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A certain logmodular algebra and its Gleason parts

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Abstract. Let A be a weak-* Dirichlet algebra on a nontrivial probability measure space $(X, \mathscr{A}, m)$ and let $H^\infty = H^\infty(m)$ be the weak-* closure of A in $L^\infty(m)$. The first objective of this paper is to study the maximal ideal space $M(H^\infty)$ of H^∞ with a special regard to the algebraic direct sum decomposition $H^\infty = \mathscr{H}^\infty \oplus I^\infty$, where I^∞ is an ideal of H^∞ apeared in [14].

The second objective of this paper is to study a certain logmodular algebra A on a compact space X and its maximal ideal space $M(A)$ in connection with an abstract Hardy algebra H^∞ associated with A.

§ 1. Introduction.

We denote by B a complex commutative Banach algebra with a unit, and by B^{-1} the group of invertible elements in B. We denote by $M(B)$ the maximal ideal space of B. We denote by $\hat{f}$ the Gelfand transform of $f \in B$, by $\hat{B}$ the set $\{\hat{f} : f \in B\}$, and by $\Gamma(B)$ the Shilov boundary of B. We often write f for $\hat{f}$, since the meaning will be clear from the context.

In § 3 and § 4, we denote by A a weak-* Dirichlet algebra on a nontrivial probability measure space $(X, \mathscr{A}, m)$, and by $H^\infty = H^\infty(m)$ the weak-* closure of A in $L^\infty(m)$. We will often denote by m the complex homomorphism of H^∞ which is determined by the measure m. Let J^∞ be the weak-* closed linear span of all functions in H^∞, each of which vanishes on some set of positive measure. Then J^∞ is an ideal of H^∞ which is contained in $H_m^\infty = \{f \in H^\infty : \int f\, dm = 0\}$. In [14], we call J^∞ the typical ideal. In [14], we have established a decomposition $H^\infty = \mathscr{H}^\infty \oplus I^\infty$ with I^∞, a spacific ideal of H^∞ with $I^\infty \subset J^\infty$, where $\oplus$ denotes the algebraic direct sum (see § 2). Let $\mathscr{L}^\infty$ (resp. N^∞) be the weak-* closure of $\mathscr{H}^\infty + \overline{\mathscr{H}^\infty}$ (resp. $I^\infty + \overline{I^\infty}$) (the bar denotes conjugation). Let $\tilde{X} = M(L^\infty(m))$, $Y = \Gamma(H^\infty | \text{hull } I^\infty)$ and let $E(I^\infty)$ be the support set of I^∞. For $\phi \in Y$, let $\mathscr{X}(\phi) = \{\tilde{x} \in \tilde{X} : f(\tilde{x}) = \phi(f)\ \forall f \in \mathscr{L}^\infty\}$. For any measurable set E of X, χ_E denotes the characteristic function of E. For any set E of a topological space X, $\bar{E}$ denotes the closure of E in X.

In § 3, we obtain the following. (i) $\phi \in \text{hull } I^\infty$ belongs to Y if and only if $|\phi(f)| = 1$ for every inner function f in $\mathscr{H}^\infty$. (ii) Theorem 3.5.

$\chi_{E(I^\infty)} \in \mathscr{L}^\infty$ and $F=\tilde{X}\cap Y=\tilde{X}\cap \text{hull } I^\infty=\{\tilde{x}\in\tilde{X}: \hat{\chi}_{E(I^\infty)}(\tilde{x})=0\}=\{\phi\in Y: \hat{\chi}_{E(I^\infty)}(\phi)=0\}$. (iii) If $\Phi\in M(H^\infty)\backslash\text{hull } I^\infty$, then there is a (unique) point $\phi\in Y$ such that $\Phi(f)=\phi(f)$ for every $f\in\mathscr{H}^\infty$. The map defined by $\Phi\mapsto\phi$ is a continuous map of $M(H^\infty)\backslash\text{hull } I^\infty$ onto $Y\backslash F$. (iv) Suppose that $H^\infty_m \supsetneqq J^\infty$. Then $\tilde{X}\supset Y$ if and only if H^∞ is a maximal weak-* closed subalgebra of L^∞. (v) $\tilde{X}\cap Y=\emptyset$ if and only if there is an inner function h in I^∞. (vi) Theorem 3.9. hull I^∞ is connected. hull $I^\infty\backslash Y$ is an open set in $M(H^\infty)$, $\overline{\text{hull } I^\infty\backslash Y}\supset Y$ and $\overline{(M(H^\infty)\backslash\text{hull } I^\infty)}\cap\text{hull } I^\infty\subset Y$. If $I^\infty\supsetneqq\{0\}$, then $M(H^\infty)\backslash\text{hull } I^\infty$ is disconnected and hence $M(I^\infty)$ is disconnected. (vii) If the Gleason part $P(m)$ of m is nontrivial, then $\cup\{S(\tilde{\phi}): \tilde{\phi}\in Y\}$ is dense in $\tilde{X}$, where $S(\tilde{\phi})$ denotes the compact support of the representing measure of $\tilde{\phi}\in Y$. (viii) If the Gleason part $P(m)$ of m is nontrivial, then $\log|(C'+I^\infty)^{-1}|=R+N^\infty\cap L^\infty_R$, where C and R are the complex and the real fields respectively and $C'=C\backslash\{0\}$. In § 3 we will generalize some results in [13] to more general cases.

In § 4, we obtain the following. (i) If $\phi\in(M(H^\infty)\backslash\text{hull}I^\infty)\cup Y$, then $S(\phi)\subset K(\phi_0)$ for some $\phi_0\in Y$. (ii) Theorem 4.2. hull $I^\infty\backslash Y$ is a union of Gleason parts. (iii) Theorem 4.4. $\mathscr{K}(\phi)$ $(\phi\in Y)$ is a weak peak set of H^∞, $\widetilde{\mathscr{K}(\phi)}$ $(=H^\infty\text{-convex hull of }\mathscr{K}(\phi))\cap\text{hull}I^\infty=\{\phi\}$, $(M(H^\infty)\backslash\text{hull}I^\infty)\cup Y=\cup\{\widetilde{\mathscr{K}(\phi)}: \phi\in Y\}$ is a union of Gleason parts, and $\widetilde{\mathscr{K}(\phi)}\cap\widetilde{\mathscr{K}(\psi)}=\emptyset$ for $\phi\neq\psi$. (iv) Theorem 4.5. Let B_1 and B_2 be weak-* closed subalgebras of $L^\infty(m)$ with $H^\infty\subsetneqq B_1\subsetneqq B_2\subset L^\infty(m)$. Let $I^\infty_{B_i}=\{h\in L^\infty(m): \int hf\,dm=0\ \ \forall f\in B_i\}$ $(i=1, 2)$, and let $\mathscr{H}^\infty_{B_i}=(B_i\cap\overline{B}_i)\cap H^\infty$ $(i=1, 2)$. Then we obtain $I^\infty_{B_1}\supsetneqq I^\infty_{B_2}$, $\mathscr{H}^\infty_{B_1}\supsetneqq\mathscr{H}^\infty_{B_2}$, $\text{hull}I^\infty_{B_1}\supsetneqq\text{hull}I^\infty_{B_2}$, and some properties of hull $I^\infty_{B_2}\backslash\text{hull}I^\infty_{B_1}$. In § 4 we will generalize some results in [12] to more general cases.

In § 5 and § 6, we denote by A a strongly logmodular algebra on a compact Hausdorff space X. For each $\phi\in M(A)$ we denote its (unique) representing measure by ϕ. Let $m\in M(A)$, and let $H^\infty=H^\infty(m)$ be the weak-* closure of A in $L^\infty(m)$. Then A is a weak-* Dirichlet algebra on $(X, \mathscr{A}, m)$, and hence we have a decomposition $H^\infty=\mathscr{H}^\infty\oplus I^\infty$. Let $J=J^\infty\cap C(X)$, $I=I^\infty\cap C(X)$, $\mathscr{H}=\mathscr{H}^\infty\cap C(X)$, $\mathscr{L}=\mathscr{L}^\infty\cap C(X)$ and $\mathscr{L}_R=\mathscr{L}^\infty\cap C_R(X)$. For $\phi\in M(\mathscr{L})$ let $K(\phi)=\{x\in X: f(x)=\phi(f)\ \ \forall f\in\mathscr{L}\}$. Suppose that $X=S(m)$.

In § 5, we obtain the following. (i) Theorem 5.4. If $\phi\in M(A)$, then $S(\phi)$ is a weak peak set for A. (ii) Theorem 5.5. $A=H^\infty(m)\cap C(X)$, $A^{-1}=(H^\infty(m))^{-1}\cap C(X)$. (iii) If the Gleason part $P(m)$ of m is nontrivial, then $I=J$ is a primary ideal of A and $I=\{f\in A: \phi(f)=0\ \forall\phi\in P(m)\}$. (iv) Theorem 5.8. There is a continuous map η of Y onte $M(\mathscr{L})$, and for $\phi\in$

$M(\mathscr{L})$, we have $\tilde{X}\cap\pi^{-1}(K(\phi))=\cup\{\mathscr{K}(\tilde{\theta}):\tilde{\theta}\in\eta^{-1}(\phi)\}$ and $\pi[\cup\{\widetilde{\mathscr{K}(\tilde{\theta})}:\tilde{\theta}\in\eta^{-1}(\phi)\}]\subset\widetilde{K(\phi)}$, where $\widetilde{\mathscr{K}(\tilde{\theta})}$ and $\widetilde{K(\phi)}$ are H^{∞}-convex hull of $\mathscr{K}(\tilde{\theta})$ and A-convex hull of $K(\phi)$ respectively (for π see § 5). If $\pi(M(H^{\infty}))=M(A)$, then $M(A)=\mathrm{hull}\,I\cup(\cup\{\widetilde{K(\phi)}:\phi\in M(\mathscr{L})\})$. (v) Theorem 5.9. If $M(\mathscr{L})$ is totally disconnected, then I is contained in the uniformly closed linear span of all functions in A, each of which vanishes on some set of positive measure. (vi) I^{∞} is contained in the uniformly closed linear span of all functions in $H^{\infty}(m)$, each of which vanishes on some set of positive measure. (vii) If $f\in L^{\infty}(m)$ is constant on $\cup\{\mathscr{K}(\tilde{\theta}):\tilde{\theta}\in\eta^{-1}(\phi)\}$ for every $\phi\in M(\mathscr{L})$, then $f\in\mathscr{L}$.

In § 6, we obtain the following. Theorem 6.3. If $P(m)$ is nontrivial, then $\mathscr{H}\oplus I$ and $\mathscr{L}_R\oplus N_R$ are both uniformly closed, and we have $\log|(C'+I)^{-1}|=R+N_R$ and $\log|(\mathscr{H}\oplus I)^{-1}|=\mathscr{L}_R\oplus N_R$, where $N_R=N^{\infty}\cap C_R(X)$.

In § 7, we denote by A a logmodular algebra on a compact Hausdorff space X. Let $m\in M(A)$ and let $P=P(m)\supsetneqq\{m\}$. Let τ be an analytic map of the open unit disc D onto P such that $f\circ\tau\in H^{\infty}(D)$, where $H^{\infty}(D)$ is the Banach algebra of bounded analytic functions on D. Let $\Gamma=\Gamma(A|\bar{P})$. In § 7, we obtain the following. (i) Theorem 7.1. If $A\circ\tau=\{f\circ\tau:f\in A\}=H^{\infty}(D)$, then $A|\Gamma$ is a strongly logmodular algebra on Γ and, roughly speaking, $A|\Gamma$ has the same properties as the function algebra $\widehat{H^{\infty}(D)}$. (ii) If $A\circ\tau=H^{\infty}(D)$, then a nontrivial Gleason part for $A|\Gamma$ is also a nontrivial Gleason part for A. (iii) We give some conditions to be $A\circ\tau=H^{\infty}(D)$.

In § 8, we will give some examples. Example 1 is related to the cases 2.1 and 2.3 in the section 2. Example 2 is related to Theorems 4.2, 4.4 and Corollary 7.2. Examples 3 and 4 are related to the sections 5, 6 and 7.

In § 2, we will give some preliminaries concerning uniform algebras, weak-* Dirichlet algebras, an algebraic direct sum decomposition $H^{\infty}(m)=\mathscr{H}^{\infty}\oplus I^{\infty}$, etc.

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§ 2. Preliminaries.

First we will give some preliminaries concerning uniform algebras. Let X be a compact Hausdorff space and let $C(X)$ (resp. $C_R(X)$) be the Banach algebra of complex (resp. real) valued continuous functions on X with the supremum norm. A closed subalgebra A of $C(X)$ is said to be a uniform algebra if A contains the constants, and A separates the points of X. A uniform algebra A is said to be a logmodular algebra on X if the set $\log|A^{-1}|$

$=\{\log|f|: f\in A^{-1}\}$ is dense in $C_R(X)$. A logmodular algebra A which satisfies $\log|A^{-1}|=C_R(X)$ is said to be a strongly logmodular algebra.

A representing measure of $\phi\in M(A)$ for a uniform algebra A is a probability measure μ on X such that $\phi(f)=\int f\,d\mu$ for all $f\in A$. We denote by $S(\mu)$ the compact support of μ i. e., the complement of the largest open set of μ-measure zero. When $\phi\in M(A)$ has a unique representing measure, we denote its measure by ϕ, μ_ϕ or λ_ϕ as the case may be. For ϕ and ψ in $M(A)$ let

(2.1) $\quad d_A(\phi, \psi)=\sup\{|\phi(f)|: f\in A,\ \|f\|\leqq 1,\ \psi(f)=0\}$,

where $\|f\|=\sup\{|f(x)|: x\in X\}$. We define ϕ and ψ to be $\phi\sim\psi$ when $d_A(\phi, \psi)<1$ (or, equivalently, $\|\phi-\psi\|<2$). Then $\sim$ is an equivalence relation in $M(A)$, and $P(m)=\{\phi\in M(A): m\sim\phi\}$ ($\supsetneqq\{m\}$) is said to be the (nontrivial) Gleason part of m for A.

Let A be a logmodular algebra on X, and let m be a point of $M(A)$. Let $H^\infty=H^\infty(m)$ be the weak-* closure of A in $L^\infty(m)$. A function f in $H^\infty(m)$ is said to be inner if $|f|=1$ a. e. (m). If $P=P(m)\supsetneqq\{m\}$, then there is an inner function Z known as the Wermer embedding function which satisfies $ZH^\infty=H_m^\infty$, where $H_m^\infty=\{f\in H^\infty: \int f\,dm=0\}$. And, there is an analytic map τ such that τ is a one-to-one continuous map of the open unit disc D in the complex plane onto P, and for every f in H^∞ $f\circ\tau$ is analytic in D (*cf.* [5], p. 158).

A compact Hausdorff space S is said to be Stonian (*or*, extremally disconnected) if disjoint open subsets of S have disjoint closures. A positive measure μ on S is said to be normal if it vanishes on all nowhere dense Borel sets in S (cf. [1], § 7, § 8). Let A be a logmodular algebra on X, let $m\in M(A)$ and let $\tilde{X}=M(L^\infty(m))$. Then $\tilde{X}$ is Stonian, and there is a probability normal measure $\tilde{m}$ on $\tilde{X}$ such that $S(\tilde{m})=\tilde{X}$ and

$$\int_X f\,dm=\int_{\tilde{X}}\hat{f}\,d\tilde{m},\quad f\in L^\infty(m).$$

This measure $\tilde{m}$ is said to be the Radonization of m. If $P(m)\supsetneqq\{m\}$, then the Gleason pare $\mathscr{P}=\mathscr{P}(\tilde{m})$ of $\tilde{m}\in M(\hat{H}^\infty)$ for $\hat{H}^\infty$ is also nontrivial (cf. [11], Proposition). It is known that $\phi\in M(H^\infty)$ belongs to $\tilde{X}$ if and only if $|\phi(f)|=1$ for every inner function f in $H^\infty(m)$. The reader is referred to Gamelin [5] and Leibowitz [15] for basic definitions and properties about uniform algebras.

Secondly we will give some preliminaries about weak-* Dirichlet algebras. Let $(X, \mathscr{A}, m)$ be a fixed nontrivial probability measure space. A weak-* Dirichlet algebra, which was introduced by Srinivasan and Wang [23], is an algebra A of essentially bounded measurable functions on $(X, \mathscr{A}, m)$ such that (i) the constant functions lie in A; (ii) $A+\bar{A}$ is weak-* dense in $L^\infty=L^\infty(m)$ (the bar denotes conjugation); (iii) for all f and g in A, $\int_X fg\,dm=\int_X f\,dm\int_X g\,dm$. The abstract Hardy spaces $H^p=H^p(m)$, $1\leqq p\leqq\infty$, associated with A are defined as follows. For $1\leqq p<\infty$, H^p is the L^p $(=L^p(m))$-closure of A, while H^∞ is defined to be the weak-* closure of A in $L^\infty(m)$. For $1\leqq p\leqq\infty$, let $H^p_m=\{f\in H^p : \int_X f\,dm=0\}$. It is known that $\hat{H}^\infty$ is a strongly logmodular algebra on $\tilde{X}=M(L^\infty(m))$.

Let $L(m)=L(m|\mathscr{A})$ be the set of equivalence classes modulo m of the measurable complex valued functions on X. Let B be a weak-* closed subalgebra of $L^\infty(m)$ which contains H^∞ properly. Let

$$\Delta=\{D\in\mathscr{A} : \chi_D\in B\}.$$

Then $\Delta\subset\mathscr{A}$ is a sigma-algebra which contains the sigma-algebra $\mathscr{A}_m$ of the m-null sets and their complements. We define $f\in L(m)$ to be Δ-measurable if and only if some and hence all $\mathscr{A}$-measurable functions which represent f are Δ-measurable. Let $L(m|\Delta)$, $L^\infty(m|\Delta)$, $L^p(m|\Delta)$, etc. denote the respective function classes.

