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On interpolation of weakly compact operators

Mieczysław MASTYŁO
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1. Introduction

In this paper we investigate whether the weak compactness property of the bounded linear operators can be interpolated. In particular, it is shown that the real interpolation functors generated by the K-functional of Peetre and the reflexive Banach lattices on the set of integers $\boldsymbol{Z}$ are stable for the weakly compact operators.

First of all, we recall some notations from interpolation theory. A pair $\bar{A}=(A_0, A_1)$ of Banach spaces is called a *Banach couple* if A_0 and A_1 are continuously embedded in some Hausdorff topological vector space V.

For a Banach couple $\bar{A}=(A_0, A_1)$ we can form the *intersection* $\Delta(\bar{A})=A_0\cap A_1$ and the *sum* $\Sigma(\bar{A})=A_0+A_1$. They are both Banach spaces, in the natural norms $\|a\|_{\Delta(\bar{A})}=\max\{\|a\|_{A_0}, \|a\|_{A_1}\}$ and $\|a\|_{\Sigma(\bar{A})}=K(1, a; \bar{A})$ respectively (whenever possible we suppress the "unnecessary" $\bar{A}$, writing Δ and Σ), where for $t>0$

$$K(t, a; \bar{A})=\inf\{\|a_0\|_{A_0}+t\|a_1\|_{A_1}: a=a_0+a_1, a_0\in A_0, a_1\in A_1\}$$

is the K-*functional* of Peetre.

A Banach space A is called an *intermediate space* between A_0 and A_1 (or with respect to $\bar{A}$) if $\Delta(\bar{A})\subset A\subset\Sigma(\bar{A})$ with continuous inclusions.

Let $\bar{A}=(A_0, A_1)$ and $\bar{B}=(B_0, B_1)$ be two Banach couples. We denote by $\mathscr{L}(\bar{A}, \bar{B})$ the Banach space of all linear operators $T: A_0+A_1\rightarrow B_0+B_1$ such that the restriction of T to the space A_i is a bounded operator from A_i into B_i, $i=0, 1$, with the norm

$$\|T\|_{\mathscr{L}(\bar{A}, \bar{B})}=\max\{\|T\|_{A_0\rightarrow B_0}, \|T\|_{A_1\rightarrow B_1}\}.$$

We say that two intermediate spaces A and B are *interpolation spaces* with respect to $\bar{A}$ and $\bar{B}$ and we will write $(A, B)\in Int(\bar{A}, \bar{B})$ if every operator from $\mathscr{L}(\bar{A}, \bar{B})$ maps A into B.
If A coincides with B and $A_i=B_i$, $i=0, 1$, then A is called an *interpolation space* between A_0 and A_1 (or with respect to $\bar{A}$).

We say that $\mathscr{F}$ is an *exact interpolation functor* if $\mathscr{F}(\bar{A})$ is an intermediate Banach space with respect to $\bar{A}$ for any Banach couple $\bar{A}$, and $(\mathscr{F}(\bar{A}), \mathscr{F}(\bar{B})) \in Int(\bar{A}, \bar{B})$ with

$$\| T \|_{\mathscr{F}(\bar{A}) \to \mathscr{F}(\bar{B})} \leq \max \{\| T \|_{A_0 \to B_0}, \| T \|_{A_1 \to B_1}\}$$

for each $T \in \mathscr{L}(\bar{A}, \bar{B})$.
The *characteristic function* $\varphi = \varphi_{\mathscr{F}}$ of an exact interpolation functor $\mathscr{F}$ is defined by $\mathscr{F}(\boldsymbol{R}, t^{-1}\boldsymbol{R}) = \varphi(t)^{-1}\boldsymbol{R}$ (see [7]), where for $a > 0$, $a\boldsymbol{R}$ is $\boldsymbol{R}$ with the norm $\| x \|_{a\boldsymbol{R}} = a|x|$. It is easily seen that φ is a *quasi-concave function,* i. e., $\varphi : \boldsymbol{R}_+ \to \boldsymbol{R}_+$ and $\varphi(s) \leqq \max \{1, s/t\} \varphi(t)$ for all $s, t > 0$. In the sequel if a quasi-concave function Ψ is such that $\min \{1, 1/t\} \Psi(t) \to 0$ as $t \to 0, \infty$, then we write $\Psi \in \mathscr{P}_0$.

