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Existence of a global solution to a semi-linear wave equation with initial data of non-compact support in low space dimensions

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§1. Introduction

We study the existence of global (in time) solutions to the Cauchy problem of the form

$$(1.1)\qquad \begin{aligned} &u_{tt}-\Delta u=A|u|^{p} \quad \text{in} \quad \boldsymbol{R}^{n}\times[0,\infty),\\ &u(x,0)=f(x),\ u_{t}(x,0)=g(x) \quad \text{for} \quad x\in\boldsymbol{R}^{n}, \end{aligned}$$

where $n=2$ or $n=3$, A and p are constants with $A>0$, $p>1$, $f\in C^{3}(\boldsymbol{R}^{n})$ and $g\in C^{2}(\boldsymbol{R}^{n})$.

First suppose the supports of the initial data f, g are compact. For $n=3$, John has shown in [7], [8] that there exists a C^{2} global solution of (1.1) provided

$$p>p_{0}(3)=1+\sqrt{2}$$

and the initial data are small, where $p_{0}(n)$ denotes the positive root of the quadratic

$$(n-1)p^{2}-(n+1)p-2=0,$$

and that nontrivial C^{2} solutions of (1.1) blow up at finite time provided

$$1<p<p_{0}(3).$$

For $n=2$, Glassey has obtained the similar results in [4], [5], where $p_{0}(3)$ is replaced by $p_{0}(2)=(3+\sqrt{17})/2$. He has also proven in [4] that the critical value $p=p_{0}(3)$ in three space dimensions is the case where C^{2} solutions of (1.1) blow up at finite time provided the initial data are appropriately prescribed. Similar results for $n=2$ and 3 are obtained also by Schaeffer [10].

Now consider the case where the supports of the initial data f, g are not necessarily compact. Assume

$$(1.2)_n \qquad p>p_0(n)=\frac{n+1+\sqrt{(n+1)^2+8(n-1)}}{2(n-1)}$$

and

$$(1.3) \qquad \sum_{|\alpha|\leqq 3}|D_x^\alpha f(x)|+\sum_{|\alpha|\leqq 2}|D_x^\alpha g(x)|\leqq\frac{\varepsilon}{(1+r)^{1+\kappa}} \quad \text{for} \quad x\in\boldsymbol{R}^n$$

hold, where $r=|x|$, and κ, ε are positive constants. For $n=3$, Asakura has shown in [3] that there exists a C^2 global solution of (1.1) provided

$$(1.4) \qquad \kappa>\frac{2}{p-1}, \quad \text{i.e.,} \quad p\kappa>\kappa+2$$

holds and the constant ε is small, and that C^2 solutions of (1.1) blow up at finite time provided

$$(1.5) \qquad f(x)=0, \quad g(x)\geqq\frac{\varepsilon}{(1+r)^{1+\kappa}} \quad \text{for} \quad x\in\boldsymbol{R}^n$$

with $\varepsilon>0$ and

$$0<\kappa<\frac{2}{p-1}.$$

Thus $\kappa=2/(p-1)$ is a critical value.

For the two dimensional case $n=2$, recently, Tsutaya [11] and Agemi and Takamura [2] have independently obtained the similar blow-up results under (1.5). Moreover, Tsutaya [11] and the present author [9] have independently shown that there exists a C^2 global soluion of (1.1) provided $(1.2)_2$ and (1.3) hold with (1.4) and small $\varepsilon>0$.

The main purpose of this paper is to show that the results of [9] imply the global existence also for the critical value $\kappa=2/(p-1)$, unless $p=5$ for $n=2$. Thus it remains open the case where

$$p=5 \quad \text{and} \quad \kappa=\frac{1}{2} \quad \text{for} \quad n=2$$

(see Remark 1.3 below and Theorem 5.4 in section 5).

Now, the main results of this paper are the following two theorems.

THEOREM 1.1. *Let* $n=2$. *Assume*

$$(1.2)_2 \qquad p>p_0(2)=(3+\sqrt{17})/2$$

and (1.3) *hold with*

$$(1.6)\qquad \begin{cases} \varkappa \geqq \dfrac{2}{p-1} & \text{for} \quad p \neq 5, \\ \varkappa > \dfrac{2}{p-1} = \dfrac{1}{2} & \text{for} \quad p = 5 \end{cases}$$

and the constant ε small enough according as A, p and $\varkappa$. Then there exists uniquely a C^2 global solution of (1.1).

THEOREM 1.2. *Let $n=3$. Assume*

$$(1.2)_3 \qquad p > p_0(3) = 1 + \sqrt{2}$$

and (1.3) *hold with*

$$(1.7)\qquad \begin{cases} \varkappa \geqq \dfrac{2}{p-1} & \text{for} \quad p \neq 3, \\ \varkappa > \dfrac{2}{p-1} = 1 & \text{for} \quad p = 3 \end{cases}$$

and ε small enough according as A, p and $\varkappa$. Then there exists uniquely a C^2 global solution of (1.1).

REMARK 1.3. The presence of the irregular value $p=5$ in (1.6) is due to the fact that the decay rate of the soltion to the linear wave equation (i.e. (1.1) with $A=0$) has an extra factor $\log((1+t+r)/(1+|t-r|))$ for $\varkappa=1/2$. (See Proposition 2.1 in section 2).

One can easily generalize Theorem 1.1 as follows. Consider the following Cauchy problem

$$(1.8)\qquad \begin{aligned} &u_{tt} - \Delta u = F(u) \quad \text{in} \quad \boldsymbol{R}^2 \times [0, \infty), \\ &u(x, 0) = f(x), \quad u_t(x, 0) = g(x) \quad \text{for} \quad x \in \boldsymbol{R}^2, \end{aligned}$$

where $f \in C^3(\boldsymbol{R}^2)$ and $g \in C^2(\boldsymbol{R}^2)$. We impose the following hypothesis (H) on F.

(H) $\qquad F \in C^2(\boldsymbol{R}^1), \quad F(0) = F'(0) = F''(0) = 0$

and there exist constants $p > p_0(2)$, $A > 0$ such that

$$\begin{aligned} &|F''(u) - F''(v)| \\ &\leqq Ap(p-1)|u-v|(\max\{|u|, |v|\})^{p-3} \qquad \text{for} \quad |u|, |v| \leqq 1. \end{aligned}$$

Note that the function $F(u) = A|u|^p$ in (1.1) satisfies this hypothesis if $p > p_0(2)$. Moreover (H) implies

$$|F'(u)| \leqq Ap|u|^{p-1}, \quad |F(u)| \leqq A|u|^p \quad \text{for} \quad |u| \leqq 1.$$

We now state a generalization of Theorem 1.1.

THEOREM 1.4. *Assume* (H) *and* (1.3) *hold with* (1.6) *and* ε *small enough according as* A, p *and* $\varkappa$. *Then there exists uniquely a* C^2 *global solution of* (1.8).

One can also generalize analogously Theorem 1.2, where $F(u)$ satisfies the same hypothesis as in Asakura [3].

Now, the proof of Theorem 1.1 is fairly long. The main task is to examine the decay rate of the solution to the linear wave equation (see Proposition 2.1 below) and to estalish a basic a priori estimate for a linear operator L defined by (1.11) below (see Lemma 4.2). A basic tool in doing so is, as in [1], a fundamental identity (Lemma 2.3) for spherical means which follows from the fundamental identity for the integral of a plane wave function (see John [6], page 8)

$$(1.9)\qquad \int_{|\omega|=1} g(y\cdot\omega)dS_\omega=\omega_{n-1}\int_{-1}^{1} g(|y|\eta)(1-\eta^2)^{\frac{n-3}{2}}d\eta,$$

where $g(s)$ is a continuous function of $s\in(-\infty,\infty)$ and

$$\omega_k=2\sqrt{\pi^k}/\Gamma\left(\frac{k}{2}\right)\quad (k\geqq 1)$$

the surface measure of the unit sphere in $\boldsymbol{R}^k$.

REMARK 1.5. Tsutaya [11] has employed Kovalyov's result instead our Lemma 2.3, so that there is a loss: $\log(2+|t-r|)$ for $0<\varkappa<1$ in the decay rate of the solution to the linear wave equation.

The plan of this paper is as follows. In the next section we study the decay rate of the linear wave equaion in two space dimension $n=2$.

As is well known, a solution of (1.1) is furnished by a solution to the following integral equation

$$(1.10)\qquad u=u_0+L(A|u|^p),$$

where u_0 is the solution of the linear wave equation and L a positive linear operator on $C^0(\boldsymbol{R}^n\times[0,\infty))$ defined by

$$(1.11)\qquad L(w)(x,t)=\frac{1}{2\pi}\int_0^t(t-\tau)d\tau\int_{|\xi|<1}\frac{w(x+(t-\tau)\xi,\tau)}{\sqrt{1-|\xi|^2}}d\xi\quad \text{for}\quad n=2$$

or

$$(1.12)\qquad L(w)(x,t)=\frac{1}{4\pi}\int_0^t(t-\tau)d\tau\int_{|\xi|=1}w(x+(t-\tau)\xi,\tau)dS_\xi\quad \text{for}\quad n=3.$$

In section 3 we research how the decay rate of $L(|u|^p)(x,t)$ depends on that of $u(x, t)$ in two space dimensions. In section 4 we prove theorem 1.4 by applying the results obtained in sections 2 and 3. We also examine in section 5 lower bounds for the lifespan of the solutions to (1.1) in the case where either $(1.2)_2$ or (1.6) is violated. In section 6 we prove Theorem 1.2. Finally we show in Appendix that the decay rate obtained in section 2 is optimal inside the characteristic cone.

§ 2. The linear wave equation in two space dimensions

The goal of this section is the following estimate for solutions of the linear wave equation with $n=2$, which has been obtained in [9].

PROPOSITION 2.1. *Let $u(x, t)$ be the solution of the Cauchy problem*

$$(2.1)\qquad \begin{aligned} &u_{tt}-\Delta u=0 \quad in \quad \boldsymbol{R}^2\times[0, \infty), \\ &u(x, 0)=f(x), \quad u_t(x, 0)=g(x) \quad for \quad x\in\boldsymbol{R}^2, \end{aligned}$$

where $f\in C^3(\boldsymbol{R}^2)$ and $g\in C^2(\boldsymbol{R}^2)$. Suppose (1.3) holds for $n=2$. Then

$$(2.2)\qquad \sum_{|\alpha|\leqq 2}|D_x^\alpha u(x, t)|\leqq C_0\varepsilon\Phi(r, t) \quad for \quad (x, t)\in\boldsymbol{R}^2\times[0, \infty),$$

where $r=|x|$, C_0 is a constant depending only on κ and $\Phi(r, t)$ a positive valued function on $[0, \infty)\times[0, \infty)$ defined by dividing into five cases:

$$\Phi(r, t)=\begin{cases} \dfrac{1}{\sqrt{1+t+r}\sqrt{1+|t-r|}} & if \quad \kappa>1, \\ \dfrac{\log(2+|t-r|)}{\sqrt{1+t+r}\sqrt{1+|t-r|}} & if \quad \kappa=1, \\ \dfrac{1}{\sqrt{1+t+r}\,(1+|t-r|)^{\kappa-(1/2)}} & if \quad \dfrac{1}{2}<\kappa<1, \\ \dfrac{1}{\sqrt{1+t+r}}\left(1+\log\dfrac{1+t+r}{1+|t-r|}\right) & if \quad \kappa=\dfrac{1}{2}, \\ \dfrac{1}{(1+t+r)^\kappa} & if \quad 0<\kappa<\dfrac{1}{2}. \end{cases}$$

REMARK 2.2. In the appendix below we will show that the decay rate (2.2) is optimal for $0\leqq r\leqq t$ and $t\geqq 1$ if $f(x)=0$ and

$$g(x)\geqq\frac{\varepsilon}{(1+r)^{1+\kappa}}.$$

Moreover we note that the irregular value $\kappa=1/2$ in (2.2) corresponds to $\kappa=1$ in three space dimensions (see Proposition 6.0 in section 6).

Now, as is well known, the Cauchy problem (2.1) has a unique C^2 solution which is represented as

(2.3) $$u(x,t)=\frac{t}{2\pi}\int_{|\xi|<1}\frac{g(x+t\xi)}{\sqrt{1-|\xi|^2}}d\xi+\frac{\partial}{\partial t}\left(\frac{t}{2\pi}\int_{|\xi|<1}\frac{f(x+t\xi)}{\sqrt{1-|\xi|^2}}d\xi\right)$$

provided $f\in C^3(\boldsymbol{R}^2)$ and $g\in C^2(\boldsymbol{R}^2)$. Hence by virtue of (1.3) we have

(2.4) $$\sum_{|\alpha|\leqq 2}|D_x^\alpha u(x,t)|\leqq\frac{\varepsilon}{2\pi}(1+t)\int_{|\xi|<1}\frac{1}{\sqrt{1-|\xi|^2}(1+|x+t\xi|)^{1+\kappa}}d\xi$$
for $(x,t)\in\boldsymbol{R}^2\times[0,\infty)$.

First of all we shall show

(2.5) $$\sum_{|\alpha|\leqq 2}|D_x^\alpha u(x,t)|\leqq 2\varepsilon\left(\frac{4}{1+t+r}\right)^{1+\kappa}\quad\text{for}\quad (x,t)\in\boldsymbol{R}^2\times[0,1).$$

Let $0\leqq t\leqq 1$. If $r\leqq 2$, then (2.4) implies

$$\sum_{|\alpha|\leqq 2}|D_x^\alpha u(x,t)|\leqq\frac{\varepsilon}{\pi}\int_{|\xi|<1}\frac{1}{\sqrt{1-|\xi|^2}}d\xi=2\varepsilon\leqq 2\varepsilon\left(\frac{4}{1+t+r}\right)^{1+\kappa}.$$

If $r\geqq 2$, then $2r\geqq 1+t+r$ and hence

$$1+|x+t\xi|\geqq 1+r-t\geqq r\geqq\frac{1+t+r}{2},$$

so (2.4) yields

$$\sum_{|\alpha|\leqq 2}|D_x^\alpha u(x,t)|\leqq\frac{\varepsilon}{\pi}\left(\frac{2}{1+t+r}\right)^{1+\kappa}\int_{|\xi|<1}\frac{1}{\sqrt{1-|\xi|^2}}d\xi,$$

Therefore (2.5) follows.

From now on we assume

$$t\geqq 1.$$

Then, changing variables in (2.4) by

$$y=x+t\xi$$

and switching to polar coordinates

$$y=x+\rho\omega,\quad |\omega|=1,$$

we have

(2.6) $$\sum_{|\alpha|\leqq 2}|D_x^\alpha u(x,t)|\leqq\frac{\varepsilon}{\pi}\int_0^t\frac{\rho}{\sqrt{t^2-\rho^2}}d\rho\int_{|\omega|=1}\frac{1}{(1+|x+\rho\omega|)^{1+\kappa}}dS_\omega.$$

We shall apply the following identity to the integral over the unit sphere.

LEMMA 2.3. *Let $b(\lambda)$ be a continuous function of $\lambda\in[0,\infty)$. Let*

$x\in\boldsymbol{R}^n\backslash 0$ $(n\geqq 2)$ and $\rho>0$. Then

$$(2.7)\qquad \int_{|\omega|=1} b(|x+\rho\omega|)dS_\omega=2^{3-n}\omega_{n-1}(r\rho)^{2-n}\int_{|\rho-r|}^{\rho+r}\lambda b(\lambda)h(\lambda,\ \rho,\ r)d\lambda,$$

where

$$(2.8)\qquad h(\lambda,\ \rho,\ r)=(\lambda^2-(\rho-r)^2)^{(n-3)/2}((\rho+r)^2-\lambda^2)^{(n-3)/2}.$$

PROOF: Since

$$|x+\rho\omega|=\sqrt{r^2+\rho^2+2\rho x\cdot\omega},$$

we set

$$g(s)=\begin{cases} b(\sqrt{r^2+\rho^2+s}) & \text{for}\quad s\geqq -r^2-\rho^2 \\ b(0) & \text{for}\quad s<-r^2-\rho^2. \end{cases}$$

Then $g(s)$ is a continuous function of $s\in(-\infty,\infty)$. Applying (1.9) with $y=2\rho x$, we get therefore

$$\int_{|\omega|=1} b(|x+\rho\omega|)dS_\omega=\omega_{n-1}\int_{-1}^{1} b(\sqrt{r^2+\rho^2+2r\rho\eta})(1-\eta^2)^{(n-3)/2}d\eta.$$

Moreover we introduce a variable of integration λ instead of η by

$$\lambda=\sqrt{r^2+\rho^2+2r\rho\eta}.$$

Then a simple calculation yields (2.7). The roof is complete.

REMARK. When $n=2$, the lemma has been implicitly proven in [5]. For $n\geqq 3$, see the proof of Lemma 4.1 in [1], where $g(s)$ is not necessarily continuous.

