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Lower bounds for the life span of solutions of semilinear wave equations with data of non compact support

Kimitoshi TSUTAYA

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1 Introduction

We consider the life span of classical solutions of the Cauchy problem for the semilinear wave equation

$$\begin{cases} \partial_t^2 u-\Delta u=|u|^{p-1}u, & (x, t)\in R^n\times(0, \infty), \\ u(x, 0)=\varepsilon f(x),\ \partial_t u(x, 0)=\varepsilon g(x), & x\in R^n, \end{cases} \tag{1.1}$$

where $p>1$, $n=2,3$ and $\varepsilon>0$ is a small parameter.

Given f, g, p, we define the life span $T^*=T^*(\varepsilon)$ as the supremum of all T such that a C^2-solution of (1.1) exists for all $x\in R^n$ and $0\leq t<T$. Existence of global solutions and blow-up of local solutions in finite time correspond to $T^*(\varepsilon)=\infty$, $T^*(\varepsilon)<\infty$, respectively. In the case $T^*(\varepsilon)<\infty$ the basic problem consists in estimating $T^*(\varepsilon)$ in terms of ε both from above and from below, namely, upper and lower bounds for $T^*(\varepsilon)$. Upper and lower bounds are related to the construction of blow-up and local solutions, respectively, and it is expected that both bounds have the same growth rate as $\varepsilon\rightarrow 0$. There is a large literature on this problem [1-16], though there still remains a gap between the upper and lower bounds for $T^*(\varepsilon)$. The purpose of this paper is to improve the previous lower bound estimates and reduce that gap. Before we state our result more precisely, we summarize a number of available results on the life span of solutions of (1.1). In the sequel it is convenient to introduce the critical power $p_0(n)$ defined by the positive root of the quadratic equation

$$(n-1)p^2-(n+1)p-2=0 \ (n=2,3).$$

We note that $p_0(3)=1+\sqrt{2}$ and $p_0(2)=(3+\sqrt{17})/2$. It is well understood that in most cases the situation is remarkably different between the super critical case $p>p_0(n)$, the critical case $p=p_0(n)$, and the subcritical case $p<p_0(n)$. Moreover, recent developments reveal that the compactness of the data f, g has a definitive influence on the problem. Therefore we describe the previous results by dividing into two cases, whether f, g have

compact support or not. First, we assume that the data $f(x)$, $g(x)$ are smooth and "compactly supported".

John [5, 6], Glassey [3, 4] and Schaeffer [10] have proved in two and three space dimensions that if $p>p_0(n)$ and ε is sufficiently small, then $T^*(\varepsilon)=\infty$, and also that if $1<p\leq p_0(n)$, $T^*(\varepsilon)<\infty$ for small data.

In the case that blow-up occurs in finite time, that is, $T^*(\varepsilon)<\infty$, there arises the problem of estimating the life span $T^*(\varepsilon)$ in terms of ε. John [5] has shown that

$$A\varepsilon^{-2}\leq T^*(\varepsilon)\leq B\varepsilon^{-2}$$

with certain constants A, B in the case $n=3$ and $p=2$. Recently, Lindblad [8] has proved the following results:

(i) if $n=3$ and $1<p<p_0(3)$,

$$\lim_{\varepsilon\to 0}\varepsilon^{p(p-1)/(1+2p-p^2)}T^*(\varepsilon)=T>0 \quad \text{exists,}$$

(ii) if $n=2$ and $p=2$,

$$\lim_{\varepsilon\to 0}\varepsilon T^*(\varepsilon)=T>0 \text{ exists, provided } \int g(x)dx=0,$$

and

$$\lim_{\varepsilon\to 0}a(\varepsilon)^{-1}T^*(\varepsilon)=T>0 \text{ exists, provided } \int g(x)dx\neq 0,$$

where $a(\varepsilon)$ satisfies

$$\varepsilon^2 a(\varepsilon)^2\ln(1+a(\varepsilon))=1.$$

In [15, 16], Zhou has proved the following:

