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An F. and M. Riesz theorem on locally compact transformation groups

Hiroshi YAMAGUCHI

Dedicated to Professor TSUYOSHI ANDO on his sixtieth birthday

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§ 1. Introduction.

Helson and Lowdenslager extended the classical F. and M. Riesz theorem as follows.

THEOREM A (cf. [12, 8.2.3. Theorem]). *Let G be a compact abelian group with ordered dual, i. e., there exsits a semigroup P in $\hat{G}$ such that (i) $P\cup(-P)=\hat{G}$ and (ii) $P\cap(-P)=\{0\}$. Let μ be a measure in $M(G)$ such that $\hat{\mu}(\gamma)=0$ for $\gamma<0$. Then*

(I) $\hat{\mu}_a(\gamma)=\hat{\mu}_s(\gamma)=0$ *for* $\gamma<0$;

(II) $\hat{\mu}_s(0)=0$.

Theorem A (I) was extended, by the author ([13]) and Hewitt-Koshi-Takahashi ([7]), to LCA groups as follows.

THEOREM B (cf. [13, Corollary], [7, Theorem D]). *Let G be a LCA group and P a semigroup in $\hat{G}$ such that $P\cup(-P)=\hat{G}$. Let μ be a measure in $M_p(G)$, where $M_p(G)=\{\nu\in M(G):\hat{\nu}=0$ on $P^c\}$. Then μ_a and μ_s also belong to $M_p(G)$.*

In Theorem B we can not expect "$\hat{\mu}_s(0)=0$" in general. As pointed out in the proof of [13, Corollary], Theorem B follows from the following theorem.

THEOREM C (cf. [13, Main Theorem]). *Let G be a LCA group and P a closed semigroup in $\hat{G}$ such that $P\cup(-P)=\hat{G}$. Let μ be a measure in $M_{p^c}(G)$. Then μ_a and μ_s also belong to $M_{p^c}(G)$.*

On the other hand, Forelli obtained the following theorem ([5]).

THEOREM D (cf. [5, Theorem 5]). *Let $(\boldsymbol{R}, X)$ be a (topological) transformation group, in which the reals $\boldsymbol{R}$ acts on a locally compact Hausdorff space X. Let σ be a positive Radon measure on X that is quasi-invariant. Let $\mu\in M(X)$, and let $\mu=\mu_a+\mu_s$ be the Lebesgue decomposi-*

tion of μ *with respect to* σ. *Suppose* μ *is an analytic measure, i. e., the spectrum of* μ *is in* $[0, \infty)$. *Then* μ_a *and* μ_s *are also analytic measures.*

We need a comment on Theorem D. Analytic measures treated in [5, Theorem 5] are bounded complex Baire measures. But every bounded complex Baire measures on a locally compact Hausdorff space is uniquely extended to a bounded complex, regular Borel measure. Thus Theorem D follows from [5, Theorem 5].

Moreover the author extended Theorem A (I) to a compact transformation group as follows.

THEOREM E (cf. [15, Theorem 2. 1]). *Let* (G, X) *be a transformation group, in which a compact abelian group* G *acts on a locally compact Hausdorff space* X. *Let* P *be a semigroup in* $\hat{G}$ *such that* $P \cup (-P) = \hat{G}$. *Let* σ *be a positive Radon measure on* X *that is quasi-invariant. Let* μ *be a measure in* $M(X)$, *and let* $\mu = \mu_a + \mu_s$ *be the Lebesgue decomposition of* μ *with respect to* σ. *Suppose* $sp(\mu) \subset P$. *Then* $sp(\mu_a)$ *and* $sp(\mu_s)$ *are also contained in* P.

In this paper, we shall extend Theorems D and E to a transformation group, in which a locally compact abelian (LCA) group acts on a locally compact Hausdorff space. In section 2, we state definitions and our theorems. In section 3, we give some results on a compact transformation group, and we give the proofs of our theorems in section 4. We also show that Theorem C follows from our theorems in section 4 (Remark 4. 2).

§ 2. Notations and results.

Let (G, X) be a (topological) transformation group, in which G is a LCA group and X is a locally compact Hausdorff space. Suppose that the action of G on X is given by $(g, x) \to g \cdot x$, where $g \in G$ and $x \in X$. Let $C_0(X)$ and $C_c(X)$ be the Banach space of continuous functions on X which vanish at infinity and the space of continuous functions on X with compact supports respectively. Let $M(X)$ be the Banach space of bounded regular Borel measures on X with the total variation norm. Let $M^+(X)$ be the set of nonnegative measures in $M(X)$. For $\mu \in M(X)$ and $f \in L^1(|\mu|)$, we often write $\mu(f) = \int_X f(x) d\mu(x)$. Let X' be another locally compact Hausdorff space, and let $S: X \to X'$ be a continuous map. For $\mu \in M(X)$, let $S(\mu) \in M(X')$ be the continuous image of μ under S. We denote by $\mathscr{B}(X)$ the σ-algebra of Borel sets in X. $\mathscr{B}_0(X)$ stands for the

σ-algebra of Baire sets in X. That is, $\mathscr{B}_0(X)$ is the σ-algebra generated by compact G_δ-sets in X. A (Borel) measure σ on X is called quasi-invariant if $|\sigma|(F)=0$ implies $|\sigma|(g\cdot F)=0$ for all $g\in G$.

Let $\hat{G}$ be the dual group of G. $M(G)$ and $L^1(G)$ denote the measure algebra and the group algebra respectively. For $\lambda\in M(G)$, $\hat{\lambda}$ denotes the Fourier-Stieltjes transform of λ, i.e., $\hat{\lambda}(\gamma)=\int_G (-x, \gamma)d\lambda(x)$. m_G stands for the Haar measure of G. Let $M_a(G)$ be the set of measures in $M(G)$ which are absolutely continuous with respect to m_G. Then by the Radon-Nikodym theorem we can identify $M_a(G)$ with $L^1(G)$. For a subset E of $\hat{G}$, $M_E(G)$ denotes the space of measures in $M(G)$ whose Fourier-stieltjes transforms vanish off E. E^- denotes the closure of E. A closed subset E of $\hat{G}$ is called a Riesz set if $M_E(G)\subset L^1(G)$. For a closed subgroup H of G, $H^\perp$ denotes the annihilator of H.

Let f be a Borel measurable function on X. Then

(2. 1) $(g, x)\to f(g\cdot x)$ is a Borel measurable function on $G\times X$.