We shall now return to (2.6). Appying Lemma 2.3 with $n=2$, we have

$$(2.9)\qquad \int_{|\omega|=1}\frac{1}{(1+|x+\rho\omega|)^{1+\kappa}}dS_\omega=4\int_{|\rho-r|}^{\rho+r}\frac{\lambda}{(1+\lambda)^{1+\kappa}}h(\lambda,\ \rho,\ r)d\lambda.$$

Since

$$\frac{\lambda}{(1+\lambda)^{1+\kappa}}\leqq\frac{1}{(1+\lambda)^{\kappa}}\quad\text{for}\quad \lambda\geqq 0,$$

by virtue of (2.6) we obtain

$$(2.10)\qquad \sum_{|\alpha|\leqq 2}|D_x^\alpha u(x,\ t)|\leqq 2\varepsilon I(r,\ t)$$

with

(2.11) $$I(r, t)=\frac{2}{\pi}\int_0^t \frac{\rho}{\sqrt{t^2-\rho^2}}d\rho\int_{|\rho-r|}^{\rho+r}\frac{1}{(1+\lambda)^\kappa}h(\lambda, \rho, r)d\lambda.$$

Moreover, since condition

$$|\rho-r|\leqq\lambda\leqq\rho+r \quad \text{and} \quad \rho\geqq 0$$

is equivalent to

$$|\lambda-r|\leqq\rho\leqq\lambda+r \quad \text{and} \quad \lambda\geqq 0,$$

one can rewrite (2.8) with $n=2$ as

(2.12) $$h(\lambda, \rho, r)=\frac{1}{\sqrt{\rho^2-(\lambda-r)^2}\sqrt{(\lambda+r)^2-\rho^2}}.$$

Furthermore we shall invert the order of the (ρ, λ) integral in (2.11). First suppose

$$r<t.$$

Then condition

$$0\leqq\rho\leqq t \quad \text{and} \quad |\rho-r|\leqq\lambda\leqq\rho+r$$

is equivalent to

$$0\leqq\lambda\leqq t+r \quad \text{and} \quad |\lambda-r|\leqq\rho\leqq\min\{t, \lambda+r\}.$$

Hence the right hand side of (2.11) is divided into two parts as follows.

(2.13) $$I(r, t)=\frac{2}{\pi}\int_{t-r}^{t+r}\frac{d\lambda}{(1+\lambda)^\kappa}\int_{|\lambda-r|}^{t}\frac{\rho}{\sqrt{t^2-\rho^2}}h(\lambda, \rho, r)d\rho$$
$$+\frac{2}{\pi}\int_0^{t-r}\frac{d\lambda}{(1+\lambda)^\kappa}\int_{|\lambda-r|}^{\lambda+r}\frac{\rho}{\sqrt{t^2-\rho^2}}h(\lambda, \rho, r)d\rho \quad \text{if} \quad r\leqq t.$$

Next suppose

$$r>t.$$

Then $\rho<r$ and $t<\lambda+r$ for $0\leqq\rho\leqq t$ and $\lambda\geqq 0$. Therefore condition

$$0\leqq\rho\leqq t \quad \text{and} \quad |r-\rho|\leqq\lambda\leqq r+\rho$$

is equivalent to

$$r-t\leqq\lambda\leqq r+t \quad \text{and} \quad |\lambda-r|\leqq\rho\leqq t.$$

Thus (2.11) becomes

(2.14) $\quad I(r, t)=\dfrac{2}{\pi}\displaystyle\int_{r-t}^{r+t}\frac{d\lambda}{(1+\lambda)^{\kappa}}\int_{|\lambda-r|}^{t}\frac{\rho}{\sqrt{t^2-\rho^2}}h(\lambda, \rho, r)d\rho \quad \text{if} \quad r\geqq t.$

Notice that (2.14) and the first term on the right hand side of (2.13) are unified, because $t-r=|t-r|$ if $r\leqq t$, and $r-t=|t-r|$ if $r\geqq t$. Thus we set

(2.15) $\quad I_1(r, t)=\dfrac{2}{\pi}\displaystyle\int_{|t-r|}^{t+r}\frac{d\lambda}{(1+\lambda)^{\kappa}}\int_{|\lambda-r|}^{t}\frac{\rho}{\sqrt{t^2-\rho^2}}h(\lambda, \rho, r)d\rho$

and when $r<t$,

(2.16) $\quad I_2(r, t)=\dfrac{2}{\pi}\displaystyle\int_{0}^{t-r}\frac{d\lambda}{(1+\lambda)^{\kappa}}\int_{|\lambda-r|}^{\lambda+r}\frac{\rho}{\sqrt{t^2-\rho^2}}h(\lambda, \rho, r)d\rho,$

where $h(\lambda, \rho, r)$ is given by (2.12). Then (2.13) and (2.14) imply

(2.17) $\quad I(r, t)=I_1(r, t)+I_2(r, t) \quad \text{for} \quad r\leqq t$

and

(2.18) $\quad I(r, t)=I_1(r, t) \quad \text{for} \quad r\geqq t.$

In order to estimate the ρ-integrals in (2.15) and (2.16) we will also employ the following identity which refines [5], Lemma 3.

LEMMA 2.4. *Let a, b be real nubers such that $0\leqq a<b$. Then*

$$\int_a^b\frac{\rho}{\sqrt{\rho^2-a^2}\sqrt{b^2-\rho^2}}d\rho=\frac{\pi}{2}.$$

PROOF: For convenience we denote the left hand side by J. Changing variable by $x=\rho^2$, we have

$$2J=\int_{a^2}^{b^2}\frac{1}{\sqrt{(x-a^2)(b^2-x)}}dx.$$

Since

$$(x-a^2)(b^2-x)=\left(\frac{b^2-a^2}{2}\right)^2-\left(x-b^2+a^2\right)^2,$$

we obtain

$$2J=\left[\arcsin\left(\left(x-\frac{b^2+a^2}{2}\right)\Big/\frac{b^2-a^2}{2}\right)\right]_{x=a^2}^{x=b^2}=\arcsin 1-\arcsin(-1)=\pi$$

Hence the desired identity follows.

We are now in a position to prove Proposition 2.1 which is a direct consequence of (2.5) and the following two lemmas.

LEMMA 2.5. *Let* $t\geqq 1$. *Then*

(2.19) $I_1(r, t)\leqq C_1\phi_1(r, t)$,

where

$$\phi_1(r, t)=\begin{cases}\dfrac{1}{\sqrt{1+t+r}\,(1+|t-r|)^{\kappa-(1/2)}} & \text{if } \kappa>\dfrac{1}{2},\\ \Phi(r, t) \quad \text{if} \quad 0<\kappa\leqq\dfrac{1}{2}\end{cases}$$

and C_1 *is a constant depending only on* κ.

LEMMA 2.6. *Let* $r<t$ *and* $t\geqq 1$. *Then*

(2.20) $I_2(r, t)\leqq C_2\phi_2(r, t)$,

where

$$\phi_2(r, t)=\begin{cases}\Phi(r, t) \quad \text{if} \quad \kappa>\dfrac{1}{2}\\ \dfrac{1}{\sqrt{1+t+r}}(1+t-r)^{(1/2)-\kappa} & \text{if} \quad 0<\kappa\leqq\dfrac{1}{2}\end{cases}$$

and C_2 *is a constant depending only on* κ.

PROOF OF LEMMA 2.5: We shall apply Lemma 2.4 with $a=|\lambda-r|$ and $b=t$ to the ρ-integral in (2.15). In view of (2.12) we must estimate $(\lambda+r)^2-\rho^2$ from below by a quantity independent of ρ. Since

$$\begin{aligned}&(\lambda+r)^2-\rho^2=(\lambda+r+\rho)(\lambda+r-\rho)\geqq(\lambda+r)(\lambda+r-t) \quad \text{for} \quad 0\leqq\rho\leqq t,\\ &\lambda+r\geqq(t+r)/2 \quad \text{for} \quad \lambda\geqq|t-r|\end{aligned}$$

and

$$t+r\geqq(1+t+r)/2 \quad \text{for} \quad t\geqq 1,$$

it follows from (2.15) and Lemma 2.4 that

$$I_1(r,t)\leqq\frac{2}{\sqrt{1+t+r}}\int_{|t-r|}^{t+r}\frac{1}{(1+\lambda)^{\kappa}\sqrt{\lambda+r-t}}d\lambda \quad \text{for} \quad t\geqq 1. \tag{2.21}$$

Moreover, intergrating by parts, we have

$$\int_{|t-r|}^{t+r}\frac{1}{(1+\lambda)^{\kappa}\sqrt{\lambda+r-t}}d\lambda=\left[\frac{2\sqrt{\lambda+r-t}}{(1+\lambda)^{\kappa}}\right]_{|t-r|}^{t+r}+2\kappa\int_{|t-r|}^{t+r}\frac{\sqrt{\lambda+r-t}}{(1+\lambda)^{\kappa+1}}d\lambda$$

$$\leqq \frac{2\sqrt{2r}}{(1+t+r)^{\kappa}} + 2\sqrt{2}\varkappa \int_{|t-r|}^{t+r} (1+\lambda)^{-\kappa-(1/2)} d\lambda,$$

since $\sqrt{\lambda+r-t} \leqq \sqrt{\lambda+\lambda}$ for $\lambda \geqq |t-r|$. Noting that

$$\frac{\sqrt{r}}{(1+t+r)^{\kappa}} \leqq \frac{1}{(1+t+r)^{\kappa-(1/2)}}$$

and

$$\int_{|t-r|}^{t+r} (1+\lambda)^{-\kappa-(1/2)} d\lambda \leqq \begin{cases} \dfrac{2}{2\varkappa-1} \cdot \dfrac{1}{(1+|t-r|)^{\kappa-(1/2)}} & \text{if} \quad \varkappa > \dfrac{1}{2}, \\ \log \dfrac{1+t+r}{1+|t-r|} & \text{if} \quad \varkappa = \dfrac{1}{2}, \\ \dfrac{2}{1-2\varkappa}(1+t+r)^{(1/2)-\kappa} & \text{if} \quad 0 < \varkappa < \dfrac{1}{2}, \end{cases}$$

we therefore obtain (2.19) from (2.21). The proof is complete.

PROOF OF LEMMA 2.6: We shall apply Lemma 2.4 with $a=|\lambda-r|$ and $b=\lambda+r$ to the ρ-integral in (2.16). Since

$$t^2-\rho^2=(t+\rho)(t-\rho) \geqq t(t-\lambda-r) \quad \text{for} \quad 0 \leqq \rho \leqq \lambda+r,$$

it follows that

$$(2.22) \qquad I_2(r, t) \leqq \frac{1}{\sqrt{t}} \int_0^{t-r} \frac{1}{(1+\lambda)^{\kappa}\sqrt{t-r-\lambda}} d\lambda \quad \text{for} \quad r<t \quad \text{and} \quad t \geqq 1.$$

We shall now divide the integral on the right hand side as

$$\int_0^{t-r} d\lambda = \int_0^{(t-r)/2} d\lambda + \int_{(t-r)/2}^{t-r} d\lambda \equiv I_{2,1} + I_{2,2},$$

so that

$$(2.23) \qquad I_2(r, t) \leqq \frac{1}{\sqrt{t}} (I_{2,1} + I_{2,2}).$$

For $I_{2,2}$ we have

$$I_{2,2} \leqq \frac{1}{\left(1+\dfrac{t-r}{2}\right)^{\kappa}} \int_{(t-r)/2}^{t-r} \frac{1}{\sqrt{t-r-\lambda}} d\lambda.$$

Since the last integral equals

$$2\sqrt{\frac{t-r}{2}},$$

if $0<\varkappa<1/2$ then

$$I_{2,2} \leqq 2\frac{1}{\left(1+\dfrac{t-r}{2}\right)^{\varkappa-(1/2)}} \leqq 2(1+t-r)^{(1/2)-\varkappa},$$

and if $\varkappa \geqq 1/2$ then

$$I_{2,2} \leqq \frac{2^{\varkappa}\sqrt{2}}{(1+t-r)^{\varkappa-(1/2)}}.$$

Hence we obtain

$$I_{2,2} \leqq 2^{\varkappa+1}\frac{1}{(1+t-r)^{\varkappa-(1/2)}} \quad \text{for} \quad \varkappa>0. \tag{2.24}$$

Finally consider $I_{2,1}$. Suppose

$$t-r \geqq 1.$$

Then

$$I_{2,1} = \int_0^{(t-r)/2} \frac{1}{(1+\lambda)^{\varkappa}\sqrt{t-r-\lambda}} d\lambda \leqq \sqrt{\frac{2}{t-r}} \int_0^{t-r} \frac{1}{(1+\lambda)^{\varkappa}} d\lambda$$

and

$$\int_0^{t-r} \frac{1}{(1+\lambda)^{\varkappa}} d\lambda \leqq \begin{cases} \dfrac{1}{\varkappa-1} & \text{if} \quad \varkappa>1 \\ \log(1+t-r) & \text{if} \quad \varkappa=1 \\ \dfrac{1}{1-\varkappa}(1+t-r)^{1-\varkappa} & \text{if} \quad 0<\varkappa<1. \end{cases}$$

Moreover $t-r \geqq 1$ implies

$$t-r \geqq (1+t-r)/2$$

hence

$$\sqrt{\frac{2}{t-r}} \leqq \frac{2}{\sqrt{1+t-r}}.$$

Thus we obtain

$$(2.25)\qquad I_{2,1}\leqq\begin{cases}\dfrac{2}{\kappa-1}\dfrac{1}{\sqrt{1+t-r}} & \text{if}\quad \kappa>1,\\ \dfrac{2\log(1+t-r)}{\sqrt{1+t-r}} & \text{if}\quad \kappa=1\\ \dfrac{2}{1-\kappa}(1+t-r)^{(1/2)-\kappa} & \text{if}\quad 0<\kappa<1\end{cases}$$

for $t-r\geqq 1$.

Now suppose $0\leqq t-r\leqq 1$. Then

$$I_{2,1}\leqq\int_0^{t-r}\frac{1}{\sqrt{t-r-\lambda}}d\lambda=2\sqrt{t-r}\leqq\frac{2\sqrt{2}}{\sqrt{1+t-r}}.$$

Thus (2.20) follows from (2.22), (2.24) and (2.25), since

$$3t\geqq 1+t+r\quad\text{for}\quad 0\leqq r\leqq t\quad\text{and}\quad t\geqq 1.$$

The proof is complete.

End of Proof of Proposition 2.1: If $0\leqq t\leqq 1$, then (2.2) follows immediately from (2.5). Now let $t\geqq 1$. If $\kappa\geqq 1/2$, then (2.2) is a direct consequence of Lemmas 2.5 and 2.6, according to (2.10) through (2.18). If $0<\kappa<1/2$, we have

$$\frac{1}{\sqrt{1+t+r}}(1+|t-r|)^{(1/2)-\kappa}\leqq\frac{1}{(1+t+r)^{\kappa}}$$

hence (2.2) follows from (2.19) and (2.20).Thus we prove the proposition.

§ 3. The basic estimates in two space dimensions

In this section we study how the decay rate of $L(|u|^p)(x,t)$ depends on that of $u(x,t)$, where L is the linear operator defined by (1.11). To do so we assume throughout the present section that $u(x,t)\in C^0(\boldsymbol{R}^2\times[0,T))$ and

$$(3.1)\qquad |u(x,t)|\leqq\frac{M}{(1+t+r)^{\mu}(1+|t-r|)^{\nu}}\quad\text{for}\quad (x,t)\in\boldsymbol{R}^2\times[0,T),$$

where $r=|x|$, T is a positive number or $T=\infty$, and M a constant. Moreover μ, ν are real numbers with $\mu>0$, $\nu\geqq 0$ which will be chosen appropriately, according as the decay rate (2.2) or p.

As will be seen, the decay rate of $L(|u|^p)$ depends on that of u if

$$0<\kappa<\frac{1}{2}+\frac{1}{p},$$

but does not if

$$x>\frac{1}{2}+\frac{1}{p}.$$

More recisely, we will prove the following three propositions which are the main results in this section. First of all we notice that

$$\frac{1}{2}+\frac{1}{p}-\frac{2}{p-1}=\frac{p^2-3p-2}{2p(p-1)} \tag{3.2}$$

hence $(1.2)_2$ implies

$$\frac{1}{2}+\frac{1}{p}>\frac{2}{p-1}.$$

Thus (1.6) is divided into three cases provided $(1.2)_2$ holds.