2. The real interpolation spaces and weak compactness

In the theory of interpolation spaces the real interpolation spaces are particulary important (see [3, 4, 5, 7, 9]). Let us recall the definition.

Let ω denote the F-space of all real-valued sequences $(a_\nu)_{\nu \in \boldsymbol{Z}}$ topologized by means of coordinatewise convergence. A Banach space $E \subset \omega$ is called *Banach lattice* on $\boldsymbol{Z}$ if the conditions $|\alpha| = (|\alpha_\nu|)_{\nu \in \boldsymbol{Z}} \leq |\beta| = (|\beta_\nu|)_{\nu \in \boldsymbol{Z}}$ (meaning that $|\alpha_\nu| \leq |\beta_\nu|$ for all $\nu \in \boldsymbol{Z}$), $\beta = (\beta_\nu) \in E$ imply $\alpha = (\alpha_\nu) \in E$ and $\| \alpha \|_E \leq \| \beta \|_E$.
A Banach lattice E is said to be *regular* if the norm is an *order continuous*, i. e., if $(\alpha_n)_{n=1}^{\infty} \subset E$, $\alpha_n \downarrow 0$ implies $\alpha_n \to 0$ in E.

The *Köthe dual* of a Banach lattice E on $\boldsymbol{Z}$ is defined by

$$E' = \left\{ \alpha' \in \omega : \sum_{\nu=-\infty}^{\infty} |\alpha_\nu \alpha'_\nu| \angle \infty \quad \text{for every} \quad \alpha = (\alpha_\nu) \in E \right\}.$$

The space E' is a Banach lattice on $\boldsymbol{Z}$ under the norm

$$\|\alpha'\|_{E'} = \sup \left\{ \sum_{\nu=-\infty}^{\infty} |\alpha_\nu \alpha'_\nu| : \alpha = (\alpha_\nu) \in E, \| \alpha \|_E \leq 1 \right\}.$$

By $l_p(w) = l_p(w_\nu)$, $1 \leq p \leq \infty$, where $w = (w_\nu)$ is a positive sequence of ω, we denote the Banach lattice on $\boldsymbol{Z}$ defined by

$$l_p(w) = \left\{ \alpha \in \omega : \| \alpha \|_{l_p(w)} = \left(\sum_{\nu=-\infty}^{\infty} |\alpha_\nu w_\nu|^p \right)^{1/p} < \infty \right\}$$

(with usual interpretation if $p = \infty$).

Throughout the paper Φ denote a Banach lattice on $\boldsymbol{Z}$ intermediate with respect to $(l_\infty, l_\infty(2^{-\nu}))$. The *real exact interpolation functor* K_Φ is defined as follows; if $\bar{A}$ is a Banach couple, then the space $K_\Phi(\bar{A})$ (called the *real interpolation space*) to consist of all $a\in\Sigma(\bar{A})$ such that $(K(2^\nu, a; \bar{A}))_{\nu\in\boldsymbol{Z}}\in\Phi$ with the norm

$$\|a\|_{K_\Phi(\bar{A})}=\|(K(2^\nu, a; \bar{A}))_{\nu\in\boldsymbol{Z}}\|_\Phi.$$

Observe that, in particular, if $\Phi=l_p(2^{-\nu\theta})$, where $0<\theta<1$, $1\leq p\leq\infty$ the space $K_\Phi(\bar{A})$ coincides with the spaces $\bar{A}_{\theta,p}$ of Lions-Peetre (see [3] for more details).

In [6] K. Hayakawa has shown that if $\bar{A}=(A_0, A_1)$, $\bar{B}=(B_0, B_1)$ are Banach couples and T is an operator of $\mathscr{L}(\bar{A}, \bar{B})$ such that (the restrictions) $T: A_0\to B_0$ and $T: A_1\to B_1$ are compact, then $T:\bar{A}_{\theta,p}\to\bar{B}_{\theta,p}$ is compact for all $0<\theta<1$ and $1\leq p<\infty$.