Let $H^\infty_{\min}$ be the intersection of all weak-* closed subalgebras of $L^\infty(m)$ which contain H^∞ properly. Let

$$J^\infty=J^\infty(H^\infty)$$

be the weak-* closed linear span of all functions in H^∞, each of which vanishes on some set of positive measure. Then J^∞ is an ideal of H^∞ which is contained in H^∞_m. By [[20], Corollary 5] we have the following equivalence

$$H^\infty_m\supsetneqq J^\infty\Longleftrightarrow H^\infty_{\min}\supsetneqq H^\infty.$$

By [17], the following (i), (ii), (iii) and (iv) are equivalent. (i) H^∞ is a maximal weak-* closed subalgebra of $L^\infty(m)$. (ii) $H^\infty_{\min}=L^\infty(m)$. (iii) $J^\infty=\{0\}$. (iv) H^∞ is an integral domain.

Here we will state some cases of the algebraic direct sum decomposition $H^\infty=\mathscr{H}^\infty\oplus I^\infty$.

Case 2.1. We suppose $H^\infty_m\supsetneqq J^\infty$ (cf. § 6 in [14] and [10]).
Let $B=H^\infty_{\min}$, and let

$$I^\infty=\{h\in B: \int hf\,dm=0\ \forall f\in B\}$$
$$=\{h\in L^\infty(m): \int hf\,dm=0\ \forall f\in B\}.$$

Then we have

$$I^\infty=J^\infty.$$

Let $\mathscr{L}^\infty=B\cap\overline{B}(\supsetneqq C)$, where C is the complex field, and let $\Delta=\{D\in\mathscr{A}: \chi_D\in B\}$. Then we have

$$\mathscr{L}^\infty=L^\infty(m|\Delta).$$

Let $\mathscr{H}^\infty=H^\infty\cap\mathscr{L}^\infty$. Then we have

(2.2) $\quad H^\infty_{\min}=\mathscr{L}^\infty\oplus I^\infty,\ \mathscr{L}^\infty I^\infty=I^\infty,\ H^\infty_{\min}I^\infty=I^\infty$

and

(2.3) $\quad H^\infty=\mathscr{H}^\infty\oplus I^\infty,$

where the sum is orthogonal in the Hilbert space sense, and hence it is the algebraic direct sum.

$\mathscr{L}^\infty$ is the weak-* closure of $\mathscr{H}^\infty+\overline{\mathscr{H}^\infty}$, and $\mathscr{H}^\infty$ is a weak-* Dirichlet algebra in $\mathscr{L}^\infty$, and $\mathscr{H}^\infty$ is a maximal weak-* closed subalgebra of $\mathscr{L}^\infty$.

Case 2.2. We suppose $H^\infty_m=J^\infty$ (cf. § 7 in [14] and [10]).

There exsist weak-* closed subalgebras B of $L^\infty(m)$ with $H^\infty\subsetneqq B\subsetneqq L^\infty$. Let B be a fixed one of them.

Let

$$I^\infty=I^\infty_B=\{h\in B: \int hf\,dm=0\ \forall f\in B\}$$
$$=\{h\in L^\infty(m): \int hf\,dm=0\ \forall f\in B\}.$$

Then we have

$$I^\infty\subsetneqq J^\infty.$$

Let $\mathscr{L}^\infty=B\cap\overline{B}(\supsetneqq C)$, and let $\Delta=\{D\in\mathscr{A}: \chi_D\in B\}$. Then we have

$$\mathscr{L}^\infty=L^\infty(m|\Delta).$$

Let $\mathscr{H}^\infty=H^\infty\cap\mathscr{L}^\infty$. Then we have

(2.4) $\quad B=\mathscr{L}^\infty\oplus I^\infty,\ \mathscr{L}^\infty I^\infty=I^\infty,\ BI^\infty=I^\infty$

and

(2.5) $\quad H^\infty=\mathscr{H}^\infty\oplus I^\infty$,

where the sum is orthogonal in the Hilbert space sense, and hence it is the algebraic direct sum.

$\mathscr{L}^\infty$ is the weak-* closure of $\mathscr{H}^\infty+\overline{\mathscr{H}^\infty}$, and $\mathscr{H}^\infty$ is a weak-* Dirichlet algebra in $\mathscr{L}^\infty$, and $\mathscr{H}^\infty$ is not a maximal weak-* closed subalgebra of $\mathscr{L}^\infty$.

Case 2.3. We suppose $P(m)=\{\phi\in M(H^\infty):\|\phi-m\|<2\}\supsetneqq\{m\}$ (cf. [14], § 8).

By Proposition 7 in [14], we have

$$H_m^\infty\supsetneqq J^\infty.$$

Let $\mathscr{L}^\infty$, I^∞ and $\mathscr{H}^\infty$ be as in the case 2.1. Then we have

(2.6) $\quad H_{\min}^\infty=\mathscr{L}^\infty\oplus I^\infty$ and $H^\infty=\mathscr{H}^\infty\oplus I^\infty$.

On the other hand, since $P(m)\supsetneqq\{m\}$, we have $H_m^\infty=ZH^\infty$ for the Wermer embedding function Z. Let $\mathscr{H}$ be the weak-* closure of the polynomials in Z in $L^\infty(m)$, and let

$$J=\{f\in H^\infty:\phi(f)=0\ \ \forall\phi\in P(m)\}.$$

Then we have

$$\mathscr{H}=\mathscr{H}^\infty \text{ and } I^\infty=J.$$

Hence the decomposition $H^\infty=\mathscr{H}\oplus I^\infty$ coincides with the decomposition of Lemma 5 in [16]. The case 2.3 is a special case of the case 2.1 (see Example 1 in § 8).

REMARK. Let B be any weak-* closed subalgebra of $L^\infty(m)$ with $H^\infty\subsetneqq B^\infty\subset L^\infty(m)$. Let $I_B^\infty=\{h\in B:\int hf\,dm=0\quad\forall f\in B\}$, let $\mathscr{L}_B^\infty=B\cap\overline{B}(\supsetneqq C)$, let $\Delta_B=\{D\in\mathscr{A}:\chi_D\in B\}$, and let $\mathscr{H}_B^\infty=H^\infty\cap\mathscr{L}_B^\infty$. Then, as in the case 2.2, we have $\mathscr{L}_B^\infty=L^\infty(m|\Delta_B)$, $B=\mathscr{L}_B^\infty\oplus I_B^\infty$, $H^\infty=\mathscr{H}_B^\infty\oplus I_B^\infty$, $BI_B^\infty=I_B^\infty$, etc.. The results in § 3 and § 4 which are not used an asumption such as $P(m)\supsetneqq\{m\}$ or $H_m^\infty\supsetneqq J^\infty$ hold for $\mathscr{L}_B^\infty$, $\mathscr{H}_B^\infty$, I_B^∞, etc..

For $1\leqq p<\infty$ and for any subset $M\subset L^\infty(m)$, denote by $[M]_p$ the $L^p(m)$ closure of M.

PROPOSITION 2.1. *Let $\mathscr{H}^\infty$, $\mathscr{L}^\infty$ and I^∞ be defined according to the above cases 2.1, 2.2 and 2.3, and let N^∞ be the weak-* closure of $I^\infty+\overline{I^\infty}$. Then we have*

$$L^\infty(m)=\mathscr{L}^\infty\oplus N^\infty,$$

where $\oplus$ *denotes the algebraic direct sum. Moreover, for* $1\leqq p<\infty$, *we have*

$$H^p(m)=[\mathscr{H}^\infty]_p\oplus[I^\infty]_p \text{ and } L^p(m)=[\mathscr{L}^\infty]_p\oplus[N^\infty]_p.$$

PROOF. Let $f=g+h$, where $g\in\mathscr{L}^\infty$ and $h\in N^\infty$. If $1\leqq p<\infty$, then we have

$$\begin{aligned}(\int_X|g|^p dm)^{1/p}&=\sup\{|\int_X sg\,dm|: s\in[\mathscr{L}^\infty]_q, \|s\|_q<1\}\\&=\sup\{|\int_X s(g+h)\,dm|: s\in[\mathscr{L}^\infty]_q, \|s\|_q<1\}\\&\leqq(\int_X|g+h|^p\,dm)^{1/p},\end{aligned}$$

where, when $1<p<\infty$, then $\frac{1}{p}+\frac{1}{q}=1$ and $\|s\|_q=(\int|s|^q\,dm)^{1/q}$, and when $p=1$, then $q=\infty$ and $\|s\|_\infty$ stands for the essential supremum norm of s. Thus, by making $p\to\infty$, we obtain $\|g\|_\infty\leqq\|f\|_\infty$, and hence $\|g\|_\infty+\|h\|_\infty\leqq3\|f\|_\infty$. Therefore $\mathscr{L}^\infty+N^\infty$ is weak-* closed in $L^\infty(m)$ (cf. [15], p. 203). And, since $H^\infty(m)$ is a weak-* Dirichlet algebras in $L^\infty(m)$, we obtain

$$L^\infty(m)=\mathscr{L}^\infty\oplus N^\infty.$$

If $1\leqq p<\infty$, as in [16], Lemma 5, we obtain

$$L^p(m)=[\mathscr{L}^\infty]_p\oplus[N^\infty]_p.$$

By the same arguments as for $L^p(m)$, we obtain

$$H^p(m)=[\mathscr{H}^\infty]_p\oplus[I^\infty]_p. \qquad \text{Q. E. D.}$$

§ 3. Some properties of $M(H^\infty)$, Part 1.

In this section, let A be a weak-* Dirichlet algebra on a nontrivial probability measure space $(X, \mathscr{A}, m)$, and let H^∞, B, $\mathscr{H}^\infty$, J^∞, I^∞, and $\mathscr{L}^\infty=L^\infty(m|\Delta)$ be those objects as defined in the cases 2.1, 2.2 and 2.3 in § 2. Then $\mathscr{H}^\infty=H^\infty\cap L^\infty(m|\Delta)$ is a weak-* Dirichlet algebra in $\mathscr{L}^\infty$ on the probability measure space (X, Δ, m), and we can apply the results in [23] to $\mathscr{H}^\infty$. Hence we have

(3.1) $\quad \log|(\mathscr{H}^\infty)^{-1}|=\mathscr{L}^\infty_R.$

Let $\Omega=M(\mathscr{L}^\infty)$, then Ω is a Stonian space and we have

(3.2) $\quad \log|(\hat{\mathscr{H}}^\infty)^{-1}|=C_R(\Omega)$ on Ω.

There is a probability normal measure μ_m on Ω such that $\int_X f\,dm=\int_\Omega \hat{f}\,d\mu_m$ $(f\in\mathscr{L}^\infty)$, and $S(\mu_m)=\Omega$.

By ([4], Theorem 4), we have the following.

LEMMA. 3. 1. *The Shilov boundary $\Gamma(\mathscr{H}^\infty)$ of $\mathscr{H}^\infty$ can be identified with $M(\mathscr{L}^\infty)$, and $\phi\in M(\mathscr{H}^\infty)$ belongs to $\Gamma(\mathscr{H}^\infty)$ if and only if $|\phi(f)|=1$ for every inner function f in $\mathscr{H}^\infty$.*

Let $f=g+h\in H^\infty(m)$, where $g\in\mathscr{H}^\infty$ and $h\in I^\infty$. Then, by the proof of Proposition 2. 1, we have $\|g\|\leqq\|f\|$. Therefore, by the map

$$S: g+I^\infty\mapsto g\ (g\in\mathscr{H}^\infty),$$

the quotient Banach algebra H^∞/I^∞ is isometrically isomorphic to $\mathscr{H}^\infty$. Hence, under the adjoint map S^* of S, the space $M(\mathscr{H}^\infty)$ is homeomorphic to $M(H^\infty/I^\infty)$. And, under the adjoint map σ^* of the natural map $\sigma: H^\infty\mapsto H^\infty/I^\infty$, $M(H^\infty/I^\infty)$ is homeomorphic to hull $I^\infty=\{\phi\in M(H^\infty): \phi(f)=0\ \forall f\in I^\infty\}$. Let

$$\Sigma=\sigma^*\circ S^*. \tag{3.3}$$

Then Σ is a homeomorphism of $M(\mathscr{H}^\infty)$ onto hull I^∞, and for every $\phi\in M(\mathscr{H}^\infty)$ we have $\Sigma(\phi)(g)=\phi(g)$ $(\forall g\in\mathscr{H}^\infty)$. Let

$$Y=\Sigma(\Gamma(\mathscr{H}^\infty))=\Sigma(M(\mathscr{L}^\infty)). \tag{3.4}$$

Then $Y=\Gamma(H^\infty|\,\text{hull}\,I^\infty)=\Gamma(\mathscr{H}^\infty|\,\text{hull}\,I^\infty)$, and Y is a Stonian space, and $\log|(\mathscr{H}^\infty)^{-1}|=C_R(Y)$ on Y. The representing measure μ_m on Y of $m\in$hull I^∞ for $\mathscr{H}^\infty$ is a normal measure, and $S(\mu_m)=Y$.

We have the following equivalence

$$\mathrm{H}_m^\infty\supsetneqq J^\infty\Longleftrightarrow \text{hull}\,J^\infty\supsetneqq\{m\}.$$

Indeed, if $H_m^\infty\supsetneqq J^\infty$, then by the map Σ we have $\Sigma(M(\mathscr{H}^\infty))=$hull J^∞. On the other hand, $\mathscr{H}^\infty$ is a nontrivial weak-* Dirichlet algebra in $\mathscr{L}^\infty$. Hence hull $J^\infty\supsetneqq\{m\}$.

By Lemma 3. 1, we have the following.

PROPOSITION 3. 2. *A point ϕ in* hull I^∞ *belongs to* $Y=\Gamma(H^\infty|\text{hull}\,I^\infty)$ if and only if $|\phi(f)|=1$ for every inner function f in $\mathscr{H}^\infty$.

Since $\mathscr{L}^\infty\subset L^\infty(m)$ and since $\mathscr{L}^\infty$ is a self-adjoint Banach algebra, every ϕ in $M(\mathscr{L}^\infty)$ can be extended multiplicatively to $L^\infty(m)$ (cf. [6], p. 80). Hence the map $\tau: \tilde{x}\mapsto\tilde{x}|\mathscr{L}^\infty$ $(\tilde{x}\in\tilde{X})$ is a continuous map of $\tilde{X}$ onto $M(\mathscr{L}^\infty)$, where $\tilde{X}=M(L^\infty(m))$ and $\tilde{x}|\mathscr{L}^\infty$ is the restriction of $\tilde{x}$ to $\mathscr{L}^\infty$. Let

(3.5) $\quad \tilde{\pi}(\tilde{x})=\sum(\tilde{x}|\mathscr{L}^\infty),\ \tilde{x}\in\tilde{X}.$

Then $\tilde{\pi}$ is a continuous map of $\tilde{X}$ onto Y. For every $\tilde{\phi}\in Y$ let

(3.6) $\quad \mathscr{K}(\tilde{\phi})=\tilde{\pi}^{-1}(\tilde{\phi}).$

Then $\tilde{X}=\cup\{\mathscr{K}(\tilde{\phi}):\tilde{\phi}\in Y\}$, and $\mathscr{K}(\tilde{\phi})\cap\mathscr{K}(\tilde{\theta})=\emptyset$ for $\tilde{\phi}\neq\tilde{\theta}$. For every $f\in\mathscr{L}^\infty$ we have

$$f=\tilde{\phi}(f) \text{ on } \mathscr{K}(\tilde{\phi}).$$

If $\phi\in M(I^\infty)$, then there is a function $h\in I^\infty$ such that $\phi(h)=1$. We define $\Phi\in M(H^\infty)$ by $\Phi(f)=\phi(fh)$, $f\in H^\infty$. Then, by a well known fact, the map

(3.7) $\quad \Pi:\phi\mapsto\Phi$

is a homeomorphism of $M(I^\infty)$ onto $M(H^\infty)\backslash\text{hull}\, I^\infty$.

Let $B=\mathscr{L}^\infty\oplus I^\infty$. For $f=g+h$, where $g\in\mathscr{L}^\infty$ and $h\in I^\infty$, we have $\|g\|\leqq\|f\|$ and hence $\|g\|+\|h\|\leqq 3\|f\|$. Hence, by ([13], *p.* 203), B is a Banach algebra. For $\phi\in M(I^\infty)$ with $\phi(h)=1$ for a function $h\in I^\infty$ we define $\Phi'\in M(B)$ by $\Phi'(f)=\phi(fh)$, $f\in B$. The map

(3.8) $\quad \Pi':\phi\mapsto\Phi'$

is a homeomorphism of $M(I^\infty)$ onto $M(B)\backslash\text{hull}\ I^\infty$, where of course hull $I^\infty=\{\phi\in M(B):\phi(f)=0\ \forall f\in I^\infty\}$.

For $\phi\in M(I^\infty)$, let $\Phi=\Pi(\phi)$ and let $\Phi'=\Pi'(\phi)$. Then it follows from $\log|(\mathscr{H}^\infty)^{-1}|=\mathscr{L}_R^\infty$, $\Phi'|\mathscr{L}^\infty\in M(\mathscr{L}^\infty)$ and $\Phi=\Phi'$ on $\mathscr{H}^\infty$ that $\Phi|\mathscr{H}^\infty$ can be identified with a complex homomorphism of $\mathscr{L}^\infty$. For $\Phi\in M(H^\infty)\backslash\text{hull}\, I^\infty$, let

(3.9) $\quad \tilde{\pi}_1(\Phi)=\sum(\Phi|\mathscr{H}^\infty\}.$

Then the map $\tilde{\pi}_1$ is a continuous map of $M(H^\infty)\backslash\text{hull}\ I^\infty$ to Y. In particular, for $\tilde{x}\in\tilde{X}\backslash\text{hull}\, I^\infty$, we have $\tilde{\pi}(\tilde{x})=\tilde{\pi}_1(\tilde{x})$.

Let $E(I^\infty)$ be the support set of I^∞ i.e., the complement of a set of maximal measure on which all $f\in I^\infty$ are null.

LEMMA 3.3. *There is a function h in I^∞ such that* $|h|=\chi_{E(I^\infty)}$.

