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A note on the classification of nonsingular flows with transverse similarity structures

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§ 1. Introduction

The purpose of this paper is to classify nonsingular flows with transverse similarity structures satisfying certain auxiliary conditions. We consider such flows as foliations with transverse similarity structures and the classification is done in this view point. A motivation for this study is as follows. In Nishimori [7], the author investigated the qualitative properties of foliations with transverse similarity structures and gave an analogy of Sacksteder's theorem (in Sacksteder [8]) on codimension one foliations. Furthermore for such foliations, Matsuda [6] gave an analogy of a theorem of Hector and Duminy (in Hector [5] and in Cantwell and Conlon [1]) on codimension one foliations. So we are interested in concrete examples of foliations with transverse similarity structures.

Here we give the difinition of foliations with transverse similarity structures. A codimension q C^{∞} foliation $\mathscr{F}$ of a C^{∞} manifold M has a *transverse similarity structure* if there exsits an open covering $\{U_i\}_{i\in I}$ of M, a family $\{h_i : U_i \rightarrow \boldsymbol{R}^q\}_{i\in I}$ of C^{∞} submersions such that (1) $\mathscr{F}|_{U_i}=\{h_i^{-1}(t)\}_{t\in h_i(U_i)}$, and (2) for each $i,j\in I$ with $U_i\cap U_j\neq\emptyset$, there exists a similarity transformation $\gamma_{ji} : \boldsymbol{R}^q \rightarrow \boldsymbol{R}^q$ satisfying.

$$\gamma_{ji}\circ(h_i|_{U_i\cap U_j})=h_j|_{U_i\cap U_j}.$$

We call $\theta=\{U_i, h_i, \gamma_{ji}\}$ *a transverse similarity structure* of $\mathscr{F}$.

By starting from one of such submersions h_i's, we can construct the *analytic continuation* and obtain a C^{∞} submersion $D : \widetilde{M} \rightarrow \boldsymbol{R}^q$, where $\widetilde{M}$ is the universel covering of M. We call D a *developing map* of the foliation $\mathscr{F}$ with the similarity structure $\theta=\{U_i, h_i, \gamma_{ji}\}$. As is well known (see Godbillon [4] for example), there exists a homomorphism $\Phi : \pi_1(M) \rightarrow \mathrm{Sim}(q)$ such that $D\circ\gamma=\Phi(\gamma)\circ D$ for all $\gamma\in\pi_1(M)$, where $\mathrm{Sim}(q)$ is the group of similarity transformations of $\boldsymbol{R}^q$. In this paper, we work in the *oriented category* for simplicity. So we suppose that γ_{ji}'s are orientation preserving, that is, $\gamma_{ji}\in\mathrm{Sim}_+(q)$.

If the dimension of such $\mathscr{F}$ is zero, the triple $(M, \mathscr{F}, \{U_i, h_i, \gamma_{ji}\})$ is a similarity manifold, which is classified by Fried [2] in the case where M is closed. If the dimension of such $\mathscr{F}$ is one, the triple can be considered as a nonsingular flow with a transverse similarity structure. In a sense, our result may be considered as a generalization of that of Fried [2].

The plan of this paper is as follows. In § 2, we quote the results of Fried [2] on closed similarity manifold in a formulation which is convenient for our purpose. In § 3, we give examples of nonsingular flows with transverse similarity structures, which will make up the classification list. In § 4, we describe the auxiliary conditions and state our main result (Theorem 4.5) on the classification. In §5, we prove Theorem 4.5. We work in the C^∞ category, and hereafter we omit the term C^∞.

§ 2. The classification of closed oriented similarity manifolds due to Fried

In this section, we quote the results in Fried [2] in a somewhat modified form which is convenient for the later application.

Let N be a connected closed oriented similarity manifold of dimension $q \geqq 1$, and $D: \tilde{N} \to \boldsymbol{R}^q$ an orientation preserving developing map of N, where $\tilde{N}$ and $\boldsymbol{R}^q$ are naturally oriented. The following theorem is the crucial result of Fried, which makes the classification possible.

THEOREM 2.1. (Fried [2]) (1) *If* $q=1$, *then* D *is a diffeomorphism onto its image* $D(\tilde{N})$ *and one of the following cases occurs*:

(i) $D(\tilde{N})=\boldsymbol{R}$,
(ii) $D(\tilde{N})=]a, \infty[$ *or* $]-\infty, a[$ *for some* $a \in \boldsymbol{R}$.

(2) *If* $q \geqq 2$, *then* D *is a covering map onto its image* $D(\tilde{N})$ *and one of the following cases occurs*:

(i) $D(\tilde{N})=\boldsymbol{R}^q$,
(ii) $D(\tilde{N})=\boldsymbol{R}^q-\{a\}$ *for some* $a \in \boldsymbol{R}^q$.

(*Note that* D *is a diffeomorphism onto its image except the case* (ii) *with* $q=2$.)

We call N *Euclidean* if $D(\tilde{N})=\boldsymbol{R}^q$, and *radiant* otherwise. When N is radiant, we may suppose that a in Theorem 2.1 coincides with the origin 0 of $\boldsymbol{R}^q$ by modifying the developing map D.