In this section we give the similar result (excluding case $p=1$) for weakly compact operators (recall that a bounded linear operator $T: X\to Y$ between two Banach spaces is said to be *weakly compact* whenever T carries the *closed unit ball* B_X of X onto a relatively weakly compact subset of Y). By the well-known Gantmacher theorem the operator $T: X\to Y$ is weakly compact if and only if $T^{**}(X^{**})\subset Y$ (Y is naturally identified with the subspace of Y^{**}).

THEOREM 1. *Let $\bar{A}$ and $\bar{B}$ be two Banach couples and let T be an operator of $\mathscr{L}(\bar{A}, \bar{B})$ such that $T:\Sigma(\bar{A})\to\Sigma(\bar{B})$ is weakly compact. Then T is weakly compact from $K_\Phi(\bar{A})$ into $K_\Phi(\bar{B})$, provided that Φ is a reflexive Banach lattice on $\boldsymbol{Z}$.*

In order to prove this theorem we need some auxiliary results. Let $(X_\nu)_{\nu\in\boldsymbol{Z}}$ be a family of Banach spaces and let E be a Banach lattice on $\boldsymbol{Z}$. The vector space of sequences $(x_\nu)_{\nu=-\infty}^\infty$, with $x_\nu\in X_\nu$ and with $(\|x_\nu\|_{X_\nu})_{\nu=-\infty}^\infty\in E$, becomes a Banach space when equipped with the norm $\|(x_\nu)\|=\|(\|x_\nu\|_{X_\nu})\|_E$. This space shall be denoted $(\prod X_\nu)_E$. The following result will be useful in the sequel (for the proof see [12], cf. also [9, p.282-284]).

PROPOSITION 2. *Let $(X_\nu)_{\nu\in\boldsymbol{Z}}$ be a sequence of Banach spaces, and assume that a Banach lattice E on $\boldsymbol{Z}$ is regular. Then we have*

$$(\prod X_\nu)_E^*=(\prod X_\nu^*)_{E'},$$

where the duality holds subject to the duality $\langle x, x^*\rangle=\sum_{\nu=-\infty}^{\infty}x_\nu^*(x_\nu)$ for $x=(x_\nu)\in(\prod X_\nu)_E$, $x^*=(x_\nu^*)\in(\prod X_\nu^*)_{E'}$.

PROPOSITION 3. *Let E be a Banach lattice on $\boldsymbol{Z}$ and let $T_\nu: X_\nu\to Y_\nu$ for $\nu\in\boldsymbol{Z}$ be bounded linear operators between Banach spaces such that*

$\sup_\nu \|T_\nu\| = C < \infty$. *Define the operator* $\oplus T_\nu : (\prod X_\nu)_E \to (\prod Y_\nu)_E$ *by* $\oplus T_\nu(x_\nu) = (T_\nu x_\nu)$. *Then the following hold :*

(i) $\oplus T_\nu$ *is a bounded linear operator such that* $(\oplus T_\nu)^* = \oplus T_\nu^*$, *provided that* E *is regular Banach lattice.*

(ii) *If* E *is reflexive, then* $\oplus T_\nu$ *is a weakly compact operator if and only if* T_ν *is a weakly compact operator for every* $\nu \in \boldsymbol{Z}$.

PROOF. (i) Let $X = (\prod X_\nu)_E$ and $Y = (\prod Y_\nu)_E$, then obviously that $\|\oplus T_\nu\|_{X \to Y} \leq C$. Now assume that E is a regular Banach lattice and take any $x = (x_\nu) \in X$ and $y^* \in Y^*$. Then by Proposition 2, we have $y^* = (y_\nu^*) \in (\prod Y_\nu^*)_{E'}$ and

$$\begin{aligned} \langle x, (\oplus T_\nu)^* y^* \rangle &= \langle (\oplus T_\nu)(x_\nu), y^* \rangle = \langle (T_\nu x_\nu), (y_\nu^*) \rangle \\ &= \langle (x_\nu), (T_\nu^* y_\nu^*) \rangle = \langle x, (\oplus T_\nu^*) y^* \rangle. \end{aligned}$$

Thus $(\oplus T_\nu^*) = \oplus T_\nu^*$.