PROOF. By [21, Corollary 1] there is a function w in $L^\infty(m)$ such that $|w|=1$ a.e. (m) and $\chi_{E(I^\infty)}w\in I^\infty$. Q. E. D.

THEOREM 3.4. *Let $E=E(I^\infty)$, $\tilde{X}=M(L^\infty(m))$, $Y=\Gamma(H^\infty|\text{ hull } I^\infty)$ and $F=\tilde{X}\cap Y$. Then we have the following.*

(i) $\chi_E \in \mathscr{L}^\infty$.
(ii) $F = \tilde{X} \cap \text{hull}\, I^\infty$.
(iii) $F = \{\phi \in Y : \hat{\chi}_E(\phi) = 0\}$.
(iv) $F = \{\tilde{x} \in \tilde{X} : \hat{\chi}_E(\tilde{x}) = 0\}$.

PROOF. (i) We have $\int_X \chi_{E^c} f\, dm = 0$ for all $f \in I^\infty$. By [10, p. 52], we have

$$\{h \in L^\infty(m) : \int_X hf\, dm = 0, \forall f \in I^\infty\} = B.$$

Hence $\chi_{E^c} \in B$, hence $\chi_{E^c} \in \mathscr{L}^\infty = B \cap \bar{B}$, and hence $\chi_E \in \mathscr{L}^\infty$.

(ii) If $\phi \in \tilde{X} \cap \text{hull}\, I^\infty$, then $|\phi(f)| = 1$ for every inner function f in $\mathscr{H}^\infty$. Hence, by Proposition 3.2, $\phi \in Y$.

(iii) By Lemma 3.3, there is a function $h \in I^\infty$ such that $|h| = \chi_E$. Then $\hat{\chi}_E = |\hat{h}| = 0$ on F. Hence $\{\phi \in Y : \hat{\chi}_E(\phi) = 0\} \supset F$.

If $\phi_0 \in Y \backslash F$, then there is an inner function f in $H^\infty(m)$ such that $|\phi_0(f)| < 1$. Let $f = g + h$, where $g \in \mathscr{H}^\infty$ and $h \in I^\infty$. Then $|\phi_0(f)| = |\phi_0(g)| < c < 1$ for some positive constant c. Since Y is a Stonian space, there is a clopen (i.e., closed and open) neighborhood $V(\phi_0)$ in Y of ϕ_0 such that $\{\phi \in Y : |\phi(g)| < c\} \supset V(\phi_0)$. Then there is a function $\chi_G \in \mathscr{L}^\infty$ such that $V(\phi_0) = \{\phi \in Y : \hat{\chi}_G(\phi) = 1\}$. If $\tilde{x} \in \pi^{-1}(V(\phi_0))$ and $\phi = \tilde{\pi}(\tilde{x})$, then $|\tilde{x}(h)| \geqq |\tilde{x}(f)| - |\tilde{x}(g)| = 1 - |\phi(g)| > 1 - c > 0$. Hence $|\hat{h}| > 1 - c$ on $\pi^{-1}(V(\phi_0)) = \{\tilde{x} \in \tilde{X} : \hat{\chi}_G(\tilde{x}) = 1\}$, and hence $\hat{\chi}_{G^c} + \hat{\chi}_G |\hat{h}| > 1 - c$ on $\tilde{X}$. Hence $\chi_{G^c} + \chi_G |h| > 1 - c$ a.e., and hence $G \subset E$. Hence $V(\phi_0) \subset \{\phi \in Y : \hat{\chi}_E(\phi) = 1\}$. Therefore we have $F \supset \{\phi \in Y : \hat{\chi}_E(\phi) = 0\}$.

(iv) By the same argument as for (iii) we are able to prove (iv).

Q.E.D.

COROLLARY 3.5. *Let $\tilde{X} = M(L^\infty(m))$, $Y = \Gamma(H^\infty |\, \text{hull}\, I^\infty)$ and $F = \tilde{X} \cap Y$. Then we have the following.*

(i) $\tilde{\pi}(\tilde{X} \backslash F) = Y \backslash F$.
(ii) $\tilde{\pi}_1(M(H^\infty) \backslash \text{hull}\, I^\infty) = Y \backslash F$.

PROOF. (i) Let $\tilde{x} \in \tilde{X} \backslash F$ and let $E = E(I^\infty)$. Then, by Theorem 3.4, (i), (iv), $1 = \hat{\chi}_E(\tilde{x}) = \hat{\chi}_E(\tilde{\pi}(\tilde{x}))$. Therefore we have $\tilde{\pi}(\tilde{X} \backslash F) \subset Y \backslash F$.

Next, by $\tilde{\pi}(\tilde{X}) = Y$ and $\tilde{\pi}(F) = F$, for every $\tilde{\phi} \in Y \backslash F$ there is a point $\tilde{x} \in \tilde{X} \backslash F$ such that $\tilde{\pi}(\tilde{x}) = \tilde{\phi}$. Therefore we have $\tilde{\pi}(\tilde{X} \backslash F) \supset Y \backslash F$.

(ii) Let $\theta \in M(H^\infty) \backslash \text{hull}\, I^\infty$. If $\tilde{\pi}_1(\theta) = \tilde{x} \in F$, then $\tilde{x}(f) = \theta(f)$ ($\forall f \in \mathscr{H}^\infty$). Hence $|\theta(f)| = 1$ for all inner functions f in $\mathscr{H}^\infty$, and hence $\theta \in Y$. By $\tilde{x} \in Y \cap \tilde{X}$, $\tilde{x} \in Y$. Hence $\theta = \tilde{x}$. By Theorem 3.4, (ii), this is absurd. Hence $\tilde{\pi}_1(\theta)\in Y\backslash F$. Therefore we have $\tilde{\pi}_1(M(H^\infty)\backslash \mathrm{hull}\, I^\infty)\subset Y\backslash F$.

Next, if $\phi\in Y\backslash F$, then by (i) there is a point $\tilde{x}\in\tilde{X}\backslash F$ such that $\tilde{\pi}(\tilde{x})=\phi$. Therefore, by Theorem 3.4, (ii), we have $\tilde{\pi}_1(M(H^\infty)\backslash \mathrm{hull}\, I^\infty)\supset Y\backslash F$. Q.E.D.

COROLLARY 3.6. *Let $\tilde{X}$ and Y be as in Corollary 3.5. Suppose that $H_m^\infty\supsetneqq J^\infty$. Then $\tilde{X}\supset Y$ if and only if H^∞ is a maximal weak-* closed subalgebra of L^∞.*

PROOF. Assume $\tilde{X}\supset Y$. Then $\tilde{\pi}(\tilde{X}\backslash Y)=\tilde{\pi}(\tilde{X}\backslash\tilde{X}\cap Y)=Y\backslash\tilde{X}\cap Y=\emptyset$. Hence $\tilde{X}=Y$, and hence $H^\infty=\mathscr{H}^\infty$. Hence $L^\infty=\mathscr{L}^\infty$. Therefore, $H^\infty=\mathscr{H}'^\infty$ is a maximal weak-* closed subalgebra of L^∞ (see Case 2.1 in § 2).

Next, assume that H^∞ is a maximal weak-* closed subalgebra of L^∞. Then $H_{\min}^\infty=L^\infty(m)$, and hence $I^\infty=\{0\}$, and hence $H^\infty=\mathscr{H}^\infty$. By $\tilde{X}=\Gamma(H^\infty)$ and $Y=\Gamma(H^\infty|\mathrm{hull}\, I^\infty)$, we have $\tilde{X}=Y$. Q.E.D.

COROLLARY 3.7 *Let $\tilde{X}$ and Y be as in Corollary 3.5. Then $\tilde{X}\cap Y=\emptyset$ if and only if there is an inner function h in I^∞.*

PROOF. By Theorem 3.4, (iv), $\tilde{X}\cap Y=\emptyset$ implies that $\tilde{X}=\{\tilde{x}\in\tilde{X}: \tilde{\chi}_E(\tilde{x})=1\}$, where $E=E(I^\infty)$. Hence $\chi_E=1$ a.e.. Hence, by Lemma 3.3, there is an inner function h in I^∞.

Next, if there is an inner function h in I^∞, then $|\hat{h}|=0$ on Y and $|\hat{h}|=1$ on $\tilde{X}$. Thus we have $\tilde{X}\cap Y=\emptyset$. Q.E.D.

THEOREM 3.8. *Let $Y=\Gamma(H^\infty|\mathrm{hull}\, I^\infty)$. Then we have the following.*

(i) *The space* hull I^∞ *is connected.*

(ii) hull $I^\infty\backslash Y$ *is an open set in $M(H^\infty)$, and* $\overline{\mathrm{hull}\, I^\infty\backslash Y}\supset Y$.

(iii) $\overline{M(H^\infty)\backslash \mathrm{hull}\, I^\infty}\cap \mathrm{hull}\, I^\infty\subset Y$.

(iv) *If $I^\infty\supsetneqq\{0\}$, then the space $M(H^\infty)\backslash \mathrm{hull}\, I^\infty$ is disconnected, and hence $M(I^\infty)$ is disconnected.*

PROOF. (i) Since $M(\mathscr{H}^\infty)$ is connected (cf. [15], p. 167, Theorem 10), hull $I^\infty=\Sigma(M(\mathscr{H}^\infty))$ is connected.

(ii) Let $\phi_0\in \mathrm{hull}\, I^\infty\backslash Y$. By Proposition 3.2, there is an inner function f in $\mathscr{H}^\infty$ with $|\phi_0(f)|<1$. On the other hand, when we defined the map $\tilde{\pi}_1$ (see (3.9)), we saw that, for every $\phi\in M(H^\infty)\backslash \mathrm{hull}\, I^\infty$, $\phi|\mathscr{H}^\infty$ can be multiplicatively extended to $\mathscr{L}^\infty$. Hence we have $|\theta(f)|=1$ for every $\theta\in Y\cup(M(H^\infty)\backslash \mathrm{hull}\, I^\infty)$. Hence $(\phi_0\in)\{\psi\in M(H^\infty): |\psi(f)|<1\}$ is contained in hull $I^\infty\backslash Y$. Therefore hull $I^\infty\backslash Y$ is an open set in $M(H^\infty)$.

Next, let $\Gamma=\Gamma(\mathscr{H}^\infty)=M(\mathscr{L}^\infty)$ and let $K=\overline{M(\mathscr{H}^\infty)\backslash\Gamma}$. Suppose that K does not contain Γ. Then $\Gamma\backslash K=M(\mathscr{H}^\infty)\backslash K$ is a non-empty open set in

$M(\mathscr{H}^\infty)$. Let $x_0 \in M(\mathscr{H}^\infty)\setminus K$. Since $M(\mathscr{H}^\infty)$ is a normal space, there is an open neighborhood $U(x_0)$ of x_0 such that $U(x_0) \subset \overline{U(x_0)} \subset M(\mathscr{H}^\infty)\setminus K$, where $\overline{U(x_0)}$ is the closure of $U(x_0)$ in $M(\mathscr{H}^\infty)$. Since $\overline{U(x_0)} = \overline{U(x_0)} \cap (\Gamma\setminus K)$, $\overline{U(x_0)}$ is the closure of an open set $U(x_0)$ in the subspace $\Gamma\setminus K$. On the other hand, since Γ is a Stonian space, the subspace $\Gamma\setminus K$ also is extremally disconnected. Hence $\overline{U(x_0)}$ is open in the subspace $\Gamma\setminus K$. Since $\Gamma\setminus K = M(\mathscr{H}^\infty)\setminus K$ is an open set in $M(\mathscr{H}^\infty)$, $\overline{U(x_0)}$ is an open set in $M(\mathscr{H}^\infty)$. Hence $\overline{U(x_0)}$ is a clopen set in $M(\mathscr{H}^\infty)$. This is a contradiction. Hence $\overline{M(\mathscr{H}^\infty)\setminus\Gamma} \supset \Gamma$. Therefore, by the map Σ, we have $\overline{\text{hull}\, I^\infty \setminus Y} \supset Y$.

(iii) Let $\phi \in \overline{M(H^\infty)\setminus \text{hull}\, I^\infty} \cap \text{hull}\, I^\infty$. Then, there is a net $\{\phi_\alpha\} \subset M(H^\infty)\setminus\text{hull}\, I^\infty$ such that $\phi_\alpha \to \phi$. Since $|\phi_\alpha(f)|=1$ for every inner function f in $\mathscr{H}^\infty$, we have $|\phi(f)|=1$. By Proposition 3.2, ϕ belongs to Y.

(iv) Let $F=\tilde{X}\cap Y$, where $\tilde{X}=M(L^\infty)$. By Theorem 3.4, (iii) and Corollary 3.5, F is a clopen set with $F \subsetneqq Y$. Since Y is a Stonian space, there is a clopen set U with $\emptyset \subsetneqq U \subsetneqq Y\setminus F$. Then, by Corollary 3.5, $\emptyset \subsetneqq \tilde{\pi}_1^{-1}(U) \subsetneqq \tilde{\pi}_1^{-1}(Y\setminus F) = M(H^\infty)\setminus\text{hull}\, \mathrm{I}^\infty$, and hence $\tilde{\pi}_1^{-1}(U)$ is a nontrivial clopen set in $M(H^\infty)\setminus\text{hull}\, I^\infty$. Therefore $M(H^\infty)\setminus\text{hull}\, I^\infty$ is disconnected. Further, by the map Π (see (3.7)), we see that $M(I^\infty)$ is disconnected.

Q. E. D.

Since A is a weak-* Dirichlet algebra, H^∞ is a strongly logmodular algebra on $\tilde{X}=M(L^\infty(m))$. Hence, every $\tilde{\phi}$ in $M(H^\infty)$ has a unique representing measure $\tilde{\phi}$ on $\tilde{X}$.

PROPOSITION 3.9. *Let $\tilde{X}=M(L^\infty)$ and let $Y=\Gamma(H^\infty|\text{hull}\, I^\infty)$. Then we have the following.*

(i) *If the Gleason part $P(m)$ of $m\in M(H^\infty)$ is nontrivial, then $\tilde{\Omega} = \cup\{S(\tilde{\phi}) : \tilde{\phi}\in Y\}$ is dense in $\tilde{X}$.*

(ii) *$f \in \hat{L}^\infty(m)$ belongs to $\hat{\mathscr{L}}^\infty$ if and only if $f(\tilde{x})=\tilde{\phi}(f)$ on $\mathscr{K}(\tilde{\phi})$ for every $\tilde{\phi}\in Y$.*

PROOF. (i) Suppose $\overline{(\tilde{\Omega})} \subsetneqq \tilde{X}$. Let $\tilde{x}_0 \in \tilde{X}\setminus\overline{(\tilde{\Omega})}$, and let $V=V(\tilde{x}_0)$ $(\subset \tilde{X}\setminus\overline{(\tilde{\Omega})})$ be a clopen neighborhood of $\tilde{x}_0$. Then, by Proposition 2.1, we have $\chi_V = g+h$, where $g\in\hat{\mathscr{L}}^\infty$ and $h \in \hat{N}_\infty$. By [14, Theorem 8], hull $I^\infty = \overline{P(m)}$, and hence, for every $\tilde{\phi}\in Y$, there is a net $\{\phi_\alpha\}$ $(\subset P(m))$ such that $\tilde{\phi}_\alpha \to \tilde{\phi}$. Hence it follows from $\tilde{\phi}_\alpha(h)=0$ for all α and $h\in C_R(\tilde{X})$ that $\tilde{\phi}(h)=0$ (cf. [8], Lemma 3). Hence, for every $\tilde{\phi}\in Y$, we have $0=\int \chi_V d\tilde{\phi} = \int g\, d\tilde{\phi} = \tilde{\phi}(g)$. Since $g=\tilde{\phi}(g)$ for all $\tilde{x}\in\mathscr{K}(\tilde{\phi})$ (see (3.6)), we have $g=0$ on $\tilde{X}=\cup\{\mathscr{K}(\tilde{\phi}) : \tilde{\phi}\in Y\}$, and hence $\chi_V=h$. Hence, by $S(\tilde{m})=\tilde{X}$, we have

$0=\int_{\tilde{X}} h\, d\tilde{m}=\int_{\tilde{X}} \chi_V\, d\tilde{m}=\tilde{m}(V)>0$, which is a contradiction.

(ii) Let $f\in \widehat{L^\infty(m)}=C(\tilde{X})$ and let f be a constant on $\mathscr{K}(\tilde{\phi})$ for every $\tilde{\phi}\in Y$. Then, there is a (unique) function $g(\in C(Y))$ such that $f=g\circ\tilde{\pi}$ (cf. [1], Lemma 4.3). On the other hand, $g=G|Y$ for a (unique) function $G\in\mathscr{L}^\infty$. Hence $g\circ\pi=G$ on $\tilde{X}$, and hence $f=G$ on $\tilde{X}$. Therefore f belongs to $\mathscr{L}^\infty$. Q. E. D.

By (3.1) we have $\log|(\mathscr{H}^\infty)^{-1}|=\mathscr{L}_R^\infty$. On the other hand, since $\log|(H^\infty)^{-1}|=L_R^\infty$ and since, for every $f\in\mathscr{L}^\infty$, $f=\phi(f)$ on $\mathscr{K}(\phi)$ $(\phi\in Y)$, we have $\log|(C'+I^\infty)^{-1}|=R+N_R^\infty$ on $\mathscr{K}(\phi)$, where $N_R^\infty=N^\infty\cap L_R^\infty$ (see Proposition 2.1) and $C'=C\backslash\{0\}$. On a certain condition, this relation holds on $\tilde{X}=M(L^\infty(m))$.

PROPOSITION 3.10. *Suppose that the Gleason part $P=P(m)$ of $m\in M(H^\infty)$ is nontrivial. Let N^∞ be the weak-* closure of $I^\infty+\overline{I^\infty}$ and let $N_R^\infty=N^\infty\cap L_R^\infty$. Then we have*

$$\log|(C'+I^\infty)^{-1}|=R+N_R^\infty,$$

where C and R are the complex and the real fields respectively, and $C'=C\backslash\{0\}$.

