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A test for membership in Lorentz spaces and some applications

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1. Introduction

Throughout this paper G will denote a locally compact Abelian group with Haar measure λ. For $1\leq r\leq\infty$, $L_r(G)$ will denote the usual Lebesgue space, with norm $\|\cdot\|_r$, defined on the measure space (G, λ). Let $M(G)$ denote the space of all bounded complex-valued regular Borel measures on G. For $p=1=q$, or $1<p<\infty$ and $1\leq q\leq\infty$, let $L_{p,q}(G)$ denote the Lorentz space defined on (G, λ) with norm $\|\cdot\|_{(p,q)}$ (see Section 2 for the definition of the Lorentz spaces and some useful facts about these spaces). Let Γ denote the dual group of G and θ the Haar measure on Γ. The spaces $L_r(\Gamma)$ and $L_{p,q}(\Gamma)$ are defined similarly. The main purpose of this paper is to prove Theorem 1 below.

THEOREM 1. *Let $(e_\alpha)_{\alpha\in D}$ be an approximate identity in $L_1(G)$ with $\sup_\alpha\|e_\alpha\|_1\leq 1$.*

(i) *If $1<p, q<\infty$ or $p=1=q$, and if k is a continuous function on Γ such that each $k\hat{e}_\alpha\in L_{p,q}(\Gamma)$ and $\sup_\alpha\|k\hat{e}_\alpha\|_{(p,q)}<\infty$, then $k\in L_{p,q}(\Gamma)$ and $\|k\|_{(p,q)}=\sup_\alpha\|k\hat{e}_\alpha\|_{(p,q)}$, where in the case $p=1=q$ it is further assumed that k is bounded.*

(ii) *If $p=1=q$, $1<p<2$ and $1<q<\infty$, or $1<q\leq p=2$, and if $\mu\in M(G)$ such that each $\hat{\mu}\hat{e}_\alpha\in L_{p,q}(\Gamma)$ and $\sup_\alpha\|\hat{\mu}\hat{e}_\alpha\|_{(p,q)}<\infty$, then μ is absolutely continuous.*

Theorem 1 arises partly from our effort to fill some gaps in the proof of Theorem 1 in Burnham, Krogstad and Larsen [1] (see lines -11 and -2, p. 96), and partly from our effort to give simpler proofs of the main results in Chen and Lai [2]. In addition to obtaining the main results in Chen and Lai [2] as an easy consequence of Theorem 1 (see Theorem 2 below), we also apply Theorem 1 to obtain a partial converse of Hölder's inequality (see Theorem 3). By applying the method used in the proof of Theorem 1, we also give in the final section a characterization of the Fourier transforms of functions in $L_p(G)$ for $1<p\leq 2$ (see Theorem 4).

2. Preliminaries

DEFINITION 1. Let f be a measurable function defined on the measure space (G, λ). For $y\geq 0$, we define

$$\lambda_f(y)=\lambda\{x\in G : |f(x)|>y\}.$$

The non-increasing rearrangement f^* of f is defined by

$$\begin{aligned} f^*(x)&=\inf\{y : y>0 \text{ and } \lambda_f(y)\leq x\} \\ &=\sup\{y : y>0 \text{ and } \lambda_f(y)>x\}, \end{aligned}$$

with the conventions inf $\phi=\infty$ and sup $\phi=0$. For $x>0$, we define

$$f^{**}(x)=x^{-1}\int_0^x f^*(t)dt.$$

We also define

$$\begin{aligned} \|f\|^*_{(p,q)}&=\left\{\int_0^\infty [x^{1/p}f^*(x)]^q\frac{dx}{x}\right\}^{1/q}, \ 1\leq p<\infty,\ 1\leq q<\infty; \\ \|f\|^*_{(p,\infty)}&=\sup_{x>0} x^{1/p}f^*(x),\ 1\leq p<\infty; \\ L_{p,q}(G)&=\{f : \|f\|^*_{(p,q)}<\infty\}. \end{aligned}$$

The spaces $L_{p,q}(\Gamma)$ are defined in the same way. For $p\neq 1$, $q\neq 1$, if we replace $f^*(x)$ by $f^{**}(x)$ in the definition of $\|f\|^*_{(p,q)}$, the resulting number will be denoted by $\|f\|_{(p,q)}$; and we define $\|f\|_{(1,1)}=\|f\|^*_{(1,1)}$ which is equal to $\|f\|_1$.