For $\lambda\in M(G)$ and $\mu\in M(X)$, we can define $\lambda*\mu\in M(X)$, by virtue of [4, (7. 23) Lemma and (7. 27) Theorem], as follows.

(2. 2) $$\lambda*\mu(f)=\int_X\int_G f(g\cdot x)d\lambda(g)d\mu(x)=\int_G\int_X f(g\cdot x)d\mu(x)d\lambda(g)$$

for $f\in C_0(X)$.

REMARK 2. 1. (2. 2) holds for all bounded Borel functions f on X. For ξ, $\lambda\in M(G)$ and $\mu\in M(X)$, the following hold.

(2. 3) $\|\lambda*\mu\|\le\|\lambda\|\,\|\mu\|$,

(2. 4) $\xi*(\lambda*\mu)=(\xi*\lambda)*\mu$.

For a closed subgroup H of G, let $J(\mu:H)=\{k\in L^1(H): k*\mu=0\}$. Set $J(\mu)=\{h\in L^1(G): h*\mu=0\}$ $(=J(\mu:G))$. Then, by (2. 3) and (2. 4), $J(\mu:H)$ and $J(\mu)$ are closed ideals in $L^1(H)$ and $L^1(G)$ respectively.

DEFINITION 2. 1. For $\mu\in M(X)$, define the spectrum of μ by $\mathrm{sp}(\mu)=\bigcap_{h\in J(\mu)}\hat{h}^{-1}(0)$. For a closed subgroup H of G, we also define $\mathrm{sp}_H(\mu)$ by $\bigcap_{k\in J(\mu:H)}\hat{k}^{-1}(0)$.

For μ, $\nu\in M(X)$, it follows from (2. 4) and the definition of spectrum that $\mathrm{sp}(\mu+\nu)\subset\mathrm{sp}(\mu)\cup\mathrm{sp}(\nu)$ (cf. [5, Lemma 3]). Now we state our theorems.

THEOREM 2. 1. *Let (G, X) be a tmansformation group, in which G is* *a LCA group and X is a locally compact Hausdorff space. Let P be a closed semigroup in $\hat{G}$ such that $P\cup(-P)=\hat{G}$. Let σ be a positive Radon measure on X that is quasi-invariant. Let $\mu\in M(X)$, and let $\mu=\mu_a+\mu_s$ be the Lebesgue decomposition of μ with respect to σ. Suppose $sp(\mu)\subset P$. Then both $sp(\mu_a)$ and $sp(\mu_s)$ are also contained in P.*

THEOREM 2.2. *Let (G, X) and σ be as in Theorem 2.1. Let P be a proper closed semigroup in $\hat{G}$ such that $P\cup(-P)=\hat{G}$, and suppose that $P\cap(-P)$ is open. Let $\mu\in M(X)$, and suppose that $sp(\mu)\subset P\backslash(-P)$. Then both $sp(\mu_a)$ and $sp(\mu_s)$ are also contained in $P\backslash(-P)$.*

Before closing this section, we state several lemmas which we shall need later on. The following lemma is well-known.

LEMMA 2.1. *For $f\in C_0(X)$ and $g\in G$, define $f_g\in C_0(X)$ by $f_g(x)=f(g\cdot x)$. Then $\lim_{g\to 0}\|f-f_g\|_\infty=0$.*

LEMMA 2.2. *Let $\mu\in M(X)$, and suppose that $k*\mu=0$ for all $k\in L^1(G)$. Then $\mu=0$.*

PROOF. Suppose $\mu\neq 0$. Then there exists $f\in C_0(X)$ such that $\int_X f(x)d\mu(x)\neq 0$. For an open neighborhood V of 0 in G, let h_V be a non-negative function in $C_c(G)$ such that $\mathrm{supp}(h_V)\subset V$ and $\|h_V\|_1=1$. It follows from Lemma 2.1 that

$$\text{(1)}\qquad \lim_V\|h_V*f-f\|_\infty=0,$$

where $h_V*f(x)=\int_G f((-g)\cdot x)h_V(g)dm_G(g)$. Set $h_V^*(g)=h_V(-g)$. Then, by the hypothesis, we have

$$\begin{aligned}\int_X h_V*f(x)d\mu(x)&=\int_X\int_G f((-g)\cdot x)h_V(g)dm_G(g)d\mu(x)\\&=\int_X f(x)d(h_V^**\mu)(x)\\&=0,\end{aligned}$$

which together with (1) yields $\int_X f(x)d\mu(x)=0$. This contradicts the choice of f, and the proof is complete.

LEMMA 2.3. *Let $\lambda\in M(G)$ and $\mu\in M(X)$. Then $sp(\lambda*\mu)\subset\{\gamma\in\hat{G}: \hat{\lambda}(\gamma)\neq 0\}^-$.*

PROOF. Suppose $\gamma_0\notin\{\gamma\in\hat{G}: \hat{\lambda}(\gamma)\neq 0\}^-$. Let U be an open neighborhood of γ_0 such that $U\cap\{\gamma\in\hat{G}:\ \hat{\lambda}(\gamma)\neq 0\}^{-}=\phi$. Let h be a function in $L^1(G)$ such that $\hat{h}(\gamma_0)\neq 0$ and $\hat{h}=0$ on U^c. Then $h*\lambda=0$, and so $h*(\lambda*\mu)=(h*\lambda)*\mu=0$. Since $\hat{h}(\gamma_0)\neq 0$, we have $\gamma_0\notin \mathrm{sp}(\lambda*\mu)$. This completes the proof.

We shall use the following lemma frequently.

LEMMA 2.4 (cf. [12, 7.2.5. (a)]). *Let G be a LCA group. Let $F\in L^1(G)$, and let I be a closed ideal in $L^1(G)$. Suppose that $\bigcap_{h\in I}\hat{h}^{-1}(0)$ is in the interior of $\hat{F}^{-1}(0)$. Then $F\in I$.*

LEMMA 2.5. *Suppose $G=H\oplus K$, where H and K are closed subgroups of G. Let E and F be closed subsets of $\hat{H}$ and $\hat{K}$ respectively. Let $\mu\in M(X)$, and suppose that $sp_H(\mu)\subset E$ and $sp_K(\mu)\subset F$. Then $sp(\mu)\subset E\times F$.*

PROOF. For $x\in X$, $s\in H$ and $t\in K$, denote $(s,0)\cdot x$ and $(0,t)\cdot x$ by $s\cdot x$ and $t\cdot x$ respectively. Suppose $(\gamma,\omega)\notin E\times F$. Then $\gamma\notin E$ or $\omega\notin F$. We may assume that $\gamma\notin E$. Then there exist $h\in L^1(H)$ and $k\in L^1(K)$ such that

(1) $\hat{h}(\gamma)\neq 0$,

(2) $\hat{k}(\omega)\neq 0$, and

(3) E is in the interior of $\hat{h}^{-1}(0)$.