PROOF. Let $u=r+h$, where $r\in R$ and $h\in N_R^\infty$. Then, there is a function $f\in(H^\infty(m))^{-1}$ such that $\log|f|=u$. For every $\phi\in P$, $\log|\phi(f)|=\int\log|f|d\phi=r$, and hence we have $|\phi(f)|=e^r$ on $\bar{P}$. Here, let $f=g+h$, where $g\in\mathscr{H}^\infty$ and $h\in I^\infty$. Then $g\in(\mathscr{H}^\infty)^{-1}$ and $|\phi(g)|=e^r$ on $\bar{P}$. Hence $G=g/e^r\in(\mathscr{H}^\infty)^{-1}$ and $|\phi(G)|=1$ on Y. Hence $|G|=1$ on $\tilde{X}$. Since $\tilde{X}$ is an antisymmetric set, G and hence g are constant on $\tilde{X}$. Hence $f=c+h$, where $c\in C'$. Let $k=1/(c+h)$. Then $k\in C'+I^\infty$, hence $f\in(C'+I^\infty)^{-1}$.

Next, if $u\in\log|(C'+I^\infty)^{-1}|$, then $u=\log|c+h|$, where $c+h\in(C'+I^\infty)^{-1}$. Then, for $\phi\in Y$,

$$\int u\, d\phi=\int\log|c+h|d\phi=\log\left|\int(c+h)d\phi\right|=\log|c|.$$

Here, let $u=g+h$, where $g\in\mathscr{L}_R^\infty$ and $h\in N_R^\infty$. Then $u(\phi)=g(\phi)=\log|c|$ for every $\phi\in Y$, and hence $g=\log|c|$ on $\tilde{X}$. Therefore $u=\log|c|+h\in R+N_R^\infty$. Q. E. D.

§ 4. Some properties of $M(H^\infty)$, Part 2.

In this section, let A be a weak-* Dirichlet algebra on a nontrivial probability measure space $(X, \mathscr{A}, m)$, and let H^∞, $\mathscr{H}^\infty$, I^∞ and $\mathscr{L}^\infty=$

$L^\infty(m|\Delta)$ be those objects as defined in the cases 2.1, 2.2 and 2.3 in § 2.

PROPOSITION 4.1. *Let* $Y=\Gamma(H^\infty|\text{hull } I^\infty)$ and let $\phi\in(M(H^\infty)\backslash\text{hull } I^\infty)\cup Y$. Then, for every $f\in\mathscr{L}^\infty$, f is a constant $(=\phi(f))$ on $S(\phi)$, and hence $S(\phi)\subset\mathscr{K}(\phi_0)$, where $\phi_0=\tilde{\pi}(S(\phi))$ and $\mathscr{K}(\phi_0)=\tilde{\pi}^{-1}(\phi_0)$.

PROOF. For every inner function f in $\mathscr{H}^\infty$, $|\phi(f)|=1$ and hence $\int|f-\phi(f)|^2d\phi=0$. Hence $f=\phi(f)$ on $S(\phi)$.

By $\log|(\mathscr{H}^\infty)^{-1}|=\mathscr{L}_R^\infty$ and [4, Theorem 2], the set $Q=\{F\bar{G}: F$ is a finite linear combination of inner functions in $\mathscr{H}^\infty$ and G is an inner function in $\mathscr{H}^\infty\}$ is dense in norm in $\mathscr{L}^\infty$. Hence, for $f\in\mathscr{L}^\infty$ and any positive ε, there is a function $F\bar{G}\in Q$ such that $\|f-F/G\|<\varepsilon/2$. Hence,

$$\int|f-\phi(f)|d\phi\leqq\|f-F/G\|+\int|\phi(F/G)-\phi(f)|d\phi<\varepsilon.$$

Hence, $f=\phi(f)$ on $S(\phi)$. Q. E. D.

THEOREM 4.2. *Let* $Y=\Gamma(H^\infty|\text{hull } I^\infty)$. *Then the set* hull $I^\infty\backslash Y$ *is a union of Gleason parts for* $H^\infty(m)$.

PROOF. By Proposition 4.1, if ϕ belongs to $(M(H^\infty)\backslash\text{hull } I^\infty)\cup Y$, then $|\phi(f)|=1$ for every inner function f in $\mathscr{H}^\infty$. On the other hand, by Proposition 3.2, for $\psi\in\text{hull } I^\infty\backslash Y$, there is an inner function f_0 in $\mathscr{H}^\infty$ such that $|\psi(f_0)|<1$. Let $F=(f_0-\psi(f_0))/(1-\overline{\psi(f_0)}f_0)$. Then F is an inner function in $\mathscr{H}^\infty$ and $\psi(F)=0$. Hence $\sup\{|\phi(f)|: f\in H^\infty(m),\ \|f\|\leqq1,\ \psi(f)=0\}=1$. Hence hull $I^\infty\backslash Y$ is a union of Gleason parts for $H^\infty(m)$. Q. E. D.

It occurs that a certain point of Y belongs to some nontrivial Gleason part and a certain point of Y composes trivial Gleason part, and both two cases occur actually (see Example 2, (iii) in § 8).

By using $\|g\|\leqq\|f\|$ for $f=g+h\in H^\infty$, where $g\in\mathscr{H}^\infty$ and $h\in I^\infty$, we obtain the following.

PROPOSITION 4.3. *For* ϕ *and* θ *in* hull I^∞ *we have*

$$\begin{aligned}&\sup\{|\phi(f)-\theta(f)|: f\in H^\infty(m),\ \|f\|\leqq1\}\\ =&\sup\{|\phi(f)-\theta(f)|: f\in\mathscr{H}^\infty,\ \|f\|\leqq1\}.\end{aligned}$$

THEOREM 4.4. *Let* $Y=\Gamma(H^\infty|\text{hull } I^\infty)$ *and let* $\mathscr{K}(\phi)=\tilde{\pi}^{-1}(\phi)$ *for* $\phi\in Y$, *and let* $\widetilde{\mathscr{K}(\phi)}$ *be the* H^∞*-convex hull of* $\mathscr{K}(\phi)$. *Then we have the following.*

(i) $\mathscr{K}(\phi)$ *is a weak peak set for* H^∞, *and* $\widetilde{\mathscr{K}(\phi)}\cap\text{hull } I^\infty=\{\phi\}$.

(ii) $M(H^\infty)\backslash\text{hull } I^\infty=\cup\{\widetilde{\mathscr{K}(\phi)}\backslash\{\phi\}: \phi\in Y\}$, *and* $\widetilde{\mathscr{K}(\phi)}\cap\widetilde{\mathscr{K}(\psi)}=\emptyset$

for $\phi \neq \psi$.

(iii) *The set* $(M(H^\infty)\backslash \mathrm{hull}\, I^\infty) \cup Y = \cup\{\widetilde{\mathscr{K}(\phi)} : \phi \in Y\}$ *is a union of Gleason parts, and for every* ϕ *in* Y, $\widetilde{\mathscr{K}(\phi)}$ *is a union of Gleason parts.*

PROOF. (i) Let $\theta \in \widetilde{\mathscr{K}(\phi)} \cap \mathrm{hull}\, I^\infty$. By $\theta \in \widetilde{\mathscr{K}(\phi)}$, for every inner function f in $\mathscr{H}^\infty$, we have $|\theta(f)|=1$. Hence, by Proposition 3.2, θ belongs to Y. Hence $\theta=\phi$, and hence, $\widetilde{\mathscr{K}(\phi)} \cap \mathrm{hull}\, I^\infty = \{\phi\}$.

Let $\phi \in Y$, and let $U(\phi)$ be any clopen neighborhood of ϕ in Y, and let $E=\tilde{\pi}^{-1}(U(\phi))$. Then $E \supset \mathscr{K}(\phi)$. By Proposition 3.9, $\chi_E \in \mathscr{L}^\infty$, and hence, there is a function $F \in (\mathscr{H}^\infty)^{-1}$ such that $\log|F|=\chi_E$. By Proposition 4.1, $S(\phi) \subset \mathscr{K}(\phi)$, and hence,

$$\log|\phi(F)| = \int \log|F|\, d\phi = 1.$$

Hence $|\phi(F/e)|=1$. Since $|F/e|=1$ on $\mathscr{K}(\phi)$, we have

$$\int_{\tilde{X}} |F/e - \phi(F/e)|^2 d\phi = 0,$$

and hence, $F=\phi(F)$ on $S(\phi)$. Hence $F=\phi(F)$ on $\mathscr{K}(\phi)$.

Let $f_E = F/\phi(F)$. Then $f_E=1$ on $\mathscr{K}(\phi)$, $|f_E|=1$ on E, and $|f_E|<1$ on E^c. Hence $\|f_E\|=f_E=1$ on $\mathscr{K}(\phi)$. Hence $K_E=\{x : f_E(x)=1\}$ is a peak set for H^∞, and $\mathscr{K}(\phi) \subset K_E \subset E$. Let $\{U_\alpha(\phi)\}$ be a fundamental system of clopen neighborhoods of ϕ in Y. Then, $\bigcap_\alpha \tilde{\pi}^{-1}(U_\alpha(\phi)) = \tilde{\pi}^{-1}(\phi) = \mathscr{K}(\phi)$, and hence, $\bigcap_\alpha K_{E_\alpha} = \mathscr{K}(\phi)$, where $E_\alpha = \tilde{\pi}^{-1}(U_\alpha(\phi))$. Therefore, $\mathscr{K}(\phi)$ is a weak peak set for H^∞.

(ii) Since $\mathscr{K}(\phi) \cap \mathscr{K}(\psi) = \emptyset$ for $\phi \neq \psi$, $\widetilde{\mathscr{K}(\phi)} \cap \widetilde{\mathscr{K}(\psi)} = \emptyset$. Let $\theta \in M(H^\infty)\backslash \mathrm{hull}\, I^\infty$. By Proposition 4.1, $S(\theta) \subset \mathscr{K}(\phi)$ for some $\phi \in Y$. Hence $\theta \in \widetilde{\mathscr{K}(\phi)} \backslash \{\phi\}$, and hence, $M(H^\infty)\backslash \mathrm{hull}\, I^\infty \subset \cup\{\widetilde{\mathscr{K}(\phi)}\backslash\{\phi\} : \phi \in Y\}$.

Next, let $\theta \in \cup\{\widetilde{\mathscr{K}(\phi)}\backslash\{\phi\} : \phi \in Y\}$. Then there is a point ϕ in Y such that $\theta \in \widetilde{\mathscr{K}(\phi)}\backslash\{\phi\}$. For every inner function g in $\mathscr{H}^\infty$, $g=\phi(g)$ on $\mathscr{K}(\phi)$, and hence, $g=\phi(g)$ on $S(\theta)$. Then, by $|\theta(g)|=|\phi(g)|=1$, we have $\theta \notin \mathrm{hull}\, I^\infty \backslash Y$. Hence $\theta \in (M(H^\infty)\backslash \mathrm{hull}\, I^\infty) \cup Y$, and, by $\theta \notin Y$, we have $\theta \in M(H^\infty)\backslash \mathrm{hull}\, I^\infty$.

(iii) By Theorem 4.2, (iii) is obvious. Q.E.D.

THEOREM 4.5. *Let* B_1 *and* B_2 *be weak-* closed subalgebras of* $L^\infty(m)$ *such that* $H^\infty \subsetneqq B_1 \subsetneqq B_2 \subset L^\infty(m)$. *Let* $I_{B_i}^\infty = \{h \in L^\infty(m) : \int_X hf\, dm = 0 \ \forall f \in B_i\}$

$(i=1,2)$, *and let* $\mathscr{H}_{B_i}^{\infty}=(B_i\cap\bar{B}_i)\cap H^{\infty}$ $(i=1,2)$. *Let* $Y_i=\Gamma(H^{\infty}|\mathrm{hull}\, I_{B_i}^{\infty})$ $(i=1,2)$ *and let* $Y=\{\phi\in\mathrm{hull}\, I_{B_2}^{\infty}:|\phi(h)|=1$ *for every inner function* h *in* $\mathscr{H}_{B_1}^{\infty}\}$. *Then we have the following.*

(i) $I_{B_1}^{\infty}\supsetneqq I_{B_2}^{\infty}$ *and* $\mathscr{H}_{B_1}^{\infty}\subsetneqq\mathscr{H}_{B_2}^{\infty}$.

(ii) $\mathrm{hull}\, I_{B_1}^{\infty}\subsetneqq\mathrm{hull}\, I_{B_2}^{\infty}$.

(iii) $\mathrm{hull}\, I_{B_1}^{\infty}=(\mathrm{hull}\, I_{B_2}^{\infty}\setminus Y)\cup Y_1$ *and* $Y\supset Y_i$ $(i=1,2)$.

(iv) $Y\setminus Y_2=\mathrm{hull}\, I_{B_2}^{\infty}\setminus[(\mathrm{hull}\, I_{B_1}^{\infty}\setminus Y_1)\cup Y_2]$ *is a union of Gleason parts.*

PROOF. (i) It follows from $H^{\infty}B_i\subset B_i$ $(i=1,2)$ and [[10], Lemma 1.1] that $B_i=[B_i]_1\cap L^{\infty}(m)$ $(i=1,2)$. Hence $[B_1]_1\subsetneqq[B_2]_1$. Therefore $I_{B_1}^{\infty}\supsetneqq I_{B_2}^{\infty}$.

By $B_1\subset B_2$ we have $\mathscr{H}_{B_1}^{\infty}\subset\mathscr{H}_{B_2}^{\infty}$. If $\mathscr{H}_{B_1}^{\infty}=\mathscr{H}_{B_2}^{\infty}$, then $H^{\infty}=\mathscr{H}_{B_1}^{\infty}\oplus I_{B_1}^{\infty}\supsetneqq H^{\infty}=\mathscr{H}_{B_2}^{\infty}\oplus I_{B_2}^{\infty}$. Therefore $\mathscr{H}_{B_1}^{\infty}\subsetneqq\mathscr{H}_{B_2}^{\infty}$.

(ii) By $I_{B_1}^{\infty}\supset I_{B_2}^{\infty}$ we have $\mathrm{hull}\, I_{B_1}^{\infty}\subset\mathrm{hull}\, I_{B_2}^{\infty}$. If $\mathrm{hull}\, I_{B_1}^{\infty}=\mathrm{hull}\, I_{B_2}^{\infty}$, then $H^{\infty}|\mathrm{hull}\, I_{B_1}^{\infty}=H^{\infty}|\mathrm{hull}\, I_{B_2}^{\infty}$. Hence $\Gamma=\Gamma(H^{\infty}|\mathrm{hull}\, I_{B_1}^{\infty})=\Gamma(H^{\infty}|\mathrm{hull}\, I_{B_2}^{\infty})$. Hence $\mathscr{H}_{B_1}^{\infty}=\mathscr{H}_{B_2}^{\infty}$ on $\Gamma=\Gamma(\mathscr{H}_{B_1}^{\infty}|\,\mathrm{hull}\, I_{B_1}^{\infty})=\Gamma(\mathscr{H}_{B_2}^{\infty}|\,\mathrm{hull}\, I_{B_2}^{\infty})$. Hence, by using $\tilde{\pi}$ (see (3.5)), we have $\mathscr{H}_{B_1}^{\infty}=\mathscr{H}_{B_2}^{\infty}$ on $\tilde{X}=M(L^{\infty}(m))$. Hence $\mathscr{H}_{B_1}^{\infty}=\mathscr{H}_{B_2}^{\infty}$, which is a contradiction. Therefore $\mathrm{hull}\, I_{B_1}^{\infty}\subsetneqq\mathrm{hull}\, I_{B_2}^{\infty}$.

(iii) This part is derived from Proposition 3.2.

(iv) This part is derived from Theorem 4.2. Q.E.D.

§5. Some properties of a strongly logmodular algebra, Part 1.

Let X be a compact Hausdorff space and let $M(X)$ be the conjugate space of $C(X)$ i.e., the space of regular Borel measures on X. Given another compact Hausdorff space Z and a continuous map ρ of X onto Z, for every $f\in C(Z)$ let

(5.1) $\rho^0(f)(x)=f(\rho(x))$ for all $x\in X$.

The map ρ^0 is an isometric isomorphism of $C(Z)$ into $C(X)$. Let ρ^* be the adjoint map of ρ^0. Then, for every $\mu\in M(X)$, we have

(5.2) $\int_X f\circ\rho\, d\mu=\int_Z f\, d(\rho^*(\mu))$ for all $f\in C(Z)$,

or, equivalently,

(5.3) $\rho^*(\mu)(B)=\mu(\rho^{-1}(B))$ for every Borel set B of Z.

Let A be a logmodular algebra on a compact space X, and m be a point of $M(A)$, and $\tilde{x}_0$ be any point of $\tilde{X}=M(L^{\infty}(dm))$. For every f in $C(X)$, if we define the map $\phi: f\mapsto\hat{f}(\tilde{x}_0)$, then $\phi\in M(C(X))$. Hence there is a unique point x_0 in X such that $\hat{f}(\tilde{x}_0)=\phi(f)=f(x_0)$. Hence we have $\hat{f}(\tilde{x}_0)=f(x_0)$

for all $f \in A$. We let

$$\pi : \tilde{x}_0 \mapsto x_0, \quad \tilde{x}_0 \in \tilde{X}.$$

Then π is a continuous map of $\tilde{X}$ into X.