PROPOSITION 3.1. *Assume $p>2$ and*

$$0<x<\frac{1}{2}+\frac{1}{p}, \quad \textit{particularly} \quad p\left(x-\frac{1}{2}\right)<1. \tag{3.3}$$

Suppose (3.1) *holds with*

$$\mu=\frac{1}{2}, \ \nu=x-\frac{1}{2} \quad \textit{if} \quad \frac{1}{2}<x<\frac{1}{2}+\frac{1}{p} \tag{3.4}$$

or

$$\mu=x, \ \nu=0 \quad \textit{if} \quad 0<x\leqq\frac{1}{2}. \tag{3.5}$$

Then

$$|L(|u|^p)(x, t)|\leqq C_1M^p\Phi_1(r, t) \quad \textit{for} \quad (x, t)\in \boldsymbol{R}^2\times[0, T), \tag{3.6}$$

where C_1 is a constant depeding only on p and x (but independent of T), and $\Phi_1(r, t)$ a positive valued function on $[0, \infty)\times[0, \infty)$ defined by dividing into five cases:

$$\Phi_1(r,t)=\begin{cases} \dfrac{1}{\sqrt{1+t+r}\sqrt{1+|t-r|}} & \text{if } 3<p\kappa<\dfrac{p}{2}+1, \\ \dfrac{(\log(2+|t-r|))^2}{\sqrt{1+t+r}\sqrt{1+|t-r|}} & \text{if } p\kappa=3, \\ \dfrac{1}{\sqrt{1+t+r}(1+|t-r|)^{p\kappa-(5/2)}} & \text{if } \dfrac{5}{2}<p\kappa<3, \\ \dfrac{1}{\sqrt{1+t+r}}\left(1+\log\dfrac{1+t+r}{1+|t-r|}\right) & \text{if } p\kappa=\dfrac{5}{2}, \\ \dfrac{1}{(1+t+r)^{p\kappa-2}} & \text{if } 0<p\kappa<\dfrac{5}{2}. \end{cases}$$

PROPOSISITION 3.2. *Assume $p>2$ and*

(3.7) $$\kappa>\frac{1}{2}+\frac{1}{p}, \quad i.e., \quad p\left(\kappa-\frac{1}{2}\right)>1.$$

Suppose (3.1) *holds with $\mu=1/2$ and a positive number ν such that*

(3.8) $$\frac{1}{p}<\nu\leqq\min\left\{\kappa-\frac{1}{2},\ \frac{1}{2}\right\}.$$

Then

(3.9) $$|L(|u|^p)(x,t)|\leqq C_2M^p\Phi_2(r,t) \quad \textit{for } (x,t)\in\mathbf{R}^2\times[0,T),$$

where C_2 is a constant depending only on p, ν, and $\Phi_2(r,t)$ a function defined by

$$\Phi_2(r,t)=\begin{cases} \dfrac{1}{\sqrt{1+t+r}\sqrt{1+|t-r|}} & \text{if } p>4 \\ \dfrac{\log(2+|t-r|)}{\sqrt{1+t+r}\sqrt{1+|t-r|}} & \text{if } p=4, \\ \dfrac{1}{\sqrt{1+t+r}(1+|t-r|)^{(p-3)/2}} & \text{if } 3<p<4, \\ \dfrac{1}{\sqrt{1+t+r}}\left(1+\log\dfrac{1+t+r}{1+|t-r|}\right) & \text{if } p=3, \\ \dfrac{1}{(1+t+r)^{(p-2)/2}} & \text{if } 2<p<3. \end{cases}$$

REMARK. $\Phi_2(r,t)=\Phi_1(r,t)$ if $\kappa=\dfrac{1}{2}+\dfrac{1}{p}$ and $p\neq4$.

PROPOSITION 3.3. *Assume $p>2$ and*

(3.10) $$\varkappa=\frac{1}{2}+\frac{1}{p}, \quad \textit{i.e.}, \quad p\left(\varkappa-\frac{1}{2}\right)=1.$$

Suppose (3.1) *holds with* $\mu=1/2$ *and* $\nu=1/p$. *Then*

(3.11) $$|L(|u|^p)(x, t)|\leqq C_3M^p\Phi_3(r, t) \quad \textit{for} \quad (x, t)\in \boldsymbol{R}^2\times[0, T),$$

where C_3 *is a constant depending only on* p, *and* $\Phi_3(r, t)$ *a function defined by*

$$\Phi_3(r, t)=\begin{cases} \Phi_2(r, t) & \textit{if} \quad p>4, \\ \Phi_2(r, t)\log(2+|t-r|) & \textit{if} \quad 3<p\leqq 4, \\ \Phi_2(r, t)\log(2+t+r) & \textit{if} \quad 2<p\leqq 3 \end{cases}$$

with Φ_2 *the function in the preeding proposition.*

REMARK 3.4. Suppose the supports of the initial data f, g are compact. Then (1.3) holds for any $\varkappa>1$ and hence the first estimate of (2.2) is valid. Glassey has shown in [5] that (3.9) in Proposition 3.2 holds for $p>p_0(2)$ under (3.1) with the right hand side replaced by $M\Phi_2(r, t)$. Notice that $(p-3)/2>1/p$ for $p>p_0(2)$. Moreover it has been shown in [1] that the decay rate in (3.9) is optimal for $p>4$.

Before proving the propositions we shall state two corollaries of Proposition 3.1, the latter of which is a complement to the proposition, because the solution of (2.1) does not satisfy (3.1) with (3.5) for $\varkappa=1/2$. The former has been implicitly proven in [9].

COROLLARY 3.5. *Let the hypotheses of Proposition* 3.1 *be fulfilled. Assume* $(1.2)_2$ *and* (1.6) *holds. Moreover suppose* $\varkappa\neq 1/2$. *Then*

(3.12) $$|L(|u|^p)(x, t)|\leqq CM^p\frac{1}{(1+t+r)^\mu(1+|t-r|)^\nu} \quad \textit{for} \quad (x, t)\in \boldsymbol{R}^2\times[0, T),$$

where μ, ν *are the numbers defined by* (3.4) *or* (3.5), *and* C *is a constant depending only on* p *and* $\varkappa$.

PROOF: First suppose

$$\frac{1}{2}<\varkappa<\frac{1}{2}+\frac{1}{p}.$$

Then (1.6) implies

$$p\varkappa\geqq\varkappa+2>\frac{5}{2}$$

and

$$p\varkappa-\frac{5}{2}\geqq\varkappa-\frac{1}{2}.$$

Therefore (3. 6) yields (3. 12) with $\mu=1/2$, $\nu=\varkappa-(1/2)$ and another constant C, because

$$\varkappa-\frac{1}{2}<\frac{1}{p}\leqq\frac{1}{2}\quad\text{for}\quad p\geqq 2.$$

Next suppose

$$0<\varkappa<\frac{1}{2}.$$

Then it is clear that (3. 6) implies (3. 12) with $\mu=\varkappa$ and $\nu=0$, since $p\varkappa-2\geqq\varkappa$ under (1. 6). The proof is complete.

COROLLARY 3. 6. *Let $\varkappa=1/2$. Assume* (1. 6) *holds and*

$$|u(x, t)|\leqq M\frac{\log(2+t+r)}{\sqrt{1+t+r}}\quad\textit{for}\quad (x, t)\in\boldsymbol{R}^2\times[0, T). \tag{3. 13}$$

Then

$$|L(|u|^p)(x, t)|\leqq CM^p\frac{\log(2+t+r)}{\sqrt{1+t+r}}\quad\textit{for}\quad (x, t)\in\boldsymbol{R}^2\times[0, T), \tag{3. 14}$$

where C is a constant depending only on p.

PROOF: Notice that (1. 6) with $\varkappa=1/2$ implies $p>5$. Set

$$\varkappa'=\frac{5}{2p},$$

so that $0<\varkappa'<1/2$. Put

$$C_p=\sup_{s\geqq 1}s^{\varkappa'-(1/2)}\log(1+s).$$

Then (3. 13) yields

$$|u(x, t)|\leqq MC_p\frac{1}{(1+t+r)^{\varkappa'}}.$$

Therefore by virtue of Proposition 3.1 with $\varkappa$ replaced by $\varkappa'$ we obtain

$$|L(|u|^p)(x, t)|\leqq C_1(MC_p)^p\frac{1}{\sqrt{1+t+r}}\left(1+\log\frac{1+t+r}{1+|t-r|}\right),$$

which yields (3. 14) with

$$C=C_1(C_p)^p\left(\frac{1}{\log 2}+1\right).$$

The roof is complete.

The rest of this section will be devoted to prove Propositions 3.1, 3.2, and 3.3, although the main procedures will appear in the proof of the first.

PROOF OF PROPOSITION 3. 1: Note that (3. 4) or (3. 5) implies

$$\mu+\nu=\varkappa. \tag{3.15}$$

Moreover (1. 11) and (3. 1) yield

$$\begin{aligned} &|L(|u|^p)(x, t)| \\ &\leqq \frac{M^p}{2\pi}\int_0^t (t-\tau)d\tau \int_{|\xi|<1} \frac{d\xi}{\sqrt{1-|\xi|^2}(1+\tau+\lambda)^{p\mu}(1+|\tau-\lambda|)^{p\nu}}, \end{aligned} \tag{3.16}$$

where we have set

$$\lambda=|x+(t-\tau)\xi|.$$

First we deal with the case where t is small.

LEMMA 3. 7. *Let* $0\leqq t\leqq 1$. *Then*

$$|L(|u|^p)(x, t)|\leqq M^p\left(\frac{4}{1+t+r}\right)^{p\mu+p\nu}. \tag{3.17}$$

PROOF: It is clear that (3. 16) implies

$$|L(|u|^p)(x, t)|\leqq \frac{M^p}{2\pi}\int_0^t (t-\tau)d\tau \int_{|\xi|<1} \frac{1}{\sqrt{1-|\xi|^2}}d\xi.$$

Since

$$\int_{|\xi|<1}\frac{1}{\sqrt{1-|\xi|^2}}d\xi=\int_0^1 \frac{\rho}{\sqrt{1-\rho^2}}d\rho\int_0^{2\pi}d\theta=2\pi,$$

if $0\leqq t\leqq 1$ and $0\leqq r\leqq 2$ then

$$|L(|u|^p)(x, t)|\leqq M^p\leqq M^p\left(\frac{4}{1+t+r}\right)^{p\mu+p\nu}.$$

Next suppose $0\leqq t\leqq 1$ and $r\geqq 2$. Then $\lambda=|x+(t-\tau)\xi|\geqq r-(t-\tau)$ hence $\lambda-\tau\geqq r-t\geqq r-1$. Therefore $1+|\lambda-\tau|\geqq r$. Besides, $1+\lambda+\tau\geqq r$. Since

$$2r\geqq 1+t+r \quad \text{for} \quad r\geqq 2 \quad \text{and} \quad t\leqq 1,$$

we have

$$\frac{1}{(1+r+\lambda)^{p\mu}(1+|\tau-\lambda|)^{p\nu}}\leqq\left(\frac{2}{1+t+r}\right)^{p\mu+p\nu}.$$

Therefore it follows from (3. 16) that

$$|L(|u|^p)(x, t)\leqq M^p\left(\frac{2}{1+t+r}\right)^{p\mu+p\nu},$$

which implies (3. 17). The proof is omplete.

From now on we assume

$$t\geqq 1.$$

Changing variables in (3. 16) by

$$y=x+(t-\tau)\xi$$

and switching to polar coordinates

$$y=x+\rho\omega,\quad |\omega|=1,$$

we have

$$|L(|u|^p)(x, t)|\leqq\frac{M^p}{2\pi}\int_0^t d\tau\int_0^{t-\tau}\frac{\rho}{\sqrt{(t-\tau)^2-\rho^2}}d\rho\times$$
$$\times\int_{|\omega|=1}(1+\tau+\lambda)^{-p\mu}(1+|\tau-\lambda|)^{-p\nu}dS_\omega,$$

where we have set $\lambda=|x+\rho\omega|$. To the integral over the unit sphere we apply Lemma 2.3 with

$$b(\lambda)=(1+\tau+\lambda)^{-p\mu}(1+|\tau-\lambda|)^{-p\nu}.$$

Then, noting that $\lambda\leqq 1+\tau+\lambda$, we obtain

(3. 18) $\quad |L(|u|^p)(x, t)|\leqq M^p I(r, t),$

where

(3. 19) $$I(r, t)=\frac{2}{\pi}\int_0^t d\tau\int_0^{t-\tau}\frac{\rho}{\sqrt{(t-\tau)^2-\rho^2}}d\rho\times$$
$$\int_{|\rho-r|}^{\rho+r}(1+\tau+\lambda)^{1-p\mu}(1+|\tau-\lambda|)^{-p\nu}h(\lambda, \rho, r)d\lambda$$

with $h(\lambda, \rho, r)$ given by (2. 12).

Notice that the domain of (ρ, λ)-integration in (3. 19) coincides with the one in (2. 11) with t replaced by $t-\tau$. Therefore, the procedure by which we derived (2.13) and (2.14) yields that if $r \leqq t-\tau$, i.e., $\tau \leqq t-r$ then

$$\int_0^{t-\tau} d\rho \int_{|\rho-r|}^{\rho+r} d\lambda = \int_{|t-\tau-r|}^{t-\tau+r} d\lambda \int_{|\lambda+r|}^{t-\tau} d\rho + \int_0^{t-\tau-r} d\lambda \int_{|\lambda-r|}^{\lambda+r} d\rho,$$

and that if $r \geqq t-\tau$, i.e., $\tau \geqq t-r$ then

$$\int_0^{t-\tau} d\rho \int_{|\rho-r|}^{\rho+r} d\lambda = \int_{|t-\tau-r|}^{t-\tau+r} d\lambda \int_{|\lambda+r|}^{t-\tau} d\rho.$$

Note that, when $r>t$, the latter case only occurs. Thus, setting

$$\text{(3.20)} \quad \begin{aligned} I_1(r, t) = \frac{2}{\pi} \int_0^t d\tau \int_{|t-\tau-r|}^{t-\tau+r} (1+\tau+\lambda)^{1-p\mu} \times \\ \times (1+|\tau-\lambda|)^{-p\nu} d\lambda \int_{|\lambda-r|}^{t-\tau} \frac{\rho}{\sqrt{(t-\tau)^2-\rho^2}} h(\lambda, \rho, r) d\rho \end{aligned}$$

and, when $r<t$,

$$\text{(3.21)} \quad \begin{aligned} I_2(r, t) = \frac{2}{\pi} \int_0^{t-r} d\tau \int_0^{t-\tau-r} (1+\tau+\lambda)^{1-p\mu} \times \\ \times (1+|\tau-\lambda|)^{-p\nu} d\lambda \int_{|\lambda-r|}^{\lambda+r} \frac{\rho}{\sqrt{(t-\tau)^2-\rho^2}} h(\lambda, \rho, r) d\rho \end{aligned}$$

with $h(\lambda, \rho, r)$ given by (2.12), we have, analogously to (2.17) and (2.18),

$$\text{(3.22)} \quad I(r, t) = I_1(r, t) + I_2(r, t) \quad \text{for} \quad r \leqq t$$

and

$$\text{(3.23)} \quad I(r, t) = I_1(r, t) \quad \text{for} \quad r \geqq t.$$

Here we remark that the domain of (τ, λ)-integration in (3.20) coincides with the one in three space dimensions (see (2.12) of [3] or (6.19) below).

Now, we shall estimate I_1 and I_2 separately. From now on we will often use for convenience the following notations

$$\text{(3.24)} \quad \alpha = \tau+\lambda, \quad \beta = \tau-\lambda.$$

Besides, by C we will denote various constants depending only p and $\varkappa$.

LEMMA 3.8. *Let* $t \geqq 1$. *Then*

$$\text{(3.25)} \quad I_1(r, t) \leqq C\Phi_4(r, t),$$

where

$$\Phi_4(r,\ t)=\begin{cases}\dfrac{1}{\sqrt{1+t+r}\,(1+|t-r|)^{p\kappa-(5/2)}} & \text{if } \dfrac{5}{2}<p\kappa<\dfrac{p}{2}+1,\\ \Phi_1(r,\ t) & \text{if } 0<p\kappa\leqq\dfrac{5}{2}\end{cases}$$

and C is a constant depending only on p and κ.

LEMMA 3.9. *Let $0<r<t$ and $t\geqq 1$. Then*

(3.26) $\quad I_2(r,\ t)\leqq C\Phi_5(r,\ t),$

where

$$\Phi_5(r,\ t)=\begin{cases}\Phi_1(r,\ t) & \text{if } 3\leqq p\kappa<\dfrac{p}{2}+1,\\ \dfrac{1}{\sqrt{1+t+r}\,(1+t-r)^{p\kappa-(5/2)}} & \text{if } 0<p\kappa<3\end{cases}$$

and C is a constant depending only on p and κ.

PROOF OF LEMMA 3.8: We shall apply Lemma 2.4 with $a=|\lambda-r|$ and $b=t-\tau$ to the ρ-integral in (3.20). Since

$$(\lambda+r)^2-\rho^2=(\lambda+r+\rho)(\lambda+r-\rho)\geqq r(\lambda+r-t+\tau),$$

we have from (2.12) and (3.24)

$$\frac{2}{\pi}\int_{|\lambda-r|}^{t-r}\frac{\rho}{\sqrt{(t-\tau)^2-\rho^2}}h(\lambda,\ \rho,\ r)d\rho\leqq\frac{1}{\sqrt{r}\sqrt{\alpha+r-t}}$$

hence

$$I_1(r,t)\leqq\frac{1}{\sqrt{r}}\int_0^t d\tau\int_{|t-\tau-r|}^{t-\tau+r}(1+\alpha)^{1-p\mu}(1+|\beta|)^{-p\nu}(\alpha+r-t)^{-1/2}d\lambda. \tag{3.27}$$

Here we shall make a change of variables by (3.24). We first claim

(3.28) $\quad |t-r|\leqq\alpha\leqq t+r$

if $|t-\tau-r|\leqq\lambda\leqq t-\tau+r$ and $\tau\geqq 0$. In fact, $t-\tau-r\leqq\lambda\leqq t-\tau+r$ implies $t-r\leqq\alpha=\tau+\lambda\leqq t+r$. Moreover $\lambda\geqq-(t-\tau-r)$ and $\tau\geqq 0$ yield $\lambda+\tau\geqq r-t+2\tau\geqq r-t$. Hence (3.28) follows. In addition,

$$-\alpha\leqq\beta\leqq t-r$$

if $\lambda\geqq-(t-\tau-r)$ and $\alpha+\beta=2\tau\geqq 0$. By virtue of (3.27) we thus obtain

$$I_1(r,\ t)\leqq\frac{1}{2\sqrt{r}}\int_{|t-r|}^{t+r}(1+\alpha)^{1-p\mu}(\alpha+r-t)^{-1/2}d\alpha\int_{-\alpha}^{t-r}(1+|\beta|)^{-p\nu}d\beta. \tag{3.29}$$

Furthermore we claim

(3. 30) $\int_{-\alpha}^{t-r}(1+|\beta|)^{-p\nu}d\beta\leqq 2\int_{0}^{\alpha}(1+\beta)^{-p\nu}d\beta$ for $\alpha\geqq|t-r|$.