In order to state the classification result, we define the isomorphisms between closed oriented similarity manifolds as follows.

DEFINITION 2.2: Let N_1 and N_2 be closed oriented similarity manifolds. We say that N_1 is *isomorphic* to N_2 if there is an orientation preserving homeomorphism $g: N_1 \to N_2$ such that, if $D_2: \tilde{N} \to \boldsymbol{R}^q$ is an orientation preserving developing map and $\tilde{g}: \tilde{N}_1 \to \tilde{N}_2$ is a lift of g, then $D_2 \circ \tilde{g}: \tilde{N}_1 \to \boldsymbol{R}^q$ is an orientation preserving developing map. We call such $g: N_1 \to N_2$ an *isomorphism*. (Note that such g becomes automatically a real analytic diffeomorphism.)

We give examples of closed oriented similarity manifolds, which will form the classification list of Fried [**2**].

EXAMPLE 2.3: Let Γ be a lattice group of $\boldsymbol{R}^q$ and denote by $N^q(\Gamma)$ the Euclidean closed oriented similarity manifold obtained as the quotient $\boldsymbol{R}^q/\Gamma$. Clearly $N^q(\Gamma_1)$ and $N^q(\Gamma_2)$ are isomorphic if and only if there exists an orientation preserving similarity transformation $g: \boldsymbol{R}^q \to \boldsymbol{R}^q$ such that $g\Gamma_1 g^{-1} = \Gamma_2$.

EXAMPLE 2.4: Take $r \in]0, 1[$ and define a map $m_r: \boldsymbol{R} \to \boldsymbol{R}$ by $m_r(x) = rx$ for all $x \in \boldsymbol{R}$. Denote by $N^1(r, +)$ (respectively $N^1(r, -)$) the radiant closed oriented similarity manifold obtained as the quotient of the interval $]0, \infty[$ (respectively $]-\infty, 0[$) by the cyclic group generated by m_r. Clearly $N^1(r, +)$ and $N^1(s, -)$ are not isomorphic for all $r, s \in]0, 1[$. If $r \neq s$, then $N^1(r, +)$ and $N^1(s, +)$ (respectively $N^1(r, -)$ and $N^1(s, -)$) are not isomorphic.

EXAMPLE 2.5: Suppose that $q=2$ and identify $\boldsymbol{R}^2$ with $\boldsymbol{C}$. Consider the exponential map $\exp: \boldsymbol{C} \to \boldsymbol{C} - \{0\}$ and the oriented similarity structure $\theta = \exp^* \theta_0$ on $\boldsymbol{C}$ induced from the canonical oriented similarity structure θ_0 of $\boldsymbol{C} - \{0\}$ by the map exp. Then a homeomorphism $g: \boldsymbol{C} \to \boldsymbol{C}$ is an automorphism of oriented similarity manifold $(\boldsymbol{C}, \theta)$ if and only if g is a translation (since $\exp(z+a) = \exp(z) \cdot \exp(a)$). Hence a lattice group Γ of $\boldsymbol{C} = \boldsymbol{R}^2$ acts on $(\boldsymbol{C}, \theta)$ as an automorphism group, and determines a radiant closed oriented similarity manifold $N^2(\exp \Gamma)$ as the quotient. Note that $N^2(\exp \Gamma_1)$ and $N^2(\exp \Gamma_2)$ are isomorphic if and only if $\Gamma_1 = \Gamma_2$.

EXAMPLE 2.6: Suppose that $q \geqq 3$. Let K be a finite orientation preserving isometry group of the standard sphere S^{q-1} such that the quotient S^{q-1}/K is a manifold. (See Wolf [**9**] for the classification of such K.) One can naturally consider K as a group of similarity transformations of $\boldsymbol{R}^n$. Take an orientation preserving similarity transformation $\gamma: \boldsymbol{R}^q \to \boldsymbol{R}^q$ such that $\gamma(0) = 0$ and $\|\gamma(x)\| < \|x\|$ if $x \neq 0$. (We call such γ *contracting*.) Denote by $\langle \gamma \rangle$ the cyclic group generated by γ. Suppose that $\gamma K \gamma^{-1} = K$, and let G be the group generated by $K \cup \{\gamma\}$ (that is, $G = K \rtimes \langle \gamma \rangle$). Denote by $N^q(K \rtimes \langle \gamma \rangle)$ the radiant closed oriented similarity manifold obtained as the quotient of $\boldsymbol{R}^q - \{0\}$ by G. Note that $N^q(K_1 \rtimes$

$\langle\gamma_1\rangle)$ and $N^q(K_2 \rtimes \langle\gamma_2\rangle)$ are isomorphic if and only if there exists orientation preserving similarity transformation $g: \boldsymbol{R}^q \to \boldsymbol{R}^q$ such that $g(0)=0$ and $gG_1g^{-1}=G_2$, where $G_1=K_1 \rtimes \langle\gamma_1\rangle$ and $G_2=K_2 \rtimes \langle\gamma\rangle$.

Now we can state the classification result.