Next, let $\tilde{\phi}_0$ be any point of $M(H^\infty(m))$ with a unique representing measure $\mu_{\tilde{\phi}_0}(=\tilde{\phi}_0)$ on $\tilde{X}$. Let $\tilde{\phi}_1(\hat{f})=\int_{\tilde{X}} \hat{f}\, d\mu_{\tilde{\phi}_0}(=\int_{\tilde{X}} \hat{f}\, d\tilde{\phi}_0)$ for all $\hat{f} \in C(\tilde{X})$. Then, this linear functional $\tilde{\phi}_1$ on $C(\tilde{X})$ is a unique Hahn-Banach (norm preserving) extension to $C(\tilde{X})$ of $\tilde{\phi}_0 \in M(H^\infty(m))$. For any $f \in C(X)$ we have $\int_{\tilde{X}} \hat{f}\, d\tilde{\phi}_0=\int_{\tilde{X}} f\circ\pi\, d\tilde{\phi}_0=\int_X f\, d(\pi^*(\tilde{\phi}_0))$. *We let*

$$\phi_0 : f \mapsto \int_X f\, d(\pi^*(\tilde{\phi}_0)), \quad f \in C(X).$$

Then we have $\tilde{\phi}_1(\hat{f})=\phi_0(f)$ for all $f \in C(X)$, and hence $\tilde{\phi}_0(\hat{f})=\phi_0(f)$ for all $f \in A$. Hence $\pi^*(\tilde{\phi}_0)$ is a unique representing measure of $\phi_0 \in M(A)$. We let

$$\pi : \tilde{\phi}_0 \mapsto \phi_0, \quad \tilde{\phi}_0 \in M(H^\infty(m)).$$

The map π is a continuous map of $M(H^\infty(m))$ into $M(A)$. If $\tilde{\phi} \in M(H^\infty(m))$ and $\pi(\tilde{\phi})=\phi$, then we have $\pi(S(\tilde{\phi}))=S(\phi)$. Hence if $X=S(m)$, then we have $\pi(\tilde{X})=X$.

In the following Proposition 5. 1 and Corollaries 5. 2 and 5. 3, we suppose $X=S(m)$, and let

$$\pi_1=\pi|\tilde{X}.$$

If $f \in C(X)$, then $\hat{f}(\pi_1^{-1}(x))=f(x)$ for any x in X. This relation is extended as follows.

PROPOSITION 5. 1. *Let $f \in L^\infty(dm)$, $\alpha \in X$, and $\tilde{X}_\alpha=\{\tilde{x} \in \tilde{X} : \pi(\tilde{x})=\alpha\}=\pi_1^{-1}(\alpha)$. If f is continuous at $x=\alpha$, then $\hat{f}(\tilde{x})=f(\alpha)$ for all $\tilde{x} \in \tilde{X}_\alpha$.*

PROOF. We suppose that $\|f\| \leqq 1$ and $f(\alpha)=0$. For any $\varepsilon>0$, there is an open neighborhood $V(\alpha)$ of α such that $|f(x)|<\varepsilon$ for every $x \in V(\alpha)$. By Urysohn's lemma, there is a function $h \in C_R(X)$ such that $h(\alpha)=1$, $h(x)=0$ for every $x \in X \setminus V(\alpha)$, and $0 \leqq h(x) \leqq 1$ for every $x \in X$. Then we have $|(1-h^n)f-f|=|h^n f| \leqq |f|<\varepsilon$ on $V(\alpha)$, and $|(1-h^n)f-f|=|h^n f|=0$ on $X\setminus V(\alpha)$. Hence we have $\|(1-h^n)f-f\|<\varepsilon$, and hence $|\tilde{x}[(1-h^n)f-f]|=|\tilde{x}(f)|<\varepsilon$ for all $\tilde{x} \in \tilde{X}_\alpha$. Hence we have $f(\tilde{x})=0$ for all $\tilde{x} \in \tilde{X}_\alpha$. Q. E. D.

Let E be a measurable subset of X, and let $\hat{E}=\{\tilde{x} \in \tilde{X} : \chi_E(\tilde{x})=1\}$. Then $\hat{E}$ is a clopen set in $\tilde{X}$.

COROLLARY 5.2. *If O is an open subset of X, then we have* $\hat{O}\supset \pi_1^{-1}(O)$, $\hat{O}=\overline{\pi_1^{-1}(O)}$ *and* $\pi(\hat{O})=\bar{O}$.

PROOF. By Proposition 5.1, we have $\hat{O}\supset\pi_1^{-1}(O)$. Since $\tilde{m}$ is a normal measure on $\tilde{X}$ and since $\pi^*(\tilde{m})=m$, we have $\tilde{m}(\hat{O})=\int\hat{\chi}_O\, d\tilde{m}=\int\chi_O\, dm=m(O)=\tilde{m}(\pi_1^{-1}(O))=\tilde{m}(\overline{\pi_1^{-1}(O)})$ (see (5.3)). Hence we have $\hat{O}=\overline{\pi_1^{-1}(O)}$.

$\hat{O}\supset\pi_1^{-1}(O)$ implies $\pi(\hat{O})\supset O$. Hence $\pi(\hat{O})\supset\bar{O}$. And we have $\pi(\hat{O})=\pi(\overline{\pi_1^{-1}(O)})\subset\overline{\pi(\pi_1^{-1}(O))}=\bar{O}$. Hence $\pi(\hat{O})=\bar{O}$. Q.E.D.

COROLLARY 5.3. *If O is a clopen subset of X, then $\hat{O}=\pi_1^{-1}(O)$. If K is a compact subset of X, then $\pi(\hat{K})\subset K$.*

PROOF. If O is a clopen subset of X, then $\hat{O}=\overline{\pi_1^{-1}(O)}=\pi_1^{-1}(O)$.

Since $O=X\backslash K$ is an open subset of X, $\hat{O}\supset\pi_1^{-1}(O)$. Hence, by $\hat{O}\cap\hat{K}=\emptyset$, we obtain $\pi(\hat{K})\subset K$. Q.E.D.

THEOREM 5.4. *Let A be a strongly logmodular algebra on a compact Hausdorff space X. Then, for every $\phi\in M(A)$, $S(\phi)$ is a weak peak set for A.*

PROOF. Let x_0 be any element of $X\backslash S(\phi)$. By Urysohn's lemma, there is a function $g\in C_R(X)$ such that $g(x_0)=0$, $g(x)=1$ for every x in $S(\phi)$, and $0\leqq g(x)\leqq 1$ for every x in X. By $\log|A^{-1}|=C_R(X)$ there is a function F in A^{-1} such that $\log|F|=g$. Since the measure ϕ is an Arens-Singer measure we have $\log|\phi(F)|=\int\log|F|d\phi=\int g\, d\phi=1$, and hence $|\phi(F/e)|=1$. By $|F/e|=1$ on $S(\phi)$, we have $\int|F/e-\phi(F/e)|^2 d\phi=0$. Hence we have $F(x)=\phi(F)$ for every $x\in S(\phi)$. Let $f=F/\phi(F)\in A$. Then we have $f(x)=\|f\|=1$ for every $x\in S(\phi)$ and $|f(x_0)|=1/e<1$.

Now, let U be any open neighborhood of $S(\phi)$, and let y be any point of $X\backslash U$. Then, by what was proved above, there is a function $f\in A$ such that $f(x)=\|f\|=1$ for every $x\in S(\phi)$, and $|f(x)|<1$ for every x in some open neighborhood $V(y)$ of y. Since $X\backslash U\subset\cup\{V(y): y\in X\backslash U\}$ and since $X\backslash U$ is a compact subset, there is $\{y_i: i=1, 2, \cdots, n\}\subset X\backslash U$ such that $X\backslash U\subset\bigcup_{i=1}^{n} V(y_i)$. Let $f_i\in A$ $(i=1, 2, \cdots, n)$ be functions such that $f_i(x)=\|f_i\|=1$ for every $x\in S(\phi)$ and $|f_i(x)|<1$ for every x in $V(y_i)$. Let $f_U=\frac{1}{n}\sum_{i=1}^{n}f_i\in A$ and let $K_U=\{x: f_U(x)=1\}$. Then $\|f_U\|=f_U(x)=1$ for every $x\in S(\phi)$, and $K_U\subset U$. Thus K_U is a peak set for A, and we have $S(\phi)=\bigcap_U K_U$. Therefore, $S(\phi)$ is a weak peak set for A. Q.E.D.

THEOREM 5.5. *Let A be a strongly logmodular algebra on a compact Hausdorff space X. Let $m\in M(A)$, and we suppose that $X=S(m)$. Then we have the following.*

(i) $A=H^{\infty}(m)\cap C(X)$.
(ii) $A^{-1}=(H^{\infty}(m))^{-1}\cap C(X)$.
(iii) $f\in A^{-1}$ *if and only if* $f\in(H^{\infty}(m))^{-1}$ *and* $\log|f|\in C_R(X)$.

PROOF. (i) Let $f\in H^{\infty}(m)\cap C(X)$. Let B be the Banach algebra generated by f and the identity. Then $\log|A^{-1}|=C_R(X)\supset\log|B^{-1}|$. Hence, for any function $g\in B^{-1}$, there is a function $G\in A^{-1}$ such that $|gG^{-1}|=1$. Since $gG^{-1}\in(H^{\infty}(m))^{-1}$ and since $X=S(m)$ is an antisymetric set of $H^{\infty}(m)$, we have $G=\alpha g$ a. e. (m), where α is a constant and $|\alpha|=1$. Since αg and G belong to $C(X)$, we have $G=\alpha g$. Hence $A^{-1}\supset B^{-1}$, and hence $A\supset B$. Thus we obtain $A=H^{\infty}(m)\cap C(X)$.

(ii) Let $f\in(H^{\infty}(m))^{-1}\cap C(X)$. Then $f\in A$, and there is a function $g\in H^{\infty}(m)$ such that $fg=1$ a. e. (m). It is easy to see that $f\in C(X)^{-1}$. Hence $1/f\in H^{\infty}(m)\cap C(X)=A$, and hence $f\in A^{-1}$.

(iii) Let $f\in(H^{\infty}(m))^{-1}$ and let $\log|f|\in C(X)$. Then there is a function $g\in A^{-1}$ such that $\log|g|=\log|f|$. Hence $|fg^{-1}|=1$ and $fg^{-1}\in(H^{\infty}(m))^{-1}$. Hence $f=\alpha g$ a. e. (m). Since $X=S(m)$ and $\alpha g\in A^{-1}$, we have $f\in A^{-1}$.

Q. E. D.

In the rest of this section, let X, A, and m be as in Theorem 5.5. Let

$$H^{\infty}=H^{\infty}(m)=\mathscr{H}^{\infty}\oplus I^{\infty} \text{ and } \mathscr{L}^{\infty}$$

be the objects as defined in the cases 2.1, 2.2 and 2.3 of § 2. Let

$$J=J^{\infty}\cap C(X),\ I=I^{\infty}\cap C(X), \mathscr{H}=\mathscr{H}^{\infty}\cap C(X),$$
$$\mathscr{L}=\mathscr{L}^{\infty}\cap C(X), \text{ and } \mathscr{L}_R=\mathscr{L}^{\infty}\cap C_R(X).$$

Then J and I are closed ideals of A, and $\mathscr{H}(\subset A)$, $\mathscr{L}$ and $\mathscr{L}_R$ are Banach algebras. By $\mathscr{L}^{\infty}I^{\infty}=I^{\infty}$ (see (2.2) and (2.4)) we have

$$(5.4)\qquad \mathscr{L}I=I.$$

The following proposition is proved by the same argument as for Theorem 5.5.

PROPOSITION 5.6. *Let X, A and m be as in Theorem 5.5. Then we have the following.*

(i) $\mathscr{H}^{-1}=(\mathscr{H}^{\infty})^{-1}\cap C(X)$.
(ii) $f\in\mathscr{H}^{-1}$ *if and only if* $f\in(\mathscr{H}^{\infty})^{-1}$ *and* $\log|f|\in C_R(X)$.
(iii) $\log|\mathscr{H}^{-1}|=\mathscr{L}_R$.

PROPOSITION 5.7. *Let X, A and m be as in Theorem 5.5. Suppose that $P(m)$ is nontrivial. Then $J=I$ is a primary ideal of A and we have*

$$I=\{f\in A:\phi(f)=0\quad\forall\phi\in P(m)\}.$$

PROOF. Let $I_1=\{f\in A:\phi(f)=0\quad\forall\phi\in P(m)\}$. Then, by [14, Theorem 8], we have $I_1\subset I^\infty$, and hence $I_1\subset I^\infty\cap C(X)=I$.

On the other hand, by $I\subset A$, we have $I\subset I_1$, and hence $I=I_1$.

As we stated in § 2, there is an analytic map of D onto $P(m)$, hnd cence I is a primary ideal of A. Q. E. D.

For x_1 and x_2 in X we define x_1 and x_2 to be $x_1\sim x_2$ when $f(x_1)=f(x_2)$ for all f in $\mathscr{L}$. Then $\sim$ is an equivalence relation on X. Let $\check{X}$ be the quotient space $X/\sim$ with the quotient topology, and let $Q:X\mapsto\check{X}$ be the quotient map (cf. [15], p. 37). For every $f\in\mathscr{L}$ we define a continuous function $\check{f}$ on a compact Hausdorff space $\check{X}$ by $f=\check{f}\circ Q$. Then we have

$$C(\check{X})=\{\check{f}:f\in\mathscr{L}\},$$

and therefore $Q^0(C(\check{X}))=\mathscr{L}$. Further, since $\mathscr{L}$ is self-adjoint, we have

$$C(M(\mathscr{L}))=\{\hat{f}:f\in\mathscr{L}\}.$$

The map $\Phi:\hat{f}\mapsto\check{f}$ $(\forall f\in\mathscr{L})$ is an algebra isomorphism of $C(M(\mathscr{L}))$ onto $C(\check{X})$, and hence the adjoint map $\sigma=\Phi^*$ of Φ is a homeomorphism of $\check{X}$ onto $M(\mathscr{L})$ such that $\Phi(\hat{f})(\check{x})=\hat{f}(\sigma(\check{x}))$ $(\forall\check{x}\in\check{X})$. Let

$$(5.5)\qquad q=\sigma\circ Q.$$

Then q is a continuous map of X onto $M(\mathscr{L})$ and we have $q^0(\hat{f})=f$ $(\forall f\in\mathscr{L})$ and $q^0(\hat{\mathscr{L}})=\mathscr{L}$.

For $\phi\in M(\mathscr{L})$, let

$$K(\phi)=q^{-1}(\phi).$$

Then we have $K(\phi)=\{x\in X:f(x)=\phi(f),\ \forall f\in\mathscr{L}\}$, $X=\cup\{K(\phi):\phi\in M(\mathscr{L})\}$, and $K(\phi)\cap K(\theta)=\emptyset$ for $\phi\neq\theta$.

For every $\tilde{\phi}$ in $Y=\Gamma(H^\infty|\text{hull }I^\infty)$, let

$$(5.6)\qquad \eta:\tilde{\phi}\mapsto(\Sigma^{-1}(\tilde{\phi}))|\mathscr{L}\ \text{(see (3.4))}.$$

Then η is a continuous map of Y onto $M(\mathscr{L})$, because $\mathscr{L}$ is a selfadjoint Banach algebra (cf. [6], p. 80).

THEOREM 5.8. *Let A be a strongly logmodular algebra on a compact Hausdorff space X. Let $m\in M(A)$ and suppose $X=S(m)$. Let $K(\phi)=\{x\in X:f(x)=\phi(f),\ \forall f\in\mathscr{L}\}$ for $\phi\in M(\mathscr{L})$. Then we have the following.*

(i) *For every* $\phi\in M(\mathscr{L})$, *we have*
$\tilde{x}\cap\pi^{-1}(K(\phi))=\cup\{\mathscr{K}(\tilde{\theta}):\tilde{\theta}\in\eta^{-1}(\phi)\}$.

(ii) *For* ϕ, $\theta\in M(\mathscr{L})$ $(\phi\neq\theta)$, *we have*

$$\pi[\cup\{\widetilde{\mathscr{K}(\tilde{\theta})}:\tilde{\theta}\in\eta^{-1}(\phi)\}]\subset\widetilde{K(\phi)}$$

and

$$\widetilde{K(\phi)}\cap\widetilde{K(\theta)}=\emptyset,$$

where $\widetilde{\mathscr{K}(\tilde{\theta})}$ *and* $\widetilde{K(\phi)}$ *are* H^{∞}*-convex hull of* $\mathscr{K}(\tilde{\theta})$ *and* A*-convex hull of* $K(\phi)$ *respectively.*

(iii) *If* $\pi(M(H^{\infty}(m)))=M(A)$, *then we have*

$$M(A)=\text{hull}\,I\cup(\cup\{\widetilde{K(\phi)}:\phi\in M(\mathscr{L})\}).$$

PROOF. (i) Let $\tilde{\theta}\in\eta^{-1}(\phi)$. Then, for every f in $\mathscr{L}$, we have $\tilde{\theta}(f)=\phi(f)$. From $f=\tilde{\theta}(f)$ on $\mathscr{K}(\tilde{\theta})$ we have $f=\phi(f)$ on $\pi(\mathscr{K}(\tilde{\theta}))$. Hence we have $\pi(\mathscr{K}(\tilde{\theta}))\subset K(\phi)$, and hence $\pi[\cup\{\mathscr{K}(\tilde{\theta}):\tilde{\theta}\in\eta^{-1}(\phi)\}]\subset K(\phi)$. Hence it follws from $\tilde{X}=\cup\{\mathscr{K}(\tilde{\theta}):\tilde{\theta}\in Y\}=\bigcup_{\phi\in M(\mathscr{L})}[\cup\{\mathscr{K}(\tilde{\theta}):\tilde{\theta}\in\eta^{-1}(\phi)\}]$, $X=\cup\{K(\phi):\phi\in M(\mathscr{L})\}$ and $\pi(\tilde{X})=X$ that

$$\tilde{X}\cap\pi^{-1}(K(\phi))=\cup\{\mathscr{K}(\tilde{\theta}):\tilde{\theta}\in\eta^{-1}(\phi)\}.$$

(ii) Let $\tilde{\theta}\in\eta^{-1}(\phi)$. If $\tilde{\psi}\in\widetilde{\mathscr{K}(\tilde{\theta})}$, then $S(\tilde{\psi})\subset\mathscr{K}(\tilde{\theta})$. Let $\psi=\pi(\tilde{\psi})$. Then $S(\psi)=\pi(S(\tilde{\psi}))\subset\pi(\mathscr{K}(\tilde{\theta}))\subset K(\phi)$. Hence $\psi\in\widetilde{K(\phi)}$. Hence we have

$$\pi\{\widetilde{\mathscr{K}(\tilde{\theta})}:\tilde{\theta}\in\eta^{-1}(\phi)\}\subset\widetilde{K(\phi)}.$$

If $\theta_1\in\widetilde{K(\phi)}$ and $\theta_2\in\widetilde{K(\theta)}$, then we have $\theta_1(f)=\phi(f)$ and $\theta_2(f)=\theta(f)$ for every $f\in\mathscr{L}$. And, for some $g\in\mathscr{L}$, $\phi(g)\neq\theta(g)$. Hence we have $\widetilde{K(\phi)}\cap\widetilde{K(\theta)}=\emptyset$.

