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On defect groups of interior G-algebras and vertices of modules

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Let G be a finite group and p is a prime number. Let $\mathscr{O}$ be a complete discrete valuation ring with unique maximal ideal (π) such that the residue field $k=\mathscr{O}/(\pi)$ is characteristic p. We assume that the field k is algebraically closed. In (5), Green defines a defect group for a G-algebra A (i.e. an $\mathscr{O}$-algebra A endowed with a G-action on A as $\mathscr{O}$-algebra automorphism) such that A^G is local. After, in (8), Puig introduces the concept of a source algebra of interior G-algebra A (i.e. an $\mathscr{O}$-algebra A endowed with an unitary $\mathscr{O}$-algebra homomorphism $\rho:\mathscr{O}[G]\longrightarrow A$) such that A^G is local and proved that the algebra A and its source algebra are Morita equivalent. The interior G-algebra A is a G-algebra by the conjugate G-action. A block $B=\mathscr{O}[G]e$ (e is a central primitive idempotent of $\mathscr{O}[G]$) of $\mathscr{O}[G]$ is an interior G-algebra by the projection $\mathscr{O}[G]\longrightarrow B: x\longmapsto xe$ such that B^G is local. Then a defect group of B in Green's sense equals a defect group for a block. See (5).

Let B be a block of $\mathscr{O}[G]$ with defect group D. In block theory, it is well known that there exists an indecomposable $\mathscr{O}[G]$-module V belonging to the block B such that the vertex of the $\mathscr{O}[G]$-module V equals D, see (2) (57.10). Now we can also define "belonging $\mathscr{O}[G]$-module" for interior G-algebras just like for blocks. The purpose of this paper is to extend this for interior G-algebras of some type using the theory of source algebras. See theorem 3.5.

In this paper, we use the following notation. Whenever A, B and C are sets and $f: A\longrightarrow B$ and $g: B\longrightarrow C$ are maps, the composed map of f and g is denoted by $g\circ f$. All $\mathscr{O}$-algebras are $\mathscr{O}$-free $\mathscr{O}$-algebras of finite rank with the unit element 1 and any $\mathscr{O}$-algebra homomorphism is an unitary homomorphism. All modules over a $\mathscr{O}$-algebra A are $\mathscr{O}$-free left A-module of finite rank. Whenever M and N are A-modules, we denote by $N|M$ if the A-module N is isomorphic to a direct summand of the A-module M. Whenever H and K are subgroups of G, the sets (G/H) and $(K\backslash G/H)$ are complete sets of representatives of left cosets gH and double cosets KgH, respectively. Whenever V is an $\mathscr{O}[G]$-module and

W is an $\mathcal{O}[H]$-module, we denote by $\mathrm{Res}_H^G(V)$ and $\mathrm{Ind}_H^G(W)$ the restricted module of V and the induced module of W, respectively. We denote by V^H the set of the fixed points of V under the action of H. We employ the other usual terminology of the representation theory of finite groups as in (2) and (4).

1. Interior G-algebras

In this section, we give some results for interior G-algebras, according to Dade (3) and Watanabe's lecture at Hokkaido University. Now A is an interior G-algebra with an $\mathcal{O}$-algebra homomorphism $\rho : \mathcal{O}[G] \longrightarrow A$. Whenever A' is other interior G-algebra with $\mathcal{O}$-algebra homomorphism $\rho' : \mathcal{O}[G] \longrightarrow A'$, an $\mathcal{O}$-algebra homomorphism $\tau : A \longrightarrow A'$ is a morphism as interior G-algebra if $\rho' = \tau \circ \rho$, and the morphism τ is isomorphism as interior G-algebra if τ is an $\mathcal{O}$-algebra isomorphism.