THEOREM 2.7. (Fried [2]). *Let N be a connected closed oriented similarity manifold.*
(I) *If N is Euclidean, then N is isomorprhic to $N^q(\Gamma)$ for some lattice group Γ of $\boldsymbol{R}^q$.*
(II) *Suppose that N is radiant.*
(i) *If $q=1$, then N is isomorphic to $N^1(r, +)$ or $N^1(r, -)$ for some $r \in]0, 1[$.*
(ii) *If $q=2$, then N is isomorphic to $N^2(\exp \Gamma)$ for some lattice group Γ of $\boldsymbol{C}$.*
(iii) *If $q \geqq 3$, then N is isomorphic to $N^q(K \rtimes \langle\gamma\rangle)$ for some K and γ.*

One can easily classify the automorphisms $g: N \to N$. But we omit the details.

§ 3. Examples of nonsingular flows with transverse similarity structures

In this and the next sections, we are going to classify nonsingular flows $\phi: M \times \boldsymbol{R} \to M$ with transverse similarity structures under certain conditions. We treat only oriented underlying manifolds M. Hence such flows have the natural transverse orientation. We begin by describing the isomorphism between such flows.

DEFINITION 3.1: Let M_1 and M_2 be closed oriented manifolds of dimension $n \geqq 2$ and ϕ_1 and ϕ_2 nonsingular flows with transverse similarity structures. We say that ϕ_1 is *isomorphic* to ϕ_2 if there exists an orientation preserving diffeomorphism $f: M_1 \to M_2$ such that
(1) for all $x \in M_1$, there exists an orientation preserving homeomorphism $\alpha: \boldsymbol{R} \to \boldsymbol{R}$ with

$$f \circ \phi_1(x, t) = \phi_2(f(x), \alpha(t)) \quad \text{for all } t \in \boldsymbol{R},$$

(2) if U is an open subset of M_2 and a submersion $h: U \to \boldsymbol{R}^{n-1}$ is compatible with the transverse similarity structure of ϕ_2, then $h \circ f: f^{-1}(U) \to \boldsymbol{R}^{n-1}$ is compatible with the transverse similarity structure of ϕ_1.
We call such $f: M_1 \to M_2$ an *isomorphism* between ϕ_1 and ϕ_2.

Our intention is the classification of nonsingular flows with transverse similarity structures up to the above isomorphisms. Now we give examples.

EXAMPLE 3.2: *Suspension flows.* Let N be a closed oriented similarity manifold of dimension $n-1$ and $g: N \to N$ an automorphism of N. Consider a $\boldsymbol{Z}$-action on $N \times \boldsymbol{R}$ defined by

$$n \cdot (x, t) = (g^n(x), t-n) \quad \text{for } n \in \boldsymbol{Z} \text{ and } (x, t) \in N \times \boldsymbol{R}.$$

Since the vector field $\partial/\partial t$ on $N \times \boldsymbol{R}$ is preserved by this action, it induces a vector field X on the quotient manifold M of $N \times \boldsymbol{R}$ by this action. Furthermore this action preserves the transverse similarity structure of $\partial/\partial t$ induced from the similarity structure of N. Hence the vector field X generates a nonsingular flow $\phi: M \times \boldsymbol{R} \to M$ with a natural transverse similarity structure. We call $\phi_{(N,g)} := \phi$ the *suspension flow* of N by g. Note that $\phi_{(N_1,g_1)}$ and $\phi_{(N_2,g_2)}$ are isomorphic if and only if there exists an isomorphism $h: N_1 \to N_2$ such that the diagram

$$\begin{array}{ccc} N_1 & \xrightarrow{g_1} & N_1 \\ h \downarrow & & \downarrow h \\ N_2 & \xrightarrow[g_2]{} & N_2 \end{array}$$

commutes.

EXAMPLE 3.3: *Circle bundle flows.* Let N be a closed oriented similarity manifold, and $\xi = (M, \pi, N)$ an oriented circle bundle. Let $\phi: M \times \boldsymbol{R} \to M$ be a nonsingular flow such that the orbits of ϕ are the fibers of ξ, and the natural orientation of an orbit of ϕ coincides with that as a fiber of ξ. Then the similarity structure of N determines a transverse similarity structure of ϕ. We call $\phi_\xi := \phi$ a *circle bundle flow* over N. Note that the circle bundle flows ϕ_{ξ_1} and ϕ_{ξ_2} are isomorphic if and only if there exists an orientation preserving bundle isomorphism

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & M_2 \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ N_1 & \xrightarrow[h]{} & N_2 \end{array}$$

such that $h: N_1 \to N_2$ is an isomorphism. If ξ is a trivial bundle, then the circle bundle flow ϕ_ξ is isomorphic to the suspension flow $\phi_{(N,\mathrm{id})}$.

We can generalize the circle bundle flows as follows. Let N be a closed oriented similarity manifold of dimension 2. Then N is diffeomorphic to T^2 and any Seifert bundle M over N has a nonsingular flow $\phi: M\times\boldsymbol{R}\rightarrow M$ with a transverse similarity structure in the similar way as circle bundles.