The facts given in the following proposition are well-known (see O'Neil [5] or Yap [7]) and they are stated here for easy reference.

PROPOSITION 1. (i) *For* $1\leq p<\infty$, *we have* $L_{p,p}(G)=L_p(G)$ *and* $\|\cdot\|^*_{(p,p)}=\|\cdot\|_p\leq\|\cdot\|_{(p,p)}$.

(ii) *For* $1\leq p<\infty$ *and* $1\leq q_1\leq q_2\leq\infty$, *we have* $L_{p,q_1}(G)\subset L_{p,q_2}(G)$.

(iii) *For* $1<p<\infty$ *and* $1\leq q\leq\infty$, *we have*

$$\|\cdot\|^*_{(p,q)}\leq\|\cdot\|_{(p,q)}\leq p/(p-1)\|\cdot\|^*_{(p,q)}$$

and $L_{p,q}(G)$ *is a Banach space with respect to the norm* $\|\cdot\|_{(p,q)}$.

NOTATIONAL CONVENTION. In the sequel when we refer to the space $L_{p,q}(G)$ with $1<p<\infty$ and $1\leq q\leq\infty$, it is tacitly assumed that $L_{p,q}(G)$ is endowed with the norm $\|\cdot\|_{(p,q)}$ (not $\|\cdot\|^*_{(p,q)}$; in fact $\|\cdot\|^*_{(p,q)}$ is not a norm) and the space $L_{1,1}(G)=L_1(G)$ is given the norm $\|\cdot\|_{(1,1)}=\|\cdot\|_1$. A similar convention holds when G is replaced by Γ. If f is an extended real-valued or complex-valued function on a set X, we will use $\mathrm{supp}(f)$ to denote the set $\{x\in X : f(x)\neq 0\}$. For $1\leq r\leq\infty$, r' will denote the number such that $1/r+1/r'=1$. All terms and notation not explained in this paper are as in Hewitt and Ross [3].

Since every non-discrete locally compact topological group G that is not σ-compact contains a locally null non-null subset (see Hewitt and Ross [3, (16.14)]), the following lemma is needed in the proof of our main result.

LEMMA 1. *If $g\in L_{1,1}(\Gamma)$, or $g\in L_{p,q}(\Gamma)$ with $1<p<\infty$ and $1\leq q\leq\infty$, and if f is a complex-valued continuous function on Γ such that $g=f$ locally almost everywhere (with respect to the Haar measure θ on Γ), then $g=f$ almost everywhere.*

PROOF. It is easy to see that the set $\mathrm{supp}(g)\equiv\{x\in\Gamma : g(x)\neq 0\}$ has σ-finite measure. Define

$$A_1=\left\{x : \frac{1}{2}<|f(x)|<\infty\right\},$$
$$A_n=\left\{x : \frac{1}{n+1}<|f(x)|<\frac{1}{n-1}\right\} \text{ for } n=2,3,\dots,$$
$$B=\{x : f(x)=g(x)\neq 0\}.$$

Note that, for each positive integer n,

$$|g(x)|=|f(x)|>\frac{1}{n+1} \text{ for all } x\in A_n\cap B,$$

and so $\theta(A_n\cap B)<\infty$. Since A_n is an open set and θ is regular, we have

$$\begin{aligned}\theta(A_n)&=\sup\{\theta(F) : F\subset A_n \text{ and } F \text{ compact}\}\\ &=\sup\{\theta(F\cap B)+\theta(F\cap(\mathrm{supp}(f)\backslash B)) : F\subset A_n \text{ and } F \text{ compact}\}.\end{aligned}$$

