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# Martin boundaries and thin sets for $\Delta u=Pu$ on Riemann surfaces

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## 1. Introduction.

Let $R$ be a hyperbolic Riemann surface and $P$ a density on $R$, that is, a non-negative Hölder continuous function on $R$ which depends on the local parameter $z=x+iy$ in such a way that the partial differential equation

$$(0.1)\qquad L_{P}u\equiv-\Delta u+Pu=0,\quad \Delta=\partial^{2}/\partial x^{2}+\partial^{2}/\partial y^{2}$$

is invariantly defined on $R$. A real valued function $u$ is said to be a $P$-harmonic function (or a $P$-solution) on an open set $U$ of $R$, if $u$ has continuons partial derivatives up to the order 2 and satisfies the equation (0.1) on $U$. Throughout this paper we assume that the density $P$ is not constantly zero on $R$. A density $P$ on a Riemann surface $R$ is called a hyperbolic density if there exists a positive $P$-harmonic function on $R$ dominated by 1 on $R$. In this paper we shall consider two Martin compactifications of a hyperbolic Riemann surface $R$, the first $R_{P}^{*}$ with respect to a hyperbolic density $P$ on $R$, the second $R^{*}$ with respect to harmonic functions. Let $K^{P}(z,a)$, $K(z,b)$ be Martin kernels on the compactifications $R_{P}^{*}$, $R^{*}$ respectively. Let $G(z,w)$ be the harmonic Green's function of $R$. For a minimal boundary point $a$ of $R_{P}^{*}$ such that

$$(0.2)\qquad \int_{R}P(w)G(w,z_{1})K^{P}(w,a)dudv<+\infty$$

for some point $z_{1}$ in $R$, there exists a unique minimal boundary point of $R^{*}$. Then we may define a mapping with the domain consisting points $a$ which satisfy the condition (0.2) into the set of minimal boundary points of $R^{*}$.

In the special case where $P$ is a bounded rotation free density on the unit disk in complex plane $C$, two compactifications coincide with the closed unit disk $\{z\in C:|z|\leq 1\}$. In this case our mapping reduces to the identity mapping of the unit circle (Remark in sec. 2). The purpose of this paper is to show that a closed set $E$ in $R$ is thin at a poinnt $a$ with the condition (0.2) if and only if $E$ is thin at the minimal boundary point of $R^*$ assigned to the point $a$ by the mapping. Therefore, fine neiborhoods of points $a$ with (0.2) and of the imagae of $a$ by our mapping have the same base the complement whose element is closed in $R$ and thin at the each point, where the first neiborhood is defined with respect to $R_P^*$, the second with respect to $R^*$ (Corollary 2.17).

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## 1. Linear spaces of $P$-harmonic functions.

By a regular region we shall always mean a connected open set in $R$ whose boundary is composed of at most a countable number of analytic curves clustering nowhere in $R$. Let $\{R_n\}$ be an exhaustion of $R$, that is, a sequnece of relatively compact regular regions in $R$ such that $cl(R_n)\subset R_{n+1}$, $R=U_{n=1}^{\infty}R_n$.

The solution $e_n^P$ of (0.1) on $R_n$ with $e_n^P=1$ on $\partial R_n$ is said the $P$-elliptic measure of $R_n$. The $P$-elliptic measures $e_n^P$ form a monotone decreasing sequence of positive $P$-solutions. This sequnece converges uniformly to a non-negative $P$-solution $e^P$ on $R$, which is called the $P$-elliptic measure of $R$. The $P$-elliptic measure is either identically zero or else everywhere positive. In the second case we call the density $P$ to be hyperbolic provided $P$ is not identically zero. We shall consider only hyperbolic densities on $R$ in the following.

LEMMA 1.1. *If a density $P$ on $R$ satisfies the condition*:

$$\int_R P(w)G(z_1,w)dudv<+\infty$$

*for some point $z_1$ in $R$, then $P$ is hyperbolic, where $G(z,w)$ is the harmonic Green's function of $R$ and $w=u+iv$.*

PROOF. Let $G(R_n,z,w)$ be harmoonic Green's function of $R_n$. By Green's formula we have

$$\int_{R_n} P(w)G(R_n,z,w)e_n^P(w)dudv=2\pi(1-e_n^P(z)). \tag{1.1}$$

If $P$ is not a hyperbolic density, then Lebesque's convergence theorem gives tthe contradiction: $1=0$. Q. E. D.

