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Topologically extremal real algebraic surfaces in $\boldsymbol{P}^2\times\boldsymbol{P}^1$ and $\boldsymbol{P}^1\times\boldsymbol{P}^1\times\boldsymbol{P}^1$

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Dedicated to Professor Haruo Suzuki on his 60th birthday

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0, Introduction

In this paper, from a general viewpoint, we construct surfaces in $\boldsymbol{P}^2\times\boldsymbol{P}^1$ and $\boldsymbol{P}^1\times\boldsymbol{P}^1\times\boldsymbol{P}^1$ defined over $\boldsymbol{R}$ having topologically extremal properties. Precisely we show that for each pair of positive integers (d, r) (resp. (d, e, r)) there exists an M-surface A in $\boldsymbol{P}^2\times\boldsymbol{P}^1$ (resp. $\boldsymbol{P}^1\times\boldsymbol{P}^1\times\boldsymbol{P}^1$) of degree (d, r) (resp. (d, e, r)) such that the projection $A\rightarrow\boldsymbol{P}^1$ has the maximal number of real critical points (Theorem 0.1 and Corollary 0.5). Also, we show the existence of M-surfaces in $(\boldsymbol{P}^2\#(-\boldsymbol{P}^2))\times\boldsymbol{P}^1$, (Corollary 0.6). Furthermore, the construction of M-surfaces in $\boldsymbol{P}^3$ by O. Ya. Viro [V1] is explained by a similar argument as that of this paper (Theorem 0.7).

Harnack [H] pointed out that the number of components in the real locus of a curve in $\boldsymbol{P}^2$ of degree d defined over $\boldsymbol{R}$ does not exceed $1+(1/2)(d-1)(d-2)$ and, for each d, there exists a non-singular curve in $\boldsymbol{P}^2$ of degree d defined over $\boldsymbol{R}$, the real locus of which has exactly $1+(1/2)(d-1)(d-2)$ components.

Hilbert, in his famous 16th problem, proposed to investigate topological restrictions for hypersurfaces in $\boldsymbol{P}^n$ of fixed degree defined over $\boldsymbol{R}$, especially for $n=2, 3$. Amount of papers are devoted to this problem (see [G1], [V2], [W]). For instance, non-singular real curves in $\boldsymbol{P}^2$ of degree ≤ 7 and surfaces in $\boldsymbol{P}^3$ of degree ≤ 4 are classified topologically. To establish such classification, we first find some restrictions on topological invariants. Second, for a fixed degree, we construct real hypersurfaces of given degree, invariants of which are permitted by the ristrictions. Then, such as Harnack's result, it is the first step of the study to obtain an uniform estimate on real hypersurfaces of given degree and to show the sharpness of the estimate.

On the other hand, we may regard a real algebraic function as a 1

-parameter family of hypersurfaces defined over $\boldsymbol{R}$. Thus, it is natural to proceed to investigate topological restrictions for hypersurfaces in $\boldsymbol{P}^n\times\boldsymbol{P}^1$ of fixed degree defined over $\boldsymbol{R}$, considered as one-parameter families of hypersurfaces in $\boldsymbol{P}^n$, and the projection to $\boldsymbol{P}^1$ of the hypersurfaces.

Now let $A\subset\boldsymbol{P}^n\times\boldsymbol{P}^1$ be a real hypersurface of degree (d, r), that is, the zero-locus of a polynomial $\sum_{i=0}^r F_i(X_0, \dots, X_n)\lambda^{r-i}\mu^i$, where F_i $(0\leq i\leq r)$ is a real homogeneous polynomial of degree d.

Consider the projection $\varphi: A\to\boldsymbol{P}^1$. Then our main object is the topology of real locus $A_{\boldsymbol{R}}$ of A and singularities of the restriction $\varphi_{\boldsymbol{R}}: A_{\boldsymbol{R}}\to\boldsymbol{R}P^1$ of φ to $A_{\boldsymbol{R}}$.

We denote by $P_t(X, K)$ the Poincaré series of a topological space X over a field K with indeterminate t, and by $s(f)$ the number of critical points of a function $f: X\to R$ from an n-dimensional manifold to a 1-dimensional manifold.

It is known that, if $A\subset\boldsymbol{P}^n\times\boldsymbol{P}^1$ is non-singular, then the diffeomorphism type of A is determined by the degree (d, r) and n. For example, we see, for any K,

$$P_1(A, K)=\begin{cases}\chi(A), & (n:\text{even}),\\ 4n-\chi(A), & (n:\text{odd}),\end{cases}$$

and

$$\chi(A)=(n+1)(1-d)^n r+2\left(\frac{(1-d)^{n+1}-1}{d}+n+1\right),$$

where $\chi(A)=P_{-1}(A, K)$ is the Euler characteristic of A, (see 1.6).

We call the hypersurface A generic if A is non-singular and $\varphi: A\to\boldsymbol{P}^1$ has only non-degenerate critical points.

If A is generic, then $s(\varphi)=(n+1)(d-1)^n r$, (see 1.6).

By Harnack-Thom's inequality ([G1], [T]), and the fact that a critical point of $\varphi_{\boldsymbol{R}}$ is necessarily a critical point of φ, we have an uniform estimate:

$$\text{(0.0)}\qquad \begin{cases} P_1(A_{\boldsymbol{R}}; \boldsymbol{Z}/2)\leq P_1(A; \boldsymbol{Z}/2),\\ \quad s(\varphi_{\boldsymbol{R}}) \quad \leq s(\varphi).\end{cases}$$

Remark that the right hand sides depend only on $(n; d, r)$.

In this paper, from a general viewpoint, we show the following

THEOREM 0.1. *For $n=1, 2$ and for each (d, r), the estimate* (0.0) *is sharp (with respect to the usual real structure of $\boldsymbol{P}^n\times\boldsymbol{P}^1$), that is, there exists a generic real hypersurface of $\boldsymbol{P}^n\times\boldsymbol{P}^1$, invariant under the complex conjugation, of degree (d, r), attaining both equalities in* (0.0).

Notice that the estimate (0.0) is reduced to

$$\begin{cases} P_1(A_{\boldsymbol{R}}\,;\boldsymbol{Z}/2)\leq 2+2(d-1)(r-1), \\ \quad s(\varphi_{\boldsymbol{R}}) \quad \leq 2d(r-1), \end{cases}$$

for $n=1$, and to

$$\begin{cases} P_1(A_{\boldsymbol{R}}\,;Z/2)\leq 3+d^2+3(d-1)^2(r-1), \\ \quad s(\varphi_{\boldsymbol{R}}) \quad \leq 3(d-1)^2 r, \end{cases}$$

for $n=2$.

To consider togather with the number of real critical points of the projection is an essential idea of this paper. (See the proof of Theorem 0.3 in §3, which implies Theorem 0.1.)

In the case $r=1$, Theorem 0.1 is proved in [I]. (See also Example 2.3.2.).

A finer result is obtained in the case $n=1$. For $A\subset \boldsymbol{P}^1\times\boldsymbol{P}^1$, we denote by $\pi: A\to\boldsymbol{P}^1$ the projection to the first component.

PROPOSITION 0.2. *For non-singular real curves* $A\subset\boldsymbol{P}^1\times\boldsymbol{P}^1$ *of degree* (d, e) *such that both* φ, π *have only non-degenerate critical points, there exists the sharp estimate :*

$$\begin{cases} P_1(A_{\boldsymbol{R}}\,;\boldsymbol{Z}/2)\leq 2+2(d-1)(e-1), \\ s(\varphi_{\boldsymbol{R}})\leq 2(d-1)e,\ s(\pi_{\boldsymbol{R}})\leq 2d(e-1). \end{cases}$$

We omit the proof of Proposition 0.2. (See §4 for the method to construct M-curves with special properties.)

Hereafter we concentrate to the case $n=2$.

Now let us formulate a general theorem which implies Theorem 0.1. For notions and notations, see §§1 and 2.

Let S be compact connected M-surface and, L be a real holomorphic very ample line bundle over S(see 2.6 and 1.9).

Denote by g the genus of zero-locus of a transverse section of L (see 1.0).

Let $s_0, s_1, \ldots, s_r$ be M-sections of L (see 2.7). Consider the following condition (*):

(* i) $(s_i)_0$ and $(s_j)_0$ intersect in $\langle c_1(L)^2, [S]\rangle$ points in $S_{\boldsymbol{R}}$, $(0\leq i<j\leq r)$,

(* ii) The real locus of $(s_is_j)_0=(s_i)_0\cup(s_j)_0$ has $2g$ empty ovals, (see 2.9), $(0\leq i<j\leq r)$,

(*iii) The ratio s_js_k/s_i^2 has a constant sign on the union of interiors of g-empty ovals of $(s_j)_0$, $(0\leq i<j<k\leq r)$.

Remark that $s_js_k/s_i^2 : (S-(s_is_js_k)_0)_{\boldsymbol{R}}\to\boldsymbol{R}-0$ is well-defined.

We denote by $\boldsymbol{P}^1_1$ (or simply by $\boldsymbol{P}^1$) the real complex curve $(\boldsymbol{P}^1, \tau_1)$, where τ_1 is the compex conjugation (see 2.3). Let $[\lambda:\mu]$ be the homogeneous coordinate of $\boldsymbol{P}_1^1$.

THEOREM 0.3. *Let S be a compact connected M-surface with $H_1(S;\boldsymbol{Z}/2)=0$ and $H_0(S_{\boldsymbol{R}};\boldsymbol{Z}/2)\cong\boldsymbol{Z}/2$, L be a real holomorphic very ample line bundle with given M-sections $s_0,\dots,s_r$ of L satisfying the condition (*). Then, $A\subset S\times\boldsymbol{P}_1^1$ defined by*

$$\sum_{i=0}^{r}\epsilon_i\lambda^{r-i}\mu^i s_i(x)=0$$

is an M-manifold with $(r-1)g$ empty ovals and each critical point of $\varphi: A\to\boldsymbol{P}_1^1$ is non-degenerate and real, for some real numbers $\epsilon_0,\epsilon_1,\dots,\epsilon_r$ with $1=\epsilon_0\gg|\epsilon_1|\gg\dots\gg|\epsilon_r|>0$.

REMARK 0.4: (1) For the existence of M-sections satisfying (*), see §4, Theorem 4.0.

(2) The assumption on the topology of S is essential for our costruction. See also Lemma 2.5.

(3) We regard λ and μ as sections of $\mathcal{O}_{\boldsymbol{P}_1^1}(1)$, and $s=\sum\epsilon_i\lambda^{r-i}\mu^i s_i$ as a section of $L_r=\xi^*L\cdot\psi^*\mathcal{O}_{\boldsymbol{P}_1^1}(r)$, where $\psi: S\times\boldsymbol{P}_1^1\to\boldsymbol{P}_1^1$, and $\xi: S\times\boldsymbol{P}_1^1\to S$ the projections respectively. Then we have $A=(s)_0$ and that s is an M-section. For a transverse section s of L_r (see 1.3), denote by $\varphi: A\to\boldsymbol{P}_1^1$. Then, associated to s, there is a natural section of Hom $(TA,\varphi^*T\boldsymbol{P}_1^1)$ induced from the tangent map of φ. Theorem 0.3 states that this section is also an M-section.

(4) Since (* i) implies that $(s_i)_0$ and $(s_j)_0$, $(i\neq j)$, intersect transversely, the condition (*) is preserved by small perturbation of $s_0,\dots,s_r$ in the space of real sections of L.

Setting $S=\boldsymbol{P}^1\times\boldsymbol{P}^1(=\boldsymbol{P}_1^1\times\boldsymbol{P}_1^1)$ and $L=\mathcal{O}_{\boldsymbol{P}^1}(d)\cdot\mathcal{O}_{\boldsymbol{P}^1}(e)$ over S, we see Theorem 0.3 implies

COROLLARY 0.5. *For non-singular real surfaces $A\subset\boldsymbol{P}^1\times\boldsymbol{P}^1\times\boldsymbol{P}^1$ of degree (degree (d,e,r) such that $\varphi: A\to\boldsymbol{P}^1$ has only non-degenerate critical points, there exists the sharp estimate :*

$$\begin{cases} P_1(A_{\boldsymbol{R}};\boldsymbol{Z}/2)\le 6der-4de-4er-4rd+4d+4e+4r,\\ \quad s(\varphi_{\boldsymbol{R}})\quad\le r(6de-4d-4e+4).\end{cases}$$

Let $S\subset\boldsymbol{P}^2\times\boldsymbol{P}^1$ be a generic real surface of degree $(1,1)$. Then S is the blowing up of $\boldsymbol{P}^2$ along a real point in $\boldsymbol{P}^2$, (see Example 2.3.2). We denote it by $\boldsymbol{P}^2\#(-\boldsymbol{P}^2)$.