Since $g=f$ l. a. e., $\mathrm{supp}(f)\backslash B$ is a locally null set. Hence

$$\theta(A_n)=\sup\{\theta(F\cap B) : F\subset A_n \text{ and } F \text{ compact}\}\leq\theta(A_n\cap B)<\infty.$$

Since we obviously have $\mathrm{supp}(f)=\bigcup_{n=1}^{\infty}A_n$, $\mathrm{supp}(f)\backslash B$ is a locally null set with σ-finite measure. Hence $\mathrm{supp}(f)\backslash B$ has measure zero. The set $\mathrm{supp}(g)\backslash B$ is also a locally null set with σ-finite measure, and so it has measure zero. Hence $g=f$ a. e.

3. Proof of Theorem 1

Before we give our proof of Theorem 1, we recall that Hunt [4, (2.7)] has proved the following theorem for $L_{p,q}$ space, $1<p$, $q<\infty$: If $g\in L_{p',q'}$

and T_g is defined on $L_{p,q}$ by $T_g(f)=\int fg$, then T_g is a bounded linear functional on $L_{p,q}$; conversely, if T is a bounded linear functional on $L_{p,q}$, then there exists a function g in $L_{p',q'}$ such that $T=T_g$. Thus the conjugate space of $L_{p,q}$ is $L_{p',q'}$ and hence $L_{p,q}$ is reflexive. [It should be noted here that the norm used by Hunt is equivalent to ours and that his theorem is indeed applicable.]

PROOF OF THEOREM 1. (i) Our first task is to show that $k\in L_{p,q}(\Gamma)$ for the case $1<p, q<\infty$. Since $(k\hat{e}_\alpha)_{\alpha\in D}$ is a bounded net in $L_{p,q}(\Gamma)$ and $L_{p,q}(\Gamma)$ is reflexive, Alaoglu's theorem tells us that $(k\hat{e}_\alpha)_{\alpha\in D}$ has a subnet (which we continue to write as $(k\hat{e}_\alpha)_{\alpha\in D}$) such that $k\hat{e}_\alpha\rightarrow h$ weakly in $L_{p,q}(\Gamma)$ for some $h\in L_{p,q}(\Gamma)$. Hence, by Hunt's theorem [loc. cit.], we have

$$(1)\qquad \int_\Gamma k\hat{e}_\alpha g\longrightarrow\int_\Gamma hg$$

for every $g\in L_{p',q'}(\Gamma)$. We will show that $k=h$ a. e. In view of Lemma 1, it suffices to show that $k=h$ l. a. e. Now let Δ be any compact subset of Γ. Let ϕ be a function in $C_{oo}(\Gamma)$ such that $\phi=1$ on Δ and $\phi(\Gamma)\subset[0, 1]$. (Here $C_{oo}(\Gamma)$ denotes the space of all continuous functions f on Γ such that the closure of supp(f) is compact.) Let Δ_1 denote the compact closure of supp (ϕ). Since

$$\hat{e}_\alpha\longrightarrow 1 \text{ uniformly on } \Delta_1 \text{ and } \int_{\Delta_1}|k|<\infty,$$

we have

$$\int_{\Delta_1}k\hat{e}_\alpha\, sgn(h-k)\phi\longrightarrow\int_{\Delta_1}k\, sgn(h-k)\phi.$$

By (1) we also have

$$\int_{\Delta_1}k\hat{e}_\alpha\, sgn(h-k)\phi\longrightarrow\int_{\Delta_1}h\, sgn(h-k)\phi.$$

Hence we have

$$\int_{\Delta_1}h\, sgn(h-k)\phi=\int_{\Delta_1}k\, sgn(h-k)\phi,$$

and it follows that

$$\int_{\Delta_1}|h-k|\phi=0 \text{ and } \int_{\Delta}|h-k|=0.$$

Thus $h=k$ a. e. on Δ. Since Δ is an arbitrary compact subset of Γ, $h=k$ l. a. e. By Lemma 1, $k\in L_{p,q}(\Gamma)$.