(iii) By Theorem 4.4, (iii), $(M(H^{\infty})\backslash\text{hull}\,I^{\infty})\cup Y=\cup\{\widetilde{\mathscr{K}(\tilde{\theta})}:\tilde{\theta}\in Y\}$. Hence, by (ii), we have $\pi[(M(H^{\infty})\backslash\text{hull}\,I^{\infty})\cup Y]\subset\cup\{\widetilde{K(\phi)}:\phi\in M(\mathscr{L})\}$. On the other hand, if $\tilde{\phi}\in\text{hull}\,I^{\infty}$, then $\tilde{\phi}(h)=0$ for all $h\in I^{\infty}$, and hence $\pi(\tilde{\phi})(h)=0$ for all $h\in I=I^{\infty}\cap C(X)$. Hence we have $\pi[\text{hull}\,I^{\infty}]\subset\text{hull}\,I$. Thus we obtain

$$M(A)=\text{hull}\,I\cup(\cup\{\widetilde{K(\phi)}:\phi\in M(\mathscr{L})\}).$$

Q. E. D.

THEOREM 5.9. *Let A, X and m be as in Theorem* 5.8. *Suppose that $\mathscr{L} \supsetneqq C$ and $M(\mathscr{L})$ is totally disconnected. Then we have the following.*

(i) *$I = I^\infty \cap C(X)$ is contained in the uniformly closed linear span $\mathscr{J}$ of all functions in A, each of which vanishes on some set of positive measure. In particular, in the cases* 2.1 *and* 2.3 *in* § 2, *we have $I = J = \mathscr{J}$.*

(ii) *For every ϕ in $M(\mathscr{L})$, the set $K(\phi)$ is a weak peak set for A.*

PROOF. (i) Let q be the continuous map defined in (5.5), let q^0 be the map of $C(M(\mathscr{L}))$ to $C(X)$ defined in (5.1), and let q^* be the adjoint map of q^0 (see (5.2)). Let $\mu_m = q^*(m)$ and let $f \in \mathscr{L}$. Then, for every $\varepsilon > 0$ there are clopen sets D_k $(k=1, 2, \cdots, n)$ in $M(\mathscr{L})$ with $0 < \mu_m(D_k) < 1$ $(k=1, 2, \cdots, n)$ and complex numbers c_k $(k=1, 2, \cdots, n)$ such that

$$\|\hat{f} - \sum_{k=1}^{n} c_k \chi_{D_k}\| < \varepsilon.$$

Hence we have

$$(5.7) \qquad \|q^0(\hat{f} - \sum_{k=1}^{n} c_k \chi_{D_k})\| = \|f - \sum_{k=1}^{n} c_k \chi_{q^{-1}(D_k)}\| < \varepsilon,$$

where $\chi_{q^{-1}(D_k)} \in \mathscr{L}$ and $0 < m(q^{-1}(D_k)) < 1$ $(k=1, 2, \cdots, n)$.

Now, by (5.4), for every h in I, there are $f \in \mathscr{L}$ and $g \in I$ with $\|g\| < 1/2$ such that $fg = h$. Suppose that f satisfies (5.7). Then we have

$$\|h - \sum_{k=1}^{n} c_k \chi_{q^{-1}(D_k)} g\| \leqq \|g\| \varepsilon < \varepsilon/2,$$

where $\chi_{q^{-1}(D_k)} g \in I$ $(k=1, 2, \cdots, n)$. Hence we have (i).

(ii) By using Proposition 5.6, Theorem 5.8 (ii) and the map q defined in (5.5), (ii) is proved by the same argument as for Theorem 5.3. (i). Q.E.D

Although, in § 2, $J^\infty = J^\infty(H^\infty)$ is defined to be the weak-* closed linear span of all functions in $H^\infty(m)$, each of which vanishes on some set of positive measure, we see by Theorem 5.9, in the cases 2.1 and 2.3 in § 2, J^∞ is the uniformly closed linear span of those sets. That is, we have the following.

COROLLARY 5.10. *Let I^∞ be as in the cases* 2.1, 2.2 *and* 2.3 *in* § 2. *Then I^∞ is contained in the uniformly closed linear span $\mathscr{J}^\infty$ of all functions in $H^\infty(m)$, each of which vanishes on some set of positive measure. In particular, in the cases* 2.1 *and* 2.3 *in* § 2, *we have $I^\infty = J^\infty = \mathscr{J}^\infty$.*

PROPOSITION 5.11. *Let X, A and m be as in Theorem* 5.8, *and let $f \in L^\infty(m)$. Then $f \in \mathscr{L}$ if and only if $\hat{f}$ is constant on $\cup\{\mathscr{K}(\tilde{\theta}) : \tilde{\theta} \in$

$\eta^{-1}(\phi)\}$ *for every* ϕ *in* $M(\mathscr{L})$. (*For* η *see* (5.6).)

PROOF. By the definition of π and Theorem 5.8, (i), we obtain the "if" part.

Let $f\in L^{\infty}(m)$. Then $\hat{f}\in C(\tilde{X})$, and by Theorem 5.8, (i), $\hat{f}$ is constant on $\tilde{X}\cap\pi^{-1}(x)$ for every $x\in X$. Hence, by [1, Lemma 4.3] and Proposition 3.9, (ii), we see that $f\in\mathscr{L}^{\infty}\cap C(X)=\mathscr{L}$. Q.E.D.

§ 6. Some properties of a strongly logmodular algebra, Part 2.

In this section, let A be a strongly logmodular algebra on a compact Hausdorff space X and let $m\in M(A)$. We suppose that $P=P(m)\supsetneqq\{m\}$ and $X=S(m)$.

Let $H^{\infty}=H^{\infty}(m)=\mathscr{H}^{\infty}\oplus I^{\infty}$ and $\mathscr{L}^{\infty}$ be as in the case 2.3 of § 2, and let N^{∞} be the weak-* closure of $I^{\infty}+\overline{I^{\infty}}$ in $L^{\infty}(m)$. Let $\mathscr{L}=\mathscr{L}^{\infty}\cap C(X)$, $\mathscr{L}_R=\mathscr{L}^{\infty}\cap C_R(X)$, $N=N^{\infty}\cap C(X)$ and $N_R=N^{\infty}\cap C_R(X)$. Let $\mathscr{P}$ be the nontrivial Gleason part of $\tilde{m}\in M(H^{\infty})$, where $\tilde{m}$ is the complex homomorphism of H^{∞} which is defined by the Radonization of the measure m. Then $\pi\mathscr{P}=P$ and $\pi\overline{\mathscr{P}}=\overline{P}$ (cf. [12])

LEMMA 6.1. (i) *Let* $f\in L^{\infty}(m)$. *Then* $f\in N^{\infty}$ *if and only if* $\phi(f)=0$ *for all* $\phi\in\mathscr{P}$.

(ii) *Let* $f\in C(X)$. *Then* $f\in N$ *if and only if* $\phi(f)=0$ *for all* $\phi\in P$.

PROOF. (i) Let $f=g+h$, where $g\in\mathscr{L}^{\infty}$ and $h\in N^{\infty}$. The "if" part is obvious. If $\hat{f}(\phi)=0$ on $\mathscr{P}$, then $\hat{g}(\phi)=0$ on $\overline{\mathscr{P}}$. Hence $\hat{g}=0$ on $Y(=\Gamma(H^{\infty}|\text{hull } I^{\infty}))$. By $g\in\mathscr{L}^{\infty}$, $\hat{g}=0$ on $\tilde{\pi}^{-1}(Y)=\tilde{X}=M(L^{\infty})$ and hence $g=0$ a.e. (m). Thus $f\in N^{\infty}$.

(ii) By using $\pi\mathscr{P}=P$, we easily obtain (ii). Q.E.D.

LEMMA 6.2. *Let* $u\in C_R(X)$. *Then* $u\in N_R$ *if and only if there is a function* $f\in A^{-1}$ *such that* $u=\log|f|$ *and* $|\hat{f}(\phi)|=1$ *on* $\overline{P}$. *And, in this case,* $\hat{f}(\phi)$ *is constant on* $\overline{P}$.

PROOF. Let $u\in N_R$. By $\log|A^{-1}|=C_R(X)$, there is a function $f\in A^{-1}$ such that $u=\log|f|$. Then, for every $\tilde{\phi}\in\mathscr{P}$, $0=\tilde{\phi}(u)=\int_{\tilde{X}}\log|f|d\tilde{\phi}=\log|\tilde{\phi}(f)|$, and hence we have $|\tilde{\phi}(f)|=1$ on $\overline{\mathscr{P}}$. Hence $\overline{\tilde{\phi}(f)}=\tilde{\phi}(1/f)$ for every $\tilde{\phi}\in\overline{\mathscr{P}}$. Hence $\hat{f}|Y$ and $\overline{\hat{f}}|Y$ belong to $\hat{H}^{\infty}|Y$. Since Y is an antisymmetric set, $\hat{f}(\tilde{\phi})$ *is constant on* Y. *Since* $\overline{\mathscr{P}}$ *is the* $\hat{H}^{\infty}$-convex hull of Y, $\hat{f}(\tilde{\phi})$ *is constant on* $\overline{\mathscr{P}}$. *Since* $\overline{P}=\pi(\overline{\mathscr{P}})$, $|\hat{f}(\phi)|=1$ on $\overline{P}$ and $\hat{f}(\phi)$ is constant on $\overline{P}$.

Conversely, if $f\in A^{-1}$ and $|\hat{f}(\phi)|=1$ on $\overline{P}$, then for $u=\log|f|$, $\phi(u)=$

$\int \log|f|d\phi=\log|\phi(f)|=0$ on $\bar{P}$. Hence, by Lemma 6.1. (ii), $u\in N_R$.

Q. E. D.

By Proposition 2.1, $\log|(H^\infty)^{-1}|=L_R^\infty$ can be rewriten as $\log|(\mathscr{H}^\infty \oplus I^\infty)^{-1}|=\mathscr{L}_R^\infty\oplus N_R^\infty$. And, we have $\log|(\mathscr{L}^\infty+I^\infty)^{-1}|=\mathscr{L}_R^\infty\oplus N_R^\infty$ and $\log|(I^\infty+C')^{-1}|=N_R^\infty+R$ (see Proposition 3.10), where $C'=C\backslash\{0\}$. As we see in the following theorem, these formulas hold for $\mathscr{H}$, I, $\mathscr{L}$ and N_R.

THEOREM 6.3. *Let A be a strongly logmodular algebra on X and let $m\in M(A)$. We suppose that $P=P(m)\supsetneqq\{m\}$ and $X=S(m)$. Let $I=\{f\in A: \phi(f)=0\ \ \forall\phi\in P\}$. Then $\mathscr{H}+I$, $\mathscr{L}\oplus I$ and $\mathscr{L}_R\oplus N_R$ are all uniformly closed and we have the following.*

(i) $\log|(C'+I)^{-1}|=R+N_R$ *on* X,

(ii) $\log|(\mathscr{H}\oplus I)^{-1}|=\mathscr{L}_R\oplus N_R$ *on* X,

(iii) $\log|(\mathscr{L}\oplus I)^{-1}|=\mathscr{L}_R\oplus N_R$ *on* X,

where $\oplus$ denotes the algebraic direct sum, C and R are the complex and the real fields respectively, and $C'=C\backslash\{0\}$.

PROOF. For $F=g+h$, where $g\in\mathscr{H}$ and $h\in I$, we have $\|g\|\leqq\|F\|$ (see the proof of Proposition 2.1), and hence $\|h\|\leqq 2\|F\|$. Hence, if $F_n=g_n+h_n$ $(n=1,2,\cdots)$ uniformly converges to F_0, then g_n and h_n uniformly converge to $g\in\mathscr{H}$ and $h\in I$ respectively. Hence $F_0\in\mathscr{H}\oplus I$. Therefore, $\mathscr{H}\oplus I$ is uniformly closed. We can see similarly that $\mathscr{L}\oplus I$ and $\mathscr{L}_R\oplus N_R$ are both uniformly closed.

(i) By lemma 6.2, for $u\in N_R$ there is a function $f\in A^{-1}$ such that $u=\log|f|$, $|\hat{f}(\phi)|=1$ on $\bar{P}$, and $\hat{f}(\phi)$ is a constant on P. Let ϕ_0 be a fixed point in $\bar{P}$. Then,

$$f=\phi_0(f)+(f-\phi_0(f))\in C'+I.$$

There is a function $g\in A$ such that $fg=1$. By $\phi(g)=1/\phi(f)$ on $\bar{P}$, $|\hat{g}(\phi)|=1$ on $\bar{P}$ and $\hat{g}(\phi)$ is constant on $\bar{P}$. Hence,

$$g=\phi_0(g)+(g-\phi_0(g))\in C'+I.$$

Hence $f\in(C'+I)^{-1}$. Therefore $R+N_R\subset\log|(C'+I)^{-1}|$.

Conversely, if $u\in\log|(C'+I)^{-1}|$, then $u=\log|c+h|$, where $c+h\in(C'+I)^{-1}$. For every $\phi\in\bar{P}$, we have

$$\phi(u)=\int\log|c+h|d\phi=\log|\phi(c+h)|=\log|c|,$$

and hence $\phi(u-\log|c|)=0$. By Lemma 6.1, $u-\log|c|\in N_R$. Hence $u=$

$\log|c|+v$, where $v\in N_R$. Therefore, $\log|(C'+I)^{-1}|\subset R+N_R$.

(ii) By Proposition 5.6, we have

(6.1) $\log|\mathscr{H}^{-1}|=\mathscr{L}_R$.

By (i) we have

(6.2) $\log|(C'+I)^{-1}|=R+N_R$.

Here, we will prove

(6.3) $(C'+I)^{-1}\mathscr{H}^{-1}=(\mathscr{H}\oplus I)^{-1}$.

Let $c+h\in(C'+I)^{-1}$ and let $g\in\mathscr{H}^{-1}$. Then $(c+h)g=cg+hg\in\mathscr{H}\oplus I$. By $1/(c+h)g=(c'+h')(1/g)=c'/g+h'/g\in\mathscr{H}\oplus I$, where $c'\in C$ and $h'\in I$. Hence $(c+h)g\in(\mathscr{H}\oplus I)^{-1}$. Thus $(C'+I)^{-1}\mathscr{H}^{-1}\subset(\mathscr{H}\oplus I)^{-1}$.

Let $f\in(\mathscr{H}\oplus I)^{-1}$. Let $f=g+h$ and $1/f=g_1+h_1$, where g, $g_1\in\mathscr{H}$ and h, $h_1\in I$. Then $gg_1=1$ and hence $g\in\mathscr{H}^{-1}$. By $f=g+h$, we have $f/g=1+h/g=1+h_1$, where $h_1=h/g\in I$. Let $F=1/(1+h_1)$. Then $F=1-h_1F\in C'+I$. Hence $1+h_1\in(C'+I)^{-1}$. Hence $f=(1+h/g)g\in(C'+I)^{-1}\mathscr{H}^{-1}$. Thus $(\mathscr{H}\oplus I)^{-1}\subset(C'+I)^{-1}\mathscr{H}^{-1}$. Therefore, we obtain (6.3).

By (6.1), (6.2) and (6.3), we obtain

$$\log|(\mathscr{H}\oplus I)^{-1}|=\mathscr{L}_R\oplus N_R.$$

(iii) By $\mathscr{L}_R=\log|\mathscr{H}^{-1}|\subset\log|\mathscr{L}^{-1}|\subset\mathscr{L}_R$, we have

$$\log|\mathscr{L}^{-1}|=\mathscr{L}_R.$$

The following formula is proved by the same argument as for (6.3).

$$(C+I)^{-1}\mathscr{L}^{-1}=(\mathscr{L}\oplus I)^{-1}$$

Hence we obtain (iii). Q. E. D.

Let A be a strongly logmodular algebra on a compact Hausdorff Stonian space X and let $m\in M(A)$. We suppose that $X=S(m)$, $P(m)\supsetneqq\{m\}$ and the Wermer embedding function Z belongs to A. Example 4, (ii) in § 8 is such an example.

By Proposition 5.6 we have $\log|\mathscr{H}^{-1}|=\mathscr{L}_R$. Let T be the map which will be defined in (7.1) of § 7. Let $H=T(\mathscr{H})$, $L=T(\mathscr{L})$ and $L_R=T(\mathscr{L}_R)$. Then, by $T(\log|\mathscr{H}^{-1}|)=\log|(T(\mathscr{H}))^{-1}|$, we have $\log|H^{-1}|=L_R$. Hence H is a strongly logmodular algebra on $M(L)$ such that $e^{i\theta}\in H$ and $A(\partial D)\subsetneqq H\subset H^\infty(D)$, where $A(\partial D)$ is the disk algebra on the unit circle ∂D.

Problem. Does such an algebra H coincide with $H^\infty(D)$?

If this problem has an affirmative answer, then $\mathscr{H}=T^{-1}(H)=T^{-1}(H^\infty(\partial D))=\mathscr{H}^\infty$, i. e., the natural injection $\mathscr{H}\subseteqq\mathscr{H}^\infty$ is an isometric isomorphism of $\mathscr{H}$ and $\mathscr{H}^\infty$. Further we will obtain an algebraic direct sum decomposition $A=\mathscr{H}\oplus J$, where $J=J^\infty\cap C(X)$.

§ 7. A logmodular algebra A satisfying $A\circ\tau=H^\infty(D)$.