We call a surface $A\subset(\boldsymbol{P}^2\#(-\boldsymbol{P}^2))\times\boldsymbol{P}^1\subset\boldsymbol{P}^2\times\boldsymbol{P}^1\times\boldsymbol{P}^1$ of degree $(d,e,$

r), if A is the zero-locue of a transverse section of $\mathcal{O}_{P^2}(d)\mathcal{O}_{P^1}(e)\mathcal{O}_{P^1}(r)|S\times \boldsymbol{P}^1$.

COROLLARY 0.6. *For non-singular real surfaces $A\subset(\boldsymbol{P}^2\#(-\boldsymbol{P}^2))\times \boldsymbol{P}^1$ of degree (d, e, r) such that $\varphi: A\to \boldsymbol{P}^1$ has only non-degenerate critical point, there exists the sharp estimates :*

$$\begin{cases} P_1(A_{\boldsymbol{R}}\,;\boldsymbol{Z}/2)\leq 3r(2d^2+4de-5d-3e+3)-2(2d^2+4de-5d-3e), \\ \quad s(\varphi_{\boldsymbol{R}}) \quad \leq 3r(2d^2+4de-5d-3e+3). \end{cases}$$

Viro [V1] constructed M-surfaces in $\boldsymbol{P}^3$. Unfortunately, only a sketch of the construction is given in [V1]. Here, we can clarify the Viro's construction as a prototype of the proof of our Theorem 0.3. (On the other hand, we have to remark that the constructions in this paper are inspired by the original Viro's construction.)

THEOREM 0.7 (Viro). *For non-singular real surfaces A in $\boldsymbol{P}^3$ of degree d, there exists the sharp estimates :*

$$P_1(A_{\boldsymbol{R}}\,;\boldsymbol{Z}/2)\leq d^3-4d^2+6d.$$

PROOF: Let X_0, X_1, X_2, X_3 be homogeneous coordinates of $\boldsymbol{P}^3$. Set $\boldsymbol{P}^2=\{X_3=0\}$, $\boldsymbol{P}^1=\{X_2=X_3=0\}$ and $\ell=\{X_0=X_1=0\}$. Let $\varphi: \boldsymbol{P}^3-\ell\to \boldsymbol{P}^1$ be a projection. Fix a tubular neighborhood U of ℓ in $\boldsymbol{P}^3$ such that $\bar{U}\cup \boldsymbol{P}^1$ is empty.

Observe that, for each d, there exist M-sections $s_0, \dots, s_d$ of $\mathcal{O}_{P^2}(0), \dots, \mathcal{O}_{P^2}(d)$ near $X_2^0, \dots, X_2^d$ respectively such that $(s_i)_0$ and $(s_j)_0$ intersect in ij points in $\boldsymbol{R}P^2$, the real locus of $(s_is_j)_0$ has $(1/2)(i-1)(i-2)+(1/2)(j-1)(j-2)$ empty ovals, $(1\leq i<j<\leq d)$, the ratio s_js_k/s_i^2 has constant sign on the union of interiors of $(1/2)(j-1)(j-2)$ empty ovals of $(s_j)_0$, $(1\leq i<j<k\leq d)$, and $\varphi|(s_i)_0$ has $(i-1)i$ real critical points $(0\leq i\leq d)$. (For the construction, see the proof of Proposition 4.0 in §4.) Naturally each s_i is extended to a real section $\tilde{s}_i$ of $\mathcal{O}_{P^3}(i)$, $(0\leq i\leq d)$.

Set

$$s=\sum_{i=0}^{d}\epsilon_i X_2^{d-i}\,\tilde{s}_i\in H^0(\boldsymbol{P}^3, \mathcal{O}_{P^3}(d))_{\boldsymbol{R}},$$

and set $A=(s)_0$. Take real numbers $\epsilon_0, \dots, \epsilon_d$ to be $1=\epsilon_0\gg|\epsilon_1|\gg\cdots\gg|\epsilon_d|>0$ and of apropriate signs.

Now, $\varphi_{\boldsymbol{R}}: A_{\boldsymbol{R}}\to \boldsymbol{R}P^1$ defines a vector field ξ' over $A_{\boldsymbol{R}}-U$, and ξ' can be extended to a C^∞ vector field ξ over $A_{\boldsymbol{R}}$ with finite singularities.

Denote by $s^+(\xi)$ (resp. $s^-(\xi)$) the sum of positive (resp. negative) indices of singular points of ξ, and set $t_i=\dim_{\boldsymbol{Z}/2}H_i(A_{\boldsymbol{R}}\,;\boldsymbol{Z}/2)$, $(i=1, 2, 3)$.

Then we see

$$s^{+}(\xi)\geq d+\frac{1}{3}d(d-1)(d-2),$$

$$s^{-}(\xi)\geq\frac{1}{3}(d+1)d(d-1)+\frac{1}{3}d(d-1)(d-2).$$

Thus $\chi(A_{\boldsymbol{R}})=s^{+}(\xi)-s^{-}(\xi)\geq d-(1/3)(d+1)d(d-1)$. On the other hand $t_0+t_1\geq 2+(1/3)(d-1)(d-2)(d-3)$. Hence we have

$$\begin{aligned}P_1(A_{\boldsymbol{R}}\,;\boldsymbol{Z}/2)&=t_0+t_1+t_2\\&=2(t_0+t_2)-\chi(A_{\boldsymbol{R}})\\&\geq d^3-4d^2+6d\ \ (=P_1(A\,;\boldsymbol{Z}/2)).\end{aligned}$$

By Harnack-Thom's inequality, all equalities hold.

Q. E. D.

To obtain exact uniform upper estimates as (0.0), we need several standard results in complex geometry. We write down them in §1. Notice that results in §1 play an important role also to construct real algebraic manifolds with special properties in §§3, 4.

In §2, we give preliminary on real geometry mainly to show Theorem 0.3. In general, to determine the topological type of a constructed real algebraic manifold is a difficult and delicate problem. Usually, in a paper on classical real algebraic geometry, this problem is left to the reader's intuition with the help of rough figures. In this paper, we try to give a foundation to this problem as exactly and generally as possible.

We prove the main Theorem 0.3 in §3.

Sufficient conditions for the existence of a pair of M-sections satisfying (*) are studied in §4 (Proposition 4.0 and Proposition 4.2). Also in §4, we prove Theorem 0.1 and Corollaries 0.5, 0.6.

Recently, Viro introduced a powerful method of constructing real plane curves. (See [V2].) It would be very interesting to apply this method to our situation treated in this paper.

Throughout this paper, for vector bundles L, K and sections s, s', we use the following abridgements: $L\cdot K=L\otimes K$, $L^d=L\otimes\cdots\otimes L$ (d-times), $s\cdot s'=s\otimes s'$ and $s^d=s\otimes\cdots\otimes s$ (d-times).

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1. Preliminary: Complex geometry

(1.0) Let V be a compact complex manifold, $\pi: E\rightarrow V$ a holomorphic vector bundle and $s: V\rightarrow E$ be a holomorphic section. Set $(s)_0=\{x\in$

$V|s(x)=0\}$.

We call s a transverse section if $s: V\to E$ is transverse to the zero section $\zeta\subset E$, that is, for any $x\in(s)_0$, $s_*T_xV+T_{s(x)}\zeta=T_{s(x)}E$.

If s is transverse, then $(s)_0$ is a complex submanifold of V and the codimension of $(s)_0$ is equal to the dimension of fibers of E.

Denote by H the complex vector space $H^0(V,E)$ of totality of holomorphic sections of E over V, and by $PH=(H-0)/\boldsymbol{C}^*$ the projectification of H.

Set $\mathscr{X}=\{(x,[s])\in V\times PH|s(x)=0\}$ and consider the projection $\Phi:\mathscr{X}\to PH$. Then s is transverse if and only if $\mathscr{X}$ is non-singular along $\Phi^{-1}([s])$ and Φ is submersive over $[s]$.

In particular, for transverse sections $s, s'\in H$, $(s)_0$ and $(s')_0$ are diffeomorphic, since PH minus critical point set of Φ is connected.

(1.1) Let $s\in H^0(V;E)$ be transverse. Set $Z=(s)_0$. Then we have an exact sequence of complex vector bundles over Z:

$$0\to TZ\to TV|_Z\to E|_Z\to 0.$$

Therefore $c_t(TV|_Z)=c_t(TZ)c_t(E|_Z)$ for Chern polynomials. Thus the Chern classes of TZ are calculated by the formula (cf. [F])

$$c_t(TZ)=\frac{c_t(TV|_Z)}{c_t(E|_Z)}.$$

We also utilze the following (see [F], for instance):

LEMMA 1.1. *Set $n=\dim V$ and $k=\operatorname{rank} E$. Then, for any $\alpha\in H^{2(n-k)}(V,\boldsymbol{Z})$,*

$$\langle\iota^*\alpha,\ [Z]\rangle=\langle\alpha,\iota_*[Z]\rangle=\langle\alpha\cdot c_k(E),[V]\rangle,$$

where $\iota: Z\to V$ is the inclusion.

(1.2) Let L be a holomorphic line bundle over a compact complex manifold V of dimension n. Let Z be the zero-locus of a transverse section s of L and $\chi(Z)$ denote the Euler characteristic of Z.

LEMMA. *We have*

$$\chi(Z)=\langle\sum_{i+j=n-1}(-1)^j c_i(TV)c_1(L)^{j+1},[V]\rangle,$$

where $[V]\in H_{2n}(V;\boldsymbol{Z})$ is the fundamental class of V.

In particular, if $\dim V=2$, *then*

$$\chi(Z)=\langle c_1(TV)c_1(L)-c_1(L)^2,[V]\rangle.$$

Furthermore, if Z is connected, then the genus of Z

$$g(Z)=1+\frac{1}{2}\langle c_1(L)^2-c_1(TV)c_1(L),[V]\rangle.$$

PROOF OF LEMMA: By (1.1),

$$c_t(TZ)=(\sum_i \iota^* c_i(TV)t^i)/(1+\iota^* c_1(L)t).$$

Then we see, $c_{n-1}(TZ)=\iota^*\alpha$, where

$$\alpha=\sum_{i+j=n-1}(-1)^j c_i(TV)c_1(L)^j.$$

By Lemma 1. 1,

$$\chi(Z)=\langle c_{n-1}(TZ),[Z]\rangle=\langle \iota^*\alpha,[Z]\rangle$$
$$=\langle \alpha,\iota_*[Z]\rangle=\langle \alpha\cdot c_1(L),[V]\rangle.$$

(1.3) Let R be a compact non-singular curve of genus g. Denote by $\xi: V\times R\to V$ and $\psi: V\times R\to R$ the projections respectively.

Set $L_j=\xi^*L\cdot\psi^*\mathcal{O}_R(j)$ over $V\times R$ for each j, where $\mathcal{O}_R(j)$ means a line bundle of degree j over R. Let $A_j\subset V\times R$ be the zero-locus of a transverse section s_j of L_j. Then, by Lemma 1. 2, $\chi(A_j)=\langle\rho,[V]\rangle$, where

$$\rho=jc_n(TV)+\sum_{i+k=n,k>0}((k+1)j+2g-2)c_i(TV)(-c_1(L))^k,$$

as an element of $H^{2n}(V;\boldsymbol{Z})$.

For example, if dim $V=2$, then

$$\chi(A_j)=\langle jc_2(TV)-(2j+2g-2)c_1(TV)c_1(L)+(3j+2g-2)c_1(L)^2,[V]\rangle.$$

Furthermore, if $R=\boldsymbol{P}^1$, then

$$\chi(A_j)=\langle jc_2(TV)-(2j-2)c_1(TV)c_1(L)+(3j-2)c_1(L)^2,[V]\rangle.$$

(1.4) Example. Let C, C' and C'' be compact non-singular curves of genus g, g' and g'' respectively. Set $V=C\times C'\times C''$, and denote projections by p_1, p_2 and p_3 to C, C' and C'' respectively.

Let $A\subset V$ be the zero-locus of a transverse section of $L'=p_1^*\mathcal{O}_C(d)\cdot p_2^*\mathcal{O}_{C'}(d')\cdot p_3^*\mathcal{O}_{C''}(d'')$. Then we have

$$\begin{aligned}\chi(A)=6(d-1)(d'-1)(d''-1)&\\ +(2+4g'')(d-1)(d'-1)+(2+4g)(d'-1)(d''-1)&\\ +(2+4g')(d''-1)(d-1)&\\ +(2+4g'g'')(d-1)+(2+4g''g)(d'-1)+(2+4gg')(d''-1)&\end{aligned}$$

$$+6-4(g+g'+g'')+4(gg'+g'g''+g''g).$$

(1.5) In (1.3), denote by $\varphi: A_j\to R$ the projection to R. Set $\mu=$ $Hom(TA_j, \varphi^* TR)$.
Then $\langle c_n(\mu), [A_j]\rangle=\langle \eta, [V]\rangle$, where

$$\eta=(-1)^n j\sum_{i+k=n}(k+1)c_i(TV)(-c_1(L))^k,$$

as an element of $H^{2n}(V;\boldsymbol{Z})$.

