(K \backslash \bigcup_{i=1}^{\infty} K_i)=0$ and (ii) $\lambda|_{K_i}$ is continuous $(i \geq 1)$.

The following theorem is obtained from [11, 3.6. Theorem].

THEOREM 6.1. *Let (G, X) be a transformation group such that a compact metrizable group G acts freely on a locally compact Hausdorff space X. Let $\pi: X \to X/G$ be the canonical map. Let μ be a measure in $M^+(X)$, and put $\nu=\pi(\mu)$. Then there exists a map $\lambda: X/G \to M^+(X)$ $(y \to \lambda_y)$ with the following properties:*

(1) λ *is ν-Lusin measurable,*

(2) $\|\lambda_y\|=1,$

(3) $supp(\lambda_y) \subset \pi^{-1}(y),$

(4) $\mu(f)=\int_{X/G} \lambda_y(f)\, d\nu(y)$ *for* $f \in C_c(X)$.

The following theorem follows from Theorem 6.1 and Proposition 1.6.

THEOREM 6.2. *Under the assumption in the previous theorem, there exists a family $\{\lambda_x\}_{x \in X/G}$ of measures in $M^+(X)$ with the following properties:*

(1) $y \to \lambda_y(f)$ is ν-measurable for each bounded Baire function f on X,

(2) $\|\lambda_y\|=1,$

(3) $supp(\lambda_y) \subset \pi^{-1}(y),$

(4) $\mu(f)=\int_{X/G} \lambda_y(f)\, d\nu(y)$ *for each bounded Baire function f on X.*

Johnson ([11]) also obtained a uniqueness theorem of ν-Lusin-measurable disintegration. However the following uniqueness theorem holds.

THEOREM 6.3. *Let (G, X) be as in Theorem 6.1, and let $\nu \in M^+(Y)$, where $Y = X/G$. Suppose $\{\lambda_y^1\}_{y\in Y}$ and $\{\lambda_y^1\}_{y\in Y}$ are families of measures in $M(X)$ with the following properties:*

(1) $y \to \lambda_y^i(f)$ *is a ν-integrable function for each bounded Baire function f on X $(i=1, 2)$,*

(2) $supp(\lambda_y^i) \subset \pi^{-1}(y)$ $(i=1, 2)$,

(3) $\int_Y \lambda_y^1(f)\, d\nu(y) = \int_Y \lambda_y^2(f)\, d\nu(y)$ *for all bounded Baire functions f on X.*

Then $\lambda_y^1 = \lambda_y^2$ ν-a.a. $y \in Y$.

We prove Theorem 6.3 by modifying Johnson's method slightly. Before giving the proof, we prepare several lemmas.

LEMMA 6.1. *Let (G, X) be a transformation group such that a compact Lie group G acts freely on a compact Hausdorff space X. Let $\pi: X \to X/G$ be the canonical map. Then, for each $x \in X$, there exists a compact neighborhood U_x of x, which is a G_δ-set, and a compact set $F_x \subset U_x$ such that $\pi^{-1}(y) \cap F_x$ is a single point whenever $y \in \pi(U_x)$. Moreover U_x can be chosen to that $G \cdot U_x = U_x$.*

PROOF. By [11, 1.1. Theorem, p. 251], there exists a compact neighborhood U' of x and a compact $F' \subset U'$ such that $G \cdot U' = U'$ and $\pi^{-1}(y) \cap F'$ is a single point whenever $y \in \pi(U')$. Then there exists a compact neighborhood W of x, which is a G_δ-set, such that $W \subset U'$. Put $U_x = G \cdot W$ and $F_x = F' \cap G \cdot W$. Then we can easily verify that U_x and F_x are the desired sets. This completes the proof.