DEFINITION 1.2. *For a density $P$ on $R$ we denote by $P_H(R)$ the class of all those $P$-harmonic functions $u$ on $R$ which satisfies the condition*:

$$\int_R P(w)G(z_1,w)|u(w)|dudv < +\infty$$

*for some point* $z_1$ *in* $R$, *where* $w = u + iv$.

For any density $P$ the Green's function on $R$ for the equation (0.1) always exists, which is denoted by $G^P(z,w)$ for $z$, $w$ in $R$.

DEFINITION 1.3. *For a density* $P$ *on* $R$ *we denote by* $H_P(R)$ *the class of all those harmonic functions* $h$ *on* $R$ *which satisfy the condition*:

$$\int_R P(w)G^P(z_1,w)|h(w)|dudv < +\infty$$

*for some point* $z_1$ *in* $R$.

These definitions of $P_H(R)$ and $H_P(R)$ are independent of the choice of the point $z_1$ in $R$. And these class are real linear spaces with respect to the usual definitions of addition and scalar multiplication of real numbers.

THEOREM 1.4. *For a hyperbolic density* $P$ *on* $R$ *the* $P$-*elliptic measure* $e^P$ *is a function contained in the linear space* $P_H(R)$ *which is not constantly zero.*

PROOF. From the equality (1.1) in the proof of Lemma 1.1 it follows that

$$\begin{aligned}&\int_R P(w)G(z,w)e^P(w)dudv\\ &\leq \liminf_{n\to\infty}\int_{R_n} P(w)G(R_n,z,w)e_n^P(w)dudv\\ &=2\pi(1-e^P(z))\leq 2\pi.\end{aligned}$$

Q. E. D.

For an exhaustion $[R_n]$ of $R$ let $G(R_n,z,w)$ and $G^P(R_n,z,w)$ be the harmonic Green's function of $R_n$ and the Green's function with respect to the equation $L_P u = 0$ on $R_n$ respectively. For a relatively compact regular region $R_n$ in $R$, we denote the transformations $T_{PH}^n u$ and $T_{HP}^n h$ for real valued bounded continuous functions $u$ and $h$ defined on $R_n$ to functions on $R_n$ as follows:

$$T_{PH}^n u(z) = u(z) + \frac{1}{2\pi}\int_{R_n} P(w)G(R_n,z,w)u(w)dudv$$

$$T_{HP}^n h(z) = h(z) - \frac{1}{2\pi}\int_{R_n} P(w)G^P(R_n,z,w)h(w)dudv.$$

By the Green's formula for a $P$-harmonic function $u$ and Green's function of $R_n$ $T^n_{PH}u$ is a harmonic function with $T^n_{PH}u=u$ on $\partial R_n$, and for a harmonic function $h$ on $R_n$ $T^n_{HP}$ is a $P$-solution on $R_n$ with $T^n_{HP}h=h$ on $\partial R_n$.

Let $u$ be in $P_H(R)$. then, the linear transformation $T_{PH}u$ of $u$ is defined by

$$T_{PH}u(z)=u(z)+\frac{1}{2\pi}\int_R P(w)G(z,w)u(w)dudv$$

And, for $h$ in $H_P(R)$ $T_{HP}h$ is defined by

$$T_{HP}h(z)=h(z)-\frac{1}{2\pi}\int_R P(w)G^P(z,w)h(w)dudv.$$

These transformations $T_{PH}$ and $T_{HP}$ are extensively studied by many authors (for example, Lahtinen [5], Nakai [8]). The next lemma has fundamental roles in our paper, especialy in Theorem 1.8 and Theorem 2.9, which is a consequence of Lebesque's convergence theorem (for example, Lahtinen [5]).

LEMMA 1.5. *Let $P$ be a density on $R$, $u$ a $P$-solution on $R$ and $\{u_n\}$ a sequence of $P$-solutions each defined on $R_n$ so that* $\lim_{n\to+\infty}u_n=u$. *If there exists a positive function $v$ on $R$ such that $|u_n|\leq v$ for each $n$ and $v$ fulfils the condition*:

$$\int_R P(w)G(z_1,w)v(w)dudv<+\infty$$

*at some point $z_1$ in $R$. Then $T_{PH}u$ is well-defined and satisfies*

(1) $\lim_{n\to+\infty}T^n_{PH}u_n=T_{PH}u$

(2) *$T_{PH}u$ is harmonic on $R$.*

For the transformation $T_{HP}$ we may have the similar lemma.