(ii) Let E be reflexive space, then by Ogasawara's theorem (see [1, Theorem 14. 22]), it follows that E and E' are regular Banach lattices on $\boldsymbol{Z}$ and $E'' = E$. Thus $X^{**} = (\prod X_\nu^{**})_E$ and $(\oplus T_\nu)^{**} = \oplus T_\nu^{**}$, by Proposition 2 and (i). Hence, it follows that if $x^{**} \in X^{**}$, then $x^{**} = (x_\nu^{**}) \in (\prod X_\nu^{**})_E$ and

$$(\oplus T_\nu)^{**} x^{**} = (\oplus T_\nu^{**})\ (x_\nu^{**}) = (T_\nu^{**}\ x_\nu^{**}).$$

This implies that $(\oplus T_\nu)^{**}(X^{**}) \subset Y$ if and only if $T_\nu^{**}(X_\nu^{**}) \subset Y_\nu$ holds for each $\nu \in \boldsymbol{Z}$. Thus the proof of (ii) is finished, by the Gantmacher theorem.

PROOF OF THEOREM 1. Let $X = (\prod X_\nu)_\Phi$ and $Y = (\prod Y_\nu)_\Phi$, where $X_\nu = A_0 + A_1$, $Y_\nu = B_0 + B_1$ with the norms $K(2^\nu, \cdot\,; \bar{A})$ and $K(2^\nu, \cdot\,; \bar{B})$ for $\nu \in \boldsymbol{Z}$, respectively. Denote by $\boldsymbol{D}(X)$ the diagonal subspace of X, i. e.

$$\boldsymbol{D}(X) = \{(x_\nu)_{\nu \in \boldsymbol{Z}} \in X :\ x_\nu = a,\ a \in \Sigma(\bar{A})\}.$$

Define the operator $J : K_\Phi(\bar{A}) \to \boldsymbol{D}(X)$ by $J(a) = (\cdots, a, a, \cdots)$. Clearly that J is the onto linear isometry.

Now let $T \in \mathscr{L}(\bar{A}, \bar{B})$, then obviously that $T_\nu = T : X_\nu \to Y_\nu$ for each ν and $\sup_\nu \|T_\nu\|_{X_\nu \to Y_\nu} \leq \|T\|_{\mathscr{L}(\bar{A}, \bar{B})}$. This implies that the operator $\oplus T_\nu : X \to Y$ is weakly compact, whenever $T : \Sigma(\bar{A}) \to \Sigma(\bar{B})$ is weakly compact, by Proposition 3. Since K_Φ is an exact interpolation functor, the restriction of T to the space $K_\Phi(\bar{A})$ is a bounded linear operator from $K_\Phi(\bar{A})$ into $K_\Phi(\bar{B})$. Thus the proof is finished, by $T = J^{-1} \circ (\oplus T_\nu) \circ J$.

COROLLARY 4. *Let $\overline{A}$ and $\overline{B}$ be Banach couples and let $T \in \mathscr{L}(\overline{A}, \overline{B})$ be such that $T: A_i \to B_i$ $(i=0, 1)$ is weakly compact. Then $T: \overline{A}_{\theta,p} \to \overline{B}_{\theta,p}$ is weakly compact for all $0<\theta<1$ and $1<p<\infty$.*

PROOF. It is a routine matter to verify that $T: \Sigma(\overline{A}) \to \Sigma(\overline{B})$ is weakly compact. Since $\Phi = l_p(2^{-\nu\theta})$ is reflexive Banach lattice on $\boldsymbol{Z}$ if $0<\theta<1$ and $1<p<\infty$, Theorem 1 applies.