Let (G, X) be as in Lemma 6.1. For $x \in X$, let U_x and F_x be the sets obtained in Lemma 6.1, and choose sets U_{x_i} $(1 \le i \le r)$ which cover X. Put $U_i = U_{x_i}$, $F_i = F_{x_i}$ and $V_i = \pi(U_i)$. And let $A_1 = V_1$, $A_j = V_j \setminus \bigcup_{k=1}^{j-1} V_k$ $(2 \le j \le r)$ and $B_i = \pi^{-1}(V_i)$ $(1 \le i \le r)$. Thn B_i are Baire sets. In fact, since $G \cdot U_i = U_i$, we have $B_1 = U_1$ and $B_i = U_i \setminus \bigcup_{j=1}^{i-1} U_j$ $(2 \le i \le r)$. Hence B_i are Baire sets because U_i are compact G_δ-sets. Moreover $A_i (1 \le i \le r)$ are pairewise disjoint and $Y = \bigcup_{i=1}^{r} A_i$. Define maps $\tau_i: V_i \to U_i$ by $\{\tau_i(y)\} = F_i \cap \pi^{-1}(y)$, and define $\tau: Y \to X$ by $\tau|_{A_i} = \tau_i$ $(1 \le i \le r)$. Then the following lemma holds.

LEMMA 6.2 (cf. [11, 1.3. Lemma, p. 252]).
The maps $(g, x) \to g \cdot x: G \times F_i \to U_i$ and $\nu_i: (g, y) \to g \cdot \tau_i(y): G \times V_i \to U_i$ are homeomorphisms $(1 \le i \le r)$.

DEFINITION 6.2. Let $\pi_2 : X \to G$ be a map defined by

$$\pi_2(x) \cdot \tau(y) = x,$$

where $x \in X$ and $y = \pi(x)$.

LEMMA 6.3. *For each $h \in C(G)$, $h \circ \pi_2$ is a Baire measurable function on X.*

PROOF. By Lemma 6.2, we can define continuous maps $\pi_2^i : U_i \to G$ by $\pi_2^i = \pi_G^i \circ \nu_i^{-1}$, where $\pi_G^i : G \times V_i \to G$ are the projections $(1 \leq i \leq r)$. Then we have

$$h \circ \pi_2 = \sum_{i=1}^{r} \chi_{B_i} \cdot \widetilde{h \circ \pi_2^i},$$

where χ_{B_i} are the characteristic functions of B_i and $\widetilde{h \circ \pi_2^i}$ are continuous extensions of $h \circ \pi_2^i$ to X $(1 \leq i \leq r)$. Hence, by the fact that B_i are Baire sets, the lemma is obtained.

DEFINITION 6.3. A locally compact group G is said to have no small subgroups if there is a neighborhood of the identity e which has no other subgroups than $\{e\}$.

DEFINITION 6.4. Let G be a locally compact group and $\{G_l\}_{l=1}^{\infty}$ a sequence of normal closed subgroups of G. We write $G_l \downarrow e$ if the following hold:

(i) $G_l \supset G_{l+1}$ $(l = 1, 2, 3, \ldots)$;
(ii) for any neighborhood U of e, there exists G_l such that $G_l \subset U$.

The following lemma, which was stated in [11, p. 255] without proof, will be needed later on. We give its proof for completeness.

LEMMA 6.4. *Let G be a metrizable compact group. Then there exists a sequence $\{G_l\}$ of closed normal subgroups of G such that $G_l \downarrow e$ and G/G_l are Lie groups $(l = 1, 2, 3, \ldots)$.*

PROOF. First we note, by [22, Thorem 3], that a locally compact group which has no small subgroups is a Lie group. Let $\{U_n\}$ be a countable base of e. Then, for each $l \in \boldsymbol{N}$, it follows from [14, 4.6 Theorem, p. 175] that there exists a closed normal subgroup H_l of G, which is included in U_l, such that G/H_l has no small subgroups. Put $G_1 = H_1$, $G_2 = H_1 \cap H_2$, $\ldots$, $G_n = H_1 \cap \ldots \cap H_n$, $\ldots$. Then G_n are closed normal subgroups of G and $G_n \downarrow e$. Moreover, by [14, 4.7.1 Lemma, p. 177], G/G_n have no small subgroups $(n = 1, 2, 3, \ldots)$. Hence G/G_n are Lie groups $(n = 1, 2, 3, \ldots)$. This completes the proof.

We return to the proof of Theorem 6. 3. We first consider the case that X is compact and G is a compact Lie group. Let $\{\lambda_y^1\}_{y\in Y}$ and $\{\lambda_y^2\}_{y\in Y}$ be families of measures in $M(X)$ satisfying (1)-(3) in Theorem 6. 3. For each $y\in Y$, we define $\omega_y^i\in M(G)$ $(i=1,2)$ by

(4) $\omega_y^i=\pi_2(\lambda_y^i)$ (i.e., $\omega_y^i(h)=\lambda_y^i(h\circ\pi_2)$ for $h\in C(G)$).