LEMMA 1.6. *The harmonic and $P$-harmonc Green's functions have the relation*:

$$\begin{aligned}G(z,w)&=G^P(z,w)+\frac{1}{2\pi}\\&\quad\times\int_R P(\zeta)G(z,\zeta)G^P(w,\zeta)d\xi d\eta\\&=G^P(z,w)+\frac{1}{2\pi}\\&\quad\times\int_R P(\zeta)G(w,\zeta)G^P(z,\zeta)d\xi d\eta,\ \ \zeta=\xi+i\eta.\end{aligned}$$

PROOF. Applying Green's formula for $G^P(R_n,z,w)$ and $G(R_n,z,w)$, we have

$$G(R_n,z,w)=G^P(R_n,z,w)$$
$$+\frac{1}{2\pi}\int_{R_n}P(\zeta)G(R_n,z,\zeta)G^P(R_n,w,\zeta)d\eta d\xi.$$

Takilng limit as $n\rightarrow+\infty$, we get this lemma. Q. E. D.

THEOREM 1. 7. *$T_{PH}$ is a linear mapping of $P_H(R)$ into $H_P(R)$.*

PROOF. The inequality $G^P(z,w)<G(z,w)$ and the preceding lemma give the inequality for some point $z_1$ in $R$:

$$\int_R P(z)G^P(z_1,z)|T_{PH}u(z)|dxdy$$
$$\leq\int_R P(w)G(z_1,w)|u(w)|dudv.$$

This completes the proof. Q. E. D.

THEOREM 1. 8. *For a $P$-harmonic function $u$ in $P_H(R)$, we have $T_{HP}T_{PH}u=u$, that is, $T_{PH}$ is an isomorphism between linear spaces $P_H(R)$ and $T_{PH}(P_H(R))$ and $T_{HP}$ on $T_{PH}(P_H(R))$ is the inverse of $T_{PH}$.*

PROOF. Since $T_{PH}u$ is contained in $H_P(R)$ and the sequecnce of restrictions to $R_n$ of the function $T_{PH}u$ converges to $T_{PH}u$ on $R$ as $n\rightarrow+\infty$ Lemma 1.5 gives that

$$T_{HP}\ T_{PH}u=\lim_{n\rightarrow+\infty}T_{HP}^n\,T_{PH}u.$$

Since in the equality

$$T_{HP}^n(T_{PH}u)=T_{HP}^n(T_{PH}u-T_{HP}^n u)+T_{HP}^n\,T_{PH}^n\,u$$

the last term equals to $u$ for evry $n$, it is sufficient to prove that the first term converges to 0 as $n\rightarrow+\infty$.

Let $\rho_n^P$ be the $P$-harmonic measure of the region $R_n$. which represents the solution of Diriclet problem as integral. Then we have

$$T_{HP}^n(T_{PH}u-T_{PH}^n u)(z)=\int_{\partial R_n}(T_{PH}u-T_{PH}^n u)d\rho_{n,z}^P$$
$$=\int_{\partial R_n}(T_{PH}u-u)d\rho_{n,z}^P$$
$$=\frac{1}{2\pi}\int_R P(w)\left\{\int_{\partial R_n}G(\cdot,w)d\rho_{n,z}^P\right\}u(w)dudv,$$

where the definition $T_{PH}u$ and Fubini's theorem are used. The Lemma 1.6 gives

$$\begin{aligned}&\int_{\partial R_n} G(\cdot,w)d\rho_{n,z}^P \\ &\quad = \int_{\partial R_n} G^P(\cdot,w)d\rho_{n,z}^P \\ &\quad + \frac{1}{2\pi}\int_R P(\zeta\left\{\int_{\partial R_n} G^P(\cdot,\zeta)d\rho_{n,z}^P\right\}G(w,\zeta)d\xi d\eta.\end{aligned}$$

In this equality, applying

$$\int_{\partial R_n} G^P(\cdot,w)d\rho_{n,z}^P \leq G^P(z,w)$$

and

$$\lim_{n\to+\infty}\int_{\partial R_n} G^P(\cdot,w)d\rho_{n,z}^P = 0,$$

which are results of properties of Green's function, Lebesque's convergence theorem gives

$$\lim_{n\to+\infty}\int_{\partial R_n} G(\cdot,w)d\rho_{n,z}^P = 0$$

and Lemma 1.6 gives

$$\int_{\partial R_n} G(\cdot,w)d\rho_{n,z}^P \leq G(z,w).$$

Then, from Lebesque's convergence theorem it follows that

$$\lim_{n\to+\infty} T_{HP}^n(T_{PH}u - T_{PH}^n) = 0. \qquad \text{Q. E. D.}$$

LEMMA 1.9. *If $f \geq g$ on $R$ for $f$, $g$ in $P_H(R)$, then $T_{PH}f \geq T_{PH}g$ on $R$. If $u \geq v$ for $u$, $v$ in $H_P(R)$, then $T_{HP}u \geq T_{HP}v$ on $R$.*

PROOF. These are results of Lemma 1.5 and the maximum principle. Q. E. D.