Whenever A is an interior G-algebra, we set

$$xa = \rho(x)a, \quad ax = a\rho(x) \text{ and } a^x = x^{-1}ax,$$

where $x \in G$ and $a \in A$. Then by the action $a \longmapsto a^x$, the $\mathcal{O}$-algebra A is a G-algebra. Whenever H is a subgroup of G, we set

$$A^H = \{a \in A : a^x = a \text{ for any } x \in H\},$$

and define the relative trace mapping Tr_H^G by

$$Tr_H^G : A^H \longrightarrow A^G, \quad a \longmapsto \textstyle\sum_{u \in [G/H]} a^{u^{-1}}.$$

Then the image $A_H^G = Tr_H^G(A^H)$ is a two sided ideal of A^G. See (5). Whenever A° is the opposite ring of A, the $\mathcal{O}$-algebra A° is an interior G-algebra by the homomorphism

$$\rho^\circ : \mathcal{O}[G] \longrightarrow A^\circ, \quad x \longmapsto \rho(x^{-1}).$$

Note that $A^H = (A^\circ)^H$ and $A_H^G = (A^\circ)_H^G$.

1.1. Whenever A is an interior G-algebra, let $A[G]$ be a free A-module generated by the elements of G. Then $A[G]$ becomes a strongly G-graded ring by the product

$$ax \cdot by = ab^{x^{-1}}xy,$$

where ax and $by \in A[G]$.

EXAMPLE 1.2. The group algebra $\mathcal{O}[G]$ is an interior G-algebra through the identity mapping. Then we have an $\mathcal{O}$-algebra isomorphism

$$\begin{array}{ccc} \mathcal{O}[G][G] & \simeq & \mathcal{O}[G\times G] \\ x\cdot y & \longleftrightarrow & (xy, y), \end{array}$$

where x and $y\in G$. Therefore the $\mathcal{O}$-algebra homomorphism ρ introduces an $\mathcal{O}$-algebra homomorphism

$$\begin{array}{ccc} \mathcal{O}[G\times G] & \longrightarrow & A[G] \\ (x, y) & \longmapsto & \rho(xy^{-1})y. \end{array}$$

If ρ is an epimorphism, then the induced $\mathcal{O}$-algebra homomorphism is an epimorphism.

1.3. Whenever H is a subgroup of G, the interior G-algebra A is an interior H-algebra through the restricted mapping $\rho|_H$, and we can define a ring $A[H]$. Whenever M is an $A[H]$-module and $\mathrm{End}_A(M)$ is the A-endomorphism ring of M, then $\mathrm{End}_A(M)$ is an interior H-algebra by the group homomorphism

$$\begin{array}{rcl} \rho_M : H & \longrightarrow & \mathrm{End}_A(M) \\ x & \longmapsto & \rho_M(x), \end{array}$$

where

$$\begin{array}{rcl} \rho_M(x) : M & \longrightarrow & \mathrm{M} \\ m & \longmapsto & \rho(\mathrm{x}^{-1})\mathrm{x}\cdot\mathrm{m}. \end{array}$$

Then we have

$$f^*(m)=x^{-1}f(xm),$$

where $f\in\mathrm{End}_A(M)$, $x\in H$ and $m\in M$. Note $\mathrm{End}_A(M)^H=\mathrm{End}_{A[H]}(M)$, and M is an indecomposable $A[H]$-module if and only if $\mathrm{End}_A(M)^H$ is local. Therefore $A[G]$-modules have unique decomposition property. See (4) Ch. 1 corollary 11.2.

1.4. Let A be an interior G-algebra, H a subgroup of G, M an $A[G]$-module and N an $A[H]$-module. we denote by $\mathrm{Res}^G_H(M)$ the restricted $A[H]$-module of M and by $\mathrm{Ind}^G_H(N)$ the induced $A[G]$-module $A(G) \otimes_{A[H]} N$.

1.5. The symbol $\otimes$ means the tensor product over $A[H]$ in 1.5, 1.6 and 1.7. Whenever N is an $A[H]$-module, then

$$\mathrm{Ind}^G_H(N)\simeq\bigoplus_{u\in(G/H)} \rho(u^{-1})u\otimes N,$$

as $\mathcal{O}$-module. Moreover we have

$$a\rho(u^{-1})u\otimes n=\rho(u^{-1})u\otimes an$$
$$x\rho(u^{-1})u\otimes n=\rho((xu)^{-1})xu\otimes\rho(x)n,$$

where $a\in A$, $x\in G$ and $\rho(u^{-1})u\otimes n\in\mathrm{Ind}_H^G(N)$. In particular, the induced module $\mathrm{Ind}_H^G(N)$ is isomorphic to $|G:H|N$ as A-module.