Then we have, by Lemma 6. 3,

(5) $y\rightarrow\omega_y^i(h)$ is a ν-integrable function for each $h\in C(G)$.

Let $f\in C(Y)$ and $h\in C(G)$. It follows from Lemma 6. 3 that $(f\circ\pi)(h\circ\pi_2)$ is a bounded Baire function on X ; hence (3) yields

(6) $$\int_Y\lambda_y^1((f\circ\pi)(h\circ\pi_2))d\nu(y)=\int_Y\lambda_y^2((f\circ\pi)(h\circ\pi_2))d\nu(y).$$

On the other hand, we have

(7) $$\begin{aligned}\int_Y\lambda_y^i((f\circ\pi)(h\circ\pi_2))d\nu(y)&=\int_Y f(y)\lambda_y^i(h\circ\pi_2)d\nu(y)\\&=\int_Y f(y)\omega_y^i(h)d\nu(y).\end{aligned}$$

Hence, by (6) and (7), we get

$$\int_Y f(y)\omega_y^1(h)d\nu(y)=\int_Y f(y)\omega_y^2(h)d\nu(y)$$

for all $f\in C(Y)$. Since $C(G)$ is separable, it follows from (5) that

(8) $\omega_y^1=\omega_y^2$ ν-a.a. $y\in Y$.

For $y\in Y$, let $x\in\pi^{-1}(y)$. Then $x=\pi_2(x)\cdot\tau(y)$. Let $B_{\tau(y)}: G\rightarrow G\cdot\tau(y)(\subset X)$ be the homeomorphism defined by $B_{\tau(y)}(g)=g\cdot\tau(y)$. For any $F\in C(X)$, define $h\in C(G)$ by $h=F\circ B_{\tau(y)}$. Then, since $h\circ\pi_2(x)=F\circ B_{\tau(y)}(\pi_2(x))=F(\pi_2(x)\cdot\tau(y))=F(x)$, we have

$$\omega_y^i(h)=\lambda_y^i(h\circ\pi_2)=\lambda_y^i(F).$$

Hence we have, by (8),

$$\begin{aligned}\lambda_y^1(F)&=\omega_y^1(h)\\&=\omega_y^2(h)\\&=\lambda_y^2(F)\end{aligned}$$

for ν-a.a. $y\in Y$ and any $F\in C(Y)$, and so

$$\lambda_y^1=\lambda_y^2\ \nu\text{-a.a. } y\in Y.$$

Thus, in this case, the theorem is obtained.

Next we consider the case that G is a metrizable compact group and X is a compact Hausdorff space. It follows from Lemma 6.4 that there exists a sequence $\{G_l\}$ of closed normal subgroups of G such that

(9) $G_l \downarrow e$, and

(10) G/G_l are Lie groups ($l=1, 2, 3, \ldots$).

Let $\pi_l: X \to X/G_l$ be the canonical maps, and put $X_l = X/G_l$. Then, by (9) and the Stone-Weierstrass theorem, we have

(11) $\bigcup_{l=1}^{\infty}\{f \circ \pi_l : f \in C(X_l)\}$ is dense in $C(X)$.

On the other hand, (G, X) yields a new transformation group $(G/G_l, X_l)$. Evidently, G/G_l acts freely on X_l, and G/G_l is a Lie group by (10). We note that $X_l/G/G_l \cong X/G = Y$. For $y \in Y$, we define a measure $\lambda_y^{i,l} \in M(X_l)$ by

$$\lambda_y^{i,l}(f) = \lambda_y^i(f \circ \pi_l)$$

for $f \in C(X_l)$ $(i=1, 2)$. We note that

(12) $f \circ \pi_l$ is a bounded Baire function on X for every bounded Baire function f on X_l.

In fact, put

$$\mathscr{F} = \{A \subset X_l : \pi_l^{-1}(A) \text{ is a Baire set in } X\}.$$

Then we can verify that $\mathscr{F}$ is a σ-algebra containing all compact G_δ-sets in X_l. Hence $\mathscr{B}_0(X_l)$ is included in $\mathscr{F}$. Therefore $\pi_l^{-1}(A)$ is a Baire set in X for each Baire set A in X_l, which shows that (12) holds.