It does not always hold that $T_{PH}T_{HP}f = f$ for $f$ in $H_P(R)$ such that $T_{HP}f$ is in $P_H(R)$.

A positive $P$-solution $u$ on $R$ is said to be $P$-minimal provided that for any $P$-solution $v$ ,$0 < v \leq u$ implies $v = \alpha u$ where $\alpha$ is a constant.

THEOREM 1.10. *For a minimal function $h$ in $T_{PH}(P_H(R))$, $T_{HP}h$ is a $P$-minimal function contained in $P_H(R)$.*

PROOF. This is evident by Theorem 1.8 and Lemma 1.9. Q. E. D.

Let $\bar{R}$ be any metrizable compactification of the Riemann surface $R$

and denote by $\Delta$ the ideal boundary of $R$ in this compactification, that is, $\Delta=\bar{R}-R$. For a $P$-harmonic function $u$ on $R$ and a compact set $A$ in $\Delta$ the reduced function of $u$ relative to $A$ is defined as the lower envelope of the class of those $P$-superharmonic functions on $R$ which majorize $u$ on the intersection of $R$ and some neiborhood of $A$ in $\bar{R}$. This reduced function is denoted by $(u)_A^P$. For a positive harmonic function $f$ on $R$ the similar reduced function $(f)_A^H$ is also defined. For reduced function we refer to Brelot [1, 2].

Considering a decreasing sequence $\{F_n\}$ of closed neiborhoods of the compact set $A$ in $\bar{R}$ whic converges to $A$, we can prove, by Lebesque's convergence theorem, the following lemma, for whose detail proof we refer to Satō [11].

LEMMA 1.11. *Let $A$ be a compact set in the ideal boundary of a metrizable compactification $\bar{R}$. For a positive $P$-harmonic function $u$ in $P_H(R)$, we have*

$$T_{PH}(u)_A^P=(T_{PH}u)_A^H \quad \text{on } R.$$

*And, for a positive harmonic function $f$ in $H_P(R)$ we have*

$$T_{HP}(f)_A^H=(T_{HP}f)_A^P \quad \text{on } R.$$

## 2. Thin sets on P-Martin compactification.

Let $R^*$ be the Martin compactification of a Riemann surface $R$ defined by Martin([6]) and $\Delta$ be its ideal boundary $R^*-R$. The set of minimal boundary points of $\Delta$ is denoted by $\Delta_1$ and Martin's kernel function with origin $z_0$ in $R$ is represented by $K(z,b)$ or $K_b(z)$ for $(z,b)$ in $R\times R^*$.

Let $R_P^*$ be the Martin compactification of $R$ with respect to the equation $L_Pu=0$ which can be constructed in the similar way to that of harmonic case as was treated by Martin (for example, see Nakai [7]). Let $\Delta_P$ be the ideal boundary $R_P^*-R$. The set of $P$-minimal boundary points is denoted by $\Delta_{P,1}$. The $P$-Martin kernel with origin $z_0$ in $R$ is denoted by $K^P(z,a)$ or $K_a^P(z)$, $(z,a)\in R\times R_P^*$, which satisfies $K^P(z_0,a)=1$ for $a$ in $\Delta_P$, and is finitely continuous on $R\times\Delta_P$. Let $u$ be a $P$-superharmonic function on $R$ and $A$ a compact subset of $\Delta_P$. The reduced function $(u)_A^P$ of $u$ relative to $A$ is defined as the lower envelope of class of positive $P$-superharmonic functions $v$ such that $v\geq u$ on the intersection of a neiborhood of $A$ and $R$ (see, for example. Brelot [1], [2]). This reduced function will be also written by $(u)_A$ simply. For a $P$-minimal harmonic function $u$ on $R$, $(u)_A^P$ is equal to 0 or $u$ for every closed set $A$ in $\Delta_P$ and there exists at least one point $a$ in $\Delta_{P,1}$ such that $(u)_{\{a\}}^P=u$ on $R$, which is called the pole of the $P$-minimal function $u$ on $R_P^*$ (Brelot [1]). In $P$-Martin compactification every $P$-minimal function $u$ has a unique pole on $\Delta_{P,1}$ and $u=\alpha K_a^P$, where $\alpha$ is the constant and $a$ is the point in $\Delta_{P,1}$ (Brelot [1]).