For example, if dim $V=2$, then

$$\langle c_2(\mu), [A_j]\rangle=j\langle c_2(TV)-2c_1(TV)c_1(L)+3c_1(L)^2, [V]\rangle.$$

(1.6) Set $V=\boldsymbol{P}^n$. Then we have

LEMMA. *Let A be a non-singular hypersurface of $\boldsymbol{P}^n\times\boldsymbol{P}^1$ of degree (d, r). Then,*

(1) *$\chi(A)=\langle c_n(TA), [A]\rangle$ is equal to*

$$(n+1)(1-d)^n r+2\Big(\frac{(1-d)^{n+1}-1}{d}+n+1\Big).$$

(2) *$H_i(A;\boldsymbol{Z})$ is torsion free $(0\leq i\leq 2n)$, and*

$$\text{rank } H_i(A;\boldsymbol{Z})=\begin{cases}0, & \text{if } i \text{ is odd and } \neq n,\\ 2, & \text{if } i \text{ is even and } \neq 0, n, 2n,\\ 1, & \text{if } i=0 \text{ or } 2n,\end{cases}$$

$$\text{rank } H_n(A;\boldsymbol{Z})=\begin{cases}\chi(A)-2(n-1), & \text{if } n \text{ in even},\\ 2n-\chi(A), & \text{if } n \text{ in odd}.\end{cases}$$

(3)

$$P_1(A;K)=\begin{cases}\chi(A), & \text{if } n \text{ is even},\\ 4n-\chi(A), & \text{if } n \text{ is odd},\end{cases}$$

for any field L.

(4) *If $\varphi: A\to\boldsymbol{P}^1$ has only isolated critical points, then,*

$$s(\varphi)=\sum_{x\in A}\mu_x(\varphi)=\langle c_n(\mathrm{Hom}(TA, \varphi^* T\boldsymbol{P}^1)), [A]\rangle$$
$$=(n+1)(d-1)^n r,$$

where $\mu_x(\varphi)$ is the Milnor number of φ at x.

PROOF: A is the zero-locus of a transverse section of of L_r, where $L=\mathcal{O}_{\boldsymbol{P}^n}(d)$ and $R=\boldsymbol{P}^1$. Using (1.3) and the equality $c_t(T\boldsymbol{P}^n)=(1+at)^{n+1}$, where $a\in H^2(\boldsymbol{P}^n;\boldsymbol{Z})$ is the Poincaré dual of a hyperplane, we have (1).

By the Lefschetz hyperplane theorem ([GH]),

$$H_i(A;\boldsymbol{Z})\cong H_i(\boldsymbol{P}^n\times\boldsymbol{P}^1;\boldsymbol{Z}),\ H^i(A;\boldsymbol{Z})\cong H^i(\boldsymbol{P}^n\times\boldsymbol{P}^1;\boldsymbol{Z}),$$

for $i\leq n-1$. By Poincaré duality, we have (2).

(3) follows from (2), and (4) follows fron (1.5).

Q. E. D.

(1.7) In (1.3), set $V=\boldsymbol{P}^1\times\boldsymbol{P}^1$, $R=\boldsymbol{P}^1$ and $L=\mathcal{O}_{\boldsymbol{P}^1}(d)\cdot\mathcal{O}_{\boldsymbol{P}^1}(e)$, (resp. $V=\boldsymbol{P}^2\#(-\boldsymbol{P}^2)$, $R=\boldsymbol{P}^1$ and $L=\mathcal{O}_{\boldsymbol{P}^2}(d)\mathcal{O}_{\boldsymbol{P}^1}(e)|V$).

Then, for a non-singular surface $A\subset V\times\boldsymbol{P}^1$, of degree (d,e,r), we have

$$\chi(A)=6der-4de-4er-4rd+4d+4e+4r.$$
$$(\text{resp. } 3r(2d^2+4de-5d-3e+3)-2(2d^2+4de-5d-3e)).$$

If $\varphi: A\to\boldsymbol{P}^1$ has only isolated critical points, then we have, by (1.5),

$$s(\varphi)=r(6de-4d-4e+4).$$
$$(\text{resp. } 3r(2d^2+4de-5d-3e+3)).$$

(1.8) Let K be a field. Then it is easy to verify that, if A is compact complex surface with $H_1(A;K)=0$, then $P_t(A;K)=P_{-t}(A;K)$, and $P_1(A;K)=P_{-1}(A;K)=\chi(A)$.

For example, in (1.7), we see $H_1(A;\boldsymbol{Z}/2)=0$, using the Lefschetz hyperplane theorem ([GH]), and $P_1(A;\boldsymbol{Z}/2)=\chi(A)$.

(1.9) Let L be a holomorphic line bundle over a compact complex manifold V.

L is called very ample if $e; V\to PH^0(V;L)^V$ is well-defined and an embedding, where $PH^0(V;L)^V$ is the projective space of hyperplanes in $H^0(V;L)$ and e is defined by $e(x)=\{s\in H^0(V;L)|s(x)=0\}$, $(x\in V)$.

L is called ample if L^d is very ample for some $d>0$.

The following is clear:

LEMMA. *If L is ample, then $L_j=\xi^*\mathscr{L}\cdot\psi^*\mathcal{O}_{\boldsymbol{P}^1}(j)$ is an ample line bundle over $V\times\boldsymbol{P}^1$, $(j=1,2,\ldots)$.*

(1.10) For the connectivity of a zero-locus $(s)_0$, we need

LEMMA. *Let V be connected of dimension ≥ 2 and L be ample* (*see* 1.9). *Then $(s)_0$ is connected, for any $s\in H^0(V;L)$.*

PROOF: First, suppose L is very ample. Then $(s)_0\cong e(V)\cap h$ for some hyperplane h of $PH^0(V;L)^V$. Since V is connected, $(s)_0$ is also connected by the Lefschetz hyperplane theorem ([GH]). If L is ample, them L^d is very ample for some $d>0$. Then $(s^d)_0$ is connected. Therefore,

$(s)_0$ is connected.

(1.11) Next, we prepare Lemmata of Bertini type on perturbations of sections.

Let V and L be as in (1.9). Let $s, s'\in H^0(V; L)$. Denote the singular locus of $(s)_0$ by $\mathrm{Sing}(s)_0$.

LEMMA. *If $(s')_0$ is non-singular at each point of $(\mathrm{Sing}(s)_0)\cap(s')_0$ and $(s')_0$ is transverse to $(s)_0$ in a neighborhood of $(s)_0\cap(s')_0$ minus $(\mathrm{Sing}(s)_0)\cap(s')_0$, then $(s+\epsilon s')_0$ is non-singular for sufficiently small $\epsilon\in \boldsymbol{C}-0$.*

PROOF: Suppose, for each $i\in \boldsymbol{N}$, there are an $\epsilon_i\in \boldsymbol{C}$ with $0<|\epsilon_i|<1/i$ and an $x_i\in V$ such that $x_i\in \mathrm{Sing}(s+\epsilon_i s')_0$. Taking subsequence, we may suppose $x_i\to x_0\in V$ as $i\to\infty$.

Set

$$Y=\{(x, \epsilon)\in V\times \boldsymbol{C} | x\in \mathrm{Sing}(s+\epsilon s')_0\}.$$

Then Y is an analytic subset of $V\times \boldsymbol{C}$ and $(x_0, 0)\in \overline{Y-Y\times 0}$. by the curve slelction lemma [M], there exists a real analytic cutve $c(t)=(x(t), \epsilon(t))$, $(t\in[-\delta, 0])$ such that $c(0)=(x_0, 0)$, $\epsilon(t)$ is not identically zero and that $x(t)\in \mathrm{Sing}(s+\epsilon(t)s')_0$.

We regard s and s' as functions in a neighborhood of x_0 and take a system of local coordinates $X_1, \ldots, X_n$ at x_0. Then we have

(1) $$s(x(t))+\epsilon(t)s'(x(t))=0,$$
$$(\partial(s+\epsilon(t)s')/\partial X_j)(x(t))=0, \quad (1\leq j\leq n).$$

Hence we have

$$0=d(s(x(t))+\epsilon(t)s'(x(t))/dt=(d\epsilon/dt)\cdot s'(x(t)).$$

Since $d\epsilon/dt\neq 0$, we have $s'(x(t))=0$ for $t\in(-\delta, 0]$, taking δ smaller if necessary. Hence $x(t)\in(s)_0\cap(s')_0$, and $x_0\in(\mathrm{Sing}(s)_0)\cap(s')_0$ by (1).

If $x(t)\in \mathrm{Sing}(s)_0)\cap(s')_0$, for sufficiently small t, then, by (1), $x(t)\in \mathrm{Sing}(s')_0$, and $x_0\in \mathrm{Sing}(s')_0$.

If there is an arbitrarily small $t_0\neq 0$ such that $x(t_0)$ does not belong to $(\mathrm{Sing}(s)_0)\cap(s')_0$, then by (1), $(s')_0$ is not transverse to $(s)_0$ at $x(t_0)\in(s)_0\cap(s')_0$.

In any case, we are led to a contradiction.

(1.12) Set $L_j=\xi^*L\cdot\psi^*\mathcal{O}_{P^1}(j)$. Recall that $[\lambda:\mu]$ is the homogeneous coordinate of $\boldsymbol{P}^1$. Then $(\lambda)_0=V\times\{[0:1]\}$.

LEMMA. *Let s, s' and $s''\in H^0(V\times \boldsymbol{P}^1; L_{j-1})$ be transverse sections. Then we have the followings:*

(1) *If $(s')_0$ is transverse to $(\lambda)_0$ and to $(s)_0$ at each point of $(s)_0\cap(s')_0$*

$\cap(\lambda)_0$, then $(\lambda s+\epsilon\mu s')_0$ is non-singular for sufficiently small $\epsilon\neq 0$.

(2) If $(s)_0$ is transverse to $(\lambda)_0$, then, for sufficiently small $\epsilon\neq 0$, there exists $\delta_0>0$ such that, for each $\delta\in \boldsymbol{C}$ with $|\delta|\leq\delta_0$, $(\lambda s+\epsilon\mu s')_0$ is transverse to $(\lambda-\delta\mu)_0$.

(3) If $(s')_0$ is transverse to $(\lambda)_0$, then, for sufficiently small $\delta\neq 0$, there exists $\epsilon_0>0$ such that, for each $\epsilon\in \boldsymbol{C}$ with $|\epsilon|\leq\epsilon_0$, $(\lambda s+\epsilon\mu s')_0$ is trans verse to $(\lambda-\delta\mu)_0$.

(4) If $(s')_0\cap(\lambda)_0$ is transverse to $(s'')_0\cap(\lambda)_0$ in $(\lambda)_0$, then $(\lambda s+\epsilon\mu s')_0$ is transverse to $(\mu s'')_0$ in $V\times \boldsymbol{P}^1$ for sufficiently small $\epsilon\neq 0$ on $(\lambda s+\epsilon\mu s')_0\cap(\mu s'')_0\cap(\lambda)_0=(s')_0\cap(s'')_0\cap(\lambda)_0$.

PROOF: (1) Notice that $\mathrm{Sing}(\lambda s)_0=(s)_0\cap(\lambda)_0$ and $\mathrm{Sing}(\mu s')_0=(s')_0\cap(\mu)_0$. $(\mu s')_0$ is non-singular near $\mathrm{Sing}(\lambda s)_0\cap(\mu s')_0=(s)_0\cap(s')_0\cap(\lambda)_0$ and $(\mu s')_0$ is transverse to $(\lambda s)_0-((s)_0\cap(\lambda)_0)$ near $(s)_0\cap(\lambda)_0$. Thus, we can apply Lemma 1.11 to λs and $\mu s'$ as s and s' respectively

(2) Assume that there exist sequences (ϵ_i) of non-zero complex numbers and (δ_{ij}) of complex numbers and (x_{ij}) of points of V respectively such that $\epsilon_i\to 0$ as $i\to\infty$, $\delta_{ij}\to 0$ as $j\to\infty$ and that x_{ij} is a singular point of $(\lambda s+\epsilon_i\mu s')_0\cap(\lambda-\delta_{ij}\mu)_0$. Then there exists a sequence (x_i) of points in V such that x_i is a singular point of $(\lambda s+\epsilon_i\mu s')_0\cap(\lambda)_0=(s')_0\cap(\lambda)_0$. This is a contradiction.