Indeed, if $t\geqq r$ then

$$\int_{-\alpha}^{t-r}d\beta=\int_{-\alpha}^{0}d\beta+\int_{0}^{t-r}d\beta$$

and

$$\int_{-\alpha}^{0}(1+|\beta|)^{-p\nu}d\beta=\int_{0}^{\alpha}(1+\beta)^{-p\nu}d\beta.$$

If $r\geqq t$ then

$$\int_{-\alpha}^{t-r}(1+|\beta|)^{-p\nu}d\beta=\int_{r-t}^{\alpha}(1+\beta)^{-p\nu}d\beta\leqq\int_{0}^{\alpha}(1+\beta)^{-p\nu}d\beta.$$

Hence we get (3. 30).

Now, since (3. 3), (3. 4) and (3. 5) imply

(3. 31) $p\nu<1$,

it follows from (3. 30) that

$$\int_{-\alpha}^{t-r}(1+|\beta|)^{-p\nu}d\beta\leqq\frac{2}{1-p\nu}(1+\alpha)^{1-p\nu}.$$

Therefore by (3. 29) and (3. 15) we obtain

(3. 32) $I_1(r,t)\leqq\frac{1}{1-p\nu}\frac{1}{\sqrt{r}}\int_{|t-r|}^{t+r}(1+\alpha)^{2-p\kappa}(\alpha+r-t)^{-1/2}d\alpha.$

We are now in a position to prove (3. 25). First suppose

(3. 33) $t\geqq 2r$ and $t\geqq 1$,

which implies

(3. 34) $6(t-r)\geqq 1+t+r$.

If $p\kappa>2$, then

$$\int_{|t-r|}^{t+r}(1+\alpha)^{2-p\kappa}(\alpha+r-t)^{-1/2}d\alpha$$
$$\leqq(1+t-r)^{2-p\kappa}\int_{t-r}^{t+r}(\alpha+r-t)^{-1/2}d\alpha=(1+t-r)^{2-p\kappa}\cdot 2\sqrt{2r}$$

hence (3. 32) yields

$$I_1 \leqq \frac{2\sqrt{2}}{1-p\nu} \frac{1}{(1+t-r)^{p\kappa-2}}.$$

Moreover by (3. 34) we get

$$\frac{1}{(1+t-r)^{p\kappa-2}} = \frac{1}{\sqrt{1+t-r}\,(1+t-r)^{p\kappa-(5/2)}}$$
$$\leqq \frac{\sqrt{6}}{\sqrt{1+t+r}\,(1+t-r)^{p\kappa-(5/2)}}.$$

Hence we obtain

$$(3.35) \qquad I_1(r, t) \leqq C \frac{1}{\sqrt{1+t+r}\,(1+|t-r|)^{p\kappa-(5/2)}},$$

which implies (3.25) for $p\kappa > 2$, because

$$(1+|t-r|)^{\frac{5}{2}-p\kappa} \leqq (1+t+r)^{\frac{5}{2}-p\kappa} \quad \text{for} \quad p\kappa \leqq \frac{5}{2}.$$

If $0 < p\kappa \leqq 2$, we also get similarly (3. 35).

Next suppose

$$(3.36) \qquad 1 \leqq t \leqq 2r,$$

which implies

$$(3.37) \qquad 5r \geqq 1+t+r.$$

Integrating by parts we have

$$(3.38) \qquad \int_{|t-r|}^{t+r} (1+\alpha)^{2-p\kappa}(\alpha+r-t)^{-1/2}d\alpha$$
$$= \left[\frac{2\sqrt{\alpha+r-t}}{(1+\alpha)^{p\kappa-2}}\right]_{|t-r|}^{t+r} + 2(p\kappa-2)\int_{|t-r|}^{t+r}(1+\alpha)^{1-p\kappa}\sqrt{\alpha+r-t}\,d\alpha.$$
$$\leqq \frac{2\sqrt{2r}}{(1+t+r)^{p\kappa-2}} + 2(p\kappa-2)\int_{|t-r|}^{t+r}(1+\alpha)^{1-p\kappa}\sqrt{\alpha+r-t}\,d\alpha.$$

Therefore, if $0 < p\kappa \leqq 2$, we obtain (3. 25) by (3. 32).

Finally suppose

$$2 < p\kappa < \frac{p}{2} + 1$$

and (3. 36) hold. Then, since

$$\sqrt{\alpha+r-t} \leqq \sqrt{2\alpha} \quad \text{for} \quad \alpha \geqq |t-r|,$$

it follows from (3. 32) and (3. 38) that

$$I_1(r,t)\leqq\frac{2\sqrt{2}}{1-p\nu}\left(\frac{1}{(1+t+r)^{p\kappa-2}}+\frac{p\kappa-2}{\sqrt{r}}\int_{|t-r|}^{t+r}(1+\alpha)^{(3/2)-p\kappa}d\alpha\right).$$

By virtue of (3.37) we thus obtain (3.25), by dividing into two cases:

$$p\kappa>5/2,\quad p\kappa\leqq 5/2.$$

The proof is complete.

PROOF OF LEMMA 3.9: We shall apply Lemma 2.4 with $a=|\lambda-r|$ and $b=\lambda+r$ to the ρ-integral in (3.21). Noting that

$$t-\tau-\rho\geqq t-\tau-\lambda-r=t-r-\alpha$$

and

$$t-\tau+\rho\geqq t-\tau+\lambda-r=t-r-\beta,$$

we obtain

$$(3.39)\qquad I_2(r,t)\leqq\int_0^{t-r}d\tau\int_0^{t-\tau-r}(1+\alpha)^{1-p\mu}(1+|\beta|)^{-p\nu}\times \\ \times\frac{1}{\sqrt{t-r-\alpha}\sqrt{t-r-\beta}}d\lambda.$$

On the other hand, since

$$t-\tau+\rho\geqq t-\tau\geqq t-(t-r)\quad\text{for}\quad \rho\geqq 0\quad\text{and}\quad \tau\leqq t-r,$$

we also get

$$(3.40)\qquad I_2(r,t)\leqq\frac{1}{\sqrt{r}}\int_0^{t-r}d\tau\int_0^{t-\tau-r}(1+\alpha)^{1-p\mu}(1+|\beta|)^{-p\nu}\frac{1}{\sqrt{t-r-\alpha}}d\lambda.$$

Moreover, since (3.24) yields $\alpha\geqq|\beta|$, we have

$$(1+\alpha)^{1-p\mu}\leqq(1+\alpha)^{1-p\mu+\delta}(1+|\beta|)^{-\delta}$$

for $\delta\geqq 0$. Changing variables in (3.39) by (3.24), we thus get

$$I_2(r,t)\leqq\frac{1}{2}\int_0^{t-r}(1+\alpha)^{1-p\mu+\delta}(t-r-\alpha)^{-1/2}d\alpha\times \\ \times\int_{-\alpha}^{t-r}(1+|\beta|)^{-p\nu-\delta}(t-r-\beta)^{-1/2}d\beta$$

for $\delta\geqq 0$. Similarly to (3.30) we have also

$$\int_{-\alpha}^{t-r}(1+|\beta|)^{-p\nu-\delta}(t-r-\beta)^{-1/2}d\beta$$

$$\leqq 2\int_0^{t-r}(1+\beta)^{-p\nu-\delta}(t-r-\beta)^{-1/2}d\beta$$

for $0\leqq \alpha\leqq t-r$, since

$$(t-r+\beta)^{-1/2}\leqq(t-r-\beta)^{-1/2} \quad \text{for} \quad \beta\geqq 0.$$

Consequently we obtain

$$(3.41) \qquad I_2(r,t)\leqq\int_0^{t-r}(1+\alpha)^{1-p\mu+\delta}(t-r-\alpha)^{-1/2}d\alpha\times \\ \times\int_0^{t-r}(1+\beta)^{-p\nu-\delta}(t-r-\beta)^{-1/2}d\beta$$

for any $\delta\geqq 0$. Analogously we have from (3.40)

$$(3.42) \qquad I_2(r,t)\leqq\frac{1}{\sqrt{r}}\int_0^{t-r}(1+\alpha)^{1-p\mu+\delta}\times(t-r-\alpha)^{-1/2}d\alpha\int_0^{t-r}(1+\beta)^{-p\nu-\delta}d\beta$$

for any $\delta\geqq 0$.

In what follows we will often use the following estimate.

LEMMA 3.10. *Let $0\leqq r<t$ and $t\geqq 1$. Set*

$$J(r,t)=\int_0^{t-r}\frac{1}{(1+\alpha)^a\sqrt{t-r-\alpha}}d\alpha,$$

where a is an arbitrary real number. Then

$$(3.43) \qquad J(r,t)\leqq C\Phi_6(r,t),$$

where C is a constant depending only on a and

$$\Phi_6(r,t)=\begin{cases}\dfrac{1}{\sqrt{1+t-r}} & \text{if } 1<a<\infty,\\[2mm] \dfrac{\log(2+t-r)}{\sqrt{1+t-r}} & \text{if } a=1,\\[2mm] \dfrac{1}{(1+t-r)^{a-(1/2)}} & \text{if } -\infty<a<1.\end{cases}$$

PROOF: If $a\leqq 0$, we have

$$J(r,t)\leqq(1+t-r)^{-a}\int_0^{t-r}\frac{1}{\sqrt{t-r-\alpha}}d\alpha=(1+t-r)^{-a}2\sqrt{t-r},$$

which implies (3.43) with $C=2$.

In what follows we suppose $a>0$. First consider the case where $t-r\geqq 1$ so that $2(t-r)\geqq 1+t-r$. Then, dividing the integral as

$$J=\int_0^{(t-r)/2} da+\int_{(t-r)/2}^{t-r} da,$$

we have

$$J(r,t)\leqq\sqrt{\frac{2}{t-r}}\int_0^{t-r}\frac{1}{(1+a)^a}da+\frac{1}{\left(1+\frac{t-r}{2}\right)^a}\int_0^{t-r}\frac{1}{\sqrt{t-r-a}}da.$$

Hence we obtain (3.43). If $0\leqq t-r\leqq 1$, we have

$$J(r,t)\leqq\int_0^{t-r}\frac{1}{\sqrt{t-r-a}}da=2\sqrt{t-r}\leqq 2$$

hence (3.43) follows immediately. The proof is complete.

Now, we shall continue to prove (3.26). First suppose

(3.44) $\quad 0<p\kappa<3.$

Then we take the δ in (3.41) and (3.42) in such a way that

$$p\mu-2<\delta<1-p\nu \quad \text{and} \quad \delta\geqq 0,$$

which is possible according to (3.15), (3.31) and (3.44). If (3.33) holds, it follows from (3.41) and Lemma 3.10 with $a<1$ that

$$\begin{aligned} I_2(r,t)&\leqq C\frac{1}{(1+t-r)^{p\mu-1-\delta-(1/2)}}\frac{1}{(1+t-r)^{p\nu+\delta-(1/2)}}\\ &=\frac{C}{(1+t-r)^{p\mu+p\nu-2}}=\frac{C}{\sqrt{1+t-r}\,(1+t-r)^{p\kappa-(5/2)}}.\end{aligned}$$

Hence by (3.34) we obtain (3.26) with (3.44). If (3.36) holds, by (3.42) and (3.37) we get (3.26), as above.

From now on we suppose

(3.45) $\quad 3\leqq p\kappa<\frac{p}{2}+1.$

Then

(3.46) $\quad p\mu-1-p\nu>0.$

In fact, this is clear provided μ and ν are given by (3.5). Consider the case (3.4). Then, since (3.45) implies $p>4$, we see from (3.31) that

$$p\mu-1-p\nu>\frac{p}{2}-2>0.$$

Hence (3.46) follows.

First suppose (3. 33) holds. Then it follows from (3. 15) and (3. 41) with $\delta=(p\mu-1-p\nu)/2$ that

$$I_2(r, t)\leqq\left(\int_0^{t-r}\frac{1}{(1+\alpha)^{(p\kappa-1)/2}\sqrt{t-r-\alpha}}d\alpha\right)^2.$$

Hence by Lemma 3.10 with $a\geqq 1$ we have

$$I_2(r, t)\leqq\begin{cases} C\dfrac{1}{1+t-r} & \text{if} \quad p\kappa>3, \\ C\dfrac{(\log(2+t-r))^2}{1+t-r} & \text{if} \quad p\kappa=3. \end{cases}$$

By (3. 34) we thus obtain (3. 26) with (3. 45).

Next suppose (3. 36) holds. Then it follows from (3. 42) with $\delta=(p\mu-1-p\nu)/2$ that

$$I_2(r, t)\leqq\frac{1}{\sqrt{r}}\int_0^{t-r}\frac{1}{(1+\alpha)^{(p\kappa-1)/2}\sqrt{t-r-\alpha}}d\alpha\times\int_0^{t-r}\frac{1}{(1+\beta)^{(p\kappa-1)/2}}d\beta.$$

Therefore by virtue of Lemma 3.10 and (3. 37) we btain (3. 26). The proof is complete.

End of Proof of Proposition 3. 1: If $0\leqq t\leqq 1$, by Lemma 3.7 we obtain (3. 6). Now let $t\geqq 1$. If $p\kappa\geqq 5/2$, then (3. 6) is a direct consequence of Lemmas 3.8 and 3.9, because of (3. 18) through (3. 23). Let $0<p\kappa<5/2$. Then

$$\frac{1}{\sqrt{1+t+r}\,(1+|t-r|)^{p\kappa-(5/2)}}=\frac{1}{(1+t+r)^{p\kappa-2}}\left(\frac{1+|t-r|}{1+t+r}\right)^{(5/2)-p\kappa}$$
$$\leqq\frac{1}{(1+t+r)^{p\kappa-2}}.$$

Hence (3. 6) follows from (3. 25) and (3. 26). Thus we complete the proof.

Proof of Proposition 3. 2: We have only to modify a little the proof of Proposition 3.1. Note that (3. 31) is replaced by (3. 8) and that (3. 15) breaks down. Nevertheless Lemma 3.7 and (3. 18) through (3. 23) are still valid with (3. 8) and $\mu=1/2$, while Lemmas 3.8 and 3.9 are replaced by the following two lemmas.

Lemma 3. 11. *Let* $t\geqq 1$. *Then*

(3. 47) $\quad I_1(r, t)\leqq C\Phi_7(r, t),$

where

$$\Phi_7(r, t)=\begin{cases}\dfrac{1}{\sqrt{1+t+r}\,(1+|t-r|)^{(p-3)/2}} & \text{if} \quad p>3,\\ \Phi_2(r, t) & \text{if} \quad 2<p\leqq 3\end{cases}$$

and C is a constant depending only on p and ν.

LEMMA 3.12: *Let $0<r<t$ and $t\geqq 1$. Then*

(3.48) $\quad I_2(r, t)\leqq C\Phi_8(r, t),$

where

$$\Phi_8(r, t)=\begin{cases}\Phi_2(r, t) & \text{if} \quad p\geqq 4,\\ \dfrac{1}{\sqrt{1+t+r}\,(1+|t-r|)^{(p-3)/2}} & \text{if} \quad 2<p<4.\end{cases}$$

PROOF OF LEMMA 3.11: We have only to modify little the proof of Lemma 3.8. Note that $\Phi_7(r, t)=\Phi_4(r, t)$ if we set

(3.49) $\quad \varkappa=\dfrac{1}{2}+\dfrac{1}{p}$, i.e., $p\varkappa=\dfrac{p}{2}+1.$

Moreover (3.29) and (3.30) are still valid with $\mu=1/2$ and $p\nu>1$. Since now

$$\int_0^\alpha (1+\beta)^{-p\nu}d\beta\leqq\frac{1}{p\nu-1},$$

we have

$$I_1(r, t)\leqq\frac{1}{p\nu-1}\cdot\frac{1}{\sqrt{r}}\int_{|t-r|}^{t+r}(1+\alpha)^{(2-p)/2}(\alpha+r-t)^{-1/2}d\alpha,$$

which coincides, except the constant, with (3.32) for $\varkappa$ satisfying (3.49). Therefore we obtain (3.47) analogously to (3.25). The proof is complete.

PROOF OF LEMMA 3.12: We have only to modify a little the proof of Lemma 3.9. Note that $\Phi_8(r, t)=\Phi_5(r, t)$ with (3.49) if $p\neq 4$. Moreover (3.41) and (3.42) are still valid with $\mu=1/2$ and $p\nu>1$. We shall take $\delta=0$. First suppose (3.33) holds. Then by (3.41) and Lemma 10 with $a=(p-2)/2$ or $a=p\nu$ we get (3.48), because (3.34) implies

$$(1+t-r)^{-1/2}\leqq\sqrt{6}(1+t+r)^{-1/2}.$$

Next suppose (3.36) holds. Then by (3.42) and (3.37) we obtain (3.48), as above. The proof is complete.

END OF PROOF OF PROPOSITION 3.2: The desired estimate (3.9) is a direct consequence of Lemmas 3.7, 3.11 and 3.12, since $\Phi_8(r, t) \leqq \Phi_2(r, t)$ for $2<p<3$. Thus we prove the propostion.

PROOF OF PROPOSITION 3.3: We shall modify the proof of Proposition 3.1. Note that (3.31) is replaced by

$$(3.50) \qquad p\nu = 1.$$

Then (3.15) through (3.23) are still valid with $\mu=1/2$. However, Lemmas 3.8 and 3.9 are replaced by the following two lemmas.