By (3) and the hypothesis, $\mathrm{sp}_H(\mu)$ is in the interior of $\hat{h}^{-1}(0)$. It follows from Lemma 2.4 that

(4) $h\in J(\mu:H)$.

Define $F\in L^1(G)$ by $F(s,t)=h(s)k(t)$. For any $f\in C_0(X)$, we have

$$\begin{aligned}F*\mu(f)&=\int_X\int_{H\oplus K}f((s,t)\cdot x)F(s,t)\,dm_{H\oplus K}(s,t)\,d\mu(x)\\&=\int_K\int_X\int_H f_t(s\cdot x)h(s)\,dm_H(s)\,d\mu(x)k(t)\,dm_K(t)\\&=\int_K\int_X f_t(x)\,d(h*\mu)(x)k(t)\,dm_K(t)\\&=k*(h*\mu)(f)\\&=0, \qquad\qquad \text{(by (4))}\end{aligned}$$

where $f_t(x)=f(t\cdot x)$ $(t\in K, x\in X)$. This shows that $F\in J(\mu)$. On the other hand, $\hat{F}(\gamma,\omega)=\hat{h}(\gamma)\hat{k}(\omega)\neq 0$. Hence $(\gamma,\omega)\notin \mathrm{sp}(\mu)$, and we have $\mathrm{sp}(\mu)\subset E\times F$. This completes the proof.

§ 3. Some results on compact transformation groups.

In this section, let (G, X) be a transformation group, in which G is a compact abelian group and X is a locally compact Hausdorff space. For $\gamma \in \hat{G}$ and $\mu \in M(X)$, we note that $\gamma \in \mathrm{sp}(\mu)$ if and only if $\gamma * \mu \neq 0$ (cf. [14, Remark 1.1 (II.1)]). Let $\pi: X \to X/G$ be the canonical map. For $x \in X$, we define a continuous map $B_x: G \to G \cdot x$ $(\subset X)$ by $B_x(g) = g \cdot x$. Set $G_x = \{g \in G: g \cdot x = x\}$. Then G_x is a closed subgroup of G. We define a map $\tilde{B}_x: G/G_x \to G \cdot x$ by $\tilde{B}_x(g + G_x) = g \cdot x$. Then $\tilde{B}_x$ is a homeomorphism.

DEFINITION 3.1. For $x \in X$, put $\dot{x} = \pi(x)$. Define $m_{\dot{x}} \in M^+(X)$ by $m_{\dot{x}} = B_x(m_G)$.

As noted in [14, Remark 1.2], $m_{\dot{x}}$ is well-defined because $B_y(m_G) = B_x(m_G)$ for every $y \in \pi^{-1}(\dot{x})$. We state two conditions (D. I) and (D. II).

(D. I) Let (G, X) be a transformation group, in which G is a metrizable compact abelian group and X is a locally compact Hausdorff space. For any $\mu \in M^+(X)$, put $\eta = \pi(\mu)$. Then there exists a family $\{\lambda_{\dot{x}}\}_{\dot{x} \in X/G}$ of measures in $M^+(X)$ with the following properties:

(3.1) $\dot{x} \to \lambda_{\dot{x}}(f)$ is η-measurable for each bounded Baire function f on X,

(3.2) $\|\lambda_{\dot{x}}\| = 1$,

(3.3) $\mathrm{supp}(\lambda_{\dot{x}}) \subset \pi^{-1}(\dot{x})$,

(3.4) $\mu(f) = \int_{X/G} \lambda_{\dot{x}}(f)\, d\eta(\dot{x})$ for each bounded Baire function f on X.

(D, II) Let (G, X) be as in (D. I). Let $\nu \in M^+(X/G)$. Suppose $\{\lambda^1_{\dot{x}}\}_{\dot{x} \in X/G}$ and $\{\lambda^2_{\dot{x}}\}_{\dot{x} \in X/G}$ are families of measures in $M(X)$ with the following properties:

(3.5) $\dot{x} \to \lambda^i_{\dot{x}}(f)$ is ν-integrable for each bounded Baire function f on X $(i = 1, 2)$,

(3.6) $\mathrm{supp}(\lambda^i_{\dot{x}}) \subset \pi^{-1}(\dot{x})$ $(i = 1, 2)$,

(3.7) $\int_{X/G} \lambda^1_{\dot{x}}(f)\, d\nu(\dot{x}) = \int_{X/G} \lambda^2_{\dot{x}}(f)\, d\nu(\dot{x})$ for each bounded Baire function f on X.

Then $\lambda^1_{\dot{x}} = \lambda^2_{\dot{x}}$ ν-a.a. $\dot{x} \in X/G$.

Let $\mu \in M(X)$ and $\eta \in M^+(X/G)$. By an η-disintegration of μ, we mean a family $\{\lambda_{\dot{x}}\}_{\dot{x} \in X/G}$ of measures in $M(X)$ satisfying (3. 1)$'$ $\dot{x} \to \lambda_{\dot{x}}(f)$

is η-integrable for each bounded Baire function f on X and (3. 3)-(3. 4) in (D. I). If, in addition, $\eta=\pi(|\mu|)$ and $\|\lambda_{\dot{x}}^{1}\|=1$ for all $\dot{x}\in X/G$, then we call $\{\lambda_{\dot{x}}\}_{\dot{x}\in X/G}$ a canonical disintegration of μ. Thus condition (D. I) says that each $\mu\in M^{+}(X)$ has a canonical disintegration $\{\lambda_{\dot{x}}\}_{\dot{x}\in X/G}$ with $\lambda_{\dot{x}}\in M^{+}(X)$.

REMARK 3. 1. Let (G, X) be a transformation group, in which G is a metrizable compact abelian group and X is a locally compact metric space. Then (G, X) satisfies conditions (D. I) and (D. II) (cf. [14, Remark 6. 1]).