EXAMPLE 3.4: *Contraction flows*. For $n\geqq 2$, take an orientation preserving contracting similarity automorphism $g:\boldsymbol{R}^{n-1}\rightarrow\boldsymbol{R}^{n-1}$ with $g(0)$, and define a map $f:\boldsymbol{R}^n\rightarrow\boldsymbol{R}^n$ by

$$f(x, t)=(g(x), rt) \qquad \text{for } x\in\boldsymbol{R}^{n-1} \text{ and } t\in\boldsymbol{R},$$

where $r>0$ means the similitude ratio of g. Then f is an orientation preserving similarity transformation of $\boldsymbol{R}^n$. Consider a vector field $\bar{X}$ on $\bar{M}:=\boldsymbol{R}^n-\{0\}$ defined by

$$\bar{X}(x, t)=\|(x, t)\|\cdot\frac{\partial}{\partial t} \qquad \text{for } (x, t)\in\boldsymbol{R}^{n-1}\times\boldsymbol{R}-\{(0, 0)\}=\bar{M}.$$

Note that $\bar{X}$ is a nonsingular vector field with the transverse similarity structure induced from the canonical similarity structure of $\boldsymbol{R}^{n-1}$ by the submersion $h:\bar{M}\rightarrow\boldsymbol{R}^{n-1}$ defined by

$$h(x, t)=x \qquad \text{for } (x, t)\in\boldsymbol{R}^{n-1}\times\boldsymbol{R}-\{(0, 0)\}=\bar{M}.$$

Since $f|\bar{M}$ is an automorphism of $\bar{X}$, we have a vector field X on the quotient manifold M of $\bar{M}$ by the cyclic group generated by $f|\bar{M}$. Then X generates a nonsingular flow $\phi: M\times\boldsymbol{R}\rightarrow M$ with the transverse similarity structure induced from that of $\bar{X}$. We call $\phi_g:=\phi$ a *contraction flow*. Clearly ϕ_g has exactly two closed orbits and M is diffeomorphic to $S^1\times S^{n-1}$. The subset $\bar{N}:=(\boldsymbol{R}^{n-1}-\{0\})\times\{0\}$ of $\bar{M}$ is invariant by $f|\bar{M}$, and submersed onto a submanifold N of M. Since N is transverse to ϕ_g, it is a closed similarity manifold. If $n=3$, then N is isomorphic to $N^2(\exp\Gamma)$, where $\Gamma=\{pz+q\cdot 2\pi\sqrt{-1}: p, q\in\boldsymbol{Z}\}$ and z is any complex number such that

$$g(x)=(\exp z)\cdot x \qquad \text{for } x\in\boldsymbol{C}=\boldsymbol{R}^2.$$

If $n\geqq 4$, then N is isomorphic to $N^{n-1}(\{\mathrm{id}\}\rtimes\langle g\rangle)$. Note that ϕ_{g_1} and ϕ_{g_2} are isomorphic if and only if there exists an orientation preserving similarity automorphism $h:\boldsymbol{R}^{n-1}\rightarrow\boldsymbol{R}^{n-1}$ such that $h\circ g_1\circ h^{-1}=g_2$.

EXAMPLE 3.5: *Generalized contraction flows*. We consider the case $n=2$ in Example 3.4. Then there exists uniquely $r\in]0, 1[$ such that $g(x)=rx$ for all $x\in\boldsymbol{R}$, that is, $g=m_r$. We see that N is isomorphic to the disjoint union $N^1(r, +)\cup N^1(r, -)$. Take $\nu\in\boldsymbol{N}$ and consider the ν-fold covering $\pi:\hat{M}\rightarrow M$ such that $\pi^{-1}(N)$ has 2ν connected components. The lift $\phi_{(r;\nu)}$ of ϕ_g by π is naturally a nonsingular flow with a transverse similarity structure. We call $\phi_{(r;\nu)}$ a *generalized contraction flow*. Note that $\phi_{(r;\nu)}$ has exactly 2ν closed orbits and M is diffeomorphic to T^2. Clearly $\phi_{(r;\nu)}$ and $\phi_{(s;\mu)}$ are isomorphic if and only if $r=s$ and $\nu=\mu$.

§ 4. Statement of the results

In this section, we give a classification of nonsingular C^∞ flows with transverse similarity structures satisfying certain conditions. We begin by giving definitions, which are needed in order to describe the conditions. Let M be a closed oriented manifold of dimension $\geqq 2$ and $\phi: M\times \boldsymbol{R}\to M$ a nonsingular flow with a transverse similarity structure.

DEFINITION 4.1: A submanifold N of M is a *closed transversal* to ϕ if N is a codimension one closed submanifold and N is transverse to ϕ.

REMARK 4.2: Since a closed transversal N to ϕ has the canonical similarity structure induced from the transverse similarity structure of ϕ, we can find N in the classification table of the closed similarity manifolds (due to Fried [2]) described in § 2. This is the starting point of our research.

REMARK 4.3: All the suspension flows $\phi_{(N,g)}$ in Example 3.2, all the contraction flows ϕ_g in Example 3.4, and all the generalized contraction flows $\phi_{(r;\nu)}$ in Example 3.5 have closed transversals. On the other hand, some circle bundle flows ϕ_ξ in Example 3.3 have no closed transversal. For example, take a circle bundle $\xi=(M, \pi, N_0)$ such that N_0 is a closed oriented similarity manifold of dimension 2 (which implies that N_0 is diffeomorphic to the 2-torus T^2), and the Euler number $\mathrm{eu}(\xi)$ of ξ is not zero. Then the circle bundle flow ϕ_ξ has no closed transversal. For, otherwise, the restriction $\pi|_N: N\to N_0$ is a covering map, where N is a closed transversal to ϕ. Since the induced bundle $(\pi|_N)^*\xi$ has a section, its Euler number $\mathrm{eu}((\pi|_N)^*\xi)$ is zero. This implies that $\mathrm{eu}(\xi)=0$. A contradiction.