LEMMA 3.13. *Let $t \geqq 1$. Then*

$$(3.51) \qquad I_1(r, t) \leqq C\Phi_9(r, t),$$

where

$$\Phi_9(r, t) = \begin{cases} \dfrac{\log(2+|t-r|)}{\sqrt{1+t+r}\,(1+|t-r|)^{(p-3)/2}} & \text{if } p>3, \\ \Phi_3(r, t) & \text{if } 2<p\leqq 3 \end{cases}$$

and C is a constant depending only on p.

LEMMA 3.14. *Let $0<r<t$ and $t \geqq 1$. Then*

$$(3.52) \qquad I_2(r, t) \leqq C\Phi_{10}(r, t),$$

where

$$\Phi_{10}(r, t) = \begin{cases} \Phi_3(r, t) & \text{if } p \geqq 4, \\ \dfrac{\log(2+t-r)}{\sqrt{1+t+r}\,(1+t-r)^{(p-3)/2}} & \text{if } 2<p<4 \end{cases}$$

and C is a constant depending only on p.

PROOF OF LEMMA 3.13: We shall modify the proof of Lemma 3.8. First we observe that (3.29) and (3.30) are still valid with $\mu=1/2$ and $p\nu=1$. Moreover (3.30) and (3.50) imply

$$\int_0^{t-r} (1+|\beta|)^{-1} d\beta \leqq 2\log(1+\alpha).$$

Therefore we have from (3.29)

$$(3.53) \qquad I_1(r, t) \leqq \frac{1}{\sqrt{r}} \int_{|t-r|}^{t+r} (1+\alpha)^{(2-p)/2} (\log(1+\alpha))(\alpha+r-t)^{-1/2} d\alpha.$$

First suppose (3.33) holds. Then

$$I_1(r, t) \leqq \frac{1}{\sqrt{r}}(1+t-r)^{(2-p)/2}(\log(1+t+r))\int_{t-r}^{t+r}(\alpha+r-t)^{-1/2}d\alpha,$$

since $p>2$. By (3. 34) we therefore obtain

$$(3.54) \qquad I_1(r, t) \leqq C\frac{\log(1+t+r)}{(1+t+r)^{(p-2)/2}}$$

provided $p>2$ and (3. 33) holds. If $2<p\leqq 3$, then (3. 54) implies (3. 51). If $p>3$, we have

$$(3.55) \qquad \frac{\log(1+t+r)}{(1+t+r)^{(p-3)/2}} \leqq C\frac{\log(2+|t-r|)}{(1+|t-r|)^{(p-3)/2}},$$

because the function

$$[1, \infty) \ni s \longmapsto s^{(3-p)/2}\log s$$

is decreasing for $s>\exp(2/(p-3))$. Therefore (3. 54) yields (3. 51) for $p>3$.

From now on we suppose (3. 36) holds. Integrating by parts we have

$$\begin{aligned}&\int_{|t-r|}^{t+r}(1+\alpha)^{(2-p)/2}(\log(1+\alpha))(\alpha+r-t)^{-1/2}d\alpha\\&=[(1+\alpha)^{(2-p)/2}(\log(1+\alpha))2\sqrt{\alpha+r-t}\,]_{|t-r|}^{t+r}\\&-\int_{|t-r|}^{t+r}\{(1+\alpha)^{(2-p)/2}\log(1+\alpha)\}'2\sqrt{\alpha+r-t}\,d\alpha.\end{aligned}$$

Since

$$-\{(1+\alpha)^{(2-p)/2}\log(1+\alpha)\}' \leqq \frac{p-2}{2}(1+\alpha)^{-(p/2)}\log(1+\alpha),$$

by (3. 53) we get, as before,

$$\begin{aligned}(3.56) \qquad I_1(r, t) \leqq\ & 2\sqrt{2}\frac{\log(1+t+r)}{(1+t+r)^{(p-2)/2}}\\&+\sqrt{2}(p-2)\frac{1}{\sqrt{r}}\int_{|t-r|}^{t+r}(1+\alpha)^{(1-p)/2}\log(1+\alpha)d\alpha.\end{aligned}$$

If $2<p\leqq 3$, by (3. 37) we get (3. 51), since the last integral is dominated by

$$\log(1+t+r)\int_{|t-r|}^{t+r}(1+\alpha)^{(1-p)/2}d\alpha.$$

Now suppose

$$p>3.$$

Integrating by parts we have

$$\begin{aligned}
&\int_{|t-r|}^{t+r}(1+\alpha)^{(1-p)/2}\log(1+\alpha)d\alpha \\
&=\frac{2}{3-p}[(1+\alpha)^{(3-p)/2}\log(1+\alpha)]_{|t-r|}^{t+r}+\frac{2}{p-3}\int_{|t-r|}^{t+r}(1+\alpha)^{(1-p)/2}d\alpha \\
&\leqq\frac{2}{p-3}(1+|t-r|)^{(3-p)/2}\log(1+|t-r|)-\left(\frac{2}{p-3}\right)^2[(1+\alpha)^{(3-p)/2}]_{|t-r|}^{t+r} \\
&\leqq\frac{2}{p-3}(1+|t-r|)^{(3-p)/2}\left\{\log(1+|t-r|)+\frac{2}{p-3}\right\}.
\end{aligned}$$

Therefore by virtue (3.37), (3.55) and (3.56) we obtain (3.51) for $p>3$. The roof is complete.

PROOF OF LEMMA 3.14: We have only to modify a little the proof of Lemma 3.9. Note that (3.10) implies

$$p\kappa=\frac{p}{2}+1.$$

Moreover (3.41) and (3.42) are still valid with (3.50) and $\mu=1/2$.

First suppose

$$2<p\leqq 4.$$

Then we take $\delta=0$ hence (3.41) and (3.42) become, respectively,

$$(3.41)' \quad I_2(r,t)\leqq\int_0^{t-r}\frac{1}{(1+\alpha)^{(p-2)/2}\sqrt{t-r-\alpha}}d\alpha\times\int_0^{t-r}\frac{1}{(1+\beta)\sqrt{t-r-\beta}}d\beta$$

and

$$(3.42)' \quad I_2(r,t)\leqq\frac{1}{\sqrt{r}}\int_0^{t-r}\frac{1}{(1+\alpha)^{(p-2)/2}\sqrt{t-r-\alpha}}d\alpha\int_0^{t-r}\frac{1}{1+\beta}d\beta.$$

Therefore, if $2<p<4$ and (3.33) holds, by virtue of (3.41)′ and Lemma 3.10 with $a\leqq 1$ we obtain

$$I_2(r,t)\leqq C\frac{\log(2+t-r)}{\sqrt{1+t-r}\,(1+t-r)^{(p-3)/2}},$$

which together with (3.34) yields (3.52). If $2<p<4$ and (3.36) holds, using (3.42)′ and (3.37) instead of (3.41)′ and (3.34) we get (3.52). Analogously we obtain (3.52) for $p=4$.

Next suppose $p>4$. Then (3.46) is still valid with $\mu=1/2$ and $p\nu=1$. Therefore, analogously to the proof of (3.26) we obtain

$$I_2(r,t)\leqq C\frac{1}{\sqrt{1+t+r}\sqrt{1+t-r}},$$

because now

$$\frac{p\varkappa-1}{2}=\frac{p}{4}>1.$$

The proof is complete.

END OF PROOF OF PROPOSITION 3.3: If $0\leqq t\leqq 1$, then (3.11) follows immediately from Lemma 3.7. If $t\geqq 1$, then (3.11) is a direct consequence of Lemmas 3.13 and 3.14, because $\Phi_{10}(r, t)\leqq\Phi_3(r, t)$ for $2<p<3$. Thus we prove the proposition.

§ 4. Proof of Theorem 1.4

For a continuous function $u(x, t)\in C^0(\boldsymbol{R}^2\times[0, \infty))$ we define a norm of u by

$$(4.1)\qquad \|u\|=\sup_{(x, t)\in\boldsymbol{R}^2\times[0, \infty)}|u(x, t)|\Psi(r, t),$$

where $\Psi(r, t)$ is a positive valued function on $[0, \infty)\times[0, \infty)$ defined by dividing into five cases.

CASE 1: Let

$$0<\varkappa\leqq\frac{1}{2}+\frac{1}{p}\quad\text{and}\quad\varkappa\neq\frac{1}{2}.$$

Then

$$(4.2)\qquad \Psi(r, t)=(1+t+r)^{\mu}(1+|t-r|)^{\nu},$$

where μ, ν are the numbers given by (3.4) or (3.5) with

$$\mu=\frac{1}{2},\quad \nu=\frac{1}{p}\quad\text{for}\quad \varkappa=\frac{1}{2}+\frac{1}{p}.$$

Note that $(1.2)_2$ implies $p>2$ and hence

$$(4.3)\qquad \frac{1}{2}+\frac{1}{p}<1.$$

CASE 2: Let

$$\frac{1}{2}+\frac{1}{p}<\varkappa<1.$$

Then

$$(4.4)\qquad \Psi(r, t)=\sqrt{1+t+r}\,(1+|t-r|)^{\nu}$$

with

$$\nu=\min\left\{\kappa-\frac{1}{2},\ \frac{p-3}{2}\right\}.$$

Note that $(1.2)_2$ implies

(4.5) $$\frac{1}{p}<\frac{p-3}{2},$$

which yields (3.8) for the above ν.

Case 3: Let $\kappa>1$. Then

(4.6) $$\Psi(r, t)=\frac{1}{\Phi_2(r, t)},$$

where Φ_2 is the function in (3.9).

Case 4: Let $\kappa=1/2$. Then

(4.7) $$\Psi(r, t)=\frac{\sqrt{1+t+r}}{\log(2+t+r)}.$$

Case 5: Let $\kappa=1$. Then

(4.8) $$\Psi(r, t)=\begin{cases}\dfrac{\sqrt{1+t+r}\sqrt{1+|t-r|}}{\log(2+|t-r|)} & \text{if } p\geqq 4,\\ \sqrt{1+t+r}(1+|t-r|)^{(p-3)/2} & \text{if } p_0(2)<p<4.\end{cases}$$

We now introduce a Banach space X defined by

(4.9) $$X=\{u\in C^0(\boldsymbol{R}^2\times[0,\infty)\,;\ D_x^\alpha u\in C^0(\boldsymbol{R}^2\times[0,\infty)) \text{ and } \|D_x^\alpha u\|<\infty \text{ for } |\alpha|\leqq 2\}.$$

Then Proposition 2.1 implies

LAMMA 4.1. *Let u_0 be the solution of* (2.1). *Assume $p>2$ and* (1.3) *holds. Then $u_0\in X$.*

The following lemma will play a basic role in the proof of Theorem 1.4.

LEMMA 4.2. *Let L be the linear operator defined by* (1.11). *Assume* $(1.2)_2$ *and* (1.6) *hold. Suppose $u\in C^0(\boldsymbol{R}^2\times[0,\infty))$ and $\|u\|<\infty$. Then*

(4.10) $$\|L(|u|^p)\|\leqq C_4\|u\|^p,$$

where C_4 is a constant depending only on p and κ.

PROOF: *CASE* 1: If

$$0<\varkappa<\frac{1}{2}+\frac{1}{p} \quad \text{and} \quad \varkappa\neq\frac{1}{2},$$

then (4.10) is a direct consequence of (4.1), (4.2) and Corollary 3.5 with $M=\|u\|$ and $T=\infty$. If

$$\varkappa=\frac{1}{2}+\frac{1}{p},$$

then (4.10) follows immediately from Proposition 3.3, because of (4.5).

CASE 2: Since (4.4), (4.5) and assumption

$$\varkappa>\frac{1}{2}+\frac{1}{p}$$

imply (3.8), by virtue of Proposition 3.2 with $M=\|u\|$ we obtain

$$|L(|u|^p)(x,t)|\Psi(r,t)\leqq C\|u\|^p\Phi_2(r,t)\Psi(r,t) \quad \text{for} \quad (x,t)\in\boldsymbol{R}^2\times[0,\infty), \tag{4.11}$$

where C is the constant in (3.9). Moreover (4.4) yields that $\Phi_2(r,t)$ $\Psi(r,t)$ is bounded. Hence (4.10) follows from (4.11).

CASE 3: Since (4.3) and assumption $\varkappa>1$ imply (3.7), we take a positive number ν satisfying (3.8) in such a way that

$$\frac{1}{p}<\nu<\min\left\{\frac{1}{2},\ \frac{p-3}{2}\right\},$$

which is possible according to (4.5), and set

$$C_\nu=1+\sup_{s\geqq1}s^{\nu-(1/2)}\log(1+s).$$

By (4.6) we have then

$$\sqrt{1+t+r}\,(1+|t-r|)^\nu\leqq C_\nu\Psi(r,t)$$

hence (4.1) yields

$$|u(x,t)|\leqq C_\nu\|u\|\frac{1}{\sqrt{1+t+r}\,(1+|t-r|)^\nu}$$

for $(x,t)\in\boldsymbol{R}^2\times[0,\infty)$. By virtue of Proposition 3.2 with $M=C_\nu\|u\|$ we therefore obtain (4.10).

CASE 4: The desired estimate (4.10) is a direct consequence of (4.7) and Corollary 3.6.

CASE 5: If $p_0(2)<p<4$, we take $\nu=(p-3)/2$ in Proposition 3.2. Then (4.8) yields

$$|u(x, t)|\leqq\|u\|\frac{1}{\sqrt{1+t+r}(1+|t-r|)^{\nu}}$$

for $(x, t)\in\boldsymbol{R}^2\times[0, \infty)$. Noting that (4.5) implies (3.8), we obtain (4.10), since

$$\Phi_2(r, t)\Psi(r, t)=1 \quad \text{for} \quad 3<p<4.$$

Next suppose $p\geqq 4$. Then (4.8) yields

$$|u(x, t)|\leqq\|u\|\frac{\log(2+|t-r|)}{\sqrt{1+t+r}\sqrt{1+|t-r|}}$$

for $(x, t)\in\boldsymbol{R}^2\times[0, \infty)$. Hence, taking $\nu=1/3$ so that (3.8) holds, we have

$$|u(x, t)|\leqq C_1\|u\|\frac{1}{\sqrt{1+t+r}(1+|t-r|)^{1/3}},$$

where

$$C_1=\sup_{s\geqq 1}s^{-1/6}\log(1+s).$$

By virtue of Proposition 3.2 we obtain therefore

$$|L(|u|^p)(x, t)|\leqq C(C_1\|u\|)^p\Phi_2(r, t),$$

which yields (4.10) for $p\geqq 4$. Thus we prove the lemma.

PROOF OF THEOREM 1.4: The procedure is analogous to Glassey [5] or Asakura [3], because of Lemmas 4.1 and 4.2. It is well known that, if a function $u(x, t)\in C^2(\boldsymbol{R}^2\times[0, \infty))$ satisfies the following integral equation, like (1.10),

$$u=u_0+L(F(u)) \tag{4.12}$$

with the operator L defined by (1.11), then u is a solution of the Cauchy problem (1.8) (see for instance [3], Proposition 2.2). Besides, the uniqueness of a solution to (1.8) is also valid (see for instance John [8], Appendix 1).

To look for a solution of (4.12) we define a sequence of functions by

$$u_{k+1}=u_0+L(F(u_k)), \quad k\geqq 0,$$

where u_0 is the solution of the Cauchy problem (2.1). Note that operators D_x and L commute. Moreover it follows from (4.2), (4.4) and (4.6) through (4.8) that $\Psi(r, t)\geqq 1$, because $\log(1+s)\leqq s$ for $s\geqq 1$. Hence (4.1) yields

$$|u(x, t)|\leqq\|u\| \quad \text{for} \quad (x, t)\in\boldsymbol{R}^2\times[0, \infty).$$

In addition, we have

$$\|\,|u|^\theta|v|^{1-\theta}\|\leqq\|u\|^\theta\|v\|^{1-\theta} \quad \text{for} \quad 0\leqq\theta\leqq 1.$$

Thus, if the norm of u_0 is so small that

$$\text{(4.13)} \qquad p2^pAC_4\|u_0\|^{p-1}\leqq 1 \quad \text{and} \quad \|u_0\|\leqq\frac{1}{2},$$

where C_4 is the constant in Lemma 4.2, one can exactly follow [5], pp. 257-260 or [3], pp. 1477-1480 and hence find a solution $u\in X$ of the integral equation (4.12), where X is the Banach space defined by (4.9). Furthermore, by virtue of Proposition 2.1 we get

$$\|u_0\|\leqq C_0C_p\varepsilon,$$

where C_p is a constant depending only on p. Therefore we obtain (4.13), taking ε small enough according as A, p and κ. Thus we prove Theorem 1.4 and hence Theorem 1.1.