Our next task is to show that $\|k\|_{(p,q)}=\sup_\alpha\|k\hat{e}_\alpha\|_{(p,q)}$ for $1<p, q<\infty$. Since $\|\hat{e}_\alpha\|_\infty\leq\|e_\alpha\|_1\leq 1$ for all α, it is clear that $\sup_\alpha\|k\hat{e}_\alpha\|_{(p,q)}\leq\|k\|_{(p,q)}$. Next we write $\mathrm{supp}(k)=\bigcup_{n=1}^{\infty}K_n\cup B$, where B is a set of measure zero, and $(K_n)_{n=1}^{\infty}$ is an increasing sequence of compact sets of positive measure. Since $|k\chi_{K_n}|\uparrow|k|$, where χ_{K_n} denotes the characteristic function of K_n, it is easy to see that $\|k\|_{(p,q)}=\lim\|k\chi_{K_n}\|_{(p,q)}$. Now let $\varepsilon>0$. Then there exists a positive integer N such that

$$\|k\|_{(p,q)}\leq\|k\chi_{K_N}\|_{(p,q)}+\frac{\varepsilon}{2}. \tag{1}$$

Since $k\hat{e}_\alpha\to k$ uniformly on K_N, there exists $\alpha_N\in D$ such that

$$|k\hat{e}_{\alpha_N}-k|\leq\delta \text{ on } K_N,$$

where

$$\delta=(q/pp')^{1/q}\frac{\varepsilon}{2\theta(K_N)^{1/p}}.$$

Now put

$$g=|k\hat{e}_{\alpha_N}\chi_{K_N}-k\chi_{K_N}|.$$

A straightforward computation shows that

$$g^{**}(t)\leq\begin{cases}\delta & \text{if } 0<t\leq\theta(K_N),\\ \dfrac{\delta\theta(K_N)}{t} & \text{if } \theta(K_N)<t,\end{cases}$$

and

$$\|g\|_{(p,q)}<\varepsilon/2.$$

Hence, by (1), we have

$$\begin{aligned}\|k\|_{(p,q)}&\leqq\|k\chi_{K_N}\|_{(p,q)}+\varepsilon/2\\ &\leq\|g\|_{(p,q)}+\|k\hat{e}_{\alpha_N}\chi_{K_N}\|_{(p,q)}+\varepsilon/2\\ &<\varepsilon/2+\|k\hat{e}_{\alpha_N}\|_{(p,q)}+\varepsilon/2\\ &\leq\sup_\alpha\|k\hat{e}_\alpha\|_{(p,q)}+\varepsilon.\end{aligned}$$

This completes the proof that $\|k\|_{(p,q)}\leq\sup_\alpha\|k\hat{e}_\alpha\|_{(p,q)}$ and hence $\|k\|_{(p,q)}=\sup_\alpha\|k\hat{e}_\alpha\|_{(p,q)}$ for $1<p,\ q<\infty$.

We now turn to the case $p=1=q$. Recall that $L_{1,1}(\Gamma)=L_1(\Gamma)$ and this space is given the norm $\|\cdot\|_1$. Since $\sup_\alpha\|k\hat{e}_\alpha\|_1\equiv C<\infty$, we have $\|k\hat{e}_\alpha\|_2^2\leq C\|k\|_\infty$ for all α. Hence, by Proposition 1, $\sup_\alpha\|k\hat{e}_\alpha\|_{(2,2)}<\infty$ and so $k\in L_{2,2}(\Gamma)=L_2(\Gamma)$. Since k is continuous and Haar measure is regular, we can write $\mathrm{supp}(k)$ as $\bigcup_{n=1}^{\infty}K_n\cup B$, where each K_n is compact, B has measure zero, and $K_n\subset K_{n+1}$. For each positive integer n, $k\hat{e}_\alpha\to k$ uniformly on K_n

and so there exists $\alpha_n \in D$ such that

$$|k\hat{e}_\alpha - k| < \frac{1}{n} \text{ on } K_n \text{ for all } \alpha \geq \alpha_n.$$