Indeed, since $\rho(u^{-1})$ is an unit of A the first isomorphism is evident. The second equality is followed from

$$\begin{aligned}a\rho(u^{-1})u\otimes n&=\rho(u^{-1})\rho(u)a\rho(u^{-1})u\otimes n\\&=\rho(u^{-1})a^{u^{-1}}u\otimes n\\&=\rho(u^{-1})ua\otimes n\\&=\rho(u^{-1})u\otimes an.\end{aligned}$$

The third equality is followed from

$$\begin{aligned}x\rho(u^{-1})u\otimes n&=\rho(x)\rho((xu)^{-1})xu\otimes n\\&=\rho((xu)^{-1})xu\otimes\rho(x)n.\end{aligned}$$

1.6. Whenever H is a subgroup of G and N is an $A[H]$-module, N is $A[H^{x^{-1}}]$-module by

$$ah^{x^{-1}}\cdot n=a^xh\cdot n,$$

where $ah^{x^{-1}}\in A[H^{x^{-1}}]$ and $n\in N$, and we denote this $A[H^{x^{-1}}]$-module by $x\otimes N$. Then by the similar argument for $\mathcal{O}[G]$-module, we have Mackey decomposition theorem for $A[G]$-modules. See (4) Ch. 2 theorem 2.9.

1.7. Whenever H and K are subgroups of G and N is a $A[H]$-module, then

$$\mathrm{Res}_K^G(\mathrm{Ind}_H^G(N))\simeq\bigoplus_{u\in[K\backslash G/H]}\mathrm{Ind}_{uHu^{-1}\cap K^K}(\mathrm{Res}_{uHu^{-1}\cap k}^{uHu^{-1}}(u\otimes N)),$$

as $A[K]$-module.

1.8. Whenever M is an $A[G]$-module and H is a subgroup of G, we call M is H-projective if there exists an $A[H]$-module N such that

$$M|\mathrm{Ind}_H^G(N).$$

Then Higmann's criteria for relative projectivity is extended for $A[G]$-modules similarly for $\mathcal{O}[G]$-modules. See (4) Ch. 2 theorem 3.8.

1.9. An $A[G]$-module M is H-projective if and only if

$$\mathrm{End}_A(M)^G=\mathrm{End}_A(M)_H^G$$

for the interior G-algebra $\mathrm{End}_A(M)$. In particular, any $A[G]$-module is S-projective, where S is a p-Sylow subgroup of G.

1.10. By 1.7, we can define a vertex for an indecomposable $A[G]$-module. That is, whenever M is an indecomposable $A[G]$-module, the minimal subgroups H satisfying M is H-projective are G-conjugate, and we call this subgroups the vertex of M and denote by $vtx_G(M)$. Whenever P is a vertex of M, then there exists an indecomposable $A[P]$-module such that

$$N|\mathrm{Res}_P^G(M) \text{ and } vtx_P(N)=P.$$

These $A[P]$-modules satisfying the above condition are $N_G(P)$-conjugate, and we call this module source of M. Note that

$$M|\mathrm{Ind}_P^G(N).$$

By 1.9, the vertex is p-subgroup of G.

EXAMPLE 1.11. Whenever A is an interior G-algebra such that A^G is local, then the $\mathcal{O}$-algebra A is an indecomposable $A[G]$-module by the action

$$ax \cdot b = ab^{x^{-1}},$$

where $ax \in A[G]$ and $b \in A$. We call the vertex $vtx_G(A)$ a defect group of A. But we have the following isomorphism as interior G-algebra

$$\begin{array}{ccc} \mathrm{End}_A(A) & \simeq & A^\circ \\ f & \longmapsto & f(1), \end{array}$$

and by 1.9, the definition of defect group in this paper is equivalent to the definition of defect group in Green's sense (5). Let D be a defect group of A and a indecomposable $A[D]$-module L a source of A. We call source algebra of A the endomorphism ring $\mathrm{End}_A(L)$. Since

$$L|\mathrm{Res}_D^G(A),$$

for the projection p_L of A to L, the element $i=p_L(1)$ is a primitive idempotent of A^D and

$$(iAi)^\circ \simeq \mathrm{End}_A(L),$$

as $\mathcal{O}$-algebra. Thus the definition of source algebra is equivalent to the definition of source algebra in Puig's sense (8). Note that

$$L \simeq Ai,$$

as $A[D]$-module.