3. Interpolation functors and weakly compact operators

B. Beauzamy in [4] has shown that the interpolation spaces of Lions-Peetre $\overline{A}_{\theta,p}$, where $0<\theta<1$ and $1<p<\infty$ are reflexive if and only if the inclusion map $I: \Delta(\overline{A}) \to \Sigma(\overline{A})$ is weakly compact. R. D. Neidinger [15] using results of the theory of Tauberian operators has shown the similar result for more general interpolation spaces (see [11, p.218]). In fact these spaces are real interpolation spaces by the results of Yu. A. Brudnyĭ and N. Ya. Krugljak [5].

In this section we investigate those exact interpolation functors which interpolate weakly compact operators. In particular we show that Theorem 1 can be improved if we suppose that Φ satisfies some additional condition. From these results we obtain results about reflexivity of these spaces. First we give some definitions and auxiliary results.

Let X be a Banach space and let W, V be subsets of X. We say that W almost absorbs V (in X) if for every $\varepsilon>0$, there exists $t>0$ such that $V \subset tW + \varepsilon B_X$ (see [13]).

Let X, Y be Banach spaces. A bounded linear operator $T: X \to Y$ is said to be *Tauberian* [8] if $(T^{**})^{-1}(Y) \subset X$, i.e., $T^{**}x^{**} \in Y$ implies $x^{**} \in X$.
If Tauberian operator T is one-to-one we say T is *Tauberian injection*. There are interesting characterization of Tauberian operators (see [8, 14, 16]). We need the following (see [14, 16]).

THEOREM 5. *Let X, Y be Banach spaces and let $T: X \to Y$ be an injective bounded linear operator. The following are equivalent:*

(*a*) *T is Tauberian injection.*
(*b*) *For every bounded subset U of X such that TU is relatively weakly compact, U is relatively weakly compact.*
(*c*) *$T^*(Y^*)$ is norm-dense in X^* and TB_X is closed subset of Y.*

PROPOSITION 6. *Let V, W be subsets of a Banach space X. If W almost absorbs V and W is relatively weakly compact, then V is relatively* *weakly compact.*

For the proof see [13, 15] or [1, Theorem 10. 17].

COROLLARY 7. *Let $\bar{X}$ and $\bar{Y}$ be two Banach couples and let $(X, Y) \in Int(\bar{X}, \bar{Y})$. If $B_{\Delta(\bar{X})}$ almost absorbs B_X in $\Sigma(\bar{X})$ and the inclusion map $Y \subset \Sigma(\bar{Y})$ is a Tauberian operator, then $T : X \to Y$ is weakly compact operator, provided $T \in \mathscr{L}(\bar{X}, \bar{Y})$ and $T : \Delta(\bar{X}) \to \Sigma(\bar{Y})$ is weakly compact operator.*

PROOF. Let $T \in \mathscr{L}(\bar{X}, \bar{Y})$. Then $T(B_{\Sigma(\bar{X})}) \subset CB_{\Sigma(\bar{Y})}$, where $C = \|T\|_{\mathscr{L}(\bar{X}, \bar{Y})}$. Moreover $T : X \to Y$ is bounded by interpolation. Take $\varepsilon > 0$, then by the assumption there exists $t > 0$ such that $B_X \subset tB_{\Delta(\bar{X})} + C^{-1}\varepsilon B_{\Sigma(\bar{X})}$. In consequence

$$TB_X \subset tTB_{\Delta(\bar{X})} + \varepsilon B_{\Sigma(\bar{Y})}.$$

Hence $TB_{\Delta(\bar{X})}$ almost absorbs TB_X in $\Sigma(\bar{Y})$. Now suppose that $T : \Delta(\bar{X}) \to \Sigma(\bar{Y})$ is weakly compact. Then TB_X is relatively weakly compact subset of Y, by Proposition 6 and Theorem 5. Thus the proof is complete.