(3) Assume that there exist sequences (δ_i) of non-zero complex numbers and (ϵ_{ij}) of complex numbers and (x_{ij}) of points of V respectively such that $\delta_i\to 0$ as $i\to\infty$, $\epsilon_{ij}\to 0$ as $j\to\infty$ and that x_{ij} is a singular point of $(\lambda s+\epsilon_{ij}\mu s')_0\cap(\lambda-\delta_i\mu)_0$. Then there exists a sequence (x_i) of points in V such that x_i is a singular point of $(\lambda s)_0\cap(\lambda-\delta_i\mu)_0=(s)_0\cap(\lambda-\delta_i\mu)_0$. Thus there exists a singular point of $(s)_0\cap(\lambda)_0$. This is a contradiction.

(4) is clear because $(\lambda s+\epsilon\mu s')_0\cap(\lambda)_0=(s')_0\cap(\lambda)_0$ is transverse to $(\mu s'')_0\cap(\lambda)_0=(s'')_0\cap(\lambda)_0$ in $(\lambda)_0=V\times\{[0:1]\}$.

(1.13) Let $s, s'\in H^0(V\times \boldsymbol{P}^1; L_{j-1})$ be transverse sections.

LEMMA. *Assume* $\dim V\geq 2$, *L is ample and $(s')_0$ is transverse to $(\lambda)_0$ and to $(s)_0$ at each point of $(s)_0\cap(s')_0\cap(\lambda)_0$. If $H_1(V;K)\cong H_1((s)_0;K)=0$ for some field K, then $H_1((\lambda s+\epsilon\mu s')_0;K)=0$ for sufficiently small $\epsilon\neq 0$.*

PROOF: Let us denote $H_i(\,\cdot\,;K)$ by $H_i(\,\cdot\,)$, $(\lambda s+\epsilon\mu s')_0$ by A_ϵ and $(s)_0$ by A. Then $A_0=(\lambda s)_0=(s)_0\cup(\lambda)_0$. Denote $(s)_0\cap(\lambda)_0$ by A'.

Step 1: Since L is ample, $H_0(V)\cong H_0(A')\cong K$ by Lemma 1.10. Using the homology exact sequence for (A, A') and the assumption $H_1(A)=0$, we have $H_1(A_0)\cong H_1(A_0, A)\cong H_1(A, A')$. Furthermore, using the homology exact sequence for (A, A'), we have $H_1(A, A')=0$. Hence $H_1(A_0)=0$.

Step 2: Set

$$M=\{(x,[\lambda:\mu],\epsilon)\in V\times \boldsymbol{P}^1\times D_{\epsilon_0}|(x,[\lambda:\mu])\in A_\epsilon\},$$

where $D_{\epsilon_0}=\{\epsilon\in \boldsymbol{C}||\epsilon|\leq\epsilon_0\}$ for some $\epsilon_0>0$. Denote by $\varphi: M\to D_{\epsilon_0}$ the projection.

Take ϵ_0 sufficiently small such that $A_0=\varphi^{-1}(0)$ is a deformation retract of M, $\varphi: M-A_0\to D_{\epsilon_0}$ is a fibration with fiber $F\cong A_\epsilon,(\epsilon\in D_{\epsilon_0}-0)$, and that M is an oriented $2(n+1)$-dimensional C^∞ manifold with boundary ∂M. This is guaranteed by Lemma 1.11. Then ∂M is a deformation retract of $M-A_0$. By Lefshetz duality, $H_2(M,\partial M)\cong H^{2n}(M)\cong H^{2n}(A_0)\cong K$. By Step 1, $H_1(M)\cong H_1(A_0)=0$. Hence $H_1(\partial M)\cong 0$ or K.

Consider the homology exact sequence

$$0\to H_1(F)\to H_1(M-A_0)\xrightarrow{\varphi_*}H_1(D_{\epsilon_0}-0)\to H_0(F)\xrightarrow{\iota_*}H_0(M-A_0),$$

for the fibering φ. Since L is ample, L_j is also ample, by Lemma 1.9. Then $H_0(F)\cong H_0(A_\epsilon)\cong K$ by Lemma 1.10. Thus ι_* is injective and φ_* is surjective. Therefore $H_1(M-A_0)\cong H_1(\partial M)\cong K$ and φ_* is an isomorphism. Hence $H_1(A_\epsilon)\cong H_1(F)=0$ for any $\epsilon\in D_{\epsilon_0}-0$.

(1.14) We also need a result on approximations (Proposition 1.18).

Let L be a very ample holomorphic line bundle on V. Set $H=H^0(V;L)$ and $PH=(H-0)/\boldsymbol{C}^*$. Assume dim $PH\geq 1$. Set

$$\mathscr{X}=\{(x,[s])\in V\times PH|s(x)=0\}.$$

LEMMA. *$\mathscr{X}$ is non-singular.*

PROOF: Since L is very ample, e: $V\to PH^V$ is an embedding (see 1.9).

Pick up the non-singular quadratic hypersurface

$$Q=\{(I,[s])\in PH^V\times PH|[s]\in I\}.$$

Then, the projection $Q\to PH^V$ is submersive, and $e\times \mathrm{id}_{PH}: V\times PH\to PH^V\times PH$ is transverse to Q. Thus $\mathscr{X}=(e\times \mathrm{id}_{PH})^{-1}Q$ is non-singular.

(1.15) Denote by $\Phi: Z\to PH$ the projection to PH and by C the critical-locus of Φ. Set $D=\Phi(C)\subset PH$ and $\rho=\Phi|C$. Define C' to be the locus of points of C, at which ρ is not an immersion, and set

$$D'=\{[s]\in D|[s]\in\rho(C') \text{ or } \#\rho^{-1}[s]\geq 2\}.$$

LEMMA. *We have*

(1) *For $(x,[s])\in Z$, $(x,[s])\in C$ if and only if x is a singular point of*

the zero-locus $(s)_0$.

(2) *C is non-singular, and* $\dim C=\dim PH-1$.

(3) ρ *is an immersion at* $(x,[s])\in C$ *if and only if* x *is an ordinary double point of* $(s)_0$.

(4) $\dim D'\leq\dim PH-2$.

PROOF. Take $(x_0,[s_0])\in Z$.

Let $\{s_0, s_1, \ldots, s_N\}$ be a basis of H; $\dim PH=N$, and $X_0, X_1, \ldots, X_N$ be the homogeneous coordinates of PH associated to $\{s_0, s_1, \ldots, s_N\}$. We may assume $s_N(x_0)\neq 0$. Take a trivialization of $L|U$ over a neighborhood U of x_0 such that $s_N|U\equiv 1$. Over $U_0=\{x_0\neq 0\}\subset PH$, set $y_j=X_j/X_0$. Then $Z\cap U\times U_0$ is defined by

$$-y_N=s_0+\sum_{j=1}^{N-1}y_js_j,$$

and $\Phi|(Z\cap U\times U_0)$ is defined by

$$y_j\circ\Phi=y_j,\ (1\leq j\leq N-1),\ -y_N\circ\Phi=s_0+\sum_{j=1}^{N-1}y_js_j.$$

Set $f_j=\partial(-y_N\circ\Phi)/\partial x_i, (1\leq i\leq n)$, where $\{x_1, \ldots, x_n\}$ is a system of coordinates at x_0, deleting U if necessary. Since $f_i(x,[s])=(\partial s/\partial x_i)(x)$, for each $(x,[s])\in Z\cap U\times U_0$, we have (1).

Since $e: V\rightarrow PH^V$ is an immersion, the $N\times n$-matrix $((\partial s_j/\partial x_i)(x))_{0\leq j\leq N-1,1\leq i\leq n}$ is of rank n. Furthermore $(\partial f_i)/(\partial y_j)=(\partial s_j)/(\partial x_i)$, $(1\leq i\leq n, 1\leq j\leq N-1)$, and $((\partial s_0)/(\partial x_i))(x_0)=0$, $(1\leq i\leq n)$, if $(x_0,[s_0])\in C\cap U\times U_0$.

Thus $f=(f_1, \ldots, f_n): Z\cap U\times U_0\rightarrow \boldsymbol{C}^n$ is an immersion at each point of C. Hence C is non-singular and $\dim C=\dim Z-n=N-1$. This show (2).

Notice that ρ is an immersion at $(x_0,[s_0])\in C$ if and only if

$$(f_1, \ldots, f_n, y_1\circ\Phi, \ldots, y_N\circ\Phi)$$

is an immersion at $(x_0[s_0])$. This condition is equivalent to the $n\times n$-matrix

$$((\partial^2 s_0/\partial x_i\partial x_k)(x_0))_{1\leq i,k\leq n}$$

is regular, that is, x_0 is an ordinary double point of $(s_0)_0$. Thus we have (3).

For each $(x,[s])\in C-C'$, $T\rho(T_{(x,[s])}C)$ is identified with the hyperplane $e(x)=\{[s']\in PH|s'(x)=0\}$. Since e is injective, for any disjoint $x,x'\in V$ with $(x,[s]),(x',[s])\in\rho^{-1}[s]-C'$, $T\rho(T_{(x,[s])}C)$ and $T\rho(T_{(x',[s])}C)$ are disjoint, so are transverse. Therefore $\{[s]\in D-\rho(C')|\#\rho^{-1}[s]\geq 2\}$ is a proper analytic set of $D-\rho(C')$. Hence we have dim $D'\leq \dim C-1=N-2$. This shows (4).

(1.16) A hypersurface $A\subset V\times \boldsymbol{P}^1$ is called generic if A is non-singular and the projection $\varphi: A\to \boldsymbol{P}^1$ has only non-degenerate critical points.

A holomorphic map $a: \boldsymbol{P}^1\to PH$, $H=H^0(V, L)$, is called a Lefschetz family if $Z_a=(\mathrm{id}_V\times a)^{-1}\mathscr{X}$ is non-singular in $V\times \boldsymbol{P}^1$ and the projection φ: $Z_a\to \boldsymbol{P}^1$ has only and at most one non-degenerate critical point in each fiber, (see 1.14).

If a is a Lefschetz family, then $Z_a\subset V\times \boldsymbol{P}^1$ is generic.

LEMMA. *a is a Lefschetz family if and only if a is transverse to $D-D'$ and $a(\boldsymbol{P}^1)\cap D'=\emptyset$.*

PROOF. Notice that Z_a is the fiber product of $\Phi: \mathscr{X}\to PH$ and a: $\boldsymbol{P}^1\to PH$. Thus Z_a is non-singular if and only if Φ and a are transverse. This condition is also equivalent to that ρ and a are transverse. Under this condition, φ has only and at most one non-degenerate critical point in each fiber if and only if $a(\boldsymbol{P}^1)\cap D'=\emptyset$.

(1.17) LEMMA. *There exists a proper algebraic subset $B'\subset H^{r+1}(=H\times\cdots\times H(r+1\text{-times}))$ such that, for any $(s_0, \ldots, s_r)\in H^{r+1}-B'$, if $\sum_{i=0}^{r}\lambda^{r-i}\mu^i s_i=0$, then $(\lambda, \mu)=(0,0)$ in $\boldsymbol{C}^2$.*

PROOF: Set

$$B=\{(s_0, \ldots, s_r; [\lambda:\mu])\in H^{r+1}\times \boldsymbol{P}^1|\sum_{i=0}^{r}\lambda^{r-i}\mu^i s_i=0\}.$$

Then B is of codimension $N+1$ in $H^{r+1}\times \boldsymbol{P}^1$, where $N=\dim PH$. Set $B'=p(B)$, where $p: H^{r+1}\times \boldsymbol{P}^1\to H^{r+1}$ is the projection. Then B' is of codimension $N\geq 1$.

(1.18) For an $s=(s_0, \ldots, s_r)\in H^{r+1}-B'$, define $a(s): \boldsymbol{P}^1\to PH$ by

$$a(s)([\lambda:\mu])=[\sum_{i=0}^{r}\lambda^{r-i}\mu^i s_i].$$

PROPOSITION. *There exists a proper algebraic subset $B''\subset H^{r+1}$ such that, for any $s\in H^{r+1}-B''$, $a(s)$ is a Lefschetz family (see 1.16).*

PROOF: Define $\alpha: (H^{r+1}-B')\times \boldsymbol{P}^1\to PH$ by $\alpha(s, [\lambda:\mu])=a(s)([\lambda:\mu])$. Then α is a submersion. We see codim $\alpha^{-1}D'\geq 2$ and codim $\beta\alpha^{-1}D'\geq 1$, where $\beta: (H^{r+1}-B')\times \boldsymbol{P}^1\to H^{r+1}-B'$ is the projection.