LEMMA 3. 1. *Let (G, X) be a transformation group, in which G is a metrizable compact abelian group and X is a locally compact Hausdorff space. Suppose (G, X) satisfies conditions (D. I) and (D. II). Let P be a semigroup in $\widehat{G}$ such that* (i) *$P\cup(-P)=\widehat{G}$ and* (ii) *$P\cap(-P)=\{0\}$. Let σ be a positive Radon measure on X that is quasi-invariant. Let ν be a measure in $M(X)$ such that $sp(\nu)\subset P$ and $\nu\perp\sigma$. Then $m_{G}*\nu\perp\sigma$.*

PROOF. Since ν is bounded and regular, we may assume that $\sigma\in M^{+}(X)$ (cf. [14, the proof of Theorem 1. 1, p. 311]). Put $\eta=\pi(|\nu|)$. By (D. I), $|\nu|$ has a canonical disintegration $\{\lambda_{\dot{x}}\}_{\dot{x}\in X/G}$ with $\lambda_{\dot{x}}\in M^{+}(X)$. Let $\eta=\eta_{a}+\eta_{s}$ be the Lebesgue decomposition of η with respect to $\pi(\sigma)$. We define ω_{1}, $\omega_{2}\in M^{+}(X)$ as follows:

$$(1)\qquad \omega_{1}(f)=\int_{X/G}\lambda_{\dot{x}}(f)d\eta_{a}(\dot{x}),$$
$$\omega_{2}(f)=\int_{X/G}\lambda_{\dot{x}}(f)d\eta_{s}(\dot{x})$$

for $f\in C_{0}(X)$. We note that (1) holds for all bounded Baire functions f on X. It is easy to verify that

(2) $\quad |\nu|=\omega_{1}+\omega_{2}$, and

(3) $\quad \pi(\omega_{2})\perp\pi(\sigma)$.

It follows from (2) and the hypothesis that $\omega_{1}\perp\sigma$. Hence, by [14, Lemma 2. 5 (II)], we have

$$(4)\qquad \lambda_{\dot{x}}\perp m_{\dot{x}}\ \ \eta_{a}\text{-a.a.}\,\dot{x}\in X/G.$$

Let h be a unimodular Baire function on X such that $\nu=h|\nu|$, and define measures ν_{1}, $\nu_{2}\in M(X)$ as follows:

$$(5)\qquad \nu_{1}(f)=\int_{X/G}\nu_{\dot{x}}(f)\ d\eta_{a}(\dot{x}),$$

$$\nu_2(f)=\int_{X/G}\nu_{\dot{x}}(f)d\eta_s(\dot{x})$$

for $f\in C_0(X)$, where $\nu_{\dot{x}}=h\lambda_{\dot{x}}$. Then

(6) $\qquad \nu=\nu_1+\nu_2.$

We note that $\{\nu_{\dot{x}}\}_{\dot{x}\in X/G}$ is a canonical disintegration of ν. Since $\mathrm{sp}(\nu)\subset P$, it follows from [14, Lemma 2.6] that

(7) $\qquad \mathrm{sp}(\nu_{\dot{x}})\subset P \ \ \eta\text{-a.a. } \dot{x}\in X/G;$

hence

(8) $\qquad \mathrm{sp}(\nu_{\dot{x}})\subset P \ \ \eta_a\text{-a. a. } \dot{x}\in X/G.$

Since $\mathrm{supp}(\nu_{\dot{x}})\subset\pi^{-1}(\dot{x})$ and $\widetilde{B}_x$: $G/G_x\rightarrow G\cdot x(=\pi^{-1}(\dot{x}))$ is a homeomorphism, there exists a measure $\xi_{\dot{x}}\in M(G/G_x)$ such that $\widetilde{B}_x(\xi_{\dot{x}})=\nu_{\dot{x}}$, where $x\in\pi^{-1}(\dot{x})$. By(8) and [14, Proposition 1.2], we have

(9) $\qquad \xi_{\dot{x}}\in M_{P\cap G_x^{\perp}}(G/G_x) \ \ \eta_a\text{-a.a. } \dot{x}\in X/G.$

It follows from (4) and [14, Proposition 1.5] that

(10) $\qquad \xi_{\dot{x}}\perp m_{G/G_x} \ \ \eta_a\text{-a.a. } \dot{x}\in X/G.$

Hence we have, by (9), (10) and Theorem A (II),

(11) $\qquad \xi_{\dot{x}}\in M_{(P\setminus\{0\})\cap G_x^{\perp}}(G/G_x) \ \ \eta_a\text{-a.a. } \dot{x}\in X/G,$

which yields

(12) $\qquad \mathrm{sp}(\nu_{\dot{x}})\subset P\setminus\{0\} \ \ \eta_a\text{-a.a. } \dot{x}\in X/G$

by [14, Proposition 1.2]. Hence

$$m_G*\nu_{\dot{x}}=0 \ \ \eta_a\text{-a.a. } \dot{x}\in X/G,$$

which together with [14, Lemma 2.3] yields

$$m_G*\nu_1(f)=\int_{X/G}m_G*\nu_{\dot{x}}(f)d\eta_a(\dot{x})=0$$

for all $f\in C_0(X)$. This shows that

(13) $\qquad m_G*\nu_1=0.$

On the other hand, it is easy to verify that

(14) $\qquad \pi(m_G*\omega_2)=\pi(\omega_2).$

(3) and (14) imply that $m_G*\omega_2\perp\sigma$, and so $m_G*\nu_2\perp\sigma$ because $m_G*\nu_2<<$

$m_G * \omega_2$. By (6) and (13), we have $m_G * \nu = m_G * \nu_2$. Hence we get $m_G * \nu \perp \sigma$, and the proof is complete.