(iii) when $n=3$,

$$A\varepsilon^{-p(p-1)/(1+2p-p^2)}\leq T^*(\varepsilon)\leq B\varepsilon^{-p(p-1)/(1+2p-p^2)}, \text{ if } 2\leq p<p_0(3), \tag{1.2}$$

and

$$\exp\{A\varepsilon^{-p(p-1)}\}\leq T^*(\varepsilon)\leq\exp\{B\varepsilon^{-p(p-1)}\}, \text{ if } p=p_0(3),$$

(iv) when $n=2$,

$$\lim_{\varepsilon\to 0}\varepsilon^{2p(p-1)/(2+3p-p^2)}T^*(\varepsilon)=T>0 \text{ exists, if } 2<p<p_0(2),$$

and

$$\exp\{A\varepsilon^{-p(p-1)}\}\leq T^*(\varepsilon)\leq\exp\{B\varepsilon^{-p(p-1)}\}, \text{ if } p=p_0(2),$$

where A and B are positive constants. Concerning the above results due to Zhou for the case $p=p_0(n)(n=2,3)$, the simplified proofs of the upper bound of $T^*(\varepsilon)$ have been recently given by Takamura [11]. He has proved that if $p=p_0(n)(n=2,3)$
and

$$f(x)=0,\ g(x)\geq 0,\ g(x)\not\equiv 0, \tag{1.3}$$

then

$$T^*(\varepsilon)\leq \exp\{B\varepsilon^{-p(p-1)}\}, \tag{1.4}$$

even if the data are not compactly supported. In [1], Agemi and Takamura have proved similar results to Lindblad and Zhou's for the case $n=2$. Especially, the upper bound of $T^*(\varepsilon)$ in [1]

$$T^*(\varepsilon)\leq B\varepsilon^{-2p(p-1)/(2+3p-p^2)},\ \text{if } 1<p<p_0(2) \tag{1.5}$$

holds even with the data of non compact support, under the condition (1.3). In the case $n=3$ and $1<p<p_0(3)$, we can also derive for the data of non compact support, the upper bound of $T^*(\varepsilon)$ in (1.2),

$$T^*(\varepsilon)\leq B\varepsilon^{-p(p-1)/(1+2p-p^2)} \tag{1.6}$$

under (1.3).

We next assume that the data do not have compact support. Asakura [2], Kubota [7] and the author [12, 13, 14] have proved the following:

Let the data $f(x)\in C^3(R^n)$, $g(x)\in C^2(R^n)$ satisfy

$$D_x^\alpha f(x),\ D_x^\beta g(x)=O(|x|^{-1-k}) \text{ as } |x|\to\infty,\ |\alpha|\leq 3,\ |\beta|\leq 2.$$

If $k\geq 2/(p-1)$, $p>p_0(3)$ for $n=3$ and $p>p_0(2)$ for $n=2$, and ε is small, then $T^*(\varepsilon)=\infty$ (see also Pecher [9]).

On the other hand, Asakura [2], Agemi and Takamura [1] and the author [12] have also proved that if the data satisfy

$$f(x)=0,\ g(x)\geq\frac{1}{(1+|x|)^{1+k}}$$

and $0<k<2/(p-1)$, then $T^*(\varepsilon)<\infty$, even with $p>p_0(n)(n=2,3)$. In this case, the upper bound for $T^*(\varepsilon)$ is

$$T^*(\varepsilon)\leq B\varepsilon^{-(p-1)/\{2-k(p-1)\}} \text{ for } n=2,3, \tag{1.7}$$

where B is a constant depending only on p and k.

We note here that when $1<p<p_0(n)(n=2,3)$, if $k>1/2+1/p$ (resp. $0<k<1/2+1/p$) for $n=2$ and $k>1+1/p$ (resp. $0<k<1+1/p$) for $n=3$, then the upper bounds of $T^*(\varepsilon)$ in (1.5), (1.6) (resp. in (1.7)) are sharper than those in (1.7) (resp. in (1.5), (1.6)), respectively.