We pick up $R=\alpha^{-1}(D-D')\subset(H^{r+1}-B')\times \boldsymbol{P}^1$ and set

$$B''=\text{Zariski closure of } (\beta(C(\beta|R))\cup\beta\alpha^{-1}D')\cup B'$$

in H^{r+1}, where $C(\beta|R)$ is the critical locus of $\beta|R$.

To complete the proof of Proposition 1.18, it sufficies to show the following

LEMMA. *$\alpha(s)$ is a Lefschetz family if and only if $s\in H^{r+1}-B'$ is a regular value of $\beta|R$ and s does not belong to $\beta\alpha^{-1}D'$.*

PROOF: $s\in H^{r+1}-B'$ is a regular value of $\beta|R$ if and only if, for any $[\lambda:\mu]\in \boldsymbol{P}^1$, $(\alpha,\beta):(H^{r+1}-B')\times\boldsymbol{P}^1\to PH\times(H^{r+1}-B')$ is transverse to $(D-D')\times\{s\}$ at $(s,[\lambda:\mu])$. This is equivalent to that $\alpha|\{s\}\times\boldsymbol{P}^1$ is transverse to $D-D'$. Further, $s\notin\beta\alpha^{-1}(D')$ means $\alpha(s)(\boldsymbol{P}^1)\cap D'=\emptyset$.

By Lemma 1.16, we have Lemma 1.18. This completes the proof of Proposition 1.18.

2. Preliminary: Real geometry

(2.1) A real structure on a complex manifold V is an anti-holomorphic involution $\tau: V\to V$. The pair (V,τ) is called a real complex manifold. Two real complex manifolds (V,τ), (V',τ') are called isomomphic if there is an isomorphism $\sigma: V\to V'$ of complex manifolds satisfying $\sigma\circ\tau=\tau'\circ\sigma$(cf. [S]).

(2.2) Let (V,τ) be a real complex manifold. We denote by $V_{\boldsymbol{R}}$ or $\boldsymbol{R}V$ the space V^τ of fixed points of τ in V, and call it the real locus of V(with respect to τ). Then $V_{\boldsymbol{R}}$ is a real analytic submanifold of V and $\dim_{\boldsymbol{R}} V_{\boldsymbol{R}}=\dim_{\boldsymbol{C}} V$, provided $V_{\boldsymbol{R}}\neq\emptyset$.

DEFINITION: A real complex manifold (V,τ) is called an *M-manifold* if $P_1(V_{\boldsymbol{R}};\boldsymbol{Z}/2)=P_1(V;\boldsymbol{Z}/2)$(cf. [G1]). An M-manifold (V,τ) of dimension 1 (resp. 2) is called an *M-curve* (resp. an *M-surface*).

(2.3) Here we give some fundamental examples.

EXAMPLE: (1) The number of equivalence classes of real structures on $\boldsymbol{P}^n$ is one if n is even and two if n is odd. (See [F], p. 240.)

The anti-holomorphic involution $\tau':\boldsymbol{P}^{2m+1}\to\boldsymbol{P}^{2m+1}$ defined by $\tau'[X_0: X_1:\cdots:X_{2i}:X_{2i+1}:\cdots:X_{2m}:X_{2m+1}]=[-\overline{X_1}:\overline{X_0}:\cdots:-\overline{X_{2i+1}}:\overline{X_{2i}}:\cdots:-\overline{X_{2m+1}}:\overline{X_{2m}}]$ gives the structure not equivalent to the usual structure defined by the complex conjugation $(\boldsymbol{P}^{2m+1},\tau_{2m+1})$. We often write $\boldsymbol{P}_0^{2m+1}=(\boldsymbol{P}^{2m+1},\tau')$ and $\boldsymbol{P}_1^{2m+1}=(\boldsymbol{P}^{2m+1},\tau_{2m+1})$.

Then $\boldsymbol{P}^{2m}$ and $\boldsymbol{P}_1^{2m+1}$ are M-manifolds, but $\boldsymbol{P}_0^{2m+1}$ is not an M-manifold.

(2) Let $\{\lambda F+\mu G|[\lambda:\mu]\in\boldsymbol{P}_1^1\}$ be a pencil of real plane cirves in $\boldsymbol{P}^2$ of

degree d. (This corresponds to the case $r=1$ in Theorem 0.1.) Then $A=(\lambda F+\mu G)_0\subset \boldsymbol{P}^2\times \boldsymbol{P}_1^1$ is non-singular if and only if $(F)_0$ and $(G)_0$ intersect transversely in $\boldsymbol{P}^2$. If A is non-singular, then A is diffeomorphic to

$$\boldsymbol{P}^2\#\underbrace{(-\boldsymbol{P}^2)\#\cdots\#(-\boldsymbol{P}^2)}_{d^2\text{-times}},$$

where $-\boldsymbol{P}^2$ means $\boldsymbol{P}^2$ with the reverse orientation. In this case, if $(F)_0$ and $(G)_0$ intersect in k-points in $\boldsymbol{R}P^2$, $(0\leq k\leq d^2, k\equiv d, \text{ mod. } 2)$, then $\boldsymbol{R}A$ is diffeomorphic to $\#_{1+k}\boldsymbol{R}P^2$. Hence, A is an M-surface, that is, $P_1(\boldsymbol{R}A\,;\boldsymbol{Z}/2)=3+d^2$ if and only if $k=d^2$.

(2.4) From properties of Poincaré series, we easily see

LEMMA. *Let* (V,τ), (V',τ') *be M-manifolds. Then* $(V\amalg V',\tau\amalg\tau')$ *and* $(V\times V',\tau\times\tau')$ *are also M-manifolds.*

Example. $\boldsymbol{P}_1^1\times\boldsymbol{P}_1^1$, $\boldsymbol{P}^2\times\boldsymbol{P}_1^1$ and $\boldsymbol{P}_1^1\times\boldsymbol{P}_1^1\times\boldsymbol{P}_1^1$ are all M-manifolds.

(2.5) We need the following

LEMMA. *Let* (V,τ) *be a connected compact M-surface. Then the followings are equivalent :*

(1) $\chi(V)+\chi(V_{\boldsymbol{R}})=4$.

(2) $H_2(V\,;\boldsymbol{Z}/2)\cong H_1(V_{\boldsymbol{R}}\,;\boldsymbol{Z}/2)$.

(3) $H_1(V\,;\boldsymbol{Z}/2)=0$ *and* $H_0(V_{\boldsymbol{R}}\,;\boldsymbol{Z}/2)\cong\boldsymbol{Z}/2$.

PROOF: First remark that $V_{\boldsymbol{R}}\neq\emptyset$.

Set $b_i=\dim_{\boldsymbol{Z}/2}H_i(V\,;\boldsymbol{Z}/2)$ and $b_i'=\dim_{\boldsymbol{Z}/2}H_i(V_{\boldsymbol{R}}\,;\boldsymbol{Z}/2)$. Then, by the Poincaré duality,

$$\chi(V)=1-b_1+b_2-b_3+1=4+2b_2-P_1(V,\boldsymbol{Z}/2)=P_1(V,\boldsymbol{Z}/2)-4b_1,$$
$$\chi(V_{\boldsymbol{R}})=b_0'-b_1'+b_2'=P_1(V_{\boldsymbol{R}},\boldsymbol{Z}/2)-2b_2'=4b_0'-P_1(V_{\boldsymbol{R}},\boldsymbol{Z}/2).$$

Since V is an M-surface, $P_1(V_{\boldsymbol{R}},\boldsymbol{Z}/2)=P_1(V,\boldsymbol{Z}/2)$. Therefore

$$\chi(V)+\chi(V_{\boldsymbol{R}})=4+2(b_2-b_2')=4(b_1+b_0').$$

Hence (1), (2) and (3) are equivalent in each other.

(2.6) Let $\pi:E\to V$ be a holomorphic vector bundle over a real complex manifold (V,τ). A real structure of π is a real structure $T:E\to E$ of E as a complex manifold (see (2.1)) such that $\pi\circ T=\tau\circ\pi$ and that the ristriction $T_x:E_x\to E_{\tau(x)}$ to each fiber $(x\in V)$ is anti-linear.

We call the triple $E=(\pi\,;T,\tau)$ a real holomorphic vector bundle. (See [A]).

For example, $\mathcal{O}_{\boldsymbol{P}^n}(r)=\mathcal{O}_{\boldsymbol{P}^n}(1)^r$ is a real holomorphic line bundle over $(\boldsymbol{P}^n,\tau_n)$, where $\mathcal{O}_{\boldsymbol{P}^n}(1)$ is the tautological line bundle over $\boldsymbol{P}^n$.

Notice that the restriction $\pi_{\boldsymbol{R}}: E_{\boldsymbol{R}} \rightarrow V_{\boldsymbol{R}}$ to the real locus of π is an usual real vector bundle.

A holomorphic section $s \in H^0(V, E)$ of E is called a real section if $T \circ s \circ \tau^{-1} = s$, that is, $s \in H^0(V, E)_{\boldsymbol{R}}$ with respect to the anti-linear involution of $H^0(V, E)$ defined by $s \rightarrow T \circ s \circ \tau^{-1}$.

For example, $H^0(\boldsymbol{P}^n, \mathcal{O}_{\boldsymbol{P}^n}(d))_{\boldsymbol{R}}$ is identified with the space of real homogeneous polynomials of $(n+1)$-variables of degree d.

(2.7) Our main object to construct is a real transverse section s of which zero-locus $(s)_0$ has topologically extremal properties.

DEFINITION: A holomorphic section s of real holomorphic vector bundle over a real complex manifold (V, τ) is an *M-section* if s is transverse and real, and the zero-locus $(s)_0 \subset V$ togather with the real structure $\tau|(s)_0$ is an M-manifold.

(2.8) Discussions in (1.11)-(1.18) can be applied in the situation that V is a real complex manifold and L is a real holomorphic line bundle.

For instance, B'' in Proposition 1.18 can be taken invariant under the complex conjugation. Thus we have

PROPOSITION. *There exists a proper algebraic subset $B \subset H_{\boldsymbol{R}}^{r+1}$ such that, for any $s \in H_{\boldsymbol{R}}^{r+1} - B$, $a(s): \boldsymbol{P}^1 \rightarrow PH$ is a Lefshetz family, and $a(s)$ is equivariant under the complex conjugations of $\boldsymbol{P}^1$ and PH respectively.*

(2.9) Let V be a real complex manifold of dimension n(see 2.1), and $C \subset V$ be a real hypersurface possibly with singularities. A non-singular component E of $C_{\boldsymbol{R}} \subset V_{\boldsymbol{R}}$ is called an oval (resp. an empty oval) if there exists a C^∞ embedding $i: D^n \rightarrow V_{\boldsymbol{R}}$ of an n-dimensional ball D^n such that $i(\partial D^n) = E$(and that $i(\text{int } D^n) \cap C_{\boldsymbol{R}}$ is empty). In any case, $i(\text{int D}^n)$ is called the interior of E.

We apply this definition also to a component of a subset in a C^∞ manifold.

(2.10) Let W be a compact C^∞-manifold of dimension n possibly with boundary ∂W. Denote by y the coordinate function of $\boldsymbol{R}$. Let $f: W \times \boldsymbol{R} \rightarrow \boldsymbol{R}$ be a C^∞-function and $i: D^n \rightarrow W - \partial W \times 0$ be a C^∞-embedding.

Assume that $i(\partial D^n) \subset f^{-1}(0) \cap W \times 0$, $f^{-1}(0)$ and $W \times 0$ are transverse along $i(\partial D^n)$ and that $f < 0$ in $i(\text{int } D^n)$.

Let g be a positive C^∞ function in a neighborhood of $i(D^n)$ in $W \times \boldsymbol{R}$.

LEMMA. *For any ϵ_0 and δ with $0 < \delta_0 \ll \epsilon_0 \ll 1$, the hypersurface A in $W \times \boldsymbol{R}$ defined by $y(y + \epsilon_0 f) + \delta_0 g = 0$ has an empty oval in a neighborhood of $i(D^n)$ in $W \times \boldsymbol{R}$.*

PROOF: The hypersurface $y+\epsilon f=0$ is non-singular for sufficiently small ϵ on each compact subset of $W\times\boldsymbol{R}$.