In (8), Puig prove that the module categories of A and B is Morita equivalent. We shall prove this using the above definitions and the following lemma of (8).

1.12. Whenever A and B are $\mathcal{O}$-algebras and i is an idempotent of A. Assume that the $\mathcal{O}$-algebra iAi is isomorphic to B as $\mathcal{O}$-algebra and A is directly embedded to the full matrix ring $M_n(B)$. Then we have isomorphisms

$$Ai\otimes_B iA\simeq A \text{ as } (A, A)\text{-bimodule},$$
$$iA\otimes_A Ai\simeq B \text{ as } (B, B)\text{-bimodule}.$$

THEOREM 1.13. *(Puig) Let A be an interior G-algebra such that A^G is local and $B=iAi$ a source algebra of A. Then the module categories of A and B are Morita equivalent by*

$$M\longmapsto iA\otimes_A M \text{ and } N\longmapsto Ai\otimes_B N,$$

where M is an A-module and N is a B-module.

PROOF. Let D be a defect group of A. By 1.11, the $\mathcal{O}$-algebra A is an indecomposable $A[G]$-module and we set L a source of A. Then L is an indecomposable $A[G]$-module and

$$A|\mathrm{Ind}_D^G(L)$$

and

$$B^\circ\simeq\mathrm{End}_A(L).$$

This implies that the endomorphism ring $\mathrm{End}_A(A)$ is directly embedded to $\mathrm{End}_A(\mathrm{Ind}_D^G(L))$. By 1.5,

$$\mathrm{Ind}_D^G(L)\simeq|G:D|L,$$

as A-module, and this implies A is directly embedded to the full matrix ring $M_{|G:D|}(B)$. Thus the $\mathcal{O}$-algebras A and B satisfy the condition of 1.12. Therefore By (2) (3.54), the module categories of A and B are Morita equivalent through the above correspondence.

1.14. By 1.2 and 1.11, for the $A[G]$-module A the $\mathcal{O}$-algebra A is $\mathcal{O}[G\times G]$-module through the $\mathcal{O}$-algebra homomorphism

$$\mathcal{O}[G\times G]\longrightarrow A[G]$$

in 1.2. Then the action of $G\times G$ on A is

$$(x, y)\cdot a=\rho(x)a\rho(y^{-1}),$$

where $(x, y)\in G\times G$ and $a\in A$. Note that $\mathscr{O}[G\times G]$-module A is indecomposable if ρ is an epimorphism.

1.15. Whenever M, is an $A[G]$-module, then by 1.3, the A-endomorphism ring $\mathrm{End}_A(M)$ is an interior G-algebra, and $\mathrm{End}_A(M)$ is $\mathscr{O}[G\times G]$-module. Whenever M' is an $A(G)$-module such that M' is a direct summand of M, then $\mathscr{O}[G\times G]$-module $\mathrm{End}_A(M')$ is a direct summand of $\mathrm{End}_A(M)$.

Indeed, whenever $f: M \longrightarrow M'$ is the projection, we have

$$\mathrm{End}_A(M')\simeq f\mathrm{End}_A(M)f,$$

as interior G-algebra. But it is obvious that

$$f\mathrm{End}_A(M)f|\mathrm{End}_A(M),$$

as $\mathscr{O}[G\times G]$-module, and proved.

Whenever H is a subgroup and N an $A[H]$-module, similarly the endomorphism rings $\mathrm{End}_A(N)$ and $\mathrm{End}_A(\mathrm{Ind}_H^G(N))$ become $\mathscr{O}[H\times H]$-module and $\mathscr{O}[G\times G]$-module. Then we have the following lemma.