DEFINITION 4.4: Suppose that ϕ has a closed transversal N. We say that the pair (ϕ, N) has the *lifting property* if, for any continuous map $c:[a,b]\to N$, any number $\tau\in\boldsymbol{R}-\{0\}$ and any compact codimension one submanifold T of M such that $\phi(c(a),\tau)\in \mathrm{Int}\ T$ and T is transverse to ϕ, there exists $b^*\in[a,b]$ and a continuous map $H:[a,b^*]\times J\to M$ (where $J=[0,\tau]$ if $\tau>0$ and $J=[\tau,0]$ if $\tau<0$) such that

(1) $H(s,0)=c(s)$ for all $s\in[a,b^*]$,

(2) $H(a,t)=\phi(c(0),t)$ for all $t\in J$,

(3) $H(\{s\}\times J)\subset\phi(\{c(s)\}\times\boldsymbol{R})$ for all $s\in[a, b^*]$,
(4) $H([a, b^*]\times\{\tau\})\subset T$,
(5) if $b^*<b$, then $H(b^*, \tau)\in\partial T$.

We call H a *lift-homotopy* of $\phi|_{\{c(a)\}\times[0, r]}$ along c and T.

Now we can state the result of classification, which is the main result of this paper.

THEOREM 4.5. *Let M be a closed oriented C^∞ manifold of dimension $n\geqq 2$ and ϕ a nonsingular C^∞ flow with a transverse structure. Suppose that ϕ has a closed transversal N and the pair (ϕ, N) has the lifting property. Then ϕ is isomorphic to (1) the suspension flow $\phi_{(N,g)}$ for some closed similarity manifold N and some C^∞ automorphism g of N, (2) the contraction flow ϕ_g for some contracting similarity transformation $g:\boldsymbol{R}^{n-1}\to\boldsymbol{R}^{n-1}$, or (3) the generalized contraction flow $\phi_{(r;\nu)}$ for some $r>0$ and $\nu\in N$(in the case $n=2$).*

REMARK 4.6: When $n=3$, the concept of a transverse similarity structure to a nonsingular flow coincides with that of a transverse complex affine structure. Ghys [3] treated this case, independently and without our auxiliary conditions.

§5. The proof of Theorem 4.5

Let M be a connected closed oriented manifold of dimension $n\geqq 2$ and $\phi: M\times\boldsymbol{R}\to M$ a nonsingular flow with a transverse similarity structure. Suppose that ϕ has a closed transversal N and the pair (ϕ, N) has the lifting property. We may suppose that N is connected.

Here we recall the usual notations. Let $x\in M$. We call $O(x):=\{\phi(x, t):t\in\boldsymbol{R}\}$ the *orbit* of x and $O^+(x):=\{\phi(x, t): t>0\}$ (respectively $O^-(x):=\{\phi(x, t): t<0\}$) the *positive* (respectively *negative*) *semi-orbit* of x. We call $L^+(x):=\cap_{\tau>0}\overline{\{\phi(x, t): t\geqq\tau\}}$ (respectively $L^-(x):=\cap_{\tau<0}\overline{\{\phi(x, t): t\leqq\tau\}}$ the *ω-limit set* (respectively *α-limit set*) of x. For a subset A of M, we call $\mathrm{Sat}(A):=\{\phi(x, t): x\in A, t\in\boldsymbol{R}\}$ the *saturation* of A. It is well known that $L^+(x)$, $L^-(x)$ and $\mathrm{Sat}(A)$ are *invariant* (that is, the union of a family of orbits of ϕ).

Now we divide the situation into the following two cases:
Case I. $O^+(x)\cap N\neq\emptyset$ for all $x\in N$.
Case II. $O^+(x_0)\cap N=\emptyset$ for some $x_0\in N$.
First consider Case I. For $x\in N$, let $g(x)$ be the first intersecting point of $O^+(x)$ with N. Then the obtained map $g:N\to N$ is an immer-

sion. Hence the image $g(N)$ is a compact open subset of N. Since N is Hausdorff and connected, it follows that $g(N)=N$, which implies that Sat $(N)=M$. Furthermore we see that $g:N\to N$ is an automorphism of the closed oriented similarity manifold N, and that ϕ is isomorphic to the suspension flow $\phi_{(N,g)}$.

Hereafter we consider Case II. Note that the ω-limit set $L^+(x_0)$ of x_0 is compact, connected, nonempty and invariant, and that the saturation $\mathrm{Sat}(N)$ of N is an open subset of M.