We may assume that $\alpha_n \leq \alpha_{n+1}$. It is easy to verify that $k\hat{e}_{\alpha_n} \to k$ a. e. on Γ. Hence, by Fatou's Lemma, we have

$$\int_\Gamma |k| \leq \underline{\lim} \int_\Gamma |k\hat{e}_{\alpha_n}| \leq C < \infty.$$

This shows that $k \in L_1(\Gamma)$ and $\|k\|_1 \leq \sup_\alpha \|k\hat{e}_\alpha\|_1$. Since $\sup_\alpha \|k\hat{e}_\alpha\|_1 \leq \|k\|_1$ is obvious, we have proved (i) when $p=1=q$.

We now prove part (ii). By Hewitt and Ross [3, (31.33)], it suffices to show that $\hat{\mu} \in L_2(\Gamma)$. If $p=1=q$, then the hypothesis says that $\sup_\alpha \|\hat{\mu}\hat{e}_\alpha\|_1 \equiv C < \infty$. It follows that $\sup_\alpha \|\hat{\mu}\hat{e}_\alpha\|_2^2 \leq C\|\hat{\mu}\|_\infty < \infty$, and so $\sup \|\hat{\mu}\hat{e}_\alpha\|_{(2,2)} < \infty$ by Proposition 1. Hence, by part (i), we conclude that $\hat{\mu} \in L_2(\Gamma)$. The case $1 < q \leq p = 2$ is an easy consequence of part (i) and the fact that $L_{p,q}(\Gamma) \subset L_{2,2}(\Gamma) = L_2(\Gamma)$. It remains to consider the case $1 < p < 2$ and $1 < q < \infty$. Since $\hat{\mu} \in L_{p,q}(\Gamma) \subset L_{p,\infty}(\Gamma)$ (by Proposition 1) and the non-increasing rearrangement $\hat{\mu}^*$ of $\hat{\mu}$ is bounded, there exist constants C_1 and C_2 such that $\hat{\mu}^* t^{1/p} \leq C_1$ and $\hat{\mu}^* \leq C_2$. Hence

$$\int_\Gamma |\hat{\mu}|^2 = \int_0^\infty \hat{\mu}^*(t)^2 dt \leq \int_0^1 C_2^2 + \int_1^\infty C_1^2 t^{-2/p} dt < \infty.$$

REMARK 1. Theorem 1(i) with $p=q$ provides the details which justify the assertion in line -11 of [1, p. 96], while Lemma 1 fills the gap in line -2 of [1, p. 96].

4. Applications

In this section we give two consequences (see Theorems 2 and 3 below) of Theorem 1: Theorem 2 contains the main results in Chen and Lai [2, Theorems 3.12 and 3.13]; Theorem 3 gives a partial converse of Hölder's inequality.

Throughout this section, let

$$A(p, q)(G) = \{f \in L_1(G) : \hat{f} \in L_{p,q}(\Gamma)\},$$
$$M(p, q)(G) = \{\mu \in M(G) : \hat{\mu} \in L_{p,q}(\Gamma)\},$$

where $1 < p, q < \infty$, or $p=1=q$. We define a norm $\|\cdot\|_{A(p,q)}$ in $A(p, q)(G)$ by

$$\|f\|_{A(p,q)} = \max\{\|f\|_1, \|\hat{f}\|_{(p,q)}\}.$$

Similarly, we define a norm $\|\cdot\|_{M(p,q)}$ in $M(p, q)(G)$ by

$$\|\mu\|_{M(p,q)}=\max\{\|\mu\|, \|\hat{\mu}\|_{(p,q)}\}.$$

By a multiplier from $L_1(G)$ to $A(p, q)(G)$ we mean a bounded linear operator from $L_1(G)$ to $A(p, q)(G)$ that commutes with convolution. The collection of all multipliers from $L_1(G)$ to $A(p, q)(G)$ will be denoted by $\mathfrak{M}(L_1(G), A(p, q)(G))$. The space $\mathfrak{M}(L_1(G), M(p, q)(G))$ is defined in the same way.