Let us consider the equation $y+\epsilon f(i(x), y)=0$, for $(x, y, \epsilon)\in D^n\times\boldsymbol{R}\times\boldsymbol{R}$. Then, by the implicite function theorem, there exsists a unique C^∞ map-germ $\varphi:(D^n\times\boldsymbol{R}, D^n\times 0)\to\boldsymbol{R}$ such that $\varphi(x, \epsilon)+\epsilon f(i(x), \varphi(x, \epsilon))=0$ as germ at $D^n\times 0$ in $D^n\times\boldsymbol{R}$ and that $\varphi(x, 0)=0$ for any $x\in D^n$. We see, for some $\epsilon_0>0$, $\varphi(x, \epsilon)=0$, $(x\in\partial D^n, \epsilon\in[-\epsilon_0, \epsilon_0])$, and $(\partial\varphi/\partial\epsilon)(x, \epsilon)>0$, $(x\in \text{int } D^n, \epsilon\in[0, \epsilon_0])$.

Define by $\alpha: D^n\times[0, \epsilon_0]\to W\times\boldsymbol{R}$ by $\alpha(x, \epsilon)=(i(x), \varphi(x, \epsilon))$. Then α is a local diffeomorphism of $(\text{int } D^n)\times[0, \epsilon_0]$. Since $\varphi(x, \epsilon)$ is an increasing function with respect to ϵ, we see α is diffeomorphism of $(\text{int } D^n)\times[0, \epsilon_0]$ onto the image.

We can take ϵ_0 sufficiently small such that g is defined and positive on $\alpha(D^n\times[0, \epsilon_0])$.

Define $\delta: \alpha(D^n\times[0, \epsilon_0])\to\boldsymbol{R}$ by $\delta=y(y+\epsilon_0 f)/(-g)$. Then

$$\alpha^*\delta(x, \epsilon)=\epsilon(\epsilon_0-\epsilon)\cdot\alpha^*(f^2/g)(x, \epsilon).$$

Thus $\alpha^*\delta(x, \epsilon)=0$ if and only if $(x, \epsilon)\in\partial(D^n\times[0, \epsilon_0])$. Furthermore $\alpha^*\delta>0$ in $\text{int}(D^n\times[0, \epsilon])$. Then, for sufficiently small $\delta_0>0$, there exist a diffeomorphism $\{\alpha^*\delta\geq\delta_0\}\cong D^{n+1}$. Set

$$E=\{y(y+\epsilon_0 f)+\delta_0 g=0\}\cap\alpha(D^n\times[0, \epsilon_0]).$$

Then $E\cong\{\alpha^*\delta=\delta_0\}\cong\partial D^{n+1}$ and E is an empty oval.

REMARK: (1) In the proof of Lemma 2.10, the mapping $\beta: D^n\to W\times\boldsymbol{R}$ defind by $\beta(x)=\alpha(x, \epsilon_0)$, $(x\in D^n)$, is an embedding.

(2) We apply Lemma 2.10 to study a manifold of type $\lambda(\lambda s+\epsilon_0\mu s')+\delta_0\mu^2 s''=0$ in the proof of Theorem 0.3 in §3. On a domain where $s\neq 0$, $\mu\neq 0$, set $y=\lambda/\mu$, $f=s'/s$ and $g=s''/s$. Then the equation is reduced to $y(y+\epsilon_0 f)+\delta_0 g=0$, which is treated in Lemma 2.10.

(3) This Lemma is also utilized implicitely in §4, in the case $n=2$.

(2.11) Now, we recall the Poincaré-Hopf-Pugh formula.

Let M be a compact C^∞ manifold of dimension n with boundary ∂M.

A tangent vector ξ to M at a point x_0 of M is external if $df_{x_0}(\xi)$ is positive for some C^∞ function f defined in a neighborhood U of x_0 such that $f^{-1}(0)=\partial M\cap U$, f takes negative values in $(M-\partial M)\cap U$ and $df|\partial M\cap U$ does not vanish.

Let $v:\partial M\to TM|\partial M$ be a C^∞ section over ∂M to the tangent bundle TM.

Assume that (a): for each $x_0\in\partial M$, $v(x_0)\neq 0$.

First set $M_0=M$. Next set

$$M_1'=\{x\in\partial M\,|\,v(x)\text{ is external}\},$$

and set $M_1=\overline{M_1'}$, and $\partial M_1=M_1-M_1'$.

Inductively, if M_k is a C^∞ manifold with boundary ∂M_k, $(k\geq 0)$, then set

$$M_{k+1}'=\{x\in\partial M_k\,|\,v(x)\text{ is external w.r.t. }M_k\},$$
$$M_{k+1}=\overline{M_{k+1}'}\text{ and }\partial M_{k+1}=M_{k+1}-M_{k+1}'.$$

Assume that (b): M_k is a C^∞ manifold with boundary ∂M_k, $(k=1,2,\dots,n-1)$.

LEMMA ([P]). *Let v satisfy two assumptions* (a), (b) *stated in above. Then for any C^∞ extension $w: M\to TM$ with isolated singularities, we have*

$$\text{(c)}: \operatorname{ind} w=\sum_{i=0}^{n}(-1)^i\chi(M_i).$$

REMARK: (0) We adopt the following definition of index of a vector field: Let $x_0\in M$ be an isolated singular point of w. Take a system of coordinates $x_1,\dots,x_n$ centered at x_0, and write locally

$$w(x)=a_1(x)(\partial/\partial x_1)+\cdots+a_n(x)(\partial/\partial x_n).$$

Then define $\operatorname{ind}_{x_0} w=\deg_0(-a)$, where $a=(a_1,\dots,a_n)$, and set $\operatorname{ind} w=\sum\operatorname{ind}_{x_0}w$, where the sum runs over isolated singular points x_0 of w.

(1) If ∂M is empty, then (c) is the Poincaré-Hopf formula.

(2) For a C^∞ vector field w over M with only isolated singular points, there exists a non-negative C^∞ function $f: U\to\boldsymbol{R}$ on a collar of $(M,\partial M)$ with the following properties: (i) $f^{-1}(0)=\partial M$. (ii) For any sufficiently small $\epsilon>0$, $w|f^{-1}(\epsilon)$ satisfies two assumptions (a), (b).

(2.12) Let W be a compact C^∞ manifold with boundary, W' be a compact submanifold of codimension 1 of W with $\partial W'=\partial W\cap W'$ and W'' be a compact submanifold of codimension 1 of W' with $\partial W''=\partial W'\cap W''$.

A compact C^∞ manifold $\widetilde{W}$ with boundary is called a modification of W along (W',W'') if $\widetilde{W}$ is constructed as follows: First, consider the disjoint union of closures of connected components of $W-W'$. Second, attach a $[0,1]$-bundle over a tubular neighborhood U of W'' in W'. Third, make its corner smooth.

Then, remark that $\chi(\widetilde{W})=\chi(W)+\chi(W')-\chi(W'')$.

(2.13) In the situation of (1.12), further assume V and L are real and $s, s' \in H^0(V \times \boldsymbol{P}_1^1 ; L_{j-1})_{\boldsymbol{R}}$. Identify V with $(\lambda)_0 = V \times \{[0:1]\}$.

LEMMA. *Assume that $(s)_0$ is transverse to V and that $(s)_0, (s')_0$ and V are in general position along $V \cap (s)_0 \cap (s')_0$. For real numbers δ_0 and ϵ_0 with $0 \leq |\epsilon_0| \ll \delta_0 \ll 1$, set $\tilde{V}_{\boldsymbol{R}} = (\lambda s + \epsilon_0 \mu s')_0 \cap V_{\boldsymbol{R}} \times [-\delta_0, \delta_0]$, where $[-\delta_0, \delta_0] = \{[\lambda : \mu] \in \boldsymbol{RP}_1^1 | -\delta \leq \lambda/\mu \leq \delta_0\}$. Then $\tilde{V}_{\boldsymbol{R}}$ is diffeomorphic to a modification of $V_{\boldsymbol{R}}$ along $(V_{\boldsymbol{R}} \cap (s)_0,\ V_{\boldsymbol{R}} \cap (s)_0 \cap (s')_0)$.*

PROOF: Since $(s)_0$ is transverse to $V_{\boldsymbol{R}}$, $\boldsymbol{R}(\lambda s)_0$ is transverse to $\boldsymbol{R}(\lambda \pm \delta_0 \mu)_0$ for a sufficiently small $\delta > 0$. Therefore $\boldsymbol{R}(\lambda s + \epsilon_0 \mu s')_0$ is transverse to $\boldsymbol{R}(\lambda \pm \delta_0 \mu)_0$ for a sufficiently small ϵ_0 relatively to δ_0, and then, $\tilde{V}_{\boldsymbol{R}}$ is a C^∞ manifold of dimension n with boundary.

Set $y = \lambda/\mu$ on $V_{\boldsymbol{R}} \times [-\delta_0, \delta_0]$. Then $\tilde{V}_{\boldsymbol{R}}$ is defined by $ys + \epsilon_0 s' = 0$.

Take a point $p \in V_{\boldsymbol{R}} \subset V_{\boldsymbol{R}} \times [-\delta_0, \delta_0]$. There are three cases: (i) $p \in V_{\boldsymbol{R}} \cap (s)_0 \cap (s')_0$, (ii) $p \in V_{\boldsymbol{R}} \cap (s)_0 - (s')_0$ and (iii) $p \in V_{\boldsymbol{R}} - (s)_0 \cup (s')_0$.

In the case (i), (resp. (ii)), since $V_{\boldsymbol{R}}$, $\boldsymbol{R}(s)_0$ and $\boldsymbol{R}(s')_0$ are in general point at p, (resp. $V_{\boldsymbol{R}}$ and $\boldsymbol{R}(s)_0$ are transversal at p), there is a system of local coordinates $y ; x_1, \ldots, x_n$ of $V_{\boldsymbol{R}} \times \boldsymbol{RP}_1^1$ centered at p such that $s = x_1$, $s' = x_2$, (resp. $s/s' = x_1$), with respect to a local trivization of L and $\mathcal{O}_{P^1}(1)$. Then, locally, $\tilde{V}_{\boldsymbol{R}}$ is defined by $yx_1 + \epsilon_0 x_2 = 0$, (resp. $yx_1 + \epsilon_0 = 0$). Take a small ball B with center p in $V_{\boldsymbol{R}}$ and set

$$W = V_{\boldsymbol{R}} \cap B,\ W' = (s)_0 \cap W,\ W'' = (s')_0 \cap W'.$$

Then $\tilde{V}_{\boldsymbol{R}} \cap B \times [-\delta_0, \delta_0]$ is diffeomorphic to a modification of W along (W', W''), (resp. $(W', \emptyset)$).

In the case (iii), the projection maps $\tilde{V}_{\boldsymbol{R}} \cap B \times [-\delta_0, \delta_0]$ to $V_{\boldsymbol{R}} \cap B$ diffeomorphically. By the compactness of $V_{\boldsymbol{R}}$, we can glue togather the above diffeomorphisms, and we have required result.

(2.14) Disjoint points $p_1, \ldots, p_m$, considered with order, of a (topological) circle are called cyclic if $m \leq 2$ or, for each i, $(1 \leq i \leq m)$, an arc from p_i to p_{i+1} does not contain other points than p_i, p_{i+1}.

Disjoint non-void sets $P_1, \ldots, P_m$ of a circle are called cyclic if, for any choice of $p_i \in P_i$, $(1 \leq i \leq m)$, $p_1, \ldots, p_m$ are cyclic.

3. Non-linear systems of real sections

In the situation of Theorem 0.3, set $Z = (s_r)_0$. Then $Z \cong (s_i)_0$, $(0 \leq i \leq r)$, by (0.1). Set

$$s^{(j)} = \sum_{i=0}^{j} \epsilon_i s_i \lambda^{j-i} \mu^i \text{ and } A^{(j)} = (s^{(j)})_0 \subset S \times \boldsymbol{P}_1^1,$$

$(0 \leq i < r)$. Then $s^{(0)} = s_0$. If we set $s = s^{(j-1)}$ and $s' = \mu^{j-1} s_j$, then $s, s' \in$

$H^0(S, L_{j-1})_{\boldsymbol{R}}$ and $s^{(j)}=\lambda s+\epsilon_j\mu s'$, $(1\leq j\leq r)$.

Using Lemma 1.12 iteratively, we can choose ϵ_i, $(1\leq i\leq r)$, such that each $A^{(j)}$ is non-singular, and any critical points of $\varphi^{(j)}=\psi|A^{(j)}$ are not on $S\times D_{\delta j}$ for some $\delta_j>0$, $(0\leq j\leq r)$, where $D_{\delta_j}=\{[\lambda:\mu]\in\boldsymbol{P}^1 \| \lambda/\mu|\leq\delta_j\}$, Furthermore, by Propositions 1.18 and 2.8, $A^{(j)}$, $(0\leq j\leq r)$, is generic in the sense of (1.16), perturbing $s_0,\ldots,s_r$ in $H^0(S, L)_{\boldsymbol{R}}$ if necessary. By Remark 0.4.4, the condition (*) does not change by a small perturbation.