LEMMA 5.1. $L^+(x_0)\subset\partial\,\mathrm{Sat}(N):=\overline{\mathrm{Sat}(N)}-\mathrm{Sat}(N)$

PROOF: It is clear that $L^+(x_0)\subset\overline{\mathrm{Sat}(N)}$. Suppose that there exists a point $y\in L^+(x_0)\cap\mathrm{Sat}(N)$. Since $y\in\mathrm{Sat}(N)$, there exists $x\in N$ and $t\in\boldsymbol{R}$ such that $y=\phi(x,t)$. Since $L^+(x_0)$ is invariant and $y\in L^+(x_0)$, it follows that $x\in L^+(x_0)$. Hence the positive semi-orbit $O^+(x_0)$ must intersect N infinitely many times, which contradicts the assumption $O^+(x_0)\cap N=\emptyset$. Hence $L^+(x_0)\cap\mathrm{Sat}(N)=\emptyset$. $\square$

Let $\iota:N\to M$ be the inclusion map, and $\pi:\tilde{M}\to M$ and $\pi_N:\tilde{N}\to N$ the universal covering maps. We can construct a developing map $D:\tilde{M}\to\boldsymbol{R}^{n-1}$ of the nonsingular flow ϕ with the transverse similarity structure in a natural way. We have the following diagram:

$$\begin{array}{ccccc} \tilde{N} & & \tilde{M} & \xrightarrow{\ D\ } & \boldsymbol{R}^{n-1} \\ {\scriptstyle\pi_N}\downarrow & & {\scriptstyle\pi}\downarrow & & \\ N & \xrightarrow[\ \iota\ ]{} & M & & \end{array}$$

We are going to take three compact disks $T\subset M$, $\tilde{T}\subset\tilde{M}$ and $\hat{T}\subset\boldsymbol{R}^{n-1}$, and six points $y,z\in\mathrm{Int}\,T$, $\tilde{y},\tilde{z}\in\mathrm{Int}\,\tilde{T}$ and $\hat{y},\hat{z}\in\mathrm{Int}\,\hat{T}$. First take a point $z\in L^+(x_0)\subset M$. Choose a point $\tilde{z}\in\pi^{-1}(z)\subset\tilde{M}$ and put $\hat{z}=D(\tilde{z})\in\boldsymbol{R}^{n-1}$. Next take a sufficiently small compact $(n-1)$-disk $\tilde{T}$ in $\tilde{M}$ such that $\tilde{z}\in\mathrm{Int}\,\tilde{T}$ and $\tilde{T}$ is transverse to the lift $\tilde{\phi}:\tilde{M}\times\boldsymbol{R}\to\tilde{M}$ of ϕ. Put $T=\pi(\tilde{T})$ $(\subset M)$ and $\hat{T}=D(\tilde{T})$ $(\subset\boldsymbol{R}^{n-1})$. We may suppose that $\pi|_{\tilde{T}}:\tilde{T}\to T$ and $D|_{\tilde{T}}:\tilde{T}\to\hat{T}$ are diffeomorphisms. Then T and $\hat{T}$ are compact disks. Furthermore $z\in\mathrm{Int}\,T$ and $\hat{z}\in\mathrm{Int}\,\hat{T}$. Last we take the points y, $\tilde{y}$ and $\hat{y}$ as follows. Since $z\in L^+(x_0)$, there exists $\tau>0$ with $y:=\phi(x_0,\tau)\in\mathrm{Int}\,T$. Put $\tilde{y}=(\pi|_{\tilde{T}})^{-1}(y)\in\mathrm{Int}\,\tilde{T}$ and $\hat{y}=D(\tilde{y})\in\mathrm{Int}\,\hat{T}$.

Define a curve $\omega:[0,\tau]\to M$ by $\omega(t)=\phi(x_0,t)$ for all $t\in[0,\tau]$, and let $\tilde{\omega}:[0,\tau]\to\tilde{M}$ be the lift of ω with $\tilde{\omega}(\tau)=\tilde{y}$. Put $\tilde{x}_0=\tilde{\omega}(0)$. Note that $D(\tilde{x}_0)=D(\tilde{y})=\hat{y}$ because the points $\tilde{x}_0$ and $\tilde{y}$ are on the same orbit of the flow ϕ. Since $\tilde{x}_0\in\pi^{-1}(x_0)$, there exists an immersion $\tilde{\iota}:\tilde{N}\to\tilde{M}$ such that

$\tilde{x}_0 \in \tilde{\iota}(\tilde{N})$ and the diagram

$$\begin{array}{ccc} \tilde{N} & \xrightarrow{\tilde{\iota}} & \tilde{M} \\ {\scriptstyle \pi_N}\downarrow & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow[\iota]{} & M \end{array}$$

commutes. Choose a point $\tilde{x}_N \in \tilde{\iota}^{-1}(\tilde{x}_0)$.

Since the similarity structure of N comes from the transverse similarity structure of ϕ, the composition $D_N := D\tilde{\iota} : \tilde{N} \to \boldsymbol{R}^{n-1}$ is a developing map of N. By Theorem 2.1, the map $D_N : \tilde{N} \to D_N(\tilde{N})$ is a covering map and the image $D_N(\tilde{N})$ is one of the following :

(a) the whole $\boldsymbol{R}^{n-1}$,

(b) $\boldsymbol{R}^{n-1}$ deleted a point ($n-1 \geqq 2$),

(c) a connected component of $\boldsymbol{R}$ deleted a point.