LEMMA 3.2. *Let (G, X) be a transformation group, in which G is a compact abelian group and X is a locally compact metric space. Then the conclusion of Lemma* 3.1 *holds.*

PROOF. Let ν be a measure in $M(X)$ such that $\mathrm{sp}(\nu) \subset P$ and $\nu \perp \sigma$. Since ν is bounded and regular, we may assume that X is σ-compact and $\sigma \in M^+(X)$. Suppose $m_G * \nu$ and σ are not mutually singular. Let $m_G * \nu = \omega + \zeta$ be the Lebesgue decomposition of $m_G * \nu$ with respect to σ, where $\omega << \sigma$ and $\zeta \perp \sigma$. Then $\omega \neq 0$. By [14, Lemmas 2.11 and 2.13], there exists a countable subgroup Γ of $\widehat{G}$ such that

(1) $\pi_H(\omega) \neq 0$,
(2) $\pi_H(|\nu|) \perp \pi_H(\sigma)$, and
(3) $\pi_H(|\zeta|) \perp \pi_H(\sigma)$,

where $H = \Gamma^\perp$ and $\pi_H : X \to X/H$ is the canonical map. Let $(G/H, X/H)$ be the transformation group induced by (G, X). By [14, Lemma 2.9], $\pi_H(m_G * \nu) = m_{G/H} * \pi_H(\nu)$. Since $\pi_H(\omega) << \pi_H(\sigma)$, it follows from (3) that $m_{G/H} * \pi_H(\nu) = \pi_H(\omega) + \pi_H(\zeta)$ is the Lebesgue decomposition of $m_{G/H} * \pi_H(\nu)$ with respect to $\pi_H(\sigma)$. Since σ is quasi-invariant, $\pi_H(\sigma)$ is quasi-invariant. By [14, Lemma 2.10], we have $\mathrm{sp}(\pi_H(\nu)) \subset P \cap \Gamma$. Since G/H and X/H are metrizable, it follows from Remark 3.1 that $(G/H, X/H)$ satisfies conditions (D. I) and (D. II). Hence, by (2) and Lemma 3.1, we have $m_{G/H} * \pi(\nu) \perp \pi_H(\sigma)$, which yields $\pi_H(\omega) = 0$. This contradicts (1), and the proof is complete.

PROPOSITION 3.1. *Let (G, X) be a transformation group, in which G is a compact abelian group and X is a locally compact Hausdorff space. Let P be a semigroup in $\widehat{G}$ such that* (i) $P \cup (-P) = \widehat{G}$ *and* (ii) $P \cap (-P) = \{0\}$. *Let σ be a positive Radon measure on X that is quasi-invariant. Let ν be a measure in $M(X)$ such that $sp(\nu) \subset P$ and $\nu \perp \sigma$. Then $m_G * \nu \perp \sigma$.*

PROOF. Since ν is bounded and regular, we may assume that X is σ-compact and $\sigma \in M^+(X)$. Suppose that $m_G * \nu$ and σ are not mutually singular. Let $m_G * \nu = \omega + \zeta$ be the Lebesgue decomposition of $m_G * \nu$ with respect to σ, where $\omega << \sigma$ and $\zeta \perp \sigma$. Then $\omega \neq 0$. By [15, Lemma 3.1], there exists an equivalence relation " $\sim$ " on X with the following properties:

(1) $X/\sim$ is a σ-compact metrizable, locally compact Hausdorff space with respect to the quotient topology;
(2) $(G, X/\sim)$ becomes a transformation group by the action $g\cdot\tau(x)=\tau(g\cdot x)$ for $g\in G$ and $x\in X$, where $\tau: X\to X/\sim$ is the canonical map;
(3) $\tau(\omega)\neq 0$;
(4) $\tau(|\nu|+|\zeta|)\perp\tau(\sigma)$.

By [15, Lemma 2.1], $\tau(\sigma)$ is quasi-invariant. By (4), $\tau(\nu)$ and $\tau(\sigma)$ are mutually singular. It follows from [15, Lemma 2.2] that $\mathrm{sp}(\tau(\nu))\subset\mathrm{sp}(\nu)\subset P$. Hence, by (1) and Lemma 3.2, we have $m_G*\tau(\nu)\perp\tau(\sigma)$. On the other hand, $m_G*\tau(\nu)=\tau(m_G*\nu)=\tau(\omega)+\tau(\zeta)$. And (3)-(4) implies that $0\neq\tau(\omega)<<\tau(\sigma)$ and $\tau(\zeta)\perp\tau(\sigma)$. Thus we have a contradiction. This completes the proof.

PROPOSITION 3.2. *Let (G, X), P and σ be as in the previous proposition. Let μ be a measure in $M(X)$ such that $sp(\mu)\subset P\backslash\{0\}$. Let $\mu=\mu_a+\mu_s$ be the Lebesgue decomposition of μ with respect to σ. Then both $sp(\mu_a)$ and $sp(\mu_s)$ are also contained in $P\backslash\{0\}$.*

PROOF. We may assume that $\sigma\in M^+(X)$. It suffices to prove that $\mathrm{sp}(\mu_s)\subset P\backslash\{0\}$ because of [14, Remark 1.1 (II)]. Suppose $\mathrm{sp}(\mu_s)\not\subset P\backslash\{0\}$. Since $\mathrm{sp}(\mu)\subset P$, it follows from [15, Theorem 2.1] that $\mathrm{sp}(\mu_s)\subset P$; hence $0\in\mathrm{sp}(\mu_s)$. Thus $m_G*\mu_s\neq 0$. Since $0=m_G*\mu=m_G*\mu_a+m_G*\mu_s$, it follows from [14, Lemma 1.1] that

$$0\neq m_G*\mu_s=-m_G*\mu_a<<m_G*\sigma<<\sigma.$$

This contradicts Proposition 3.1, and the proof is complete.

§ 4. Proofs of Theorems 2.1 and 2.2.

In this section we prove Theorems 2.1 and 2.2. Let (G, X) be a transformation group, in which G is a LCA group and X is a locally compact Hausdorff space. Suppose $\hat{G}$ is ordered, i.e., there exists a closed semigroup P in $\hat{G}$ such that (i) $P\cup(-P)=\hat{G}$ and (ii) $P\cap(-P)=\{0\}$ (cf. [12, 8.1]). If G is noncompact and not isomorphic with $\boldsymbol{R}$, G is isomorphic with $\boldsymbol{R}\oplus H$ and $P=\{(x, d)\in\boldsymbol{R}\oplus\hat{H}: d>0, \text{ or } d=0 \text{ and } x\geq 0\}$, where H is a compact connected subgroup of G (cf. [12, 8.1.5. Theorem and 8.1.6. Theorem]).

LEMMA 4.1. *Suppose $\hat{G}$ is ordered and G is isomorphic with $\boldsymbol{R}\oplus H$, where H is a compact connected abelian group. Let P be the closed semigroup in $\hat{G}$ which induces an order on $\hat{G}$, and set $P_H=\{d\in\hat{H}: d\geq 0\}$. Let $\mu\in M(X)$, and suppose $sp(\mu)\subset P$. Then $sp_H(\mu-m_H*\mu)\subset P_H\backslash\{0\}$.*

PROOF. Evidently $P \subset \boldsymbol{R} \times P_H$.