THEOREM 8. *Let $\bar{X} = (X_0, X_1)$ and $\bar{Y} = (Y_0, Y_1)$ be two Banach couples and let $\mathscr{F}$ be an exact interpolation functor with the fundamental function $\varphi \in \mathscr{P}_0$. If the inclusion map $\mathscr{F}(\bar{Y}) \subset \Sigma(\bar{Y})$ is a Tauberian operator, then for every $T \in \mathscr{L}(\bar{X}, \bar{Y})$, we have $T : \mathscr{F}(\bar{X}) \to \mathscr{F}(\bar{Y})$ is weakly compact operator if and only if $T : \Delta(\bar{X}) \to \Sigma(\bar{Y})$ is weakly compact.*

PROOF. By Theorem 15 of [7], we have

$$(*) \qquad K(t, x ; \bar{X}) \le \varphi(t) \|x\|_X$$

for every $x \in X = \mathscr{F}(\bar{X})$ and $t > 0$. Suppose that $\varphi \in \mathscr{P}_0$. First we show that $B_{\Delta(\bar{X})}$ almost absorbs B_X in $\Sigma(\bar{X})$. Let $\varepsilon > 0$. Since $\varphi \in \mathscr{P}_0$, there exist $i, j \in \boldsymbol{N}$ such that $\varphi(2^{-i}) < \varepsilon/4$ and $\varphi(2^j)/2^j < \varepsilon/4$. If $x \in B_X$, then by $(*)$ we obtain.

$$K(2^{-i}, x ; \bar{X}) \le \varphi(2^{-i}), \quad K(2^j, x ; \bar{X}) \le \varphi(2^j).$$

Hence we can find the decompositions $x = x_0 + x_1 = x_0' + x_1'$ such that $x_0, x_0' \in X_0$, $x_1, x_1' \in X_1$ and

$$\|x_0\|_{X_0} + 2^{-i}\|x_1\|_{X_1} < \varphi(2^{-i}) + \varepsilon/4 < \varepsilon/2,$$
$$\|x_0'\|_{X_0} + 2^j\|x_1'\|_{X_1} < \varphi(2^j) + \varepsilon/4 < 2^j\varepsilon/2.$$

Thus $\|x_0\|_{X_0} < \varepsilon/2$, $\|x_1\|_{X_1} < 2^i\varepsilon/2$ and $\|x_0'\|_{X_0} < 2^j\varepsilon/2$, $\|x_1'\|_{X_1} < \varepsilon/2$. Let $y = x - x_0 - x_1'$, then $y = x_0' - x_0 \in X_0$ and $\|y\|_{X_0} \le \|x_0\|_{X_0} + \|x_0'\|_{X_0} < 2^j\varepsilon$. Further $y = x_1 -$

$x_1' \in X_1$ and $\|y\|_{X_1} \leq \|x_1\|_{X_1} + \|x_1'\|_{X_1} < 2^i \varepsilon$. Hence $y \in \Delta$ and $\|y\|_\Delta \leq \max(2^i, 2^j)\varepsilon$. Since $x - y = x_0 + x_1'$ and $\|x-y\|_\Sigma \leq \|x_0\|_{X_0} + \|x_1'\|_{X_1} < \varepsilon$, $x \in tB_\Delta + \varepsilon B_\Sigma$, where $t = \max(2^i, 2^j)\varepsilon$. Finally $B_X \subset tB_{\Delta(\bar{X})} + \varepsilon B_{\Sigma(\bar{X})}$. Now, if the inclusion map $\mathscr{F}(\bar{Y}) \subset \Sigma(\bar{Y})$ is a Tauberian operator and $T \in \mathscr{L}(\bar{X}, \bar{Y})$ is such that $T: \Delta(\bar{X}) \to \Sigma(\bar{Y})$ is weakly compact, then $T: \mathscr{F}(\bar{X}) \to \mathscr{F}(\bar{Y})$ is weakly compact operator by Corollary 7. The converse is obvious.