THEOREM 2. *Let $F(p, q)(G)=A(p, q)(G)$ or $M(p, q)(G)$. For $1<p$, $q<\infty$ or $p=1=q$, the space $\mathfrak{M}(L_1(G),F(p, q)(G))$ is isometrically isomorphic to $M(p, q)(G)$. More precisely, if $\mu\in M(p, q)(G)$ and T_μ is defined on $L_1(G)$ by $T_\mu(f)=\mu*f$, then $T_\mu\in\mathfrak{M}(L_1(G), F(p, q)(G))$ with $\|T_\mu\|_{\mathfrak{M}(L_1,F(p,q))}=\|\mu\|_{M(p,q)}$; and, conversely, if $T\in\mathfrak{M}(L_1(G), F(p, q)(G))$, then $T=T_\mu$ for some $\mu\in M(p, q)(G)$.*

PROOF. Let $(e_\alpha)_{\alpha\in D}$ be an approximate identity in $L_1(G)$ such that $\sup_\alpha\|e_\alpha\|_1\leq 1$. We note that for $\mu\in M(G)$, we have

$$(1)\qquad \|\mu\|=\sup_\alpha\|\mu*e_\alpha\|_1.$$

We now prove that $\mathfrak{M}(L_1(G), A(p, q)(G))\approx M(p, q)(G)$, where $\approx$ means that the two spaces are isometrically isomorphic under the correspondence $T_\mu\leftrightarrow\mu$ with T_μ defined by $T_\mu(f)=\mu*f$ for all $f\in L_1(G)$. It is clear that if $\mu\in M(p, q)(G)$, then $T_\mu\in\mathfrak{M}(L_1(G), A(p, q)(G))$. Conversely, by using $\mathfrak{M}(L_1(G), A(p, q)(G))\subset\mathfrak{M}(L_1(G), L_1(G))\approx M(G)$ and Theorem 1, we see that every $T\in\mathfrak{M}(L_1(G), A(p, q)(G))$ is of the form T_μ for some $\mu\in M(p,q)(G)$. Thus it remains to verify that, for $\mu\in M(p, q)(G)$, the operator norm $\|T_\mu\|_{\mathfrak{M}(L_1,A(p,q))}$ is equal to $\|\mu\|_{M(p,q)}$. For $\mu\in M(p, q)(G)$, we have

$$\begin{aligned}\|T_\mu\|_{\mathfrak{M}(L_1,A(p,q))}&=\sup\{\|\mu*f\|_{A(p,q)} : \|f\|_1\leq 1\}\\&\geq\sup_\alpha\|\mu*e_\alpha\|_{A(p,q)}\\&=\sup_\alpha\max\{\|\mu*e_\alpha\|_1, \|\hat{\mu}\hat{e}_\alpha\|_{(p,q)}\}\\&\geq\max\{\|\mu\|, |\hat{\mu}\|_{(p,q)}\}\ \text{(by (1) above and Theorem 1)}\\&=\|\mu\|_{M(p,q)}.\end{aligned}$$

On the other hand, it is easy to verify that

$$\|T_\mu(f)\|_{A(p,q)}=\|\mu*f\|_{A(p,q)}\leq\|\mu\|_{M(p,q)}\|f\|_1$$

for $\mu\in M(p, q)(G)$ and $f\in L_1(G)$. Thus $\|T_\mu\|_{\mathfrak{M}(L_1,A(p,q))}\leq\|\mu\|_{M(p,q)}$. This completes the proof that $\mathfrak{M}(L_1(G), A(p, q)(G))\approx M(p, q)(G)$. The proof of

$\mathfrak{M}(L_1(G), M(p, q)(G)) \approx M(p, q)(G)$ is similar: one uses $\mathfrak{M}(L_1(G), M(G)) \approx M(G)$ instead of $\mathfrak{M}(L_1(G), L_1(\dot{G})) \approx M(G)$.