Step 1. $\mathrm{sp}_H(\mu) \subset P_H$.

Suppose $d \in \hat{H} \setminus P_H$. For any $\omega \in L^1(\boldsymbol{R})$, define $F_\omega \in L^1(\boldsymbol{R} \oplus H)$ by $F_\omega(s, t) = \omega(s)\,(t, d)$. Then $\mathrm{sp}(\mu)$ is in the interior of $\hat{F}_\omega^{-1}(0)$. It follows from Lemma 2.4 that $F_\omega * \mu = 0$. For any $f \in C_0(X)$, put $f_s(x) = f(s \cdot x)$ $(s \in \boldsymbol{R},\ x \in X)$. Then

$$
\begin{aligned}
0 = F_\omega * \mu(f) &= \int_X \int_{\boldsymbol{R} \oplus H} f((s, t) \cdot x) F_\omega(s, t)\, dm_{\boldsymbol{R} \oplus H}(s, t)\, d\mu(x) \\
&= \int_{\boldsymbol{R}} \int_X \int_H f_s(t \cdot x)\ (t, d)\, dm_H(t)\, d\mu(x)\, \omega(s)\, ds \qquad (1) \\
&= \int_{\boldsymbol{R}} (d m_H) * \mu(f_s)\, \omega(s)\, ds.
\end{aligned}
$$

By Lemma 2.1, $s \to (dm_H) * \mu(f_s)$ is a bounded continuous function on $\boldsymbol{R}$. By (1), we have

$$(dm_H) * \mu(f_s) = 0 \quad \text{a.a. } s \in \boldsymbol{R}\,;$$

hence

$$(dm_H) * \mu(f) = 0 \quad \text{for all } f \in C_0(X).$$

Hence $(dm_H) * \mu = 0$, and so $d \notin \mathrm{sp}_H(\mu)$. This shows that Step 1 holds.

Step 2. $0 \notin \mathrm{sp}_H(\mu - m_H * \mu)$.

We note that $m_H * (\mu - m_H * \mu) = m_H * \mu - (m_H * m_H) * \mu = 0$. Since $\hat{m}_H(0) = 1$, we have $0 \notin \mathrm{sp}_H(\mu - m_H * \mu)$. Thus Step 2 is obtained.

Since $J(\mu : H) \subset J(m_H * \mu : H)$, we have $\mathrm{sp}_H(m_H * \mu) \subset \mathrm{sp}_H(\mu)$. Hence, by Steps 1 and 2, we have $\mathrm{sp}_H(\mu - m_H * \mu) \subset P_H \setminus \{0\}$. This completes the proof.

THEOREM 4.1. *Let $(G. X)$ be a transformation group, in which G is a LCA group and X is a locally compact Hausdorff space. Suppose there exists a closed semigroup in $\hat{G}$ such that* (i) $P \cup (-P) = \hat{G}$ *and* (ii) $P \cap (-P) = \{0\}$. *Let σ be a positive Radon measure on X that is quasi-invariant. Let $\mu \in M(X)$, and let $\mu = \mu_a + \mu_s$ be the Lebesgue decomposition of μ with respect to σ. Suppose $sp(\mu) \subset P$. Then both $sp(\mu_a)$ and $sp(\mu_s)$ are also contained in P.*

PROOF. By [12, 8.1.5. Theorem], we have

(a) $G \cong \boldsymbol{R}$, (b) G is compact, or (c) $G \cong \boldsymbol{R} \oplus H$, where H is a compact connected subgroup of G.

Case 1. $G \cong \boldsymbol{R}$.

In this case, the theorem follows from Theorem D.

Case 2. G is compact.

In this case, the theorem follows from Theorem E.

Case 3. $G \cong \boldsymbol{R} \oplus H$, where H is a compact connected subgroup of G.

As seen at the begining of this section, we have $P=\{(x, d) \in \boldsymbol{R} \oplus \hat{H} : d>0$, or $d=0$ and $x \geq 0\}$. Put $P_H=\{d \in \hat{H} : d \geq 0\}$. Set

(1) $\qquad \mu=(\mu-m_H*\mu)+m_H*\mu.$

By Lemma 4.1, $\mathrm{sp}_H(\mu-m_H*\mu) \subset P_H \backslash \{0\}$. Hence Proposition 3.2 implies that

(2) $\qquad \mathrm{sp}_H((\mu-m_H*\mu)_a),\ \mathrm{sp}_H((\mu-m_H*\mu)_s) \subset P_H \backslash \{0\}.$

Since $\boldsymbol{R} \times (P_H \backslash \{0\}) \subset P$, it follows from (2) and Lemma 2.5 that

(3) $\qquad \mathrm{sp}((\mu-m_H*\mu)_a),\ \mathrm{sp}((\mu-m_H*\mu)_s) \subset P.$

Claim. $\mathrm{sp}_{\boldsymbol{R}}(m_H*\mu) \subset \boldsymbol{R}^+$, where $\boldsymbol{R}^+$ is the nonnegative real numbers. For any $x_0 \in \boldsymbol{R} \backslash \boldsymbol{R}^+$, there exists a function k in $L^1(\boldsymbol{R})$ such that $\hat{k}(x_0) \neq 0$ and $\boldsymbol{R}^+$ is in the interior of $\{x \in \boldsymbol{R} : \hat{k}(x)=0\}$. Define $F \in L^1(\boldsymbol{R} \oplus H)$ by $F(s, t)=k(s)$. Then $\mathrm{sp}(\mu)$ is in the interior of $\hat{F}^{-1}(0)$. It follows from Lemma 2.4 that

(4) $\qquad F*\mu=0.$

We note that $k*(m_H*\mu)=F*\mu$ (see the proof of Lemma 2.5). Hence (4) yields $k*(m_H*\mu)=0$. Since $\hat{k}(x_0) \neq 0$, x_0 does not belong to $\mathrm{sp}_{\boldsymbol{R}}(m_H*\mu)$. This shows that the claim holds.
By Claim and Theorem D, we have

(5) $\qquad \mathrm{sp}_{\boldsymbol{R}}((m_H*\mu)_a),\ \mathrm{sp}_{\boldsymbol{R}}((m_H*\mu)_s) \subset \boldsymbol{R}^+.$

By Lemma 2.3, we have $\mathrm{sp}_H(m_H*\mu) \subset \{0\}$, and $\{0\}$ is a Riesz set in $\hat{H}$. Hence, by [15, Theorem 2.4], we have

(6) $\qquad \mathrm{sp}_H((m_H*\mu)_a),\ \mathrm{sp}_H((m_H*\mu)_s) \subset \{0\},$

which together with (5) and Lemma 2.5 yields

(7) $\quad \mathrm{sp}((m_H*\mu)_a),\ \mathrm{sp}((m_H*\mu)_s)\subset \boldsymbol{R}^+\times\{0\}\subset P.$

It follows from (1), (3) and (7) that $\mathrm{sp}(\mu_a)\subset P$ and $\mathrm{sp}(\mu_s)\subset P$. This completes the proof.