In this section, let A be a logmodular algebra on a compact Hausdorff space X and let $m\in M(A)$. We suppose that $P=P(m)$ is nontrivial. Let Z be the Wermer embedding function, let $\mathscr{H}^\infty$ be the weak-* closure of the polynomials in Z in $L^\infty(m)$, and let $\mathscr{L}^\infty$ be the weak-* closure of the polynomials in Z and $\bar{Z}$ in $L^\infty(m)$. Let $I^\infty=\{f\in H^\infty(m): \phi(f)=0\ \forall\phi\in P\}$. Then, as we stated in the case 2.3 of § 2, we have $H^\infty=H^\infty(m)=\mathscr{H}^\infty\oplus I^\infty$.

The correspondence

$$(7.1)\qquad T: \sum_{k=-n}^{n} a_k Z^k \mapsto \sum_{k=-n}^{n} a_k e^{ik\theta},\ a_k\in C$$

induces an isometric isomorphism T of $\mathscr{L}^\infty$ onto $L^\infty(d\theta)$, which carries $\mathscr{H}^\infty$ onto $H^\infty(D)$, where n ranges over all integers and $(1/2\pi)d\theta$ is the normalized Haar measure on the unit circle (cf. [16]). By Fatou's theorem, every function in $H^\infty(D)$ is identified with its boundary function on the unit circle ∂D.

THEOREM 7.1. *Let A be a logmodular algebra on a compact Hausdorff space X, let $P=P(m)$ be the nontrivial Gleason part of m for A, and let τ be an analytic map of D onto P. Let $\Gamma=\Gamma(A|\bar{P})$ be the Shilov boundary of $A|\bar{P}$. Suppose that $A\circ\tau=H^\infty(D)$. Then we have the following.*

(i) *$\Gamma=\pi(Y)$, and Γ is a compact Hausdorff Stonian space, where $Y=\Gamma(H^\infty|\text{hull } I^\infty)$ (see (3.4)).*

(ii) *$M(A|\Gamma)=\bar{P}=\text{hull } I$, where $I=\{f\in A: \phi(f)=0$ for all $\phi\in P\}$.*

(iii) *$A|\Gamma$ is a strongly logmodular algebra on Γ.*

(iv) *Let λ_m be a (unique) representing measure on Γ of m for $A|\Gamma$, and let $H^\infty(\lambda_m)$ be the weak-* closure of $A|\Gamma$ in $L^\infty(\lambda_m)$. Then, λ_m is a normal measure on Γ such that $S(\lambda_m)=\Gamma$, and the natural injection $A|\Gamma\subseteqq H^\infty(\lambda_m)$ is an isometric isomorphism of $A|\Gamma$ and $H^\infty(\lambda_m)$.*

(v) *If the Wermer embedding function Z belongs to A, then $A|\Gamma$ is the weak-* closure of the set of polynomials in Z in $L^\infty(\lambda_m)$.*

PROOF. (i), (ii) By the map

$$f|\Gamma\mapsto f|\bar{P}\mapsto f|P\mapsto f\circ\tau\in H^\infty(D),\ f\in A$$

the algebra $A|\Gamma$ is isometrically isomorphic to the Banach algebra $H^\infty(D)$. Hence $A|\Gamma$ is a uniform algebra on Γ. By a general theory, we have $M(A|\Gamma)=\{\phi\in M(A): |\phi(f)|\leqq\|f\|_\Gamma$ for all $f\in A\}$ (cf. [15], p. 166). By $A\circ\tau=$

$H^{\infty}(D)$, we have hull $(I)=\bar{P}$ (cf. [12], Theorem 4. 4). Hence, by $M(A|\Gamma)\subset$ hull I and $\bar{P}\subset M(A|\Gamma)$, we have

(7.2) $\quad M(A|\Gamma)=\text{hull } I=\bar{P}.$

From $H^{\infty}(D)=\{f\circ\tau : f\in A\}\subset\{f\circ\tau : f\in H^{\infty}(m)\}=\{f\circ\tau : f\in \mathscr{H}^{\infty}\}\subset H^{\infty}(D)$, we obtain $\{f\circ\tau : f\in A\}=\{f\circ\tau : f\in\mathscr{H}^{\infty}\}$. Hence we have

(7.3) $\quad A|\bar{\mathscr{P}}=\mathscr{H}^{\infty}|\bar{\mathscr{P}}.$

The map π is a homeomorphism of $\bar{\mathscr{P}}$ onto $\bar{P}$ (cf. [12], Theorem 4.1), and, for every f in A, we have

(7.4) $\quad f(\tilde{\phi})=f(\pi(\tilde{\phi})),\ \tilde{\phi}\in\bar{\mathscr{P}}.$

Further, by $\bar{\mathscr{P}}=\text{hull } I^{\infty}$ (cf. [12], Theorem 3. 1), we have

(7.5) $\quad Y=\Gamma(\mathscr{H}^{\infty}|\bar{\mathscr{P}}).$

Hence, by (7. 3), (7.4) and (7. 5), we obtain

$$\Gamma=\pi(Y).$$

Since Y is a compact Hausdorff Stonian space, Γ is a compact Hausdorff Stonian space too.

(iii) Let $\pi_Y=\pi|Y$. Then π_Y^0 is a one-to-one map of $C_R(\Gamma)$ onto $C_R(Y)$ (see (5. 1)), and we have

$$\begin{aligned}&\pi_Y^0(\log|(A|\Gamma)^{-1}|)=\log|(A|Y)^{-1}|\\&=\log|(\mathscr{H}^{\infty}|Y)^{-1}|=C_R(Y) \text{ on } Y.\end{aligned}$$

Hence we have

$$\log|(A|\Gamma)^{-1}|=C_R(\Gamma).$$

Since $A|\Gamma$ is a uniform algebra on Γ, the algebra $A|\Gamma$ is a strongly log-modular algebra on Γ.

(iv) Let $\lambda_{\tilde{m}}=\tilde{\pi}^*(\tilde{m})$ (see (5. 2)). Then $\lambda_{\tilde{m}}$ is a normal measure on Y (cf. [13], p. 77). For any $f\in A$, we have

$$\begin{aligned}m(f)&=\int_X f\,dm=\int_{\tilde{X}}\hat{f}\,d\tilde{m}=\int_{\tilde{X}}\hat{g}\,d\tilde{m}\\&=\int_{\tilde{X}}\hat{g}\circ\pi\,d\tilde{m}=\int_Y\hat{g}\,d\lambda_{\tilde{m}}=\int_Y(\hat{g}+\hat{h})\,d\lambda_{\tilde{m}}\\&=\int_Y\hat{f}\,d\lambda_{\tilde{m}}=\int_Y f\circ\pi\,d\lambda_{\tilde{m}}=\int_\Gamma f\,d(\pi^*(\lambda_{\tilde{m}}))\\&=\int_\Gamma f\,d\lambda_m,\end{aligned}$$

where $f=g+h$ a.e. (m), $g\in\mathscr{H}^\infty$ and $h\in I^\infty$. Hence we see that λ_m is a normal measure on the compact Stonian space Γ and $S(\lambda_m)=\Gamma$. Hence, the natural injection $C(\Gamma)\subseteqq L^\infty(\lambda_m)$ is an isometric isomorphism of $C(\Gamma)$ and $L^\infty(\lambda_m)$. So, by Theorem 5.5, we have $A|\Gamma=H^\infty(\lambda_m)\cap C(\Gamma)=H^\infty(\lambda_m)$.

(v) For every function $f\in H^\infty(D)$, there is a sequence $\{f_n\}\subset A(\partial D)$ such that $\|f_n\|<\|f\|$ for all n and $f_n\to f$ a.e. $(d\theta)$, where $A(\partial D)$ is the disc algebra on the unit circle ∂D. Then, from

$$\begin{aligned}|\int_{\partial D} gf\,d\theta| &= \lim_n |\int_{\partial D} gf_n\,d\theta| \\ &\leqq \varliminf_n \|f_n\| \int_{\partial D} |g|\,d\theta,\ g\in L^1(d\theta),\end{aligned}$$

we have $\|f\|\leqq\varliminf_n\|f_n\|\leqq\varlimsup_n\|f_n\|\leqq\|f\|$, and hence $\lim_n\|f_n\|=\|f\|$.

For every $n=1, 2, \cdots$, let ε_n be a number such that $0<\varepsilon_n<\|f\|-\|f_n\|$. For every f_n, there is a polynomial $P_n(e^{i\theta})$ in $e^{i\theta}$ on the unit circle ∂D such that $\|P_n-f_n\|<\varepsilon_n$. Then we have $\|P_n\|<\|f\|$ $(n=1, 2, \cdots)$ and $P_n\to f$ a.e. $(d\theta)$.

Let $F=T^{-1}(f)$ and let $P_n(Z)=T^{-1}(P_n(e^{i\theta}))$ (for T see (7.1)). Then, by [16, p. 464], we have

$$\begin{aligned}\int_{\partial D}|P_n(e^{i\theta})-f(e^{i\theta})|\,d\theta &= \int_X |P_n(Z)-F|\,dm \\ &= \int_{\tilde X}|\widehat{P_n(Z)}-\hat F|\,d\tilde m = \int_{\tilde X}|(\widehat{P_n(Z)}-\hat F)\circ\tilde\pi|\,d\tilde m \\ &= \int_Y |\widehat{P_n(Z)}-\hat F|\,d\lambda_{\tilde m}\to 0\ (n\to\infty).\end{aligned}$$

Hence (by passing to a subsequence) there is a sequence $\{\widehat{P_n(Z)}\}$ such that $\widehat{P_n(Z)}\to\hat F$ a.e. $(\lambda_{\tilde m})$ and $\|\widehat{P_n(Z)}\|_Y\leqq\|\hat F\|_Y$.

For any $g\in A|\Gamma$ there is a function $f\in A$ such that $g=f|\Gamma$. Let $f=G+H$ a.e. (m), where $G\in\mathscr{H}^\infty$ and $H\in I^\infty$. Then, for every $\tilde\phi\in Y$, we have $[g\circ\pi](\tilde\phi)=g(\phi)=f(\phi)=f(\tilde\phi)=G(\tilde\phi)$, where $\phi=\pi(\tilde\phi)$. That is, we have $g\circ\pi=G$ on Y. Then, there is a sequence of polynomials $\{P_n(Z)\}$ $(=\{\widehat{P_n(Z)}\})$ in Z such that $P_n(Z)\to G$ a.e. $(\lambda_{\tilde m})$ and $\|P_n(Z)\|_Y\leqq\|G\|_Y$. Since Z belongs to A we have

$$\begin{aligned}\int_Y|P_n(Z)-G|\,d\lambda_{\tilde m} &= \int_Y |P_n(Z)\circ\pi-g\circ\pi|\,d\lambda_{\tilde m} \\ &= \int_\Gamma |P_n(Z)-g|\,d\lambda_m\to 0\ (n\to\infty).\end{aligned}$$

Hence (by passing to a subsequence) there is a sequence of polynomials

$\{P_n(Z)\}$ in Z such that $P_n(Z)\to g$ a. e. (λ_m) and $\|P_n(Z)\|_\Gamma\leqq\|g\|_\Gamma$. Hence $A|\Gamma$ is the weak-* closure the set of polynomials in Z in $L^\infty(\lambda_m)$. Q. E. D

For the examples such that $A\circ\tau=H^\infty(D)$, see Example 4, (ii) in § 8.

COROLLARY 7. 2. *Let X, A, $P=P(m)$, Γ, and τ be as in Theorem 7. 1, and suppose that $A\circ\tau=H^\infty(D)$. Then we have the following.*

(i) *If $P(\phi)$ $(\phi\in\bar{P})$, the Gleason part of ϕ for $A|\Gamma$, is nontrivial, then $P(\phi)$ is a nontrivial Gleason part of ϕ even for A.*

(ii) *There is a one-to-one map from the set of nontrivial Gleason parts for $H^\infty(D)$ onto the set of nontrivial Gleason parts for $A|\Gamma$.*

(iii) *$G=\cup\{P(\phi)\,;P(\phi)$ is the nontrivial Gleason part for $A|\Gamma\}$ is an open dense subset in the subspace $\bar{P}$.*

PROOF. (i) By Theorem 7. 1, (i)$\sim$(iii), there is an analytic map σ of D onto $P(\phi)$. Let $G(\phi)$ be the Gleason part of ϕ for A. Then, by $d_{A|\Gamma}(\phi_1,\phi_2)\geqq d_A(\phi_1,\phi_2)$, we have $P(\phi)\subset G(\phi)$ (see (2, 1)). Let ρ be an analytic map of D onto $G(\phi)$. The map σ (resp. ρ) is an isometry of D with the pseudo-hyperbolic metric onto $P(\phi)$ (resp. $G(\phi)$) with the metric $d_{A|\Gamma}(\phi_1,\phi_2)$ (resp. $d_A(\phi_1,\phi_2)$) (cf. [11], Theorem 3). Hence $\xi=\rho^{-1}\circ\sigma$ is a one-to-one continuous map of D into D.

Let $K=\{z\in D:|z|\leqq 1/2\}$. Then $\xi(K)$ is a compact subset of D. Hence there is a sequence of distinct points $\{z_n\}(\subset\xi(K))$ which converges to some point in $\xi(K)$. There is a point $\lambda_n\in K$ such that $z_n=\xi(\lambda_n)$ for every n. For every $f\in I$, we have $f\circ\rho\in H^\infty(D)$ and $(f\circ\rho)(z_n)=(f\circ\rho)(\rho^{-1}\circ\sigma)(\lambda_n)$ $=(f\circ\sigma)(\lambda_n)=0$ $(n=1,2,\cdots)$. Hence we have $f\circ\rho=0$ in D, and hence $G(\phi)$ $=\rho(D)\subset$ hull I. From hull $I=\bar{P}=M(A|\Gamma)$, we have $f\circ\rho\in H^\infty(D)$ for every f in $A|\Gamma$. Hence we have $G(\phi)\subset P(\phi)$, and hence $P(\phi)=G(\phi)$.

(ii) Let $f\in A$, $F^*(\lambda)=f(\tau(\lambda))$, and F be the boundary value on ∂D of F^*. Let $\eta:f|\Gamma\to\hat{F}$ be the map derived by the chain of maps

$$f|\Gamma\mapsto f|\bar{P}\mapsto f|P\mapsto F^*\mapsto F\mapsto\hat{F}\ (\in\widehat{H^\infty(D)}|M(L^\infty(\partial D))).$$

Then η is an isometric isomorphism of $A|\Gamma$ onto $\widehat{H^\infty(D)}|M(L^\infty(\partial D))$. Let η^* be the adjoint of η. Then η^* is a one-to-one map of the set of Gleason parts for $H^\infty(D)$ onto the set of Gleason parts for $A|\Gamma$.

(iii) Since the set G_1 of nontrivial Gleason parts for $H^\infty(D)$ is an open dense subset of $M(H^\infty(D))$, $G=\eta^*(G_1)$ is an open dense subset in the subspace $\bar{P}$ (cf. [9], p. 89). Q. E. D.

Under the same condition as in Corollary 7. 2, we could not decide that, when $\phi\in\bar{P}\backslash\Gamma$ and $P(\phi)$ is a trivial Gleason part of ϕ for $A|\Gamma$, then $P(\phi)$ is a trivial Gleason part of ϕ for A. (See Example 2, (iii) in § 8.)

COROLLARY 7.3. *Let A be a logmodular algebra on a compact Hausdorff space X, and let the Gleason part $P=P(m)$ of m for A be nontrivial, and let τ be an analytic map of D onto P. Then $A\circ\tau=H^{\infty}(D)$ if and only if $\mathscr{H}^{\infty}=\{g : f\in A,\ f=g+h$ a.e. (m), where $g\in\mathscr{H}^{\infty}$ and $h\in I^{\infty}\}$.*

PROOF. Let $H=\{g : f\in A,\ f=g+h$ a.e. (m), where $g\in\mathscr{H}^{\infty}$ and $h\in I^{\infty}\}$. If $A\circ\tau=H^{\infty}(D)$, then by (7.3) we have $A|\bar{\mathscr{P}}=\mathscr{H}^{\infty}|\bar{\mathscr{P}}$. Hence we have $H|\bar{\mathscr{P}}=H^{\infty}|\bar{\mathscr{P}}$, and hence $H|Y=\mathscr{H}^{\infty}|Y$, and hence $H|\tilde{X}=\mathscr{H}^{\infty}|\tilde{X}$. Therefore we have $H=\mathscr{H}^{\infty}$.

Conversely let $\tau=\Sigma\circ T^{*}$. Then we have $\tau(D)=\mathscr{P}$, and for every $F=g+h\in H^{\infty}(m)$, where $g\in\mathscr{H}^{\infty}$ and $h\in I^{\infty}$, we have

$$\begin{aligned}F(\tau(\lambda))&=g(\tau(\lambda))=g(\Sigma\circ T^{*}(\lambda))=g(T^{*}(\lambda))\\&=T(g)(\lambda)\in H^{\infty}(D),\ \text{where } \lambda\in D.\end{aligned}$$

Hence τ is an analytic map of D onto $\mathscr{P}$. Since T is an isometric isomorphism of $\mathscr{H}^{\infty}$ onto $H^{\infty}(D)$, for every function F in $H^{\infty}(D)$, there is a function $g\in\mathscr{H}^{\infty}$ such that $Tg=F$. By the assumption, there is a function $f\in A$ such that $f=g+h$ a.e. (m), where $g\in\mathscr{H}^{\infty}$ and $h\in I^{\infty}$. Hence we have $F(\lambda)=(Tg)(\lambda)=g(T^{*}(\lambda))=g(\Sigma(T^{*}(\lambda)))=g(\tau(\lambda))=f(\tau(\lambda))$. Hence we have $A\circ\tau=H^{\infty}(D)$. Q.E.D.

A function f is called a bounded analytic function on P if f is a complex valued function defined on P and $f\circ\tau\in H^{\infty}(D)$, where τ is an analytic map of D onto $P=P(m)$. Let $H^{\infty}(P)$ be the set of bounded analytic functions on P.