REMARK 4.3. Let $u\in X$ be the solution of the integral equation (1.10), where X is the Banach space defined by (4.9). Suppose the assumptions of Theorem 1.1 are fulfilled. Then one can see that the decay rate of $u-u_0$ is better than that of u. For instance consider the case where

$$\frac{1}{2}<\kappa<\frac{1}{2}+\frac{1}{p}.$$

Define another norm by

$$|||v|||=\sup_{(x, t)\in\boldsymbol{R}^2\times[0, \infty)}(|v(x, t)|/\Phi_1(r, t)),$$

where Φ_1 is the function in Proposition 3.1. In view of (4.1) and (4.2) with $\mu=1/2$ and $\nu=\kappa-(1/2)$ we find from (3.6) with $M=\|u\|$ that

$$|||u-u_0|||\leqq AC_1\|u\|^p.$$

Note that the ratio of the decay rate of $(u-u_0)(x, t)$ to that of $u(x, t)$ is given by $\Phi_1(r, t)\ \Psi(r, t)$. For example, if $5/2<p\kappa<3$, then

$$\Phi_1(r, t)\Psi(r, t)=\frac{1}{(1+|t-r|)^{(p-1)\kappa-2}}.$$

§ 5. The life span in two space dimensions

In this section we study lower bounds for the lifespan of solutions to the Cauchy Problem (1. 1) with $n=2$ and $p>2$. To this end we assume throughout the present section that (1. 3) holds for $n=2$. By the lifespan we mean the least upper bound of the set of positive numbers T such that there exists a C^2 solution of (1. 1) with the time interval $[0, \infty)$ replaced by $[0, T)$. We also denote the lifespan by $T(\varepsilon)$. Agemi and Takamura have shown in [2] that $T(\varepsilon)$ is finite under (1. 5). More precisely, they have given an upper bound for $T(\varepsilon)$ by

$$T(\varepsilon)\leqq C\left(\frac{1}{\varepsilon}\right)^{1/\left(\frac{2}{p-1}-\kappa\right)} \quad \text{for} \quad p>1.$$

(See also Remark 5.5 below).

Now, we have already shown that $T(\varepsilon)$ is infinite provided the hypotheses of Theorem 1.1 are fulfilled. So, we shall study lower bounds for $T(\varepsilon)$ in the case where $(1.2)_2$ or (1. 6) is violated. Note that the case can be divided into the following four ones.

(5. 1) $$2<p\leqq p_0(2),\quad 0<\kappa<\frac{1}{2}+\frac{1}{p} \quad \text{and} \quad \kappa\neq\frac{1}{2},$$

(5. 2) $$2<p\leqq p_0(2) \quad \text{and} \quad \kappa\geqq\frac{1}{2}+\frac{1}{p},$$

(5. 3) $$p>p_0(2),\quad 0<\kappa<\frac{2}{p-1} \quad \text{and} \quad \kappa\neq\frac{1}{2}$$

and

(5. 4) $$\kappa=\frac{1}{2}\leqq\frac{2}{p-1} \quad (\text{hence} \quad 2<p\leqq 5).$$

Note that (3. 2) yields

(5. 5) $$\frac{1}{2}+\frac{1}{p}\leqq\frac{2}{p-1} \quad \text{for} \quad p\leqq p_0(2)$$

and

(5. 6) $$\frac{1}{2}+\frac{1}{p}>\frac{2}{p-1} \quad \text{for} \quad p>p_0(2).$$

Moreover (5. 4) with $p=5$ is related to the irregular value in (1. 6).

The main results in this section are the following four theorems.

THEOREM 5. 1. *Let* (5. 1) *hold. Then*

$$(5.7)\qquad \begin{cases} T(\varepsilon)^{2-(p-1)\kappa} \geqq C\left(\dfrac{1}{\varepsilon}\right)^{p-1} & \text{if} \quad p\kappa \neq \dfrac{5}{2}, \\ T(\varepsilon)^{(5-p)/(2p)} \log T(\varepsilon) \geqq C\left(\dfrac{1}{\varepsilon}\right)^{p-1} & \text{if} \quad p\kappa = \dfrac{5}{2} \end{cases}$$

for $0<\varepsilon \leqq \varepsilon_0$, *where* C *and* ε_0 *are positive constants depending only on* A, p *and* κ

THEOREM 5.2. *Let* (5.2) *hold. Then*

$$(5.8)\qquad \begin{cases} T(\varepsilon)^{q(p)} \log T(\varepsilon) \geqq C\left(\dfrac{1}{\varepsilon}\right)^{p-1} & \text{if} \quad p \neq 3, \\ T(\varepsilon)^{1/3} (\log T(\varepsilon))^2 \geqq C\left(\dfrac{1}{\varepsilon}\right)^{2} & \text{if} \quad p=3 \end{cases}$$

for $0<\varepsilon \leqq \varepsilon_0$, *where*

$$(5.9)\qquad q(p)=\frac{1}{p}-\frac{p-3}{2}=-\frac{p^2-3p-2}{2p}$$

and C, ε_0 *are positive cnstants depending only on* A *and* p.

REMARK. If

$$\kappa=\frac{1}{2}+\frac{1}{p},$$

then $2-(p-1)\kappa=q(p)$. Moreover we have $q(p)=0$ for $p=p_0(2)$.

THEOREM 5.3. *Let* (5.3) *hold. Then* (5.7) *is valid.*

THEOREM 5.4. *Let* (5.4) *hold. Then*

$$(5.10)\qquad \begin{cases} \log T(\varepsilon) \geqq C\left(\dfrac{1}{\varepsilon}\right)^{4/5} & \text{if} \quad p=5, \\ T(\varepsilon)^{(5-p)/2} (\log T(\varepsilon))^{p-1} \geqq C\left(\dfrac{1}{\varepsilon}\right)^{p-1} & \text{if} \quad 2<p<5 \end{cases}$$

for $0<\varepsilon \leqq \varepsilon_0$, *where* C, ε_0 *are positive constants depending only on* A *and* p.

REMARK 5.5. It follows from Theorem 5.1 and [2], Theorem 3 that

$$C\left(\frac{1}{\varepsilon}\right)^{1/\left(\frac{2}{p-1}-\kappa\right)} \leqq T(\varepsilon) \leqq C'\left(\frac{1}{\varepsilon}\right)^{1/\left(\frac{2}{p-1}-\kappa\right)}$$

for small $\varepsilon>0$ and some positive constants C, C', provided (5.1), (1.5) hold and $p\kappa \neq 5/2$. Moreover (5.8) is the same lower bound as in [2], Theorem 1 except the constant C, since

$$q(2, p)=\frac{p^2-3p-2}{2}.$$

Notice that the upper bound in [2], Theorem 2 is better than the one in Theorem 3 when $1<p<p_0(2)$ and

$$\frac{1}{2}+\frac{1}{p}<\varkappa<\frac{2}{p-1},$$

because (3. 2) implies

$$\frac{1}{2}+\frac{1}{p}-\frac{2}{p-1}=\frac{q(2, p)}{p(p-1)}$$

hence

$$-\frac{p(p-1)}{q(2, p)}<\left(\frac{2}{p-1}-\varkappa\right)^{-1}$$

for such $\varkappa$.

The rest of this section will be devoted to prove the above theorems. The procedure is similar to the one in the proof of Theorem 1.4. For a continuous function $u(x, t)\in C^0(\boldsymbol{R}^2\times[0, T))$ we define, as (4. 1), a norm of u by

$$\|u\|=\sup_{(x, t)\in\boldsymbol{R}^2\times[0, T)}|u(x, t)|\Psi(r, t), \tag{5.11}$$

where $\Psi(r, t)$ is a positive valued function on $[0, \infty)\times[0, \infty)$ defined by

$$\Psi(r, t)=\begin{cases}(1+t+r)^{\varkappa} & \text{if } 0<\varkappa<\dfrac{1}{2},\\ \sqrt{1+t+r}/\log(2+t+r) & \text{if } \varkappa=\dfrac{1}{2},\\ \sqrt{1+t+r}(1+|t-r|)^{\varkappa-(1/2)} & \text{if } \dfrac{1}{2}<\varkappa<\dfrac{1}{2}+\dfrac{1}{p},\\ \sqrt{1+t+r}(1+|t-r|)^{1/p} & \text{if } \varkappa\geqq\dfrac{1}{2}+\dfrac{1}{p}.\end{cases} \tag{5.12}$$

Note that Proposition 2.1 implies

$$\|D_x^\alpha u_0\|<\infty \quad \text{for} \quad |\alpha|\leqq 2 \quad \text{and} \quad T>0,$$

where u_0 is the solution of (2. 1).

The following four lemmas are essential to prove the theorems.

LEMMA 5. 6. *Let* (5. 1) *hold. Then* $p\varkappa<3$ *and*

$$\Phi_1(r, t)\Psi(r, t)\leqq N_1(r, t) \tag{5.13}$$

for $(r, t)\in[0, \infty)\times[0, \infty)$, *where* $\Phi_1(r, t)$ *is the function in Proposition* 3.1 *and*

$$N_1(r, t)=\begin{cases}(1+t+r)^{2-(p-1)\kappa} & \text{if} \quad p\kappa\neq\frac{5}{2},\\ \frac{2}{\log 2}(1+t+r)^{(5-p)/(2p)}\log(2+t+r) & \text{if} \quad p\kappa=\frac{5}{2}.\end{cases}$$

PROOF: Since $p_0(2)<4$, we have $p\kappa<3$ from (5.1). First suppose $0<\kappa<1/2$. Then $p\kappa<2$. Therefore it follows from (5.12) and the definition of Φ_1 that

$$\Phi_1(r, t)\Psi(r, t)=(1+t+r)^{2-p\kappa+\kappa}. \tag{5.14}$$

Next suppose

$$\frac{1}{2}<\kappa<\frac{1}{2}+\frac{1}{p},$$

so that (5.12) implies

$$\Psi(r, t)=\sqrt{1+t+r}(1+|t-r|)^{\kappa-(1/2)}.$$

If $5/2<p\kappa<3$, we have

$$\Phi_1(r, t)\Psi(r, t)=(1+|t-r|)^{2-p\kappa+\kappa}\leqq(1+t+r)^{2-p\kappa+\kappa}, \tag{5.15}$$

since $2-p\kappa+\kappa>0$ according to (5.5) and the assumption

$$\kappa<\frac{1}{2}+\frac{1}{p}.$$

If $p\kappa<5/2$, then

$$\Phi_1(r, t)\Psi(r, t)=(1+t+r)^{(5/2)-p\kappa}(1+|t-r|)^{\kappa-(1/2)}\leqq(1+t+r)^{2-p\kappa+\kappa}, \tag{5.16}$$

since $\kappa>1/2$. Finally, if $p\kappa=5/2$, we have

$$\begin{aligned}\Phi_1(r, t)\Psi(r, t)&=\left(1+\log\frac{1+t+r}{1+|t-r|}\right)(1+|t-r|)^{\kappa-(1/2)}\\ &\leqq\frac{2}{\log 2}(\log(2+t+r))(1+t+r)^{(5-p)/(2p)},\end{aligned} \tag{5.17}$$

since

$$0<\kappa-\frac{1}{2}=\frac{5-p}{2p} \quad \text{for} \quad p\kappa=\frac{5}{2}.$$

Thus we prove the lemma.

LEMMA 5.7. *Let* (5.2) *hold. Then*

$$\Phi_3(r, t)\Psi(r, t) \leqq N_2(r, t)$$

for $(r, t) \in [0, \infty) \times [0, \infty)$, *where* $\Phi_3(r, t)$ *is the function in Proposition* 3.3 *and*

$$N_2(r, t) = \begin{cases} (1+t+r)^{q(p)} \log(2+t+r) & \text{if } p \neq 3, \\ \dfrac{2}{\log 2}(1+t+r)^{1/3}(\log(2+t+r))^2 & \text{if } p = 3. \end{cases}$$

PROOF: Recall that (5.12) implies

$$\Psi(r, t) = \sqrt{1+t+r}(1+|t-r|)^{1/p} \quad \text{for} \quad \varkappa \geqq \frac{1}{2} + \frac{1}{p}.$$

Moreover $q(p) \geqq 0$, because of (5.9) and assumption $p \leqq p_0(2)$. Hence, if $3 < p \leqq p_0(2) (< 4)$, it follows from the definition of Φ_3 that

$$\begin{aligned} \Phi_3(r, t)\Psi(r, t) &= (1+|t-r|)^{q(p)} \log(2+|t-r|) \\ &\leqq (1+t+r)^{q(p)} \log(2+t+r). \end{aligned}$$

If $2 < p < 3$, we have

$$\begin{aligned} \Phi_3(r, t)\Psi(r, t) &= (1+t+r)^{\frac{1}{2} - \frac{p-2}{2}} (1+|t-r|)^{\frac{1}{p}} \log(2+t+r) \\ &\leqq (1+t+r)^{\frac{1}{p} - \frac{p-3}{2}} \log(2+t+r). \end{aligned}$$

If $p = 3$, then

$$\begin{aligned} \Phi_3(r, t)\Psi(r, t) &= (1+|t-r|)^{1/p} (\log(2+t+r)) \left(1 + \log \frac{1+t+r}{1+|t-r|}\right) \\ &\leqq (1+t+r)^{1/p} \frac{2}{\log 2} (\log(2+t+r))^2. \end{aligned}$$

The proof is complete.

LEMMA 5.8. *Let* (5.3) *hold. Then* $p\varkappa < 3$ *and* (5.13) *is valid.*

PROOF: If $p \geqq 5$, then condition $\varkappa < 2/(p-1)$ implies that $\varkappa < 1/2$ and

$$p\varkappa < 2 + \frac{2}{p-1} \leqq \frac{5}{2}.$$

Therefore we obtain (5.13) from (5.14).

Next suppose $p_0(2) < p < 5$. If $\varkappa < 1/2$, then $p\varkappa < 5/2$ hence (5.14) implies (5.13). If

$$\frac{1}{2} < \varkappa < \frac{2}{p-1},$$

we have $p\varkappa<2+\dfrac{2}{p-1}<3$, since $p>p_0(2)>3$. Therefore we obtain (5.13) by virtue of (5.6), (5.15), (5.16) and (5.17). The proof is complete.

LEMMA 5.9. *Let* (5.4) *hold. Then*

(5.18) $$|L(|u|^p)(x,t)|\Psi(r,t)\leqq C_1\|u\|^p N_3(r,t)$$

for $(x,t)\in\boldsymbol{R}^2\times[0,T)$, *where* C_1 *is the constant in* (3.6) *and*

$$N_3(r,t)=\begin{cases}(1+t+r)^{(5-p)/2}(\log(2+t+r))^{p-1} & \text{if}\quad 2<p<5,\\ \dfrac{2}{\log 2}(\log(2+t+r))^5 & \text{if}\quad p=5.\end{cases}$$

PROOF: Since (5.11) and (5.12) imply

$$|u(x,t)|\leqq\|u\|\frac{\log(2+t+r)}{\sqrt{1+t+r}}\quad\text{for}\quad(x,t)\in\boldsymbol{R}^2\times[0,T),$$

we observe that the estimate (3.16), at the opening of the proof of Proposition 3.1, is still valid with $\mu=1/2$, $\nu=0$ and

$$M=\|u\|\log(2+t+r),$$

noting that

$$\lambda=|x+(t-\tau)\xi|\leqq r+t-\tau.$$

By (3.6) and (5.12) with $\varkappa=1/2$ we therefore obtain

$$|L(|u|^p)(x,t)|\Psi(r,t)\leqq C_1\|u\|^p\Phi_1(r,t)\sqrt{1+t+r}(\log(2+t+r))^{p-1}.$$

Thus, if $2<p<5$ so that $p\varkappa<5/2$, we have

$$\Phi_1(r,t)=(1+t+r)^{2-(p/2)}$$

hence (5.18) follows. If $p=5$, then

$$\Phi_1(r,t)\sqrt{1+t+r}=1+\log\frac{1+t+r}{1+|t-r|}.$$

Consequently we obtain (5.18). The proof is complete.

To prove the theorems we need also another estimate for $r\gg t$.

LEMMA 5.10. *Let* $(x,t)\in\boldsymbol{R}^2\times[0,T)$ *and* $r\geqq 2t$. *Then*

(5.19) $$|L(|u|^p)(x,t)|\Psi(r,t)\leqq C\|u\|^p N_4(t),$$

where C is a constant depending only on p, $\varkappa$ and

$$N_4(r,t)=\begin{cases}(1+t)^{2-(p-1)\kappa} & \text{if} \quad 0<\kappa<\dfrac{1}{2}+\dfrac{1}{p} \quad \text{and} \quad \kappa\neq\dfrac{1}{2},\\ (1+t)^{q(p)} & \text{if} \quad \kappa\geqq\dfrac{1}{2}+\dfrac{1}{p},\\ (1+t)^{(5-p)/2}(\log(2+t))^{p-1} & \text{if} \quad \kappa=\dfrac{1}{2} \quad \text{and} \quad 2<p\leqq 5.\end{cases}$$

PROOF: First suppose

$$0<\kappa<\frac{1}{2}+\frac{1}{p} \quad \text{and} \quad \kappa\neq\frac{1}{2}.$$

Then it follows from (5.11) and (5.12) that (3.1) holds with $M=\|u\|$ and the numbers μ, ν given by (3.4) or (3.5). Therefore by (3.16) we get

$$\begin{aligned}|L(|u|^p)(x,t)|\Psi(r,t)&\leqq\|u\|^p t^2(1+t+r)^{\kappa}\frac{1}{\left(1+\frac{r}{2}\right)^{p\kappa}}\\ &\leqq\left(\frac{3}{2}\right)^{\kappa}2^{p\kappa}\|u\|^p t^2(1+r)^{\kappa-p\kappa} \quad \text{for} \quad r\geqq 2t,\end{aligned}$$

because $\Psi(r,t)\leqq(1+t+r)^{\kappa}$ and

$$\lambda=|x+(t-\tau)\xi|\geqq r-(t-\tau)$$

hence

$$|\lambda-\tau|\geqq r-t\geqq\frac{r}{2}.$$

Noting that $\kappa-p\kappa<0$ and $r\geqq t$, we thus obtain (5.19) with $C=3^{\kappa}2^{p\kappa-\kappa}$.