1.16. Whenever N is an $A[H]$-module, we have

$$\mathrm{Ind}_{H\times H}^{G\times G}(\mathrm{End}_A(N))\simeq \mathrm{End}_A(\mathrm{Ind}_H^G(S)),$$

as $\mathscr{O}[G\times G]$-module.

Indeed, by 1.5, we have

$$\mathrm{Ind}_H^G(N)\simeq \oplus_{u\in[G/H]}\ \rho(u^{-1})u\otimes_H N.$$

Whenever s, $t\in G$ and $h\in\mathrm{End}_A(N)$, we define the mapping $f_{ts}\otimes h: \mathrm{Ind}_H^G(N) \longrightarrow \mathrm{Ind}_H^G(N)$ by

$$\begin{aligned} f_{ts}\otimes h &: \rho(u^{-1})u\otimes_H n \\ &\longmapsto \begin{cases} \rho(t^{-1})t\otimes_H h(\rho((su)^{-1})su\cdot n) & \text{if } su\in H, \\ 0 & \text{otherwies.} \end{cases} \end{aligned}$$

The mapping $f_{ts}\otimes h$ is in $\mathrm{End}_A(\mathrm{Ind}_H^G(N))$. Then the following mapping,

$$\begin{aligned} \mathrm{Ind}_{H\times H}^{G\times G}(\mathrm{End}_A(N)) &\longrightarrow \mathrm{End}_A(\mathrm{Ind}_H^G(N)), \\ (t, s)\otimes h &\longmapsto f_{ts^{-1}}\otimes_{H\times H} h \end{aligned}$$

introduces an $\mathscr{O}[G\times G]$-module isomorphism.

2. Source algebras and source of modules

In this section, we define $\mathscr{O}[G]$-modules belonging to interior G-

algebras and show that the source of a module belonging to an interior G-algebra can be introduced from a module belonging to its source algebra.

Let A be an interior G-algebra with $\mathcal{O}$-homomorphism ρ satisfying the subalgebra A^G is local. Let D be a defect group of A and $B=iAi$ is source algebra, where i is a primitive idempotent of A^D. We define an $\mathcal{O}$-algebra homomorphism ρ_i by

$$\rho_i : \mathcal{O}[D] \longrightarrow B,\ X \longmapsto \rho(x)i.$$

Then B is an interior D-algebra through ρ_i.

Whenever M is an A-module, then M is an $\mathcal{O}[G]$-module through ρ. Similarly, any B-module N is an $\mathcal{O}[D]$-module through ρ_i.

2.1. Let V be an indecomposable $\mathcal{O}[G]$-module. We call the $\mathcal{O}[G]$-module V is belonging to A if there exists an A-module M such that

$$V|M,$$

as $\mathcal{O}[G]$-module. The $\mathcal{O}$-endomorphism ring $\mathrm{End}(M)$ is an interior G-algebra by the representation afforded by the $\mathcal{O}[G]$-module M, and the representation $A \longrightarrow \mathrm{End}(M)$ afforded by the A-module M is a morphism as interior G-algebra. Therefore since D is a defect group of A the $\mathcal{O}[G]$-module M is D-projective, and an indecomposable $\mathcal{O}[G]$-module V belonging to A is D-projective. Of course, for block algebra this definition of "belonging" is equivalent to one of block theory.

In this case, we obtain the following propositions.

PROPOSITION 2.2. *Whenever N is a B-module and U is an indecomposable $\mathcal{O}[D]$-module such that U is an indecomposable direct summand of N as $\mathcal{O}[D]$-module satisfying*

$$vtx_D(U)=D.$$

Then the A-module $Ai \otimes_B N$ has indecomposable direct summand V as $\mathcal{O}[G]$-module satisfying

$$vtx_G(V)=D$$

and U is a source of V.

PROOF. Let $M=Ai \otimes_B N$. Then because i is primitive idempotent of A^D, we have

$$iM|\mathrm{Res}_D^G(M),$$

as $\mathcal{O}[D]$-module. But by theorem 1.13,

$$iM \simeq N,$$

as B-module and so we have

$$U | iM,$$

as $\mathcal{O}[D]$-module. Therefore there exists an indecomposable direct summand V of M as $\mathcal{O}[G]$-module such that

$$U | \mathrm{Res}_D^G(V).$$

Since $vtx_D(U)=D$ the vertex of the indecomposable $\mathcal{O}[G]$-module equals D and U is a source of V.