In the last part of the preceding section we recalled the reduced function in a general metrizable compactification $\overline{R}$ of $R$. In this case, for a $P$-minimal function $u$ on $R$ there exists at least one pole of $u$ on the boundary $\Delta$, but $u$ may have many poles on $\Delta$. About the uniqueness of $u$ we can say the following, if we take the Martin's compactification $R^*$.

THEOREM 2.1. *A $P$-harmonic function $u$ in $P_H(R)$ has a pole uniquely determined on the set $\Delta_1$ of minimal boundary points of Martin's compactification $R^*$. The harmonic function $T_{PH}u$ is a minimal harmonic function with the same pole as that of $u$.*

PROOF. The reduced function of the positive harmonic function $T_{PH}u$ with respect to a compact set $A$ is represented by the canonical measure supported by $A$. Letting $A$ be the set {b} of any pole $b$ of $u$ on the Martin's boundary we have

$$(T_{PH}u)_{\{b\}}=\alpha K_b,$$

where $\alpha$ is a positive constant. From Lemma 1.11, it follows that

$$\begin{aligned} T_{PH}u &= T_{PH}(u)_{\{b\}}^P \\ &=(T_{PH}u)_{\{b\}}=\alpha K_b. \end{aligned}$$

Let $b_1,b_2$ be two poles of $u$ on $\Delta_1$. Then $\alpha K_{b_1}=\beta K_{b_2}$, from which it follows that $b_1=b_2$, where $\alpha$ and $\beta$ are positive constants. Q. E. D.

About pole of a minimal harmonic function we may say the following by the same way as above.

THEOREM 2.2. *A minimal harmonic function $f$ in $H_P(R)$ has a pole uniquely determined in $\Delta_{P,1}$, provided that $T_{HP}f$ is positive. The $P$-harmonic function $T_{HP}f$ is $P$-minimal with the same pole as that of $f$.*

Therefore we can make the following definitions.

DEFINITION 2.3. *We define the subsets $\Delta_{PH}$ of $\Delta_{P,1}$ and $\Delta_{HP}$, $\Delta_{HP}^0$ pf $\Delta_1$ by*

$$\Delta_{PH}=\{a\in\Delta_{P,1} : K_a^P \text{ belongs to } P_H(R)\},$$
$$\Delta_{HP}=\{b\in\Delta_1 ; K_b \text{ belongs to } H_P(R)\},$$

$$\Delta_{HP}^{0} = \{b \in \Delta_{HP} : T_{HP}K_b > 0\}.$$

For a point $a$ in $\Delta_{PH}$ the transform $T_{PH}K_a^P$ is a minimal function contained in $T_{PH}(P_H(R))$ by Theorem 2.1. Then there exists a point $b \in \Delta_1$ which is the pole of the function $T_{PH}K_a^P$.

DEFINITION 2.4. *To a point $a$ in $\Delta_{PH}$ assigning the point $b$ in $\Delta_1$ which is a pole of $T_{PH}K_a^P$ in $\Delta_{HP}$ we define a mapping*

$$t_{PH} : \Delta_{PH} \to \Delta_1$$

*Let $\Delta(P)$ be the image of the mapping $t_{PH}$.*

LEMMA 2.5. *$\Delta(P)$ is a subset of $\Delta_{HP}^{0}$.*

PROOF. This is a result of Theorem 1.7 and Theorem 1.8. Q.E.D.

It is evident from the definition that

$$T_{PH}K_a^P = T_{PH}K_a^P(z_0)K_{t_{PH}(a)} \text{ on } R.$$

DEFINITION 2.6. *For a point $b$ in $\Delta_{HP}^{0}$ assinging a point $a$ which is the pole of the $P$-minimal function $T_{HP}K_b$ in $\Delta_{P,1}$ we define the mapping*

$$t_{HP} : \Delta_{HP}^{0} \to \Delta_{P,1}.$$

LEMMA 2.7. $\Delta(P) = \{b \in \Delta_1 : K_b \in T_{PH}(P_H(R))\}$.

PROOF. This is evident from the definition of $T_{PH}(P_H(R))$ and Theorem 1.12. Q.E.D.

LEMMA 2.8. *The restriction of the mapping $t_{HP}$ to $\Delta(P)$ is the inverse mapping of $t_{PH} : \Delta_{PH} \to \Delta(P)$.*

PROOF. This follows from Theorem 1.8 and definitions of $t_{PH}$, $t_{HP}$. Q.E.D.