Fix an orientation of $\boldsymbol{R}P^1\cong S^1$. Then denote by $\gamma_i^{(j)}$ the number of real critical points of $\varphi^{(j)}=\psi|A^{(j)}$ of index i, by $t_i^{(j)}$ the dimension of $H_i(A_{\boldsymbol{R}}^{(j)};\boldsymbol{Z}/2)$ over $\boldsymbol{Z}/2$ and by $e^{(j)}$ the number of empty ovals of $A_{\boldsymbol{R}}^{(j)}$, $(i=1,2,3; 0\leq j\leq r)$.

Identify $H^4(S;\boldsymbol{Z})$ with $\boldsymbol{Z}$ by the fundamental class $[S]$ of S.

By Lemmata 1.2 and 1.10, the genus $g=g(Z)$ is equal to

$$1+(1/2)(c_1(L)^2-c_1(L)c_1(TS)).$$

Consider the following inequality and equalities:

$$(\boldsymbol{A}_j'):\gamma_1^{(j)}\geq j(c_1(L)^2+2g-\chi(S_{\boldsymbol{R}})).\quad (\boldsymbol{A}_j):\gamma_1^{(j)}=j(c_1(L)^2+2g-\chi(S_{\boldsymbol{R}})).$$
$$(\boldsymbol{B}_j'):\gamma_0^{(j)}+\gamma_2^{(j)}\geq 2jg.\quad (\boldsymbol{B}_j):\gamma_0^{(j)}+\gamma_2^{(j)}=2jg.$$
$$(\boldsymbol{C}_j):s(\varphi_{\boldsymbol{R}}^{(j)})=s(\varphi^{(j)}).\quad (\boldsymbol{D}_j):H_1(A^{(j)};\boldsymbol{Z}/2)=0.$$
$$(\boldsymbol{E}_j'):e^{(j)}\geq(j-1)g.\quad (\boldsymbol{E}_j):e^{(j)}=(j-1)g.$$
$$(\boldsymbol{F}_j'):t_0^{(j)}+t_2^{(j)}\geq 2(j-1)g+2.\quad (\boldsymbol{F}_j):t_0^{(j)}+t_2^{(j)}=2(j-1)g+2.$$
$$(\boldsymbol{HT}_j):P_1(A_{\boldsymbol{R}}^{(j)};\boldsymbol{Z}/2)=P_1(A^{(j)};\boldsymbol{Z}/2).$$

Clearly, we have $(\boldsymbol{A}_0)$, $(\boldsymbol{B}_0)$, $(\boldsymbol{C}_0)$, $(\boldsymbol{D}_0)$ and $(\boldsymbol{HT}_0)$.

Further, we have the following implications:

LEMMA 3.1.

(1) $(\boldsymbol{A}_j')\&(\boldsymbol{B}_j')\Rightarrow(\boldsymbol{A}_j)\&(\boldsymbol{B}_j)\&(\boldsymbol{C}_j)$, $(0\leq j\leq r)$.

(2) $(\boldsymbol{A}_j)\&(\boldsymbol{B}_j)\&(\boldsymbol{D}_j)\&(\boldsymbol{F}_j')\Rightarrow(\boldsymbol{F}_j)\&(\boldsymbol{HT}_j)$, $(1\leq j\leq r)$.

(3) $(\boldsymbol{E}_j')\&(\boldsymbol{F}_j)\Rightarrow(\boldsymbol{E}_j)$, $(1\leq j\leq r)$.

(4) $(\boldsymbol{E}_j')\Rightarrow(\boldsymbol{F}_j')$, $(1\leq j\leq r)$.

PROOF: (1): By $(\boldsymbol{A}_j')$ and $(\boldsymbol{B}_j')$, we have

$$\begin{aligned} s(\varphi_{\boldsymbol{R}}^{(j)}) &= \gamma_0^{(j)}+\gamma_1^{(j)}+\gamma_2^{(j)} \\ &\geq j(3c_1(L)^2-2c_1(L)c_1(TS)+4-\chi(S_{\boldsymbol{R}})). \end{aligned}$$

By Lemma 2.5, we have $4-\chi(S_{\boldsymbol{R}})=\chi(S)=c_2(TS)$. Thus, by (1.5), the right hand side is equal to $s(\varphi^{(j)})$. Since $s(\varphi_{\boldsymbol{R}}^{(j)})\leq s(\varphi^{(j)})$, we have $(\boldsymbol{C}_j)$, and therefore $(\boldsymbol{A}_j)$ and $(\boldsymbol{B}_j)$ at the same time.

(2): By $(\boldsymbol{A}_j)$, $(\boldsymbol{B}_j)$ and Lemma 2.5, we see

$$\begin{aligned}\chi(A_{\boldsymbol{R}}^{(j)}) &= \gamma_0^{(j)}-\gamma_1^{(j)}+\gamma_2^{(j)}\\ &= j(-c_1(L)^2-c_2(TS)+4).\end{aligned}$$

Therefore, by $(\boldsymbol{F}_j')$, we have

$$\begin{aligned}P_1(A_{\boldsymbol{R}}^{(j)}\,;\boldsymbol{Z}/2) &= t_0^{(j)}+t_1^{(j)}+t_2^{(j)}\\ &= 2(t_0^{(j)}+t_2^{(j)})-\chi(A_{\boldsymbol{R}}^{(j)})\\ &\geq (3j-2)c_1(L)^2-(2j-2)c_1(L)c_1(TS)+jc_2(TS).\end{aligned}$$

By (1.3), the right hand side is equal to $\chi(A^{(j)})$. By $(\boldsymbol{D}_j)$ and the Poincaré duality, we see $\chi(A^{(j)})=P_1(A^{(j)}\,;\boldsymbol{Z}/2)$, (see (1.8)). On the other hand, by Harnack-Thom's inequality ([G1], [T]), $P_1(A_{\boldsymbol{R}}^{(j)}\,;\boldsymbol{Z}/2)\leq P_1(A^{(j)}\,;\boldsymbol{Z}/2)$, we have $(\boldsymbol{HT}_j)$, and therefore $(\boldsymbol{F}_j)$ at the same time.

(3)&(4): If $A_{\boldsymbol{R}}^{(j)}$ has $e^{(j)}$ empty ovals (and necessarily at least one other components), we have

$$t_0^{(j)}+t_2^{(j)}\geq 2e^{(j)}+2.$$

Therefore, $(\boldsymbol{F}_j)$ implies $e^{(j)}\leq(j-1)g$. Hence $(\boldsymbol{E}_j')$&$(\boldsymbol{F}_j)$ implies $(\boldsymbol{E}_j)$. On the other hand, $(\boldsymbol{E}_j')$ implies $(\boldsymbol{F}_j')$.

PROOF OF THEOREM 0.3: To prove Theorem 0.3, that is, to show $(\boldsymbol{C}_r)$, $(\boldsymbol{E}_r)$ and $(\boldsymbol{HT}_r)$, it is sufficient to show $(\boldsymbol{A}_j')$, $(\boldsymbol{B}_j')$, $(\boldsymbol{D}_j)$ and $(\boldsymbol{E}_j')$, $(1\leq j\leq r)$, by Lemma 3.1.

First we show $(\boldsymbol{A}_j')$ and $(\boldsymbol{B}_j')$ by the induction on j.

We consider the gradient of $\varphi_{\boldsymbol{R}}$. Precisely, let $w: A_{\boldsymbol{R}}^{(j)}\to \mathrm{Hom}(TA_{\boldsymbol{R}}^{(j)}, T\boldsymbol{R}P^1)$ be the section defined by $w(x)=T_x\varphi_{\boldsymbol{R}}$, $x\in A_{\boldsymbol{R}}^{(j)}$. By an identification

$$\mathrm{Hom}(TA_{\boldsymbol{R}}^{(j)}, T\boldsymbol{R}P^1)\cong T^*A_{\boldsymbol{R}}^{(j)}\cong TA_{\boldsymbol{R}}^{(j)},$$

we regard w as a vector field over $A_{\boldsymbol{R}}^{(j)}$.

We see w does not tangent to $A_{\boldsymbol{R}}^{(j)}\cap S_{\boldsymbol{R}}\times\{p\}$, for $p=[0:1]$, $[1:\pm\delta_{j-1}]$.

Set $\tilde{N}=A_{\boldsymbol{R}}^{(j)}\cap S_{\boldsymbol{R}}\times[-\delta_{j-1}, \delta_{j-1}]$, where $[-\delta_{j-1}, \delta_{j-1}]=\{[\lambda:\mu]\in\boldsymbol{R}P^1|\ -\delta_{j-1}\leq\lambda/\mu\leq\delta_{j-1}\}$. Then by Lemma 2.13, $\tilde{N}$ is diffeomorphic to a modification of $S_{\boldsymbol{R}}$ along $(S_{\boldsymbol{R}}\cap(s)_0, S_{\boldsymbol{R}}\cap(s)_0\cap(s')_0)=(\boldsymbol{R}(s_{j-1})_0, \boldsymbol{R}(s_{j-1})_0\cap\boldsymbol{R}(s_j)_0)$. Especially, $\tilde{N}$ has disk components $D_1', \dots, D_g'$ corresponding to g empty ovals of $\boldsymbol{R}(s_{j-1})_0$.

Denote by $D_1, \dots, D_g$ the interiors of g-empty ovals of $\boldsymbol{R}(s_j)_0$ in $S_{\boldsymbol{R}}$. Then, by Remark 2.10.1, there are open disk domains $\tilde{D}_1, \dots, \tilde{D}_g$ on $\tilde{N}$ corresponding to $D_1, \dots, D_g$ such that $\tilde{D}_i$ and D_i have common boundary $(1\leq i\leq g)$.

Set $N=\tilde{N}-\bigcup_{i=1}^g D_i'-\bigcup_{i=1}^g\tilde{D}_i$. Then

$$\chi(N)=\chi(\tilde{N})-2g=\chi(S_{\boldsymbol{R}})-c_1(L)^2-2g.$$

Since w is not tangent to ∂N, we see, by Lemma 2.11, ind $w|N=\chi(N)$. Thus, there exist at least ind$w$ critical points of $\varphi_{\boldsymbol{R}}^{(j)}$ of index 1 on N. Therefore, we have

$$\begin{aligned}\gamma_1^{(j)}-\gamma_1^{(j-1)} &\geq -\mathrm{ind}\,w\\ &=c_1(L)^2+2g-\chi(S_{\boldsymbol{R}}).\end{aligned}$$

On the other hald, there exist at least $2g$ critical points of $\varphi_{\boldsymbol{R}}^{(j)}$ of index 0 or 2 on $2g$-disks $\tilde{N}-N$. Therefore, we have

$$\gamma_0^{(j)}+\gamma_2^{(j)}-(\gamma_0^{(j-1)}+\gamma_2^{(j-1)})\geq 2g.$$

Thus $(\boldsymbol{A}'_{j-1})$ implies $(\boldsymbol{A}'_j)$ and $(\boldsymbol{B}'_{j-1})$ implies $(\boldsymbol{B}'_j)$.

Next we show $(\boldsymbol{E}'_j)$. Since $(\boldsymbol{E}'_1)$ is clear, let $j\geq 2$. Now set $s=s^{(j-2)}$, $s'=\mu^{j-2}s_{j-1}$ and $s''=\mu^{(j-2)}s_j$. Then we have

$$s^{(j)}=\lambda(\lambda s+\epsilon_{j-1}\mu s')+\epsilon_j\mu^2 s''.$$

Set $y=\lambda/\mu$, $f=s'/s$ and $g=s''/s$ in $S\times D_{\delta_{j-1}}-(\mu)_0\cup(s^{(j-2)})_0$. Then $A^{(j)}$ is defined by $y(y+\epsilon_{j-1}f)+\epsilon_j g=0$, (see Remark 2.10.2). Notice that $(s^{(j-2)})_0$ restricted to S equals to $(s_{j-2})_0$. On $S_{\boldsymbol{R}}-(s_{j-2})_0$, we have $fg=s_{j-1}s_j/s_{j-2}^2$. By Lemma 2.10 and (*iii), if we choose the sign of ϵ_j, then $(s^{(j)})_0$ has g-empty ovals in $S_{\boldsymbol{R}}\times[-\delta_{j-1}, \delta_{j-1}]$. Therefore we see

$$e^{(j)}-e^{(j-1)}\geq g,$$

$(2\leq j\leq r)$.

Thus we see $(\boldsymbol{E}'_{j-1})$ implies $(\boldsymbol{E}'_j)$.