Next suppose

$$\kappa\geqq\frac{1}{2}+\frac{1}{p}.$$

Then

$$|u(x,t)|\leqq\|u\|\frac{1}{\sqrt{1+t+r}\,(1+|t-r|)^{1/p}},$$

namely, (3.1) holds with $\mu=1/2$ and $\nu=1/p$. Therefore, as above, we have

$$\begin{aligned}|L(|u|^p)(x,t)|\Psi(r,t)&\leqq\|u\|^p t^2(1+t+r)^{\frac{1}{2}+\frac{1}{p}}\left(1+\frac{r}{2}\right)^{-\frac{p}{2}-1}\\ &\leqq\left(\frac{3}{2}\right)^{\frac{1}{2}+\frac{1}{p}}2^{\frac{p}{2}+1}\|u\|^p t^2(1+r)^{\frac{1}{p}-\frac{p}{2}-\frac{1}{2}} \quad \text{for} \quad r\geqq 2t.\end{aligned}$$

Since $p>1$ and

$$2+\left(\frac{1}{p}-\frac{p}{2}-\frac{1}{2}\right)=\frac{1}{p}-\frac{p-3}{2},$$

we thus obtain (5. 19).

Finally suppose $\varkappa=1/2$. Then

$$|u(x,t)|\leqq\|u\|\frac{\log(2+t+r)}{\sqrt{1+t+r}}.$$

Hence, similarly to the proof of the preceding lemma we have

$$\begin{aligned}|L(|u|^p)(x,t)|\Psi(r,t)&\leqq\|u\|^p t^2\sqrt{1+t+r}\left(1+\frac{r}{2}\right)^{-p/2}(\log(2+t+r))^{p-1}\\&\leqq C\|u\|^p t^2\left(\frac{\log(2+t+r)}{\sqrt{1+t+r}}\right)^{p-1}\quad\text{for}\quad r\geqq 2t.\end{aligned}$$

Moreover, since the function

$$[1,\infty)\ni s\longmapsto\frac{\log(1+s)}{\sqrt{s}}$$

is decreasing for large s, we get

$$\frac{\log(2+t+r)}{\sqrt{1+t+r}}\leqq\text{const.}\frac{\log(2+t)}{\sqrt{1+t}}.$$

Therefore we obtain (5. 19). The proof is complete.

Proof of Theorem 5. 1: Suppose (5. 1) holds. Then it follows from Proposition 3.1, (5. 11), (5. 12), (5. 13) and (5. 19) that

$$\|L(|u|^p)\|\leqq C_5\phi_1(T)\|u\|^p$$

where

$$\phi_1(T)=\begin{cases}(1+T)^{2-(p-1)\kappa} & \text{if}\quad p\varkappa\neq\frac{5}{2},\\ (1+T)^{(5-p)/(2p)}\log(2+T) & \text{if}\quad p\varkappa=\frac{5}{2}\end{cases}$$

and C_5 is a constant depending only on p and $\varkappa$. Moreover by virtue of Proposition 2.1 we have

$$\|u_0\|\leqq C_0\varepsilon,$$

where u_0 is the solution of (2. 1) and C_0 the constant in (2. 2). Therefore, if T and ε satisfy the following relation, like (4. 13),

$$\begin{cases} p2^pAC_5\phi_1(T)(C_0\varepsilon)^{p-1}\leqq 1, \\ C_0\varepsilon\leqq\dfrac{1}{2}, \end{cases}$$

one can find a solution of the integral equation (1.10) in $\boldsymbol{R}^2\times[0, T)$, as in the proof of Theorem 1.4. Thus we prove the theorem.

REMARK. The proofs of Theorems 5.2, 5.3 and 5.4 are analogous to that of Theorem 5.1 hence left for the readers.

§ 6. Proof of Theorem 1.2

As is seen from the proof of Theorem 1.4 we have only to establish estimates analogous to Lemmas 4.1 and 4.2. The following proposition is due to Asakura [3].

PROPOSITION 6.0. *Let $u(x, t)$ be the solution of the Cauchy problem*

$$\text{(6.1)}\qquad \begin{array}{l} u_{tt}-\Delta u=0 \quad in \quad \boldsymbol{R}^3\times[0, \infty), \\ u(x, 0)=f(x), \quad u_t(x, 0)=g(x) \quad for \quad x\in\boldsymbol{R}^3, \end{array}$$

where $f\in C^3(\boldsymbol{R}^3)$ and $g\in C^2(\boldsymbol{R}^3)$. Suppose (1.3) holds for $n=3$. Then

$$\text{(6.2)}\qquad \sum_{|\alpha|\leqq 2}|D_x^\alpha u(x, t)|\leqq C_0\varepsilon\Phi(r, t) \quad for \quad (x, t)\in\boldsymbol{R}^3\times[0, \infty),$$

where

$$\Phi(r, t)=\begin{cases} \dfrac{1}{(1+t+r)(1+|t-r|)^{\kappa-1}} & if \quad \kappa>1, \\ \dfrac{1}{1+t+r}\left(1+\log\dfrac{1+t+r}{1+|t-r|}\right) & if \quad \kappa=1, \\ \dfrac{1}{(1+t+r)^{\kappa}} & if \quad 0<\kappa<1 \end{cases}$$

and C_0 is a constant depending only on κ.

The rest of this section will be devoted to study how the decay rate of $L(|u|^p)(x, t)$ depends on that of $u(x, t)$ and to establish an estimate analogous to (4.10), where L is the operator defined by (1.12). Assume $u(x, t)$ $\in C^0(\boldsymbol{R}^3\times[0, \infty))$ and

$$\text{(6.3)}\qquad |u(x, t)|\leqq\frac{M}{(1+t+r)^{\mu}(1+|t-r|)^{\nu}} \quad \text{for} \quad (x, t)\in\boldsymbol{R}^3\times[0, \infty),$$

where M, μ and ν are constants. The following three lemmas correspond to Propositions 3.1, 3.2 and 3.3 in two space dimensions.

LEMMA 6.1. *Let $p>1$ and*

(6.4) $\quad 0<\varkappa<1+\frac{1}{p}$, *particularly*, $p(\varkappa-1)<1$.

Suppose (6.3) *holds with*

(6.5) $\quad \mu=1, \quad \nu=\varkappa-1 \quad \text{if} \quad 1<\varkappa<1+\frac{1}{p}$

or

(6.6) $\quad \mu=\varkappa, \quad \nu=0 \quad \text{if} \quad 0<\varkappa\leqq 1$.

Then

(6.7) $\quad |L(|u|^p)(x, t)|\leqq C_1 M^p \Phi_1(r, t) \quad \text{for} \quad (x, t)\in \boldsymbol{R}^3\times[0, \infty)$,

where

$$\Phi_1(r, t)=\begin{cases} \dfrac{1}{(1+t+r)(1+|t-r|)^{p\varkappa-3}} & \text{if} \quad 3<p\varkappa<p+1, \\ \dfrac{1}{1+t+r}\left(1+\log\dfrac{1+t+r}{1+|t-r|}\right) & \text{if} \quad p\varkappa=3, \\ \dfrac{1}{(1+t+r)^{p\varkappa-2}} & \text{if} \quad 0<p\varkappa<3 \end{cases}$$

and C_1 *is a constant depending only on* p *and* $\varkappa$

LEMMA 6.2. *Let* $p>1$ *and*

(6.8) $\quad \varkappa>1+\frac{1}{p}$, *i.e.*, $p(\varkappa-1)>1$.

Suppose (6.3) *holds with* $\mu=1$ *and a positive number* ν *such that*

(6.9) $\quad \frac{1}{p}<\nu\leqq\varkappa-1$.

Then

(6.10) $\quad |L(|u|^p)(x, t)|\leqq C_2 M^p \Phi_2(r, t) \quad \text{for} \quad (x, t)\in \boldsymbol{R}^3\times[0, \infty)$,

where

$$\Phi_2(r, t)=\begin{cases} \dfrac{1}{(1+t+r)(1+|t-r|)^{p-2}} & \text{if} \quad p>2, \\ \dfrac{1}{1+t+r}\left(1+\log\dfrac{1+t+r}{1+|t-r|}\right) & \text{if} \quad p=2, \\ \dfrac{1}{(1+t+r)^{p-1}} & \text{if} \quad 1<p<2 \end{cases}$$

and C_2 is a constant depending only on p and ν.

REMARK. $\Phi_2(r, t)=\Phi_1(r, t)$ for $\varkappa=1+\dfrac{1}{p}$.

LEMMA 6.3. *Let $p>1$ and*

(6.11) $\quad \varkappa=1+\dfrac{1}{p}$, *i.e.*, $p(\varkappa-1)=1$.

Suppose (6.3) *holds with $\mu=1$ and $\nu=1/p$. Then*

(6.12) $\quad |L(|u|^p)(x, t)|\leqq C_3\Phi_3(r, t)$ *for* $(x, t)\in \boldsymbol{R}^3\times[0, \infty)$,

where

$$\Phi_3(r, t)=\begin{cases}\Phi_2(r, t)\log(2+|t-r|) & \text{if} \quad p>2,\\ \Phi_2(r, t)\log(2+t+r) & \text{if} \quad 1<p\leqq 2\end{cases}$$

and C_3 is a constant depending only on p.

REMARK. Since

$$1+\frac{1}{p}-\frac{2}{p-1}=\frac{p^2-2p-1}{p(p-1)},$$

we see that $(1.2)_3$ implies

(6.13) $\quad 1+\dfrac{1}{p}>\dfrac{2}{p-1}$.

Before proving Lemmas 6.1, 6.2 and 6.3 we shall state two corollaries of Lemma 6.1, like Corollaries 3.5 and 3.6.

COROLLARY 6.4. *Let the hypotheses of Lemma* 6.1 *be fulfilled. Assume* $(1.2)_3$ *and* (1.7) *hold. Moreover suppose $\varkappa\neq 1$. Then*

(6.14) $\quad |L(|u|^p)(x, t)|\leqq CM^p\dfrac{1}{(1+t+r)^{\mu}(1+|t-r|)^{\nu}}$
for $(x, t)\in \boldsymbol{R}^3\times[0, \infty)$,

where μ, ν are the numbers given by (6.5) *or* (6.6), *and C is a constant depending only on p and $\varkappa$.*

PROOF: First suppose

$$1<\varkappa<1+\frac{1}{p}.$$

Then (1.7) implies

$$p\varkappa \geqq \varkappa+2>3 \quad \text{and} \quad p\varkappa-3 \geqq \varkappa-1.$$

Therefore (6.7) yields (6.14) with $\mu=1$, $\nu=\varkappa-1$ and $C=C_1$. Next suppose $0<\varkappa<1$. Then it is clear that (6.7) implies (6.14) with $\mu=\varkappa$ and $\nu=0$, since $p\varkappa-2\geqq\varkappa$ under (1.7). The proof is complete.

COROLLARY 6.5. *Let $\varkappa=1$. Assume that* (1.7) *holds and that*

$$|u(x, t)| \leqq M\frac{\log(2+t+r)}{1+t+r} \quad \textit{for} \quad (x, t) \in \boldsymbol{R}^3 \times [0, \infty). \tag{6.15}$$

Then

$$|L(|u|^p)(x, t)| \leqq CM^p\frac{\log(2+t+r)}{1+t+r} \quad \textit{for} \quad (x, t) \in \boldsymbol{R}^3 \times [0, \infty), \tag{6.16}$$

where C is a constant depending only on p.

PROOF: Noting that (1.7) with $\varkappa=1$ implies $p>3$, we set

$$\varkappa'=\frac{3}{p}$$

so that $0<\varkappa'<1$, and

$$C_p=\sup_{s\geqq 1} s^{\kappa'-1}\log(1+s).$$

Then (6.15) yields

$$|u(x, t)| \leqq MC_p\frac{1}{(1+t+r)^{\kappa'}}.$$

Therefore by virtue of Lemma 6.1 we obtain

$$|L(|u|^p)(x, t)| \leqq C_1(MC_p)^p\frac{1}{1+t+r}\left(1+\log\frac{1+t+r}{1+|t-r|}\right),$$

which yields (6.16) with

$$C=C_1(C_p)^p\left(\frac{1}{\log 2}+1\right).$$

The proof is complete.

PROOF OF LEMMA 6.1: If follows from (1.12) and (6.3) that

$$\begin{aligned} &|L(|u|^p)(x, t)| \\ &\leqq \frac{M^p}{4\pi}\int_0^t (t-\tau)d\tau \int_{|\xi|=1} \frac{1}{(1+\tau+\lambda)^{p\mu}(1+|\tau-\lambda|)^{p\nu}}dS_\xi, \end{aligned} \tag{6.17}$$

where we have set

$$\lambda=|x+(t-\tau)\xi|.$$

Analogously to Lemma 3.7 we have therefore

$$(6.18)\quad |L(|u|^p)(x, t)|\leqq M^p\left(\frac{4}{1+t+r}\right)^{p\mu+p\nu} \quad \text{for} \quad (x, t)\in \boldsymbol{R}^3\times[0, 1],$$

which implies (6.7) for $0\leqq t\leqq 1$.

In what follows we assume

$$t\geqq 1.$$

We shall apply Lemma 2.3 with $n=3$ to the integral over the unit sphere in (6.17). Then we have

$$\int_{|\xi|=1}\frac{1}{(1+\tau+\lambda)^{p\mu}(1+|\tau-\lambda|)^{p\nu}}dS_\xi$$
$$=2\pi\frac{1}{r(t-\tau)}\int_{|t-\tau-r|}^{t-\tau+r}\frac{\lambda}{(1+\tau+\lambda)^{p\mu}(1+|\tau-\lambda|)^{p\nu}}d\lambda.$$

Noting that

$$\frac{\lambda}{(1+\tau+\lambda)^{p\mu}}\leqq\frac{1}{(1+\tau+\lambda)^{p\mu-1}},$$

we thus obtain

$$(6.19)\quad |L(|u|^p)(x, t)|\leqq M^p I(r, t),$$

where

$$I(r, t)=\frac{1}{2r}\int_0^t d\tau\int_{|t-\tau-r|}^{t-\tau+r}(1+\tau+\lambda)^{1-p\mu}(1+|\tau-\lambda|)^{-p\nu}d\lambda.$$

(Compare with (3.27)). Moreover, analogously to (3.29) together with (3.30) we get

$$(6.20)\quad I(r, t)\leqq\frac{1}{2r}\int_{|t-r|}^{t+r}(1+\alpha)^{1-p\mu}d\alpha\int_0^\alpha(1+\beta)^{-p\nu}d\beta.$$

Now, let the hypotheses of the lemma be fulfilled. Then, since $p\nu<1$, we have

$$\int_0^\alpha(1+\beta)^{-p\nu}d\beta\leqq\frac{1}{1-p\nu}(1+\alpha)^{1-p\nu}$$

hence, like (3.32),

$$(6.21)\qquad I(r,t)\leqq\frac{1}{1-p\nu}\cdot\frac{1}{2r}\int_{|t-r|}^{t+r}(1+\alpha)^{2-p\kappa}d\alpha.$$

First suppose (3.33) holds. If $p\kappa>2$, we have

$$I(r,t)\leqq\frac{1}{1-p\nu}\cdot\frac{1}{(1+t-r)^{p\kappa-2}},$$

because $(t+r)-|t-r|\leqq 2r$. Hence by (3.34) we get

$$(6.22)\qquad I(r,t)\leqq C\frac{1}{(1+t+r)(1+|t-r|)^{p\kappa-3}}$$

with $C=6(1-p\nu)^{-1}$, which together with (6.19) implies (6.7), because

$$(1+|t-r|)^{3-p\kappa}\leqq(1+t+r)^{3-p\kappa}\quad\text{for}\quad p\kappa<3.$$

If $0<p\kappa\leqq 2$, we have from (6.21)

$$I(r,t)\leqq\frac{1}{1-p\nu}(1+t+r)^{2-p\kappa}.$$

Next suppose (3.36) holds. Then by (3.37) and (6.21) we get (6.7), since

$$\int_{|t-r|}^{t+r}(1+\alpha)^{2-p\kappa}d\alpha\leqq\begin{cases}\dfrac{1}{p\kappa-3}\cdot\dfrac{1}{(1+|t-r|)^{p\kappa-3}} & \text{if}\quad p\kappa>3\\ \log\dfrac{1+t+r}{1+|t-r|} & \text{if}\quad p\kappa=3,\\ \dfrac{1}{3-p\kappa}(1+t+r)^{3-p\kappa} & \text{if}\quad 0<p\kappa<3.\end{cases}$$

Thus we prove the lemma.

PROOF OF LEMMA 6.2: We have only to modify little the proof of the preceding lemma. Note that (6.18), (6.19) and (6.20) are still valid with $\mu=1$ and $p\nu>1$, while (6.21) is replaced by

$$(6.21)'\qquad I(r,t)\leqq\frac{1}{p\nu-1}\cdot\frac{1}{2r}\int_{|t-r|}^{t+r}(1+\alpha)^{1-p}d\alpha.$$

Notice that (6.21)′ coincides with (6.21) except the constant if we take $p\kappa=p+1$ in the latter. Therefore we obtain (6.10), as before. Te proof is complete.

PROOF OF LEMMA 6.3: We have only to modify a little the proof of the preceding lemma. Note that (6.18), (6.19) and (6.20) are still valid with $\mu=1$ and $p\nu=1$, while (6.21)′ is replaced by

$$(6.21)'' \quad I(r, t) \leqq \frac{1}{2r}\int_{|t-r|}^{t+r}(1+\alpha)^{1-p}\log(1+\alpha)d\alpha.$$

Therefore, if $1<p\leqq 2$, we obtain (6.12) as before, since $\log(1+\alpha)\leqq\log(1+t+r)$.