This implies that the intersection Int $\hat{T} \cap D_N(\tilde{N})$ is diffeomorphic to (a) an open disk, (b) a punctured open disk of dimension $n-1 \geqq 2$, or (c) an open interval. Note that Int $\hat{T} \cap D_N(\tilde{N})$ is path-connected.

LEMMA 5.2. $\hat{z} = D(\tilde{z}) \notin D_N(\tilde{N})$.

PROOF : We suppose that $\hat{z} \in D_N(\tilde{N})$ and will bring out a contradiction. Since Int $\hat{T} \cap D_N(\tilde{N})$ is path-connected and the points $\hat{y}$ and $\hat{z}$ belong to Int $\hat{T} \cap D_N(\tilde{N})$, we can take a curve $\hat{c} : [0,1] \to \text{Int } \hat{T} \cap D_N(\tilde{N})$ such that $\hat{c}(0) = \hat{y}$ and $\hat{c}(1) = \hat{z}$. Since $D_N : \tilde{N} \to D_N(\tilde{N})$ is a covering map and

$$D_N(\tilde{x}_N) = D\tilde{\iota}(\tilde{x}_N) = D(\tilde{x}_0) = D(\tilde{y}) = \hat{y} = \hat{c}(0),$$

there exists uniquely a continutus map $\tilde{c}_N : [0,1] \to \tilde{N}$ such that $D_N \tilde{c}_N = \hat{c}$ and $\tilde{c}_N(0) = \tilde{x}_N$. Put $c = \pi_N \tilde{c}_N : [0,1] \to N$. Then it follows that

$$c(0) = \pi_N \tilde{c}_N(0) = \pi_N(\tilde{x}_N) = \pi \tilde{\iota}(\tilde{x}_N) = \pi(\tilde{x}_0) = x_0 = \phi(x_0, 0) = \omega(0).$$

Since the pair (ϕ, N) has the lifting property, there exists $b^* \in]0,1]$ and a lift-homotopy $H : [0, b^*] \times [0, \tau] \to M$ of $\phi|_{\{x_0\} \times [0, \tau]}$ along c and T. Then $H(0, \tau) = \omega(\tau) = \phi(x_0, \tau) = y$. Let $\tilde{H} : [0, b^*] \times [0, \tau] \to \tilde{M}$ be the lift of H with $\tilde{H}(0, \tau) = \tilde{y}$.

Since $\tilde{H}(0, t) = \tilde{\omega}(\tau)$ and for all $t \in [0, \tau]$

$$\pi \tilde{H}(0, t) = H(0, t) = \omega(t) = \pi \tilde{\omega}(t),$$

the unique path lifting property of covering maps implies that $\tilde{H}(0, t) = \tilde{\omega}(t)$ for all $t \in [0, \tau]$. Hence $\tilde{H}(0, 0) = \tilde{\omega}(0) = \tilde{x}_0 = \tilde{\iota} \tilde{c}_N(0)$. Next note that

$\pi\tilde{H}(s,0)=H(s,0)=c(s)=\pi_N\tilde{c}_N(s)=\pi\tilde{\iota}\tilde{c}N(s)$ for all $s\in[0,b^*]$. By the unique path lifting property, it follows that $\tilde{H}(s,0)=\tilde{\iota}\tilde{c}_N(s)$ for all $s\in[0,b^*]$. Note that $\tilde{H}(\{s\}\times[0,\tau])$ is contained in some orbit of $\tilde{\phi}$. This implies that

$$D\tilde{H}(s,\tau)=D\tilde{H}(s,0)=D\tilde{\iota}\tilde{c}_N(s)=D_N\tilde{c}_N(s)=\hat{c}(s)\in\text{Int }\hat{T}.$$

Hence $\tilde{H}(s,\tau)\in$ Int $\tilde{T}$ and $H(s,\tau)\in$ Int T for all $s\in[0,b^*]$. Therefore $b^*=1$ and $H(1,\tau)=z$. It follows that $z=H(1,\tau)\in O^+(c(1))\subset\text{Sat}(N)$, which contradicts Lemma 5.1. □

By Lemma 5.2 and Theorem 2.1, we conclude that N is radiant, and that if $n-1\geqq 2$ then $D_N(\tilde{N})=\boldsymbol{R}^{n-1}-\{\hat{z}\}$, and if $n-1=1$ then $D_N(\tilde{N})=]-\infty,\hat{z}[$ or $]\hat{z},+\infty[$.

LEMMA 5.3. *If* $n-1\geqq 2$, *then* $\text{Sat}(N)\cap T=T-\{z\}$. *If* $n-1=1$, *then* $\text{Sat}(N)\cap T$ *is a connected component of* $T-\{z\}$.