LEMMA 2.9. *$\Delta_{PH}$ is a Borel measurable subset of $\Delta_{P,1}$.*

PROOF. The function

$$\int_R P(w)G(z_1,w)K^P(w,a)dudv$$

of $a$ in $\Delta_{P,1}$ is the limit of an increasing sequence of continuous functions

$$\int_{R_n} P(w)G(R_n,z_1,w)K^P(w,a)dudv$$

of $a$. Q.E.D.

LEMMA 2.10. *The set $\Delta_{HP}^0$ is Borel measurable in $\Delta_1$.*

PROOF. This is evident. Q.E.D.

For any positive $P$-harmonic function $u$ on $R$ there exists a unique measure $\mu$ on $\Delta_{P,1}$ such that $\mu(\Delta_P-\Delta_{P,1})=0$ and

$$u(z)=\int_{\Delta_{P,1}} K^P(z,a)d\mu(a),\ z\in R.$$

This measure is called the canonical measure of $u$. Let $\chi^P$ is the canonical measure of the $P$-elliptic measure $e^P$.

THEOREM 2.11. *Let $u$ be a positive $P$-harmonic function on $R$ which belongs to the class $P_H(R)$, and let $\mu$ be the canonical measure of $u$. Then the set $\Delta_{P,1}-\Delta_{PH}$ has $\mu$-measure zero.*

PROOF. For each positive integer $n$, let $E_n$ be a set of points $a$ in $\Delta_{P,1}$ such that

$$\int_R P(w)G(z_1,w)K^P(w,a)dudv>n,$$

where $z_1$ is a fixed point in $R$ and $E_n$ is a measurable set by the proof of Lemma 2.6. By Fubini's theorem we have

$$\begin{aligned}\mu(\cap_{n=1}^{\infty}E_n)&\leq\mu(E_n)\\ &\leq\frac{1}{n}\int_{E_n}\left\{\int_R P(w)G(z_1,w)K^P(w,a)dudv\right\}d\mu(a)\\ &\leq\frac{1}{n}\int_R P(w)G(z_1,w)u(w)dudv,\end{aligned}$$

by which $\mu(\Delta_{P,1}-\Delta_{PH})=0$ is obtained. Q.E.D.

COROLLARY 2.12. *Let $P$ be a hyperbolic density on $R$. Then we have*

$$\chi^P(\Delta_{PH})=e^P(z_0)\ \textit{and}\ \chi^P(\Delta_{P,1}-\Delta_{PH})=0.$$

REMARK 2.13. Let $R$ be an open unit disk $\{z\in C:|z|<1\}$ and we take a density $P(z)=\phi(r)$ where $\phi$ is a bounded function of $r$ and $z=re^{i\theta}$, which is a hyperbolic density by Lemma 1.1. In this case $P$-Martin compactification of $R$ is the closed disk $\{z\in C:|z|\leq 1\}$ and the canonical measure $\chi^P$ of the $P$-elliptic measure $e^P$ is $kd\theta$ where $K$ is a constant dependent on $P$. Let $K^P(z,e^{i\theta})$ be $P$-Martin kernel normalized at origin $O$. Then for every $\theta$, $K^P(z,e^{i\theta})$ is contained in the class $P_H(R)$, because from the inequality(Theorem 1.4)

$$\int_R P(w)G(0,w)e^P(w)dudv$$
$$= k\int_{[0,2\pi]}\left\{\int_R P(w)G(0,w)K^P(w,e^{i\theta})dudv\right\}d\theta < +\infty$$

it follows that the rotation free function

$$\theta \to \int_R P(w)G(0,w)K^P(w,e^{i\theta})dxdy$$

is finite for every $\theta$. Therefore we have $\Delta_{PH} = \{z \in C : |z| = 1\}$.

Let $D$ be a Lipschitz domain with a point $z_1$ fixed in $D$. We say a function $u$ on $D$ is a kernel function at $a \in \partial D$ if $u$ is positive and harmonic on $D$ with $u(z_1) = 1$ and $u(z)$ vanishes continuously as $z$ approches to every boundary point $b$ of the domain $D$ with $b \neq a$. One fundamental result of Hunt and wheeden [4] is a uniform estimate for various approximations to kernel functions and another is the uniqueness of kernel functions on $D$. And, for Lipshcitz domains they showed that Martin kernels of $D$ are exactly the unique kernel function. Then they identified the Martin ideal boundary with the Euclidean boundary and showed that the Martin topology is equivalent to the Euclidean topology. On the other hand Taylor([12]) obtained the similar results for a uniformly bounded second order elliptic operator on a Lipschitz domain as Hunt and Wheeden. By these results we can show the following proposition:

PROPOSITION. *The transformation* $t_{PH} : \Delta_{PH} \to \Delta_1$ *is the identity transformation of* $\{z \in C : |z| = 1\}$.