We return to the general case. For $\mu\in M(X)$, set $J_{M(G)}(\mu)=\{\lambda\in M(G):\lambda*\mu=0\}$.

DEFINITION 4.1. For $\mu\in M(X)$, define $\mathrm{sp}_{M(G)}(\mu)$ by $\bigcap_{\lambda\in J_{M(G)}(\mu)} \mu^{\hat{\lambda}^{-1}(0)}$.

LEMMA 4.2. *For $\mu\in M(X)$, we have $sp(\mu)=sp_{M(G)}(\mu)$.*

PROOF. Since $J_{M(G)}(\mu)\supset J(\mu)$, we have $\mathrm{sp}_{M(G)}(\mu)\subset\mathrm{sp}(\mu)$. Suppose $\gamma\notin\mathrm{sp}_{M(G)}(\mu)$. Then there exists $\lambda\in J_{M(G)}(\mu)$ such that $\hat{\lambda}(\gamma)\neq 0$. Let k be a function in $L^1(G)$ such that $\hat{k}(\gamma)\neq 0$. Then $k*\lambda\in L^1(G)$, $(k*\lambda)*\mu=k*(\lambda*\mu)=0$ and $(k*\lambda)^\wedge(\gamma)\neq 0$. Hence $\gamma\notin\mathrm{sp}(\mu)$, and so $\mathrm{sp}(\mu)\subset\mathrm{sp}_{M(G)}(\mu)$. This completes the proof.

Suppose there exists a proper closed semigroup P in $\hat{G}$ such that $P\cup(-P)=\hat{G}$. Put $\Lambda=P\cap(-P)$, and let $\tau:\hat{G}\to\hat{G}/\Lambda$ be the natural homomorphism. Let $H=\Lambda^\perp$, and set $\tilde{P}=\tau(P)$. Then $\tilde{P}$ is a proper closed semigroup in $\hat{G}/\Lambda$ such that (i) $\tilde{P}\cup(-P)=\hat{G}/\Lambda$ and (ii) $\tilde{P}\cap(-\tilde{P})=\{0\}$. From Proposition 4.1 through Proposition 4.3, we assume that there exists such a proper closed semigroup P in $\hat{G}$.

PROPOSITION 4.1. *Let E be a closed set in $\hat{G}$ such that $E+\Lambda=E$, and let $\tilde{E}=\tau(E)$. Let μ be a measure in $M(X)$. Then the following are equivalent.*

(i) $\quad sp_H(\mu)\subset\tilde{E}$;
(ii) $\quad sp(\mu)\subset E$.

PROOF. Since $E+\Lambda=E$, we note that $\tilde{E}$ is a closed set in $\hat{G}/\Lambda$.

(i)$\Rightarrow$(ii): Suppose $\gamma\notin E$. Since $E+\Lambda=E$, $\tau(\gamma)\notin\tilde{E}$, and so $\tau(\gamma)\notin\mathrm{sp}_H(\mu)$. Hence there exists $f\in L^1(H)$ such that $f*\mu=0$ and $\hat{f}(\tau(\gamma))\neq 0$. We can consider f as a measure in $M(G)$. We denote it by λ_f. Then $\lambda_f*\mu=f*\mu=0$ and $\hat{\lambda}_f(\gamma)=\hat{f}(\tau(\gamma))\neq 0$. It follows from Lemma 4.2 that $\gamma\notin\mathrm{sp}_{M(G)}(\mu)=\mathrm{sp}(\mu)$. Hence we have $\mathrm{sp}(\mu)\subset E$.

(ii)$\Rightarrow$(i): Suppose $\tau(\gamma)\notin\tilde{E}$ ($\gamma\notin$E). Let $\tilde{V}$ be a compact neighborhood of $\tau(\gamma)$ such that $\tilde{V}\cap\tilde{E}=\phi$. Then there exists $f\in L^1(H)$ such that $\hat{f}(\tau(\gamma))\neq 0$ and $\mathrm{supp}(\hat{f})\subset\tilde{V}$. Let μ_f be the measure in $M(G)$ corresponding to f. Then

(1) $\quad \hat{\lambda}_f(\gamma)=\hat{f}(\tau(\gamma))\neq 0.$

We note that

(2) $\quad \operatorname{supp}(\hat{\lambda}_f) \subset \tau^{-1}(\tilde{V})$.

Since $E+\Lambda=E$, we have $\tau^{-1}(\tilde{E})=E$. This together with $\tilde{V}\cap\tilde{E}=\phi$ yields

(3) $\quad \tau^{-1}(\tilde{V})\cap E=\phi$.

It follows from (ii), (2) and (3) that

(4) $\quad \operatorname{sp}(\mu)$ is in the interior of $\hat{\lambda}_f^{-1}(0)$.

Claim. $\lambda_f*\mu=0$.

For any $h\in L^1(G)$, $h*\lambda_f\in L^1(G)$, and (4) implies that $\operatorname{sp}(\mu)$ is in the interior of $\{\gamma\in\hat{G}: (h*\lambda_f)^\wedge(\gamma)=0\}$. It follows from Lemma 2.4 that $h*(\lambda_f*\mu)=(h*\lambda_f)*\mu=0$. Hence, by Lemma 2.2, we have $\lambda_f*\mu=0$, and the claim is obtained.

By Claim, $f*\mu=0$, and $\hat{f}(\tau(\gamma))\neq 0$. Hence $\tau(\gamma)\notin \operatorname{sp}_H(\mu)$. This shows that $\operatorname{sp}_H(\mu)\subset\tilde{E}$, and the proof is complete.

The following two propositions follow from the previous proposition.

PROPOSITION 4.2. *Let μ be a measure in $M(X)$. Then the following are equivalent.*

(i) $\quad sp_H(\mu)\subset\tilde{P}$;
(ii) $\quad sp(\mu)\subset P$.