COROLLARY 9. *Let $\bar{X}$ be a Banach couple and let $\mathscr{F}$ be an exact interpolation functor with the fundamental function $\varphi \in \mathscr{P}_0$. Then the following are equivalent*:

(i) *The interpolation space $\mathscr{F}(\bar{X})$ is reflexive.*

(ii) *The inclusion map $I: \Delta \to \Sigma$ is weakly compact and the inclusion map $J: \mathscr{F}(\bar{X}) \to \Sigma(\bar{X})$ is Tauberian injection.*

REMARK. If $\bar{X}$ is a couple of Banach lattices such that $\Delta(\bar{X})^*$ has order continuous norm and $\bar{Y}$ is a Banach couple for which $\Sigma(\bar{Y})$ contains no subspace isomorphic to c_0, then every bounded linear operator $T: \Delta(\bar{X}) \to \Sigma(\bar{Y})$ is weakly compact by Grothendieck-Ghoussoub-Johnson theorem (see [1, Theorem 17.6]).

4. Applications

In this section we shall give applications of the results of section 2. First we give sufficient conditions which imply that the inclusion map $J: K_\Phi(\bar{X}) \to \Sigma(\bar{X})$ is a Tauberian injection. Note that the fundamental function φ of the functor K_Φ (called in the sequel also the fundamental function of the space Φ) satisfy $\varphi(t)^{-1} = \|(\min\{1, 2^\nu/t\})_\nu\|_\Phi$.

PROPOSITION 10. *Let $\bar{X}$ be a Banach couple. Then the following hold*:

(i) *If B_Φ is a closed subset of ω, then the closed unit ball of the space $K_\Phi(\bar{X})$ is closed subset of Σ.*

(ii) *If Φ contains no subspace isomorphic to l^1, then $J^*(\Sigma(\bar{X})^*)$ is norm-dense in $K_\Phi(\bar{X})^*$.*

PROOF. (i) Let $(x_n)_{n=1}^\infty \subset B_X$, where $X = K_\Phi(\bar{X})$ and let $x_n \to x$ in Σ. Then $a_{n\nu} = K(2^\nu, x_n; \bar{X}) \to K(2^\nu, x; \bar{X}) = a_\nu$ as $n \to \infty$, for every $\nu \in \boldsymbol{Z}$, whence $\beta_n = (a_{n\nu})_{\nu \in \boldsymbol{Z}} \to a = (a_\nu)_{\nu \in \boldsymbol{Z}}$ in ω. Since $(\beta_n)_{n=1}^\infty \subset B_\Phi$ and B_Φ is closed subset of ω, $a = (a_\nu)_{\nu \in \boldsymbol{Z}} \in B_\Phi$ and thus $x \in B_X$. For the proof of (ii) see [12].

COROLLARY 11 (cf. [5, Theorem 4.6.8]). *Let $\bar{X}$ and $\bar{Y}$ be two Banach couples and let Φ be a reflexive Banach lattice on $\boldsymbol{Z}$ with the funda-*

mental function in $\mathscr{P}_0$. Then the following hold:

(i) *If $T \in \mathscr{L}(\bar{X}, \bar{Y})$, then $T : K_\Phi(\bar{X}) \to K_\Phi(\bar{Y})$ is weakly compact operator if and only if $T : \Delta(\bar{X}) \to \Sigma(\bar{Y})$ is weakly compact.*

(ii) *The real interpolation space $K_\Phi(\bar{X})$ is reflexive if and only if the inclusion map $I : \Delta \to \Sigma$ is weakly compact operator.*

PROOF. Since $\Phi \subset \omega$ with the continuous inclusion, so reflexivity of Φ imply that B_Φ is closed subset of ω. Of course Φ contains no subspace isomorphic to l^1. Thus by Proposition 10 and Theorem 5 the inclusion map $J : K_\Phi(\bar{X}) \to \Sigma(\bar{X})$ is a Tauberian injection. Thus (i) (and (ii)) follows by Theorem 8 (Corollary 9).

The result obtained by taking $\Phi = l_p(2^{-\nu\theta})$, $0<\theta<1$, $1<p<\infty$ in Corollary 11 (ii) is an extension of the Beauzamy result [4] (see also Neidinger [15]) for the Lions-Peetre space $\bar{X}_{\theta,p}$.