Let $\pi_{X_l}: X_l \to X_l/G/G_l \cong Y$ be the canonical map. Then, by (1)-(3) and (12), we have

(13) $y \to \lambda_y^{i,l}(f)$ is a ν-integrable function for each bounded Baire function f on X_l $(i=1, 2)$,

(14) $\mathrm{supp}(\lambda_y^{i,l}) \subset \pi_{X_l}^{-1}(y) (= \pi_l(\pi^{-1}(y))) (i=1, 2)$, and

(15) $\int_Y \lambda_y^{1,l}(f)\, d\nu(y) = \int_Y \lambda_y^{2,l}(f)\, d\nu(y)$ for all bounded Baire functions f on X_l.

Since G/G_l is a compact Lie group, it follows from (13)-(15) and the last case that there exists a Borel set B_l in Y such that $\nu(B_l^c) = 0$ and $\lambda_y^{1,l} = \lambda_y^{2,l}$ for $y \in B_l$. Put $B = \bigcap_{l=1}^{\infty} B_l$. Then $\nu(B^c) = 0$. For any $f \in C(X)$, choose $f_{l_n} \in C$

(X_{l_n}) so that $\lim_{n\to\infty}\|f - f_{l_n}\circ\pi_{l_n}\|_\infty = 0$. Then, for $y\in B$, we have

$$\begin{aligned}\lambda_y^1(f) &= \lim_{n\to\infty}\lambda_y^1(f_{l_n}\circ\pi_{l_n})\\ &= \lim_{n\to\infty}\lambda_y^{1, l_n}(f_{l_n})\\ &= \lim_{n\to\infty}\lambda_y^{2, l_n}(f_{l_n})\\ &= \lim_{n\to\infty}\lambda_y^2(f_{l_n}\circ\pi_{l_n})\\ &= \lambda_y^2(f),\end{aligned}$$

which shows that $\lambda_y^1 = \lambda_y^2$ ν-a.a. $y\in Y$. Hence, in this case, the theorem is also obtained.

Finally we consider the case that G is a metrizable compact group and X is a locally compact Hausdorff space. Since ν is bounded regular, there exists a sequence $\{K_j\}$ of pairewise disjoint compact G_δ-sets in Y such that $\nu(Y\setminus\bigcup_{j=1}^\infty K_j)=0$. Put $L_j=\pi^{-1}(K_j)$. Then L_j are compact G_δ-sets in X ($j=1, 2, 3, \ldots$). For each $j\in\boldsymbol{N}$, we have, by (1)-(3),

(16) $y\to\lambda_y^i(f)$ is a $\nu|_{K_i}$-integrable function on K_j for each bounded Baire function f on L_j,

(17) $\mathrm{supp}(\lambda_y^i)\subset(\pi|_{L_j})^{-1}(y)$ for $y\in K_j$, and

(18) $\int_{K_j}\lambda_y^1(f)\,d(\nu|_{K_j})(y)=\int_{K_j}\lambda_y^2(f)\,d(\nu|_{K_j})(y)$ for all bounded Baire functions f on L_j.

Since (G, X) yields a transformation group (G, L_j) such that G acts freely on L_j, it follows from (16)-(18) and the last case that

$$\lambda_y^1=\lambda_y^2 \quad \nu|_{K_j}\text{-a.a. } y\in K_j,$$

which yields

$$\lambda_y^1=\lambda_y^2 \quad \nu\text{-a.a. } y\in Y$$

because $\nu(Y\setminus\bigcup_{j=1}^\infty K_j)=0$. This completes the proof of Theorem 6. 3.

By Theorems 6. 2 and 6. 3, we obtain the following theorem.

THEOREM 6. 4. *Let (G, X) be a transformation group such that a compact abelian group G acts freely on a locally compact Hausdorff space X. Then (G, X) satisfies conditions (C. I) and (C. II).*

REMARK 6. 1. Let (G, X) be a transformation group such that a compact abelian group G acts on a locally compact metric space X. Then (G, X) satisfies conditions (C. I) and (C. II).