COROLLARY 7.4. *Let X, A, $P=P(m)$, and τ be as in Corollary 7.3. Then $A\circ\tau=H^{\infty}(D)$ if and only if $A\circ\tau=H^{\infty}(P)$.*

PROOF. By [15, p. 154] we have $H^{\infty}(P)=H^{\infty}(m)\circ\tau$.

If $A\circ\tau=H^{\infty}(D)$, then by Corollary 7.3 we have $A\circ\tau=H^{\infty}(m)\circ\tau=H^{\infty}(P)$.

Conversely, by the proof of Corollary 7.3, we have $H^{\infty}(m)\circ\tau=H^{\infty}(D)$ and hence $A\circ\tau=H^{\infty}(D)$. Q.E.D.

COROLLARY 7.5. *Let X, A, $P=P(m)$, and τ be as in Corollary 7.3. Then $A\circ\tau=H^{\infty}(D)$ and $I=\{0\}$ if and only if $A=\mathscr{H}^{\infty}=H^{\infty}(m)$ on X.*

PROOF. If $A\circ\tau=H^{\infty}(D)$ and $I=\{0\}$, then we have $\bar{P}=\mathrm{hull}\,(I)$ and hull $I=M(A)$. Hence we have $\bar{P}\supset X$, and hence $\Gamma=\Gamma(A|\bar{P})=X$. Hence, by Theorem 7.1, (iv), $A|X=A|\Gamma$ is the weak-* closure of $A|\Gamma$ in $L^{\infty}(m)=L^{\infty}(\lambda_m)$. Hence we have $A=H^{\infty}(m)$ and hence $Z\in A$. Hence by Theorem 7.1, (v) we have $A=\mathscr{H}^{\infty}$.

Conversely, if $A=\mathscr{H}^{\infty}=H^{\infty}(m)$ on X, then $I\subset I^{\infty}=\{0\}$ implies $I=\{0\}$. And we have $A\circ\tau=H^{\infty}(m)\circ\tau=H^{\infty}(D)$. Q.E.D.

Finally we will state a proposition which is an immediate consequence of Proposition 3.9.

PROPOSITION 7.6. *Let X, A, P, and Γ be as in Theorem 7.1. Suppose that $X=S(m)$. Then $\cup\{S(\phi):\phi\in\Gamma\}$ is dense in X.*

§ 8. Examples.

EXAMPLE 1. Let K be the Bohr compactification of the real line R. Let A be the Dirichlet algebra of continuous, complex valued functions on $X=K\times K$ which are uniform limits of the polynomials in $\chi_{\tau_1}\chi_{\tau_2}$, where

$$(\tau_1, \tau_2)\in S=\{(\tau_1, \tau_2):\tau_2>0\}\cup\{(\tau_1,0):\tau_1\geqq 0\},$$

and χ_{τ_i} are the characters of K determined by $\tau_i\in R$. We denote by m the normaized Haar measure on X, and we also denote by m the complex homomorphism of A defined by the measure m. We denote by $H^\infty(m)$ the weak-* closure of A in $L^\infty(m)$.

The Gleason part $P(m)$ of $m\in M(A)$ is trivial (cf. [15], p. 149). Further $H^\infty_{\min}$ is the weak-* closure of $\bigcup_{\tau_1\geqq 0}\bar{\chi}_{\tau_1}H^\infty(m)$. Hence we have $H^\infty(m)\subsetneqq H^\infty_{\min}\subsetneqq L^\infty(m)$. (Cf. [19] p. 166.)

EXAMPLE 2. (i) Let $A(T^2)$ be the Dirichlet algebra of continuous, complex valued functions on the torus $T^2=\{(z, w):|z|=|w|=1\}$, which are uniform limits of the polynomials in z^iw^j, where $(i, j)\in S=\{(i, j):j>0\}\cup\{(i, 0):i\geqq 0\}$. Then the maximal ideal space of $A(T^2)$ can be identified with $(\{z:|z|=1\}\times\{w:|w|\leqq 1\})\cup(\{z:|z|\leqq 1\}\times\{0\})$, with the normalized Haar measure m identified with $(z, w)=(0, 0)$.

The Gleason part $P=P(m)$ of m is $\{z:|z|<1\}\times\{0\}$. For each $z\in\{z:|z|=1\}$, $D_z=\{z\}\times\{w:|w|<1\}$ is a nontrivial Gleason part. The closure $\bar{P}=\{z:|z|\leqq 1\}\times\{0\}$ of P does not meet T^2. Every point $(z_0, 0)$ of $\partial P=\bar{P}\backslash P=\{z:|z|=1\}\times\{0\}$ is a point of D_{z_0}. Therefore $\bar{P}$ is not a union of Gleason parts.

(ii) Let $(1/2\pi)d\theta$ be the normalized Haar measure on the unit circle T, and let $H^\infty(d\theta)$ be the weak-* closure of the disc algebra $A(T)$ in $L^\infty(d\theta)$.

Let $A(T^2)$ and $m=(1/4\pi^2)d\theta d\phi$ be as in (i). Let $H^\infty=H^\infty(m)$ be the weak-* closure of $A(T^2)$ in $L^\infty(m)$. As in (2.6), we have

$$H^\infty=\mathscr{H}^\infty\oplus I^\infty, \tag{8.1}$$

where $\mathscr{H}^\infty$ and I^∞ are the weak-* closure in $L^\infty(m)$ of the sets $\{\sum_{i=0}^{n}a_iz^i:a_i\in$

C $(i=1, 2, \cdots, n)\}$ and $\{\sum_{j=1}^{m} w^j g_j(z) : g_j \in L^\infty(d\theta),\ (j=1, 2, \cdots, m)\}$ respectively, where $z=e^{i\theta}$ and $w=e^{i\phi}$. Let τ be the natural homeomorphism of $M(H^\infty(d\theta))$ and $M(L^\infty(d\theta))$ onto $M(\mathscr{H}^\infty)$ and $M(\mathscr{L}^\infty)$ respectively, which is induced by the correspondence between $z=e^{i\theta}$ on T^2 and $z=e^{i\theta}$ on T. We will identify $\lambda \in M(H^\infty(d\theta))$ with $\tau(\lambda) \in M(\mathscr{H}^\infty)$.

For a fixed point w_0 with $|w_0|<1$, each element $f \in H^\infty(m)$ uniquely can be decomposed as

$$f(z, w)=f_1(z)+wf_2(z)+(w-w_0)f_3(z, w) \text{ a.e. } (m), \tag{8.2}$$

where $f_1 \in \mathscr{H}^\infty$, $f_2 \in \mathscr{L}^\infty$ and $f_3 \in I^\infty$. In fact, by (8.1), we have

$$f(z, w)=f_1(z)+f_2'(z, w),$$

where $f_1 \in \mathscr{H}^\infty$ and $f_2' \in I^\infty$. By the definition of I^∞, $f_2'(z, w)/w$ is the boundary value of a bounded analytic function in w for almost every $z=e^{i\theta}$. We define

$$f_2(z)=\int_0^{2\pi} e^{-i\phi} f_2'(z, e^{i\phi}) P_{w_0}(\phi) \frac{d\phi}{2\pi},$$

where $P_{w_0}(\phi)$ is the Poisson kernel for w_0. Then we have

$$\frac{f_2'(z, w)}{w} - f_2(z)=(w-w_0)f_3'(z, w),$$

where $f_3'(z, w)$ is the boundary value of a bounded analytic function in w for almost every $z=e^{i\theta}$. Here we difine

$$f_3(z, w)=wf_3'(z, w),$$

then $f_3 \in I^\infty$, and this gives the decomposition (8.2) of f. It is easy to see the uniqueness of the decomposition from our construction.

Now, for $\xi \in M(H^\infty)$, we difine a linear functional ϕ_ξ on $H^\infty(m)$ by

$$\phi_\xi(f)=\xi(f_1).$$

For f, $g \in H^\infty(m)$, the decomposition (8.2) of fg is given by

$$\begin{aligned}(fg)(z, w)=&(f_1g_1)(z)+w(f_1g_2+f_2g_1+w_0f_2g_2)(z)\\ &+(w-w_0)(f_1g_3+f_3g_1+wf_2g_2+wf_2g_3\\ &+wf_3g_2+(w-w_0)f_3g_3)(z, w),\end{aligned} \tag{8.3}$$

where $f(z, w)=f_1(z)+wf_2(z)+(w-w_0)f_3(z, w)$ and $g(z, w)=g_1(z)+wg_2(z)+(w-w_0)g_3(z, w)$. It followm from (8.3) that ϕ_ξ is multiplicative on $H^\infty(m)$. Hence $\phi_\xi \in M(H^\infty(m))$. Clearly the map

$$\Phi : \xi \mapsto \phi_\xi$$

is a continuous map from $M(\mathscr{H}^\infty)$ into $M(H^\infty(m))$.

Similary, for $(\xi, w_0)\in M(L^\infty(d\theta))\times D$ $(D=\{w\in C:|w|<1\})$, we define a linear functional ϕ_{ξ,w_0} on $H^\infty(m)$ by

$$\phi_{\xi,w_0}(f)=\xi(f_1)+w_0\xi(f_2).$$

By (8.3), ϕ_{ξ,w_0} is also multiplicative on $H^\infty(m)$, and hence $\phi_{\xi,w_0}\in M(H^\infty(m))$. Moreover, the map

$$\phi:(\xi, w_0)\mapsto\phi_{\xi,w_0}$$

is a continuous map from $M(L^\infty(d\theta))\times D$ into $M(H^\infty(m))$. Here we note that $\phi_{\xi,0}=\phi_\xi$ if $\xi\in M(L^\infty(d\theta))=M(\mathscr{L}^\infty)$.

Now, the Gleason part $\mathscr{P}$ of $\phi_0=\Phi(0)$ $(=\tilde{m})$ is $\Phi(D)$, where $\tilde{m}$ is the complex homomorphism defined by the Radonization of the measure m. So,

$$\bar{\mathscr{P}}=\overline{\Phi(D)}=\Phi(\bar{D})=\Phi(M(\mathscr{H}^\infty)).$$

Hence $\bar{P}\ni\phi_\xi=\phi_{\xi,0}$ for $\xi\in M(L^\infty(d\theta))$. However $\phi_{\xi,0}$ and ϕ_{ξ,w_0} are in the same Gleason part whenever $|w_0|<1$. Therefore $\bar{\mathscr{P}}$ is not a union of Gleason parts for $H^\infty(m)$.

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(iii) Let $H^\infty=\mathscr{H}^\infty\oplus I^\infty$ and $\mathscr{L}^\infty$ be as in (ii). Let χ be a function in $\mathscr{L}^\infty$ with $\chi^2=\chi$ and $\chi\neq0, 1$, and let $A_1=\mathscr{H}^\infty\oplus\chi I^\infty$. Then A_1 is a weak-* Dirichlet algebra. If we put $m_1=\tilde{m}|A_1$, then $m_1\in M(A_1)$ and the Gleason part P_1 of m_1 for A_1 is nontrivial. If τ_1 is an analytic map of D onto P_1, then $A_1\circ\tau_1=H^\infty(D)$. Let Γ_1 be the Shilov boundary of $A_1|\bar{P}_1$. Then it follows from (ii) and [[13], § 4] that a certain point of Γ_1 belongs to some nontrivial Gleason part for A_1 and a certain point of Γ_1 composes trivial Gleason part for A_1.

Example 3. Let A' be a weak-* Dirichlet algebra on a nontrivial probability measure space $(X, \mathscr{A}, m)$ and let $H^\infty=H^\infty(m)$ be the weak-* closure of A' in $L^\infty(m)$. Let $\tilde{X}=M(L^\infty(m))$ and $A=\hat{H}^\infty|\tilde{X}$. Then A is a strongly logmodular algebra on a compact Hausdorff Stonian space $\tilde{X}$.

Let ϕ be any point of $M(A)$ and let $X_1=S(\phi)$. Then X_1 is a compact Hausdorff Stonian space (cf. [22], Theorem 2.2), and by Theorem 5.4 X_1 is a weak peak set of A. Let $A_1=A|X_1$. Then A_1 is a uniform algebra on X_1. From

$$C_R(X_1)=C_R(\tilde{X})|X_1=(\log|A^{-1}|)|X_1\subset\log|A_1^{-1}|\subset C_R(X_1)$$

we have $\log|A_1^{-1}|=C_R(X_1)$. Therefore A_1 also is a strongly logmodular algebra on X_1.

If the Gleason part $P(m)$ of $m \in M(H^\infty(m))$ is nontrivial and τ is an analytic map of D onto $P(m)$, then we have $H^\infty(m)\circ\tau = H^\infty(D)$.

EXAMPLE 4. We will identify a function in $H^\infty(D)$ with its boundary function on the unit circle ∂D. Let $(1/2\pi)d\theta$ be the normalized Lebesgue measure on ∂D. Let $\{z_n\}$ be a sequence in D such that $\overline{\lim_n}|z_n|=1$. We denote by $\overline{\{z_n\}}$ the weak-* closure of $\{z_n\}$ in $M(H^\infty(D))$. $\{z_n\}$ is said to be interpolation and sparse if

$$\inf_n \prod_{m:m\neq n}\left|\frac{z_n - z_m}{1-\bar{z}_m z_n}\right| > 0 \text{ and } \lim_{n\to\infty}\prod_{m:m\neq n}\left|\frac{z_n - z_m}{1-\bar{z}_m z_n}\right| = 1$$

respectively.

(i) Let m be any point in $\overline{\{z_n\}}\backslash D$. Then $P(m)\supsetneqq\{m\}$ if and only if $\{z_n\}$ is an interpolating sequence (cf. [9], Theorem 4.3.). In this case, there is a Blaschke product B such that $\hat{B}=0$ on $P(m)$. Hence the closure $\overline{P(m)}$ of $P(m)$ does not meet the Shilov boundary $\tilde{X}=M(L^\infty(d\theta))$ of $\widehat{H^\infty(D)}$ (cf. [9], p. 102). It is known from [3] that $\overline{P(m)}$ is a union of Gleason parts. Let μ_m be the representing measure on $\tilde{X}$ of m for $\widehat{H^\infty(D)}$, and let $X= S(\mu_m)$. Then, by Example 3, $A=H^\infty(D)|X$ is a strongly logmodular algebra on a compact Hausdorff Stonian space X.

(ii) Let $\{z_n'\}$ be a sequence in D such that $\overline{\lim_n}|z_n'|=1$. Then there is a subsequence $\{z_n\}$ of $\{z_n'\}$ such that $\{z_n\}$ is sparse (cf. [7], *p*. 106). Let m be any point in $\overline{\{z_n\}}\backslash D$. As in (i), let $X=S(\mu_m)$ and let $A=H^\infty(D)|X$. In this case, the Wermer embedding function Z belongs to A i.e., if $B(z)$ is the Blaschke product with zero sequence $\{z_n\}$, then $Z=\alpha\widehat{B(z)}|X$ for some unimodular constant α (cf. [9], *p*. 106). Let f be any function in $H^\infty(D)$. Then $g=f\circ(\alpha B)$ belongs to $H^\infty(D)$. Hence for an analytic map $\tau=Z^{-1}=(\alpha\hat{B}|P(m))^{-1}$ we have $\hat{g}(\tau(t))=f(\alpha\hat{B}(\tau(t)))=f(t)$, $t\in D$ (cf. [9], Lemma 6.3). Hence we have $A\circ\tau=H^\infty(D)$.

(iii) Let m be an element of $M(H^\infty(D))\backslash D$ such that the Gleason part $P(m)$ is nontrivial and any analytic map τ of D onto $P(m)$ is not a homeomorphism (cf. [9], p. 109). As in (i), let $X=S(\mu_m)$ and let $A=\widehat{H^\infty(D)}|X$. Then $Z\notin A$. Hence $Z\notin C(X)$, because, if $Z\in C(X)$, then by Theorem 5.5, we have $Z\in\mathscr{H}^\infty\cap C(X)\subset A$. From this, $\mathscr{H}^\infty\cap C(X)\subsetneqq \mathscr{H}^\infty(m)$. Further we have $A\circ\tau\subsetneqq H^\infty(D)$, because, if $A\circ\tau=H^\infty(D)$, then $\rho=\pi\circ\Sigma\circ T^*$ is an analytic map of D onto P (see the proof of Corollary 7.3), and ρ is a homeomorphism (cf. [12], Theorem 4.1), and hence τ is a homeomorphism (cf. [11], Theorem 2).

(iv) Let m be any point in $M(H^{\infty}(D))\backslash(D\cup\tilde{X})$, let $X=S(\mu_m)$ as in (i) and let $A=\widehat{H^{\infty}(D)}|X$. Let $H^{\infty}(m)$ be the weak-* closure of A in $L^{\infty}(\mu_m)$. It follows from [[5], p. 63] and the Tietze extension theorem that $|A|=C_R^+(X)(=\{u\in C_R(X): u\geqq 0\})$. Since X is a Stonian space, X is totally disconnected. For any clopen set V in X with $0<\mu_m(V)<1$ there is a function $f\in A$ such that $|f|=\chi_V$. Then $f\in J^{\infty}=J^{\infty}(H^{\infty}(m))$ (see (2.1)). Hence $J^{\infty}\supsetneqq\{0\}$.

Let E be the set of all clopen sets V with $0<\mu_m(V)<1$. Let $V\in E$, let $J_V=\{f\in A: |f|=\chi_V\}$, let $J_{V^c}=\{f\in A: |f|=\chi_{V^c}\}$ and let $J_V+J_{V^c}=\{f+g: f\in J_V,\ g\in J_{V^c}\}$. Then $J=\cup\{J_V+J_{V^c}: V\in E\}\subset J^{\infty}$. J separates the points of X, and we have $fg\in J$ for $f\in J$ and $g\in J$. Hence the linear span $\mathscr{V}(J)$ of J is an algebra. If $P(m)\supsetneqq\{m\}$, then the uniform closure of $\mathscr{V}(J)$ is contained in $I=\{f\in A: \phi(f)=0\ \ \forall\phi\in P(m)\}$. It is not known whether the weak-* closure of $\mathscr{V}(J)$ in $L^{\infty}(\mu_m)$ coincides with J^{∞}.

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