Lastly, to see $(\boldsymbol{D}_j)$, we remark that, by the assumption, $H_1(S;\boldsymbol{Z}/2)=0$ and therefore, by Lemma 1.13, $(\boldsymbol{D}_{j-1})$ implies $(\boldsymbol{D}_j)$.

Q. E. D.

4. Construction of M-curves in a surface

Let S be a compact real complex surface, K be a real holomorphic line bundle and s be a real transverse section of K with zero-locus $C=(s)_0$.

Consider the following condition (**):

(** i): $C\cong \boldsymbol{P}_1^1$ and $C^2=\langle c_1(K)^2, [S]\rangle>0$,

(** ii): For any effective divisor α on C of degree C^2 with support in $C_{\boldsymbol{R}}$, there exists a real section s' of K such that $(s')_0|C=\alpha$.

PROPOSITION 4.0. *Let (S, K, s) satisfy the condition (**). Then, for any positive integer d and for any non-negative integer r, there exist a* *system of M-sections $s_0, s_1, \ldots, s_r$ near s^d in $H^0(S, K^d)_{\boldsymbol{R}}$ satisfying the condition* (*) *of Theorem* 0.3.

EXAMPLE 4.1: (1) Set $S=\boldsymbol{P}^2$ and $K=\mathcal{O}_{\boldsymbol{P}^2}(1)$. Let s be a real transverse section of K. Then (**) is satisfied. The construcion of an M-section of $K^d=\mathcal{O}_{\boldsymbol{P}^2}(d)$ is just the Harnack's one ([H], [G1]).

(2) Set $S=\boldsymbol{P}^2$ and $K=\mathcal{O}_{\boldsymbol{P}^2}(2)$. Let $C=(s)_0$ be a real ellipse. Then (**) is verified and Proposition 4.0 is reduced to Hilbert's construction ([G1]).

(3) Let $S\subset\boldsymbol{P}^3$ be a real hyperboloid, that is, the image of $\boldsymbol{P}^1\times\boldsymbol{P}^1$ by the Segre embedding. Set $K=\mathcal{O}_{\boldsymbol{P}^3}(1)|S$ and take a real hyperplane section C on S. Then (**) is satisfied. Especially, there exists an M-section in $H^0(S, K^d)_{\boldsymbol{R}}$, for each $d>0$ ([G2]).

PROOF OF PROPOSITION 4.0: By (**i) and Lemma 1.2, we have

$$c_1(TS)c_1(K)-c_1(K)^2=2.$$

Let $Z\subset S$ be the zero-locus of a transverse section of K^d. Then Z is connected. In fact, by (**ii), there is a section $s'=s\cdot s^{(2)}\cdots s^{(d)}$ of K^d such that $C^{(i)}=(s^{(i)})_0\cong\boldsymbol{P}^1$ and $C^{(i)}$ intersects to $C=(s)_0$ transversely, $(2\leq i\leq d)$. Then there exists a transverse section s'' of K^d, which is a perturbation of s', and $(s'')_0$ is connected, (cf. (1.0)).

The genus g of Z equals to

$$1+(1/2)(c_1(K^d)^2-c_1(K^d)c_1(TS))=(1/2)d(d-1)C^2-(d-1).$$

Remark that a real transverse section of K^d is an M-section if and only if $\boldsymbol{R}(s)_0$ has $1+g$ connected components.

Set $k=C^2>0$.

Take a sequence $(P^{\ell}_{i,j})_{1\leq j\leq i\leq d, 0\leq\ell\leq r}$ of disjoint k-points on $C_{\boldsymbol{R}}$ such that

$$P^0_{1,1}, P^0_{2,1}, P^0_{2,2}, P^0_{3,1}, P^0_{3,2}, \ldots, P^0_{d,1}, \ldots, P^0_{d,d}, P^1_{1,1}, \ldots, P^1_{d,d}, P^2_{1,1}, P^r_{1,1}, \ldots, P^r_{d,d}$$

are cyclic in the sence of (2.14).

By (** ii), for each i, j, ℓ with $1\leq j\leq i\leq d, 0\leq\ell\leq r$, there exists $s(i, j, \ell)\in H^0(S, K)_{\boldsymbol{R}}$ such that $(s(i, j, \ell))_0\cap C_{\boldsymbol{R}}=P^{\ell}_{i,j}$.

Set

$$u(1,\ell)=s+\epsilon_{1,\ell}s(1, 1, \ell)\in H^0(S, K)_{\boldsymbol{R}},$$

$\epsilon_{1,\ell}\in\boldsymbol{R}-0, 0\leq\ell\leq r$, and set inductively,

$$u(i, \ell)=u(i-1, \ell)\cdot s+\epsilon_{i,\ell}\prod_{j=1}^{i}s(i, j, \ell)\in H^0(S, K^i)_{\boldsymbol{R}},$$

$\epsilon_{i,\ell}\in \boldsymbol{R}-0, 2\leq i\leq d, 0\leq \ell\leq r$. If we choose the sign of $\epsilon_{i,\ell}$ and take $\epsilon_{i,\ell}$ sufficiently small relatively to $\epsilon_{i-1,\ell}$ and $\epsilon_{d,\ell-1}$, $(1\leq i\leq d, 0\leq \ell\leq r, \epsilon_{0,\ell}=\epsilon_{i,0}=1)$, then, we have, for each i, ℓ,

(i) $u(i, \ell)$ is a transverse section.

(ii) $\boldsymbol{R}(u(i, \ell))_0$ has $1+g_i$ connected components, where $g_i=\sum_{j=2}^{i}\{(j-1)k-1\}$, and g_i empty ovals, on the union of interiors of which $u(i, \ell)/s^i$ has a constant sign.

We have, for $(i, \ell)\neq(i', \ell')$,

(iii) $\boldsymbol{R}(u(i, \ell))_0$ and $\boldsymbol{R}(u(i', \ell'))_0$ intersect transversely in $ii'k$ points in $S_{\boldsymbol{R}}$.

(iv) $\boldsymbol{R}(u(i, \ell)u(i', \ell'))_0$ has $g_i+g_{i'}$ empty ovals.

(v) On the union of interiors of empty ovals appearing in $\boldsymbol{R}(u(i, \ell))_0$, $u(i, \ell)/u(i', \ell')$ takes a constant sign.

Further, we have, for disjoint (i, ℓ), (i', ℓ') and (i'', ℓ''),

(vi) The ratio $u(i, \ell)u(i', \ell')/u(i'', \ell'')^2$ takes a constant sign on the union of interiors of empty ovals in $\boldsymbol{R}(u(i, \ell))_0$.

Now, set $s_\ell=u(d, \ell)$, $(0\leq \ell\leq r)$. Then $s_0, \ldots, s_r$ satisfy the condition (*).

Q. E. D.

Next, we proceed to another situation: Let S be a compact real complex surface, K and J be real holomorphic line bundles and s and t be real transverse sections of K and J with zero-loci $C=(s)_0$ and $D=(t)_0$ respectively.

Consider the following condition (***):

(***0) $\dim_{\boldsymbol{C}} H^0(S, K)\geq 2$.

(***i) $C\cong \boldsymbol{P}^1_1$, $D\cong \boldsymbol{P}^1_1$, and $CD=\langle c_1(K)c_1(J), [S]\rangle=1$.

(***ii) For any point $p\in C_{\boldsymbol{R}}$, there exists a real section t' of J such that $(t')_0\cap C=\{p\}$.

PROPOSITION 4.2. *Let (S, K, J, C, D) satisfy the condition (***). Then, for any positive integers d and e, and, for any non-negative integer r, there exists a system of M-sections $s_0, s_1, \ldots, s_r$ in $H^0(S, K^dJ^e)_{\boldsymbol{R}}$ satisfying the condition (*) of Theorem* 0.3.

EXAMPLE 4.3: (1) Set $S=\boldsymbol{P}^1_1\times \boldsymbol{P}^1_1$, $K=p_1^*\mathcal{O}_{\boldsymbol{P}^1}(1)$ and $J=p_2^*\mathcal{O}_{\boldsymbol{P}^1}(1)$. Let s and t be real transverse section of K and J respectively. Then, (***) is easily verified.

(2) Let $S\subset \boldsymbol{P}^2\times \boldsymbol{P}^1_1$ be a non-singular real surface of degree (1,1). Set $K=\pi^*\mathcal{O}_{\boldsymbol{P}^2}(1)$ and $J=\varphi^*\mathcal{O}_{\boldsymbol{P}^1}(1)$, where $\pi: S\to \boldsymbol{P}^2$ and $\varphi: S\to \boldsymbol{P}^1_1$ are projections. Let s and t be real transverse sections of K and J respectively. Then (***) is satisfied. (For (***ii), notice that, for any line in $\boldsymbol{R}P^2$ not

containing the base point of S as a pencil of lines, and, for any point on the line, there exists a parameter defining a line through that point.)

PROOF OF PROPOSITION 4.2: By (*** i) and (***ii), the zero-locus of a transverse section of $K^d J^e$ is connected and of genus $1+(d-1)(e-1)$. By (***0), take $s' \in H^0(S, K)_{\boldsymbol{R}}$ such that $(s)_0 \cap (s')_0$ is a finite set. Set $Q = \boldsymbol{R}((s)_0 \cap (s')_0) \subset \boldsymbol{C}_{\boldsymbol{R}}$.

Let $(P_{i,\ell})_{0 \leq i \leq d, 0 \leq \ell \leq r}$ be a system of disjoint e-points of $\boldsymbol{C}_{\boldsymbol{R}}$. Assume

$$Q,\ P_{0,0},\ P_{1,0},\ P_{2,0},\ \ldots,\ P_{d,0},\ P_{0,1},\ P_{1,1},\ \ldots,\ P_{d,1},\ P_{0,2},\ \ldots,\ P_{d,2},\ \ldots,\ P_{d,r}$$

are cyclic on $C_{\boldsymbol{R}} \cong S^1$ in the sense of (2.14). For each i, ℓ, by (***ii), take a real transverse section $s(i, \ell)$ of J^e such that $(s(i, \ell))_0 \cap C_{\boldsymbol{R}} = P_{i,\ell}$.

Set

$$u(1, \ell) = s \cdot s(0, \ell) + \epsilon_{1,\ell} s' \cdot s(i, \ell),$$

and inductively set,

$$u(i, \ell) = s \cdot u(i-1, \ell) + \epsilon_{i,\ell} s' \cdot s(i, \ell),$$

$(2 \leq i \leq d)$, where $\epsilon_{i,\ell} \in \boldsymbol{R} - 0$.

Set $s_\ell = u(d, \ell) \in H^0(S, L)$, $(0 \leq \ell \leq r)$. If we choose the sign of each $\epsilon_{i,\ell}$ and take $\epsilon_{i,\ell}$ sufficiently small relatively to $\epsilon_{i-1,\ell}$ and $\epsilon_{d,\ell-1}$, then we see that the system $s_0, s_1, \ldots, s_r$ satisfies (*).

PROOF OF THEOREM 0.1 AND COROLLARY 0.5:

Set $S = \boldsymbol{P}^2$ and $L = \mathcal{O}_{\boldsymbol{P}^2}(d)$, (resp. $S = \boldsymbol{P}_1^1 \times \boldsymbol{P}_1^1$ and $L = p_1^* \mathcal{O}_{\boldsymbol{P}^1}(d) p_2^* \mathcal{O}_{\boldsymbol{P}^1}(e)$).

Then S is a compact connected M-surface with $H^1(S; \boldsymbol{Z}/2) = 0$ and $H^0(S_{\boldsymbol{R}}; \boldsymbol{Z}/2) \cong \boldsymbol{Z}/2$, (see Example 2.3.1 and Lemma 2.4), and L is a real holomorphic very ample vector bundle over S.

By Proposition 4.0 applied to Example 4.1.1 (resp. by Proposition 4.2 applied to Example 4.3.1), there exists a system of M-sections $s_0, \ldots, s_r$ of L satisfying (*), for $r = 0, 1, 2, \ldots$. Then, by Theorem 0.3, there exists an M-surface $A \subset S \times \boldsymbol{P}_1^1$ such that $\varphi: A \to \boldsymbol{P}_1^1$ has only non-degenerate real critical points. This means the existence of a generic surface A attaining the equality in the estimate of Theorem 0.1, (resp. Corollary 0.5), which is obtained from the formula in (1.6), (resp. (1.7) and (1.8)).

Q. E. D.

PROOF OF COROLLARY 0.6: Set K and J be as in Example 4.3.2. Set $L = K^d J^e$. Then, L is very ample. Similarly to the above proof, we only need to combine the results in (1.7), (1.8), Example 2.3.2, Example 4.3.2, Proposition 4.2 and Theorem 0.3.

Q. E. D.

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