PROPOSITION 2.3. *Whenever M is an A-module and V is an indecomposable $\mathcal{O}[G]$-module such that V is an indecomposable direct summand of M as $\mathcal{O}[G]$-module satisfying*

$$vtx_G(U)=D.$$

Then the B-module iM has indecomposable direct summand U as $\mathcal{O}[D]$-module satisfying

$$vtx_D(U)=D$$

and U is a source of V.

PROOF. Let

$$\mathrm{Res}_D^G(A)=Ai_1 \oplus Ai_2 \oplus \cdots \oplus Ai_r \oplus Aj_1 \oplus Aj_2 \oplus \cdots Aj_s$$

be an indecomposable decomposition as $A[D]$-module, where $i_1, i_2, \cdots, i_r$ and $j_1, j_2, \cdots, j_s$ are primitive idempotent of A^D. Assume that the vertices of the indecomposable $A[D]$-modules $Ai_1, Ai_2, \cdots, Ai_r$ are D (i.e. sources of indecomposable $A[G]$-module A), and the vertices of the indecomposable $A[D]$-modules $Aj_1, Aj_2, \cdots, Aj_s$ are proper subgroups of D.

Then we have the following decomposition

$$\mathrm{Res}_D^G(M)=i_1M \oplus i_2M \oplus \cdots \oplus i_rM \oplus j_1M \oplus j_2M \oplus \cdots j_sM,$$

as $\mathcal{O}[D]$-module. But whenever U_1 is a source of indecomposable $\mathcal{O}[G]$-module V, there exists an idempotent h in $(i_1, i_2, \cdots, i_r, j_1, j_2, \cdots, j_s)$ such that

$$U_1 | hM,$$

because

$$V|M,$$

as $\mathcal{O}[G]$-module.

We claim $h=i_k$ for some k. Indeed, if $h=j_m$ for some m, then

$$U_1|j_mM.$$

But j_mM is a j_mAj_m-module and interior D-algebra j_mAj_m has defect group smaller than D by assumption. Thus 2.1 implies that the vertex of the indecomposable $\mathcal{O}[D]$-module U_1 is smaller than D, and this is contradiction.

Since Ai_k is source of A there exists $x\in N_G(D)$ such that $i_k=i^x$. So we have

$$(U_1)^{x^{-1}}|iM.$$

We set $U=(U_1)^{x^{-1}}$, then the indecomposable $\mathcal{O}[D]$-module U is a source of V, and proved the proposition.

By proposition 2.2 and 2.3, the following corollary is immediate.

COROLLARY 2.4. *There exists an indecomposable $\mathcal{O}[G]$-module V belonging to A such that $vtx_G(V)=D$ if and only if there exists an indecomposable $\mathcal{O}[D]$-module U belonging to B such that $vtx_D(U)=D$.*

3. Defect groups and vertices

In this section, A is an interior G-algebra with an epimorphism $\rho: \mathcal{O}[G] \longrightarrow A$ such that A^G is local. We call this interior G-algebra A and epimorphic interior G-algebra. Let D is a defect group of A and $B=iAi$ is a source algebra of A, where $i\in A^D$ is a primitive idempotent. By 1.14, the $\mathcal{O}[G\times G]$-module A is indecomposable.

3.1. We have

$$B|\mathrm{Res}^{G\times G}_{D\times D}(A) \text{ and } A|\mathrm{Ind}^{G\times G}_{D\times D}(B).$$

In particular, whenever L is a source of the indecomposable $\mathcal{O}[G\times G]$-module, there exists an indecomposable direct summand B' of the $\mathcal{O}[D\times D]$-module B such that

$$xtx_{G\times G}(A)=vtx_{D\times D}(B')$$

and L is a source of B'.