PROOF. Since $P+\Lambda=P$, the proposition follows from Proposition 4.1.

PROPOSITION 4.3. *Let μ be a measure in $M(X)$. If Λ is open, then the following are equivalent.*

(i) $\quad sp_H(\mu)\subset\tilde{P}\backslash\{0\}$;
(ii) $\quad sp(\mu)\subset P\backslash(-P)$.

PROOF. $P\backslash(-P)$ is closed because $P\backslash(-P)=P\backslash\Lambda$. It is easy to verify that $P\backslash(-P)+\Lambda=P\backslash(-P)$ and $\tau(P\backslash(-P))=\tilde{P}\backslash\{0\}$. Hence the proposition follows from Proposition 4.1.

Now we prove Theorem 2.1. We may assume that P is a proper closed semigroup in $\hat{G}$. Put $\Lambda=P\cap(-P)$ and $H=\Lambda^\perp$. Let $\tau: \hat{G}\to\hat{G}/\Lambda$ be the natural homomorphism, and set $\tilde{P}=\tau(P)$. Then $\tilde{P}$ is a closed semigroup in $\hat{G}/\Lambda$ such that $\tilde{P}\cup(-P)=\hat{G}/\Lambda$ and $\tilde{P}\cap(-\tilde{P})=\{0\}$. Since $\operatorname{sp}(\mu)\subset P$, it follows from Proposition 4.2 that $\operatorname{sp}_H(\mu)\subset\tilde{P}$. We note that $\hat{H}\cong\hat{G}/\Lambda$. Hence Theorem 4.1 implies that

$$\operatorname{sp}_H(\mu_a),\ \operatorname{sp}_H(\mu_s)\subset\tilde{P},$$

which together with Proposition 4.2 yields that $\mathrm{sp}(\mu_a)$, $\mathrm{sp}(\mu_s)\subset P$. This completes the proof of Theorem 2.1.

Next we prove Theorem 2.2. Notations are as in the proof of Theorem 2.1. Since $\mathrm{sp}(\mu)\subset P\backslash(-P)$, it follows from Proposition 4.3 that $\mathrm{sp}_H(\mu)\subset \tilde{P}\backslash\{0\}$. Since $P\cap(-P)$ is open, H is a compact subgroup of G. It follows from Proposition 3.2 that $\mathrm{sp}_H(\mu_a)$, $\mathrm{sp}_H(\mu_s)\subset \tilde{P}\backslash\{0\}$. Hence, by Proposition 4.3, we have $\mathrm{sp}(\mu_a)$, $\mathrm{sp}(\mu_s)\subset P\backslash(-P)$. This completes the proof of Theorem 2.2.

REMARK 4.1. Let G be a LCA group. Then we get a transformation group (G, G). Let $\mu\in M(G)$, and let E be a closed subset of $\hat{G}$. The following are equivalent.

(i) $\hat{\mu}$ vanishes on E^c;
(ii) $\mathrm{sp}(\mu)\subset E$.

In fact, "(i)$\Rightarrow$(ii)" is not difficult, and "(ii)$\Rightarrow$(i)" is obtained as follows: Let $\gamma\in E^c$. Then there exists $f\in L^1(G)$ such that $\hat{f}(\gamma)=1$ and E is in the interior of $\hat{f}^{-1}(0)$. By (ii), $\mathrm{sp}(\mu)$ is in the interior of $\hat{f}^{-1}(0)$. By Lemma 2.4, we have $f*\mu=0$. Hence $0=\hat{f}(\gamma)\hat{\mu}(\gamma)=\hat{\mu}(\gamma)$. This shows that (i) holds.

REMARK 4.2. Theorem C follows from Theorems 2.1 and 2.2. In fact, let μ be a measure in $M_{P^c}(G)$. We may assume that P is a proper closed semigroup in $\hat{G}$. First we consider the case that $P\cap(-P)$ is not open. Since $\mu\in M_{P^c}(G)$, μ belongs to $M_{(-P)}(G)$. It follows from Theorem 2.1 and Remark 4.1 that μ_a, $\mu_s\in M_{(-P)}(G)$. Since $(-P)^c$ is dense in P, we have μ_a, $\mu_s\in M_{P^c}(G)$. Next we consider the case that $P\cap(-P)$ is open. Since $P^C=(-P)\backslash P$ is closed, $\mu\in M_{P^c}(G)$ implies that $\mathrm{sp}(\mu)\subset P^C=(-P)\backslash P$, by Remark 4.1. Hence, by Theorem 2.2 and Remark 4.1, we have μ_a, μ_s $\in M_{P^c}(G)$.

REMARK 4.3. Let $(\boldsymbol{R}, X)$ be a transformation group, in which the reals $\boldsymbol{R}$ acts on a locally compact Hausdorff space X. Let σ be a quasi-invariant, positive Radon measure on X. Let μ be an analytic measure on X, and let $\mu=\mu_a+\mu_s$ be the Lebesgue decomposition of μ with respect to σ. In [5, Theorem 5], Forelli showed that $\mathrm{sp}(\mu_a)$ and $\mathrm{sp}(\mu_s)$ are contained in $\mathrm{sp}(\mu)$. This result seems to depend on the fact that the semigroup $[0, \infty)$ is a Riesz set (cf. [15, Theorem 2.4]). In general, this result does not hold for another transformation group. We give an example: Let G be a compact connected abelian group which is not isomorphic to the circle group $\boldsymbol{T}$, and let P be a semigroup in $\hat{G}$ such that $P\cup$

$(-P)=\widehat{G}$ and $P\cap(-P)=\{0\}$. We consider the transformation group (G, G). We take m_G as a quasi-invariant measure. Since G is not isomorphic to $\boldsymbol{T}$, there exists a nonzero singular measure $\xi\in M_P(G)$. Then there exists $\gamma_0\in P$ such that $\hat{\xi}(\gamma_0)\neq 0$. Let $\mu=\hat{\xi}(\gamma_0)m_G-\xi$. Then $\hat{\mu}(\gamma_0)=0$ and $\hat{\mu_s}(\gamma_0)=-\hat{\xi}(\gamma_0)\neq 0$. It follows from Remark 4.1 and [14, Remark 1.1 (II. 1)] that $\mathrm{sp}(\mu)\subset P$ and $\mathrm{sp}(\mu_s)\not\subset\mathrm{sp}(\mu)$.

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Department of Mathematics
Josai University
Sakado, Saitama
Japan