PROOF: We treat only the case $n-1\geqq 2$ and omit the proof for the case $n-1=1$. It is clear that $\text{Sat}(N)\cap T\subset T-\{z\}$. Conversely take $w\in T-\{z\}$ and put $\tilde{w}=(\pi|_{\tilde{T}})^{-1}(w)\in\tilde{T}$ and $\hat{w}=D(\tilde{w})\in\hat{T}$. Choose a curve $\hat{c}:[0,1]\to\text{Int }\hat{T}-\{\hat{z}\}$ such that $\hat{c}(0)=\hat{y}$ and $\hat{c}(1)=\hat{w}$. By the similar argument as in Lemma 5.2, we see that $w\in\text{Sat}(N)$. Therefore $T-\{z\}\subset\text{Sat}(N)$. □

By Lemma 5.3, we see that the orbit $O(z)$ of z is proper, which implies that $L^+(z)\subset\overline{O(z)}-O(z)$ if $O(z)$ is not compact. Furthermore, we have the following.

LEMMA 5.4. *The orbit* $O(z)$ *is a closed orbit.*

PROOF: Consider the case $n-1\geqq 2$. Suppose that $O(z)$ is not a closed orbit. Then there exists a point $z_1\in L^+(z)(\subset\overline{O(z)}-O(z)\subset L^+(x_0))$. Take a sufficiently small $(n-1)$-disk T_1 in M such that $z_1\in\text{Int }T_1$ and T_1 is transverse to ϕ. Since Lemma 5.3 is valid for z_1, it follows that $\text{Sat}(N)\cap T_1=T_1-\{z_1\}$. This implies that $O(z)$ cannot approach z_1, a contradiction. This argument is valid for the case $n-1=1$, too. □

LEMMA 5.5. $L^+(x_0)=O(z)$.

PROOF: By Lemmas 5.3 and 5.4, we can conclude that $L^+(x_0)$ is the union of some isolated closed orbits. Since $L^+(x_0)$ is connected, it follows that $L^+(x_0)=O(z)$. □

Now consider the holonomy g along the closed orbit $O(z)$. Since $L^+(x_0)=O(z)$, the similarity transformation g of $\boldsymbol{R}^{n-1}$ must be contracting.

We can take a compact tubular neighborhood W of $O(z)$ such that ∂W is transverse to ϕ. Then the closed oriented similarity manifold ∂W is isomorphic to $N(g):=(\boldsymbol{R}^{n-1}-\{0\})/\langle g\rangle$, where $\langle g\rangle$ is the cyclic group generated by g. Note that if $n-1=1$ then $N(g)=N^{-1}(r,-)\cup N^{1}(r,+)$ (where $r\in]0,1[$ is the similitude ratio of g), that if $n-1=2$ then $N(g)=N^{2}(\exp\Gamma)$ (where we take $\alpha_0\in\boldsymbol{C}$ with $g(x)=\exp(\alpha_0)x$ and let $\Gamma=\{\mu\cdot\alpha_0+\nu\cdot 2\pi\sqrt{-1}:\mu,\nu\in\boldsymbol{Z}\}$), and that if $n-1\geqq 3$ then $N(g)=N^{n-1}\{\mathrm{id}\}\rtimes\langle g\rangle$).

LEMMA 5.6. *For all $x\in N$, the positive semi-orbit $O^{+}(x)$ intersects ∂W exactly once.*

PROOF: Clearly any orbit of ϕ intersects ∂W at most once. Since $L^{+}(x_0)=O(z)$, the positive semi-orbit $O^{+}(x_0)$ intersects ∂W. We may suppose that $y=\phi(x_0,\tau)\in\partial W$. Take a point $x\in N$ arbitrarily and consider a curve $c:[0,1]\to N$ such that $c(0)=x_0$ and $c(1)=x$. By the lifting property of (ϕ, N), we can lift $\phi|_{\{x_0\}\times[0,\tau]}$ along c and ∂W. This implies that $O^{+}(x)\cap\partial W\neq\emptyset$. □

For $x\in N$, denote by $f(x)$ the intersecting point of $O^{+}(x)$ and ∂W. Since the subset $f(N):=\{f(x):x\in N\}$ of ∂W is connected, open and closed, it is one of the connected component of ∂W. If $n-1\geqq 2$, then $f(N)=\partial W$. Furthermore the obtained map $f:N\to f(N)(\subset\partial W)$ is an isomorphism between the closed oriented similarity manifolds N and $f(N)$. Therefore N is isomorphic to $N(g)$.

Now repeat the same argument as above for the negative side of N. Then $L^{-}(x_0)$ is a closed orbit of ϕ and its holonomy h is expanding in this case. Furthermore the contracting similarity transformation $h^{-1}:\boldsymbol{R}^{n-1}\to\boldsymbol{R}^{n-1}$ determines the transverse similarity structure of the boundary ∂V of a small compact tubular neighborhood V of $L^{-}(x_0)$. Since one of the connected components of ∂V is isomorphic to N, it follows that $h^{-1}=g$. If $n-1\geqq 2$, then we have the following decomposition of M:

$$M=V\cup(N\times[-1,1])\cup W.$$

By using this decomposition, we see that ϕ is isomorphic to the contraction flow ϕ_g.

If $n-1=1$, then we see that ϕ has exactly an even number of closed orbits, whose holonomies are g or g^{-1}. Furthermore we have the similar decomposition as in the case $n-1\geqq 2$. Thus we see that ϕ is isomorphic to the generalized contraction flow $\phi_{(r;\nu)}$ for some $r>0$ and $\nu\in\boldsymbol{Z}$. This completes the proof of Theorem 4.5.

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