In fact, since every measure in $M(X)$ is bounded regular, we may assume that X is σ-compact. Then, for any closed subgroup H of G, X/H is a σ-compact metric space. Hence (G, X) satisfies conditions (C. I) and (C. II).

REMARK 6. 2. If conditions (D. I) and (D. II) are satisfied for any transformation group such that a metrizable compact abelian group acts on a locally compact Hausdorff space, then conditions (C. I) and (C. II) are satisfied for any transformation group such that a compact abelian group acts on a locally compact Hausdorff space. The author does not know whether conditions (D. I) and (D. II) are satisfied or not for any transformation group such that a metrizable compact abelian group acts on a locally compact Hausdorff space.

Finally the author wishes to express his thanks to the referee for his valuable comments and suggestions.

References

[1] N. BOURBAKI, Intégration, Éléments de Mathématique, Livre VI, ch. 6, Paris, Herman, 1959.

[2] K. DELEEUW and I. GLICKSBERG, Quasi-invariance and analyticity of measures on compact groups, Acta Math. 109 (1963), 179-205

[3] K. DELEEUW, The Fubini theorem and convolution formula for regular measures, Math. Scand. 11 (1962), 117-122.

[4] C. DELLACHERIE and P.-A. MEYER, Probabilities and Potential (North-Holland Mathematics Studies Vol. 29), North-Holland, Amsterdam-New York-Oxford, 1978.

[5] R. DOSS, On measures with small transforms, Pacific J. Math. 26 (1968), 257-263.

[6] G. B. FOLLAND, Real Analyss: Modern Techniques and Their Applications, Wiley-Interscience, New York, 1984.

[7] F. FORELLI, Analytic and quasi-invariant measures, Acta Math. 118 (1967), 33-57.

[8] C. C. GRAHAM and O. C. MCGEHEE, Essays in Commutative Harmonic Analysis, Springer-Verlag, New York-Heidelberg-Berlin, 1979.

[9] H. HELSON and D. LOWDENSLAGER, Prediction theory and Fourier series in several variables, Acta Math. 99 (1958), 165-202.

[10] E. HEWITT and K. A. ROSS, Abstract Harmonic Analysis, Volumes I and II, New York-Heidelberg-Berlin, Springer-Verlag, 1963 and 1970.

[11] R. A. JOHNSON, Disintegration of measures on compact transformation groups, Trans. Amer. Math. Soc. 233 (1977), 249-264.

[12] J. M. LÓPEZ and K. A. ROSS, Sidon sets, Marcel Dekker, New York, 1975.

[13] V. MANDREKAR and M. NADKARNI, Quasi-invariance of analytic measures on compact groups, Bull. Amer. Math. Soc. 73 (1967), 915-920.

[14] D. MONTGOMERY and L. ZIPPIN, Topological Transformation Groups, Interscience, New York, 1955.

[15] W. PARRY, Compact abelian group extension of discrete dynamical systems, Z. Wahr. Verw. Geb. 13 (1969), 95-113.

[16] L. PIGNO and S. SAEKI, Fourier-Stieltjes transforms which vanish at infinity, Math. Z.

141 (1975), 83-91.
[17] L. PIGNO and B. SMITH, On measures of analytic type, Proc. Amer. Math. Soc. 82 (1981), 541-547.
[18] Y. TAKAHASHI and H. YAMAGUCHI, On measures which are continuous by certain translation, Hokkaido Math. J. 13 (1984), 109-117.
[19] H. L. ROYDEN, Real Analysis (Second Edition), The Macmillan Company, Collier-Macmillan Ltd., London, 1968.
[20] W. RUDIN, Fourier Analysis on Groups, Interscience, New York, 1962.
[21] W. RUDIN, Real and Complex Analysis (Second Edition), Tata McGraw-Hill Publishing Company Ltd., New Delhi, 1978.
[22] H. YAMABE, A generalization of a theorem of Gleason, Ann. Math. 58 (1953), 351-365.
[23] H. YAMAGUCHI, On the product of a Riesz set and a small p set, Proc. Amer. Math. Soc. 81 (1981), 273-278.
[24] H. YAMAGUCHI, A property of some Fourier-Stieltjes transforms, Pacific J. Math. 108 (1983), 243-256.

Department of Mathematics,
Josai University