PROOF. For each integer $n$ let $N_n$ be the neiborhood of a point $e^{i\theta}$ on the unit circle:

$$N_n = \{z \in C : |z - e^{i\theta}| < 1/n\},$$

and $R_n$ be the subset of unit disk:

$$\{z \in C : |z| < 1\} - N_n.$$

Since the $P$-Martin kernel $K^P(z,e^{i\theta})$ is $P$-harmonic on $R_n$ and may be continuously extended on $\partial R_n - R$ with the boundary value zero by Proposition 4.5 in J. C. Taylor ([12]), we have that

$$T^n_{PH}K^P_{e^{i\theta}}(z) = K^P_{e^{i\theta}}(z)$$
$$+ \frac{1}{2\pi}\int_{R_n} P(w)G(R_n,z,w)K^P_{e^{i\theta}}(w)dudv$$

is harmonic on $R_n$ and continuously extended at each point on $\partial R_n - R$

with the boundary value zero. Recall that $K^P(z.e^{i\theta})$ belongs to $P_H(R)$ and from Lemma 1.5 it follows that

$$\lim_{n\to\infty} T^n_{PH} K^P_{e^{i\theta}} = T_{PH} K^P_{e^{i\theta}}$$

Then the functionn $T_{PH}K^P(z,e^{i\theta})/T_{PH}K^P(0,e^{i\theta})$ is a kernel function at $e^{i\theta}$ by Lemma 2.6 in Hunt and wheeden [4]. Since Martin kernel $K_{e^{i\theta}}$ is also a kernel function at $e^{i\theta}$, the uniqueness of kernel functions at $e^{i\theta}$ gives that

$$\begin{aligned} T_{PH}K^P_{e^{i\theta}} &= T_{PH}K^P_{e^{i\theta}}(0)\times K_{e^{i\theta}} \\ &= T_{PH}K^P_{e^{i\theta}}(0)\times K_{t_{PH}(e^{i\theta})}. \end{aligned}$$

Then we have $t_{PH}(e^{i\theta})=e^{i\theta}$ for every $\theta$, that is, $t_{PH}$ is the identity transformation of the circle $\{z\in C: |z|=1\}$. Q. E. D.

For a positive $P$-superharmonic function $u$ on $R$ the reduced function $(u)^P_E$ (denoted also by $(u)_E$) relative to a closed set $E$ in $R$ is defined as the lower envelope of the class of positive $P$-superharmonic $v$ such that $v\geq u$ on $E$ except for a set of capacity zero. For properties of reduced functions we refer to [Naïm [6], Brelot [1] and Constantinescu and Cornea [3]). For a $P$-minimal function $u$ on $R$ the reduced function $(u)_E$ relative to $E$ is $u$ or a potential with kernel $G^P$. Let $E$ be a closed set in $R$ and a point $a$ in $\Delta_{P,1}$. The set $E$ is said to be $P$-thin at $a$ provided that $(K^P_a)_E$ is a potential, that is, $(K^P_a)_E$ is not $K^P_a$. In the case where $P$ is constantly zero $E$ is said to be thin at $b$ in $\Delta_1$ provided $(K_a)_E$ is a Green potential.

THEOREM 2.14. *A closed set $E$ in $R$ is $P$-thin at a point $a\in\Delta_{PH}$ if and only if $E$ is thin at the point $t_{PH}(a)$.*

PROOF. Let a point $a$ be in $\Delta_{PH}$. For a closed set $E$ in $R$ such that

$$(K^P_a)_E=K^P_a \quad \text{on } R,$$

that is, $E$ is not $P$-thin at a point $a$, we shall show that

$$(T_{PH}K^P_a)_E=T_{PH}K^P_a \quad \text{on } R,$$

which means that $E$ is not thin at $t_{PH}(a)$.

If $z$ is an interior point of $E$ or a regular point for the open set $R$-$E$, then the equality is evident.