The following remark generalizes Theorem 3.6(i) in Chen and Lai [2]; it also settles the problem posed therein (see p. 255).

REMARK 2. We note here that if (i) $p=1=q$, (ii) $1<p<2$ and $1<q<\infty$, or (iii) $1<q\leq p=2$, then $M(p, q)(G)=A(p, q)(G)$. [Let $\mu \in M(p, q)(G)$. By Hewitt and Ross [3, (31.33)], it suffices to show that $\hat{\mu} \in L_2(\Gamma)$. Case (i) is obvious; the last two sentences in the proof of Theorem 1 verify case (ii); and case (iii) follows from $L_{p,q}(\Gamma) \subset L_{2,2}(\Gamma) = L_2(\Gamma)$.]

As another application of Theorem 1 we give the following partial converse of Hölder's inequality.

THEOREM 3. *Let $1<p<\infty$ and let f be a complex-valued continuous function defined on G such that*

$$\left|\int_G f\phi\right| \leq C\|\phi\|_{p'},$$

for all $\phi \in C_{oo}(G)$, where C is a constant. Then $f \in L_p(G)$ and $\|f\|_p \leq C$.

PROOF. Let $(e_\alpha)_{\alpha \in D}$ be an approximate identity in $L_1(\Gamma)$ such that $\|e_\alpha\| \leq 1$ for all $\alpha \in D$, and each $\hat{e}_\alpha$ has compact support. Clearly $f\hat{e}_\alpha \in L_p(G)$, and so

$$\|f\hat{e}_\alpha\|_p = \sup\{\left|\int_G f\hat{e}_\alpha \phi\right| : \phi \in C_{oo}(G) \text{ and } \|\phi\|_{p'} \leq 1\}$$

by Theoren (12.13) in Hewitt and Ross [3]. Hence $\|f\hat{e}_\alpha\|_p \leq C$ for all $\alpha \in D$. By Theorem 1, we have $f \in L_p(G)$ and $\|f\|_p = \sup_\alpha \|f\hat{e}_\alpha\|_p \leq C$.

5. Fourier transforms of $L_p(G)$ functions

In this section we use the method of Theorem 1 to give a characterization of functions on Γ which are Fourier transforms of functions in $L_p(G)$, $1<p\leq 2$.

THEOREM 4. *Let $1<p\leq 2$, and let $(e_\alpha)_{\alpha \in D}$ be a bounded approximate identity in $L_1(G)$. Suppose $k \in L_{p'}(\Gamma)$, where $1/p+1/p'=1$. Then k is the Fourier transform of some function in $L_p(G)$ if and only if $((k\hat{e}_\alpha)^\vee)_{\alpha \in D}$ is a norm-bounded net in $L_p(G)$.*

PROOF. Suppose $k=\hat{h}$ with $h \in L_p(G)$. Then we have

$$\sup_\alpha \|(k\hat{e}_\alpha)^\vee\|_p = \sup_\alpha \|h * e_\alpha\|_p \leq \sup_\alpha \|h\|_p \|e_\alpha\|_1 < \infty.$$

Now suppose that $((k\hat{e}_\alpha)^\vee)_{\alpha\in D}$ is a norm-bounded net in $L_p(G)$. Then, by Alaoglu's theorem, $((k\hat{e}_\alpha)^\vee)_{\alpha\in D}$ has a subnet $((k\hat{e}_\beta)^\vee)_{\beta\in B}$ such that $(k\hat{e}_\beta)^\vee\to h$ weakly in $L_p(G)$ for some function h in $L_p(G)$. Thus

$$\int_G (k\hat{e}_\beta)^\vee \check{g} \longrightarrow \int_G h\check{g} \text{ for all } g\in L_p(\Gamma). \tag{1}$$