Indeed, the definition implies that

$$Ai|\mathrm{Res}^G_D(A) \text{ as } A[D]\text{-module},$$

and

$$A|\mathrm{Ind}_D^G(Ai) \text{ as } A[G]\text{-module.}$$

So by 1.15, we have

$$\mathrm{End}_A(Ai)|\mathrm{End}_A(A) \text{ as } \mathcal{O}[G\times G]\text{-module,}$$

and

$$\mathrm{End}_A(A)|\mathrm{End}_A(\mathrm{Ind}_D^G(Ai) \text{ as } \mathcal{O}[G\times G]\text{-module.}$$

But $\mathrm{End}_A(A)\simeq A^\circ$ as $\mathcal{O}[G\times G]$-module, $\mathrm{End}_a(Ai)\simeq(iAi)^\circ\simeq B^\circ$ as $\mathcal{O}[D\times D]$-module and by 1.15

$$\mathrm{Ind}_{D\times D}^{G\times G}(\mathrm{End}_a(Ai))\simeq\mathrm{End}_a(\mathrm{Ind}_D^G(Ai)),$$

as $\mathcal{O}[G\times G]$-module. Thus we have

$$B|\mathrm{Res}_{D\times D}^{G\times G}(A) \text{ and } A|\mathrm{Ind}_{D\times D}^{G\times G}(B).$$

The second statement is introduced from the first part.

The following is prove in (6) and (7).

3. 2. We set

$$\Delta D=\{(d, d)\in D\times D : d\in D\}.$$

Then we have

$$\Delta D\leqq vtx_{G\times G}(A)\leqq D\times D.$$

Moreover,

$$vtx_{G\times G}(A)=(\langle 1\rangle\times Q)\cdot\Delta D,$$

where $Q=\{d\in D : (1, d)\in vtx_{G\times G}(A)\}$ is a normal subgroup of D.

3. 3. Whenever V is an $\mathcal{O}[G\times G]$-module and

$$V^{<1>\times G}=\{v\in V : (1, x)v=v \text{ for any } x\in G\},$$

then $V^{<1>\times G}$ is an $\mathcal{O}[G\times G]$-submodule of V. Note that

$$(x, 1)v=(x, x)v,$$

where $v\in V^{<1>\times G}$ and $x\in G$.

The following lemma is (4) ch. 2 lemma 3.4.

3. 4. Whenever H is a subgroup of G and W is an $\mathcal{O}[H]$-module. Then we have

$$(\mathrm{Ind}_H^G(W))^G=\{\textstyle\sum_{u\in[G/H]} u\otimes w : w\in W^H\}.$$

In particular,

$$(\mathrm{Ind}_H^G(W))^G\simeq W^H,$$

as $\mathcal{O}$-module.

The following is the main result of this paper.

THEOREM 3.5. *Whenever A is an epimorphic interior G-algebra such that A^G is local and D is a defect group of A. Let $R=(\langle 1\rangle\times Q)\cdot\Delta D$ be a vertex of indecomposable $\mathcal{O}[G\times G]$-module A and L its source. Assume that the $\mathcal{O}[\Delta D]$-module $L^{<1>\times Q}$ has an indecomposable direct summand whose vertex equals ΔD. Then there exists an indecomposable $\mathcal{O}[G]$-module V belonging to A such that the vertex of V equals D.*

PROOF. Let $B=iAi$ ($i\in A^D$: primitive idempotent) be a source algebra of A. Then B is an interior D-algebra with vertex D. By corollary 2.4, we may prove that there exists an indecomposable $\mathcal{O}[D]$-module W belonging to B such that $vtx_D(W)=D$.

The $\mathcal{O}$-module $B^{<1>\times D}$ becomes a left B-module, so becomes $\mathcal{O}[D]$-module. We shall prove that there exists an indecomposable direct summand W of the $\mathcal{O}[D]$-module B such that the vertex of W is D.