Suppose now that $p>2$. If (3.33) holds, we have, similarly to (6.22),

$$I(r, t)\leqq C\frac{\log(1+t+r)}{(1+t+r)^{p-1}},$$

where C is a constant depending only on p. Moreover analogously to (3.55) we get

$$\frac{\log(1+t+r)}{(1+t+r)^{p-2}}\leqq \text{const.}\frac{\log(2+|t-r|)}{(1+|t-r|)^{p-2}}.$$

Therefore we obtain (6.12). If (3.36) holds, integrating by parts we have

$$\begin{aligned}
&\int_{|t-r|}^{t+r}(1+\alpha)^{1-p}\log(1+\alpha)d\alpha \\
&=\frac{1}{2-p}[(1+\alpha)^{2-p}\log(1+\alpha)]_{|t-r|}^{t+r}+\frac{1}{p-2}\int_{|t-r|}^{t+r}(1+\alpha)^{1-p}d\alpha \\
&\leqq \frac{1}{(p-2)(1+|t-r|)^{p-2}}\left(\log(1+|t-r|)+\frac{1}{p-2}\right).
\end{aligned}$$

Hence by (3.37) and (6.21)$''$ we obtain (6.12). The proof is complete.

We are now in a position to establish an estimate analogous to Lemma 4.2. For $u(x, t)\in C^0(\boldsymbol{R}^3\times[0, \infty))$ we define, as (4.1), a norm of u by

$$(6.23) \quad \|u\|=\sup_{(x, t)\in \boldsymbol{R}^3\times[0, \infty)}|u(x, t)|\Psi(r, t),$$

where

$$\Psi(r, t)=\begin{cases}(1+t+r)^{\kappa} & \text{if } 0<\varkappa<1, \\ (1+t+r)/\log(2+t+r) & \text{if } \varkappa=1, \\ (1+t+r)(1+|t-r|)^{\kappa-1} & \text{if } 1<\varkappa\leqq 1+\dfrac{1}{p}, \\ (1+t+r)(1+|t-r|)^{\nu} & \text{if } \varkappa>1+\dfrac{1}{p}\end{cases}$$

with

$$\nu=\min\{\varkappa-1, \ p-2\}.$$

(This norm is the same one as in Asakura [3]). Then we have

LEMMA 6.6. *Let* $n=3$. *Assume* $(1.2)_3$ *and* (1.7) *hold. Then*

(6.24) $\|L(|u|^p)\| \leqq C_6 \|u\|^p$

for $u \in C^0(\boldsymbol{R}^3 \times [0, \infty))$ *with* $\|u\| < \infty$, *where* C_6 *is a constant depending only on* p *and* κ.

PROOF: If (6.4) holds, then (6.24) is a direct consequence of Corollaries 6.4, 6.5 and (6.23). If (6.8) holds, we get (6.24) by Lemma 6.2, because $p > p_0(3) > 2$ hence $\Phi_2(r, t)\Psi(r, t) \leqq 1$. Finally, suppose (6.11) holds. Then

$$\Phi_3(r, t)\Psi(r, t) = \frac{\log(2+|t-r|)}{(1+|t-r|)^{p-2-(1/p)}},$$

where Φ_3 is the function in (6.12). Moreover $(1.2)_3$ implies

$$p-2-\frac{1}{p}>0.$$

Therefore by virtue of Lemma 6.3 we obtain (6.24). The proof is complete.

PROOF OF THEOREM 1.2: By virtue of Proposition 6.0 and Lemma 6.6 one can prove the theorem analogously to [3] hence we omit the details.

REMARK. One can also derive lower bounds, like Theorems 5.1 through 5.4, for the lifespan of solutions to (1.1) in three space dimensions, emplying Lemmas 6.1 and 6.3.

Appendix

The purpose of this appendix is to show that the decay rate (2.2) is optimal in the region $0 \leqq r \leqq t$, $t \geqq 1$.

PROPOSITION A. *Let* $u(x, t)$ *be the solution of* (2.1) *with* $f=0$, $g \in C^2(\boldsymbol{R}^2)$ *and*

(A.1) $g(x) \geqq \dfrac{\varepsilon}{(1+r)^{1+\kappa}}$ *for* $x \in \boldsymbol{R}^2$,

where $\kappa > 0$, $\varepsilon > 0$. *Suppose* $0 \leqq r \leqq t$ *and* $t \geqq 1$. *Then*

(A.2) $u(x, t) \geqq C_0' \varepsilon \Phi(r, t)$,

where $\Phi(r, t)$ *is the same function as in Proposition* 2.1 *and* C_0' *a positive constant depending only on* κ.

PROOF: It follows from (2.3) and (A.1) that

(A. 3) $$u(x, t)\geqq\frac{\varepsilon t}{2\pi}\int_{|\xi|<1}\frac{1}{\sqrt{1-|\xi|^2}(1+|x+t\xi|)^{1+\kappa}}d\xi.$$

Hence analogously to the proof of (2. 10) with (2. 13) we have

(A. 4) $$u(x, t)\geqq\varepsilon(I_1(r, t)+I_2(r, t)),$$

where

(A. 5) $$I_1(r, t)=\frac{2}{\pi}\int_{t-r}^{t+r}\frac{\lambda}{(1+\lambda)^{1+\kappa}}d\lambda\int_{|\lambda-r|}^{t}\frac{\rho}{\sqrt{t^2-\rho^2}}h(\lambda, \rho, r)d\rho,$$

(A. 6) $$I_2(r, t)=\frac{2}{\pi}\int_0^{t-r}\frac{\lambda}{(1+\lambda)^{1+\kappa}}d\lambda\int_{|\lambda-r|}^{\lambda+r}\frac{\rho}{\sqrt{t^2-\rho^2}}h(\lambda, \rho, r)d\rho,$$

and $h(\lambda, \rho, r)$ is the function given by (2. 12). Moreover, if

$$r-\lambda\leqq\rho\leqq t \quad\text{and}\quad 0\leqq\lambda\leqq t+r,$$

then

$$\lambda+r+\rho\leqq 2(t+r) \quad\text{and}\quad \lambda+r-\rho\leqq 2\lambda$$

hence

$$h(\lambda, \rho, r)\geqq\frac{1}{2\sqrt{1+t+r}\sqrt{\lambda}}\cdot\frac{1}{\sqrt{\rho^2-(\lambda-r)^2}}.$$

Therefore by virtue of (A. 5) and Lemma 2.4 with $a=|\lambda-r|$, $b=t$ we obtain

(A. 7) $$I_1(r, t)\geqq\frac{1}{2\sqrt{1+t+r}}\int_{t-r}^{t+r}\frac{\sqrt{\lambda}}{(1+\lambda)^{1+\kappa}}d\lambda.$$

Similarly we hve from (A. 6)

(A. 8) $$I_2(r, t)\geqq\frac{1}{\sqrt{2}\sqrt{1+t+r}}\int_0^{t-r}\frac{\lambda}{(1+\lambda)^{1+\kappa}\sqrt{\lambda+t-r}}d\lambda,$$

since $t+\rho\leqq 2t$ and $t-\rho\leqq t+\lambda-r$ for $r-\lambda\leqq\rho\leqq t$.

First consider the case where

(A. 9) $$t\geqq r+2 \quad\text{and}\quad r\geqq 0.$$

Then, since $\lambda+t-r\leqq 2(t-r)$ for $\lambda\leqq t-r$, it follows from (A. 8) that

$$I_2(r, t)\geqq\frac{1}{2}C(\kappa)\frac{1}{\sqrt{1+t+r}\sqrt{1+t-r}},$$

where we have set

(A. 10) $$C(\varkappa)=\int_0^2 \frac{\lambda}{(1+\lambda)^{1+\kappa}}d\lambda.$$

Hence by (A. 4) we obtain (A. 2) with $\varkappa>1$ and $C_0'=C(\varkappa)/2$. From (A. 8) we also have

$$I_2(r,t)\geqq\frac{1}{\sqrt{2}\sqrt{1+t+r}(1+t-r)^{1+\kappa}\sqrt{2(t-r)}}\int_0^{t-r}\lambda d\lambda$$
$$=\frac{(t-r)^{3/2}}{4\sqrt{1+t+r}(1+t-r)^{1+\kappa}}.$$

Noting that (A. 9) implies $t-r\geqq(1+t-r)/2$, we obtain

(A. 11) $$I_2(r,t)\geqq\frac{1}{8\sqrt{2}\sqrt{1+t+r}(1+t-r)^{\kappa-(1/2)}} \quad \text{for} \quad \varkappa>0,$$

which yields (A. 2) with $1/2<\varkappa<1$.

Next suppose $0<\varkappa<1/2$. Since $\sqrt{\lambda}\geqq\sqrt{1+\lambda}/\sqrt{2}$ for $\lambda\geqq1$, it follows from (A. 7) and (A. 9) that

$$I_1(r,t)\geqq\frac{1}{2\sqrt{2}\sqrt{1+t+r}}\int_{t-r}^{t+r}(1+\lambda)^{-\kappa-\frac{1}{2}}d\lambda$$
$$\geqq\frac{\frac{1}{2}-\varkappa}{8\sqrt{2}\sqrt{1+t+r}}\int_{t-r}^{t+r}(1+\lambda)^{-\kappa-\frac{1}{2}}d\lambda$$
$$=\frac{1}{8\sqrt{2}\sqrt{1+t+r}}\{(1+t+r)^{\frac{1}{2}-\kappa}-(1+t-r)^{\frac{1}{2}-\kappa}\}.$$

Therefore by (A. 11) we get

$$I_1(r,t)+I_2(r,t)\geqq\frac{1}{8\sqrt{2}(1+t+r)^{\kappa}},$$

which gives (A. 2) for $0<\varkappa<1/2$.

Now suppose $\varkappa=1/2$. Then (A. 7) and (A. 9) imply

$$I_1(r,t)\geqq\frac{1}{2\sqrt{2}\sqrt{1+t+r}}\int_{t-r}^{t+r}\frac{1}{1+\lambda}d\lambda$$
$$=\frac{1}{2\sqrt{2}\sqrt{1+t+r}}\log\frac{1+t+r}{1+t-r},$$

since $\sqrt{\lambda}\geqq\sqrt{1+\lambda}/\sqrt{2}$. Hence by (A. 11) we obtain (A. 2) with $C_0'=1/(8\sqrt{2})$.

Finally suppose $\varkappa=1$. Then (A. 8) implies

$$I_2(r,t)\geqq\frac{1}{2\sqrt{1+t+r}\sqrt{1+t-r}}\int_0^{t-r}\frac{\lambda}{(1+\lambda)^2}d\lambda.$$

Moreover

$$\int_0^{t-r}\frac{\lambda}{(1+\lambda)^2}d\lambda=\left[\frac{-\lambda}{1+\lambda}\right]_0^{t-r}+\int_0^{t-r}\frac{1}{1+\lambda}d\lambda$$
$$\geqq -1+\log(1+t-r).$$

Furthermore by virtue of (A. 9) we obtain

$$\log(1+t-r)\geqq\log 3$$

hence

$$\log(1+t-r)\geqq 1+\left(1-\frac{1}{\log 3}\right)\log(1+t-r).$$

Thus

$$-1+\log(1+t-r)\geqq\left(1-\frac{1}{\log 3}\right)^2(1+\log(1+t-r)).$$

Consequently we obtain

$$I_2(r,t)\geqq\frac{1}{2}\left(1-\frac{1}{\log 3}\right)^2\frac{1}{\sqrt{1+t+r}\sqrt{1+t-r}}(1+\log(1+t-r)),$$

which yields (A. 2) with $\varkappa=1$.

Next consider the case where

(A. 12) $\quad 0\leqq t-r\leqq 2 \quad$ and $\quad t+r\geqq 2.$

Then (A. 8) implies

$$I_2(r,t)\geqq\frac{1}{2\sqrt{2}\sqrt{1+t+r}}\int_0^{t-r}\frac{\lambda}{(1+\lambda)^{1+\kappa}}d\lambda,$$

since $\lambda+t-r\leqq 2(t-r)\leqq 4$. Moreover it follows from (A. 7) that

$$I_1(r,t)\geqq\frac{1}{2\sqrt{2}\sqrt{1+t+r}}\left(\int_2^{t+r}(1+\lambda)^{-\kappa-\frac{1}{2}}d\lambda+\int_{t-r}^{2}\frac{\lambda}{(1+\lambda)^{1+\kappa}}d\lambda\right),$$

because $\sqrt{\lambda}\geqq\sqrt{1+\lambda}/\sqrt{2}$ for $\lambda\geqq 1$ and $\sqrt{\lambda}\geqq\lambda/\sqrt{2}$ for $0\leqq\lambda\leqq 2$. Therefore we obtain

(A. 13) $$I_1(r,t)+I_2(r,t)\geqq\frac{1}{2\sqrt{2}\sqrt{1+t+r}}\left(\int_2^{t+r}(1+\lambda)^{-\kappa-\frac{1}{2}}d\lambda+C(\varkappa)\right),$$

where $C(\varkappa)$ is the constant given by (A. 10). Clearly (A. 13) yields (A. 2) with $\varkappa>1/2$ and $C_0'=C(\varkappa)/(2\sqrt{2})$, since

$$\frac{\log(2+|t-r|)}{\sqrt{1+|t-r|}} \leqq 1.$$

Now suppose $0<\varkappa<1/2$. Note that (A. 10) implies

$$C(\varkappa) \geqq \frac{1}{3^{1+\kappa}} \int_0^2 \lambda d\lambda = \frac{2}{3^{1+\kappa}} \quad \text{for} \quad \varkappa>0. \tag{A. 14}$$

Moreover

$$\int_2^{t+r} (1+\lambda)^{-\kappa-\frac{1}{2}} d\lambda \geqq \frac{2}{3\sqrt{3}} \left(\frac{1}{2}-\varkappa\right) \int_2^{t+r} (1+\lambda)^{-\kappa-\frac{1}{2}} d\lambda$$

$$= \frac{2}{3\sqrt{3}} \left[(1+\lambda)^{\frac{1}{2}-\kappa} \right]_2^{t+r} = \frac{2}{3\sqrt{3}} \cdot \frac{\sqrt{1+t+r}}{(1+t+r)^{\kappa}} - \frac{2}{3^{1+\kappa}}.$$

Therefore by (A. 13) we obtain

$$I_1(r, t)+I_2(r, t) \geqq \frac{1}{3\sqrt{6}} \frac{1}{(1+t+r)^{\kappa}},$$

which yields (A. 2) for $0<\varkappa<1/2$.

Finally suppose $\varkappa=1/2$. Then it follows from (A. 13) and (A. 14) that

$$I_1(r, t)+I_2(r, t) \geqq \frac{1}{2\sqrt{2}\sqrt{1+t+r}} \left(\int_2^{t+r} \frac{1}{1+\lambda} d\lambda + \frac{2}{3\sqrt{3}} \right).$$

Moreover

$$\int_2^{t+r} \frac{1}{1+\lambda} d\lambda + \frac{2}{3\sqrt{3}} \geqq \frac{2}{3\sqrt{3}(1+\log 3)} \left(\int_2^{t+r} \frac{1}{1+\lambda} d\lambda + 1 + \log 3 \right)$$

$$= \frac{2}{3\sqrt{3}(1+\log 3)} (1+\log(1+t+r)).$$

Thus we obtain

$$I_1(r, t)+I_2(r, t) \geqq C_0' \frac{1}{\sqrt{1+t+r}} (1+\log(1+t+r)),$$

which yields (A. 2), where

$$C_0' = \frac{1}{3\sqrt{6}(1+\log 3)}.$$

Finally consider the case where $0 \leqq r \leqq 2-t$ and $t \geqq 1$. Then it follows from (A. 3) that

$$u(x, t) \geqq \frac{\varepsilon}{2\pi} \cdot \frac{1}{3^{1+\kappa}} \int_{|\xi|<1} \frac{1}{\sqrt{1-|\xi|^2}} d\xi = \frac{\varepsilon}{3^{1+\kappa}}.$$

Since $\Phi(r, t) \leqq 2$, we obtain (A. 2). Thus we prove the proposition.

REMARK. We observe from the above proof that Lemma 2.4 plays an essential role.

REMARK. For fixed $\kappa > 0$ we set

$$g(x) = C(\kappa)\frac{\varepsilon}{(1+r^2)^{(1+\kappa)/2}},$$

where

$$C(\kappa) = \{1+4(1+\kappa)+4(1+\kappa)(3+\kappa)\}^{-1}.$$

Then $g \in C^\infty(\boldsymbol{R}^2)$,

$$g(x) \geqq C(\kappa)\frac{\varepsilon}{(1+r)^{1+\kappa}} \quad \text{for} \quad x \in \boldsymbol{R}^2$$

and (1. 3) holds with $f=0$. Thus (2. 2) is optimal under (1. 3).

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Added in Proof. After the completion of this paper we received two preprints of Tsutaya [12] and [13]. The assumption (1.6) of Theorem 1.1 is relaxed as $\varkappa \geqq 2/(p-1)$ in [12] where Kovalyov's method is employed instead of Lemma 2.3 but Proposition 2.1 for the linear wave equation plays still a basic role. The assumption (1.7) of Theorem 1.2 is also relaxed as $\varkappa \geqq 2/(p-1)$ in [13].

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