For $z$ in $R-E$ the function $(T_{PH}K^P_a)_E$ is the solution of Diriclet problem on the open set $R-E$ relative to the equation $\Delta u=0$ with boundary value $T_{PH}K^P_a$ on $\partial(R-E)$. Let $\rho^P_z$ be the $P$-harmonic measure and $\rho_z$ the harmonic measure for the open set $R-E$, respectively. Then we have, for $z$ in $R-E$,

$$
(*)\quad \begin{aligned}
(T_{PH}K_a^P)_E(z) &= \int_{\partial(R-E)} T_{PH}K_a^P(\cdot)d\rho_z \\
&= \int_{\partial(R-E)} K_a^P(\cdot)d\rho_z \\
&+ \int_{\partial(R-E)} \{\frac{1}{2\pi}\int_R P(w)G(\cdot,\, w)K_a^P(w)dudv\}d\rho_z.
\end{aligned}
$$

The first term of $(*)$ may be written by Lemma 1.5 as follows:

$$
\begin{aligned}
&\text{the first term} = \int_{\partial(R-E)} K_a^P d\rho_z^P \\
&+ \frac{1}{2\pi}\int_{R-E} P(w)G(R-E,z,w)\left\{\int_{\partial(R-E)} K_a^P d\rho_w^P\right\}dudv,
\end{aligned}
$$

where we denoted the Green's function for the open set $R-E$ by $G(R-E, z,w)$ and used

$$
\int_{\partial(R-E)} K_a^P d\rho_z^P = \lim_{n\to+\infty}\int_{\partial(R_n\cap(R-E))} (K_a^P)_n d\rho_{n,z}^P
$$

each term of which is dominated by $K_a^P \in P_H(R)$, letting $(K_a^P)_n = K_a^P$ on $R_n \cap \partial(R-E)$ and $=0$ on $\partial R_n \cap (R-E)$, $\rho_{n,z}^P$ be $P$-harmonic measure for the open set $R_n \cap (R-E)$. The first terme of $(*)$ is equal to, by unthinness of $E$ at $a$,

$$
\begin{aligned}
&(K_a^P)_E(z) + \frac{1}{2\pi}\int_{R-E} P(w)G(R-E,z,w)(K_a^P)_E(w)dudv \\
&= K_a^P(z) + \frac{1}{2\pi}\int_{R-E} P(w)G(z,w)K_a^P(w)dudv \\
&\quad -\frac{1}{2\pi}\int_{R-E} P(w)\left\{\int_{\partial(R-E)} G(\cdot,\, w)d\rho_z\right\}K_a^P(w)dudv,
\end{aligned}
$$

in which we used

$$
G(R-E,z,w) = G(z,w) - \int_{\partial(R-E)} G(\cdot,\, w)d\rho_z.
$$

The second term of the equality $(*)$ is decomposed as follows:

$$
\begin{aligned}
&\frac{1}{2\pi}\int_E P(w)G(z,w)K_a^P(w)dudv \\
&+ \frac{1}{2\pi}\int_{R-E} P(w)\left\{\int_{\partial(R-E)} G(\cdot,\, w)d\rho_z\right\}K_a^P(w)dudv,
\end{aligned}
$$

Therefore we obtained that

$$
(T_{PH}K_a^P)_E = T_{PH}K_a^P \quad \text{for } z \text{ in } R-E,
$$

so that, this equality holds on $R$ except for the set of irregular points of the open set $R-E$. From this equality we have that $(T_{PH}K_a^P)_E=T_{PH}K_a^P$ on $R$, which is equivalent to

$$(K_{t_{PH}(a)})_E=K_{t_{PH}(a)} \text{ on } R.$$

Similaly it may be verified that

$$(K_a^P)_E^P<K_a^P \text{ on } R-E$$

implies

$$(T_{PH}K_a^P)_E<T_{PH}K_a^P \quad \text{on } R-E,$$

which means that if $E$ is $P$-thin at $a$, the set $E$ is thin at $t_{PH}(a)$. Q. E. D.

The class of open sets $G$ in $R$ such that $R-G$ is $P$-thin at $a\in\Delta_{P,1}$ is denoted by $\mathscr{G}^P(a)$. In the case where $P$ is identically zero, $\mathscr{G}^H(a)$ or simply $\mathscr{G}(a)$ are used in stead of $\mathscr{G}^P(a)$ (Constantinescu and Cornea [3]). $\mathscr{G}^P(a)$ is a base of the fine neiborhood of $a$.

COROLLARY 2.15. *For $a$ in $\Delta_{PH}$, $\mathscr{G}^P(a)=\mathscr{G}^H(t_{PH}(a))$.*

By changing roles $H$ and $P$ in the proof of the preceding theorem we obtain the following theorem.

THEOREM 2.16. *A closed set $E$ in $R$ is thin at a point $b$ in $\Delta_{HP}^0$ if and only if $E$ is $P$-thin at $t_{HP}(b)$.*

COROLLARY 2.17. For a point $b$ in $\Delta_{HP}^0$, $\mathscr{G}^H(b)=\mathscr{G}^P(t_{HP}(b))$.

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