REMARK. The assumption that Φ is reflexive space is essential in Corollary 11. Namely, by the result of M. Levy [10], if $\Phi = l_1(2^{-\nu\theta})$, $0< \theta<1$, then $K_\Phi(\bar{X}) = \bar{X}_{\theta,1}$ is not reflexive since it contains a subspace isomorphic to l^1 for any Banach couple $\bar{X}$ such that $\Delta(\bar{X})$ is not closed subspace of $\Sigma(\bar{X})$.

In [2] Aronszajn and Gagliardo showed that to any Banach couple $\bar{A}$ and a coresponding intermediate space A there exists a *maximal exact interpolation functor* $\mathscr{F}$ with the property $\mathscr{F}(\bar{A}) \subset A$. This functor is called the *coorbit functor* and is denoted by $Corb_{\bar{A}}(\cdot, A)$. The space $Corb_{\bar{A}}(\bar{X}, A)$ consit of all $x \in \Sigma(\bar{X})$ such that $Tx \in A$ for any $T \in \mathscr{L}(\bar{X}, \bar{A})$. We put

$$\|x\|_{Corb} = \sup\{\|Tx\|_A : \|T\|_{\mathscr{L}(\bar{X},\bar{A})} \leq 1\}.$$

In the theory of interpolation spaces many important exact interpolation functors are the coorbit functors. Let us consider the special coorbit functor $H_{1\rho}$ generated by $(A_0, A_1) = (l_1, l_1(2^\nu))$ and $A = l_1(1/\rho(2^{-\nu}))$, where ρ is a quasi-concave function. This functor was extensively studied lately (see for example [7] and [17]).

PROPOSITION 12. *Let $\bar{X} = (X_0, X_1)$ be a Banach couple such that Δ is a dense subspace of X_0, X_1 and $H_{1\rho}(\bar{X})$. Then the inclusion map $J : H_{1\rho}(\bar{X}) \to \Sigma(\bar{X})$ is Tauberian injection, provided that neither X_0 nor X_1 contain subspaces isomorphic to c_0 and ρ, $\rho_* \in \mathscr{P}_0$, where $\rho_*(t) = t/\rho(t)$ for $t>0$.*

PROOF. First we observe that $B_{\mathscr{F}(\bar{X})}$ is a closed subset of $\Sigma(\bar{X})$ for every Banach couple $\bar{X}$, where $\mathscr{F}(\bar{X})=Corb_{\bar{A}}(\bar{X}, A)$, whenever B_A is closed in $\Sigma(\bar{A})$. Thus the unit ball of $H_{1\rho}(\bar{X})$ is closed subset in $\Sigma(\bar{X})$ for every Banach couple $\bar{X}$.

Now if ρ, $\rho_* \in \mathscr{P}_0$ and $\bar{X}$ satisfies the assumptions of Proposition 12, then by Corollary 1 and Theorem 12 of [7] (see also [17, Lemma 8.3.2 and Theorem 8.8.2]), it follows that $J^*(\Sigma(\bar{X})^*)$ is norm-dense in $H_{1\rho}(\bar{X})^*$. Thus the proof is finished by Theorem 5.

REMARK. It is easily see that the fundamental function φ of the functor $H_{1\rho}$ satisfies

$$\varphi(t)^{-1}=\sup\{\|\xi\|_{l_1(1/\rho(2^{-\nu}))} : \xi\in l_1\cap l_1(2^\nu), \\ \|\xi\|_{l_1}\leq 1, \quad \|\xi\|_{l_1(2^\nu)}\leq t^{-1}\}$$

for $t>0$. Hence, we obtain $\varphi\approx\rho$. Thus $\varphi\in\mathscr{P}_0$ if and only if $\rho\in\mathscr{P}_0$. In consequence applying Proposition 12 and Theorem 8, we obtain the interpolation theorem concerning weak compactness of the operators from $\mathscr{L}(\bar{X}, \bar{Y})$ acting between $H_{1\rho}(\bar{X})$ and $H_{1\rho}(\bar{Y})$.

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Institute of Mathematics,
A. Mickiewicz University,
Matejki 48/49
60-769 Poznań, Poland