By the generalized Parseval identity (see Hewitt and Ross [3, (31.48)]), we have

$$\int_G (k\hat{e}_\beta)^\vee \check{g} = \int_\Gamma (k\hat{e}_\beta)^\# g \text{ and } \int_G h\check{g} = \int_\Gamma \hat{h}^\# g,$$

where $f^\#(\gamma)=f(-\gamma)$ for $\gamma\in\Gamma$. Hence we have

$$\int_\Gamma k\hat{e}_\beta g \longrightarrow \int_\Gamma \hat{h} g \text{ for all } g\in L_p(\Gamma). \tag{2}$$

We now show that $k=\hat{h}$ a.e. Since k and $\hat{h}$ are in $L_{p'}(\Gamma)$, it suffices to show that $k=\hat{h}$ l.a.e. Now let Δ be any compact subset of Γ. Then there is a function $\phi:\Gamma\to[0,1]$ such that $\phi\in C_{00}(\Gamma)$ and $\phi=1$ on Δ. Let Δ_1 denote the (compact) closure of supp(ϕ). Since

$$\hat{e}_\beta \longrightarrow 1 \text{ uniformly on } \Delta_1 \text{ and } \int_{\Delta_1} |k|\phi<\infty,$$

we have

$$\int_{\Delta_1} k\hat{e}_\beta\, sgn(\hat{h}-k)\phi \longrightarrow \int_{\Delta_1} k\, sgn(\hat{h}-k)\phi. \tag{3}$$

By (2) we have

$$\int_{\Delta_1} k\hat{e}_\beta\, sgn(\hat{h}-k)\phi \longrightarrow \int_{\Delta_1} \hat{h}\, sgn(\hat{h}-k)\phi. \tag{4}$$

Hence we have

$$\int_{\Delta_1} \hat{h}\, sgn(\hat{h}-k)\phi = \int_{\Delta_1} k\, sgn(\hat{h}-k)\phi$$

and so

$$\int_{\Delta_1} |\hat{h}-k|\phi=0.$$

Thus

$$\int_{\Delta} |\hat{h}-k| = \int_{\Delta} |\hat{h}-k|\phi=0.$$

Hence $\hat{h}=k$ a.e. on Δ. Since Δ is an arbitrary compact subset of Γ, we see that $\hat{h}=k$ l.a.e. and hence $\hat{h}=k$ a.e.

REMARK 3. Pigno [6] proved the following result: Let $1<p\le 2$, and let $k\in L_{p'}(\Gamma)$. Let $(e_n)_{n=1}^\infty$ be a sequence in $L_1(G)$ such that $\|e_n\|_1\le 2$, $\hat{e}_n$ has compact support and $k\hat{e}_n\to k$ a.e. Then $k=\hat{h}$ a.e. for some $h\in L_p(G)$

if $((k\hat{e}_n)^\vee)_{n=1}^\infty$ is a norm-bounded sequence in $L_p(G)$. Pigno proved his result by using regular Toeplitz summation matrices. It can also be proved as follows: There exist h in $L_p(G)$ and a subsequence $(e_{n_j})_{j=1}^\infty$ of $(e_n)_{n=1}^\infty$ such that

$$(1) \qquad \int_\Gamma k\hat{e}_{n_j} g \longrightarrow \int_\Gamma \hat{h} g$$

for every $g \in L_p(\Gamma)$ (see (2) in the proof of Theorem 4). By Hölder's inequality and the Dominated Convergence Theorem, we have

$$(2) \qquad \int_\Gamma k\hat{e}_{n_j} g \longrightarrow \int_\Gamma kg \text{ for every } g \in L_p(\Gamma).$$

It follows from (1) and (2), as in the proof of Theorem 4, that $k=\hat{h}$ a.e.

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