By 3.1, there exists an indecomposable direct summand B' of $\mathcal{O}[D\times D]$-module B such that

$$vtx_{D\times D}(B')=R$$

and L is a source of B'. Because the residue field k is an algebraically close field and D is a p-subgroup of G, the Green's indecomposablity theorem ((4) ch. 3 Theorem 3.8) implies

$$B'\simeq\mathrm{Ind}_R^{D\times D}(L).$$

So by Mackey decomposition theorem, we have

$$\mathrm{Res}^{D\times D}_{<1>\times D}\ (\mathrm{Ind}_R^{D\times D}(L))\simeq\mathrm{Ind}^{<1>\times D}_{<1>\times Q}\ (\mathrm{Res}^R_{<1>\times Q}(L)).$$

But 3.4 implies that

$$(\mathrm{Ind}^{<1>\times D}_{<1>\times Q}\ (\mathrm{Res}^R_{<1>\times Q}(L)))^{<1>\times D}\simeq L^{<1>\times Q},$$

as $\mathcal{O}$-module by

$$\textstyle\sum_{u\in[D/Q]}(1, u)\otimes 1\longleftrightarrow 1,$$

where $1\in L^{<1>\times Q}$. It is easily checked that this $\mathcal{O}$-module isomorphism is an $\mathcal{O}[\Delta D]$-module isomorphism. Thus we obtain

$$\mathrm{Res}_{\Delta D}^{D\times D}\ (B'^{<1>\times D}) \simeq \mathrm{Res}_{\Delta D}^{R}(L^{<1>\times Q}),$$

as $\mathcal{O}[\Delta D]$-module. By assumption, the $\mathcal{O}[\Delta D]$-module $\mathrm{Res}_{\Delta D}^{R}(L^{<1>\times Q})$ has an indecomposable direct summand whose vertex is ΔD, and so the $\mathcal{O}[\Delta D]$-module $\mathrm{Res}_{\Delta D}^{D\times D}(B'^{<1>\times D})$ has indecomposable direct summand whose vertex is ΔD. Note that

$$(d, 1)b = (d, d)b,$$

where $b \in B'$ and $d \in D$. Thus the $\mathcal{O}[D\times\langle 1\rangle]$-module $B'^{<1>\times D}$ has indecomposable direct summand W whose vertex is $D\times\langle 1\rangle$. W can be an indecomposable $\mathcal{O}[D]$-module whose vertex is D by

$$dw = (d, 1)w,$$

where $w \in W$ and $d \in D$. Then the indecomposable $\mathcal{O}[D]$-module W is an indecomposable direct summand of the $\mathcal{O}[D]$-module $B^{<1>\times D}$, because

$$B'|B,$$

as $\mathcal{O}[D\times D]$-module.

Therefore there exists B-module $B^{<1>\times D}$ such that

$$W|B^{<1>\times D},$$

as $\mathcal{O}[D]$-module and $vtx_D(W)=D$, and proved theorem.

COROLLARY 3.6. *Under the notation of theorem* 3.5, *if $\mathcal{O}$-rank of the source L is not larger than p and the $\mathcal{O}$-submodule $L^{<1>\times Q}$ is not* $\{0\}$, *there exists an indecomposable $\mathcal{O}[G]$-module V belonging to A such that the vertex of V equals D.*

PROOF. There occur two cases. If the $\mathcal{O}$-rank of $\mathcal{O}[\Delta D]$-module $L^{<1>\times Q}$ is smaller than p, any indecomposable direct summand of $\mathcal{O}[\Delta D]$-module $L^{<1>\times Q}$ has vertex ΔD. So the assumption of theorem 3.5 is hold.

If the $\mathcal{O}$-rank of $\mathcal{O}[\Delta D]$-module $L^{<1>\times Q}$ equals p, then we have

$$L^{<1>\times Q} = L.$$

But L is an indecomposable $\mathcal{O}[R]$-module and $R/(\langle 1\rangle\times Q) \simeq \Delta D$. So L becomes $\mathcal{O}[\Delta D]$-module and this module is isomorphic to the restriction of L to ΔD. Therefore the $\mathcal{O}[\Delta D]$-module $L^{<1>\times Q}(=L)$ is indecomposable. Since the vertex of the indecomposable $\mathcal{O}[R]$-module L is R, the vertex of the indecomposable $\mathcal{O}[\Delta D]$-module $L^{<1>\times Q}$ is ΔD.

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