For the lower bound, it has been shown by Kubota [7] and the author [13, 14] that if $p>p_0(n)(n=2,3)$ and $0<k<2/(p-1)$,

$$T^*(\varepsilon)\geq A\varepsilon^{-(p-1)/\{2-k(p-1)\}}.$$

For $n=2$, Kubota has also obtained the lower bounds in the other cases:

$$\begin{aligned} &p=p_0(2) && \text{and} && k\geq\frac{1}{2}+\frac{1}{p}, \\ &2<p\leq p_0(2) && \text{and} && 0<k<\frac{1}{2}+\frac{1}{p}, \\ &2<p<p_0(2) && \text{and} && k\geq\frac{1}{2}+\frac{1}{p}. \end{aligned}$$

In particular, for the critical case $p=p_0(2)$ and $k\geq 1/2+1/p$, the lower bound due to Kubota is the following:

$$T^*(\varepsilon)\geq\exp\{A\varepsilon^{-(p-1)}\}. \tag{1.8}$$

We note that there is a gap between (1.4) and (1.8) about the order of ε in the estimates of life span. In this paper, we shall improve the lower bound (1.8) and give an almost optimal lower bound for $T^*(\varepsilon)$ for the critical cases $p=p_0(2)$ and $k>1/2+1/p$, and $p=p_0(3)$ and $k>1+1/p$ in two and three space dimensions.

The crucial part of the proof of our theorem below is to establish the basic estimate for (1.1). We now state the difference between the proofs of the basic estimates for the data of compact support and for those of non compact support. The weight function in the norm is related with the decay estimate of the solution to the linear problem. In three space dimensions, if the data $f(x)$, $g(x)$ satisfy $supp\{f, g\}\subset\{x\in R^3: |x|\leq K\}$, then, as is well known, the solution u^0 of $\partial_t^2 u^0-\Delta u^0=0$ is equal to zero in the inside of the characteristic cone $\Gamma=\{(x, t): |x|+K\leq t\}$ by Huygens' Principle. Hence we can define the special weight function (see [5], (58b) and also [15], (3.5)). This leads to the sharp basic estimate. In two space dimensions, we need to use another method due to the lack of Huygens' Principle. In fact, Zhou [16] has applied the Friedlander radiation field technique to (1.1) to obtain the sharp estimate. However, the above methods do not seem to be applicable to our problem, since the data are not compactly supported.

The main idea in the proof of the theorem is to introduce new weight functions which are slightly different from those used in [7]. The logarithmic factor in the weight function gives more precise information on the space-time behavior of solutions and plays an important role to prove our main theorem (see (3.6)). Accordingly, our method is simpler than those by John and Zhou.

2 Assumptions and main theorem

We study the Cauchy problem for the semilinear wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = F(u), & (x, t) \in R^n \times (0, \infty), \\ u(x, 0) = \varepsilon f(x),\ \partial_t u(x, 0) = \varepsilon g(x), & x \in R^n. \end{cases} \tag{2.1}$$

Before stating the main result, we make the following hypotheses: Let $p = p_0(n)$.

(H1) $F(u) \in C^2(R)$ and there exists $\lambda > 0$ such that

$$|F^{(j)}(u)| \le \lambda |u|^{p-j} \text{ for } |u| \le 1,\ j = 0, 1, 2,$$

and

$$|F''(u) - F''(v)| \le \begin{cases} \lambda p(p-1)|u-v|^{p-2}, & \text{if } n=3, \\ \lambda p(p-1)|\Phi|^{p-3}|u-v|, & \text{if } n=2 \end{cases}$$

for $|u|$, $|v| \le 1$, where

$$\Phi = \max\{|u|, |v|\}.$$

(H2) $f(x) \in C^3(R^n)$, $g(x) \in C^2(R^n)$ $(n = 2, 3)$ satisfy

$$\sum_{|\alpha| \le 3} |D_x^\alpha f(x)| + \sum_{|\beta| \le 2} |D_x^\beta g(x)| \le \frac{1}{(1+|x|)^{1+k}}$$

with $k > 0$.

Our main theorem is the following:

THEOREM 2.1. *Let $n = 2, 3$, $p = p_0(n)$ and $k > 2/(p-1)$. Assume $(H1)$ and $(H2)$. Then there exist two sufficiently small constants A, $\varepsilon_0 > 0$ depending only on λ, p and k such that for all $0 < \varepsilon \le \varepsilon_0$,* (2.1) *has a C^2-solution for $0 \le t < T^*(\varepsilon)$, where $T^*(\varepsilon)$ satisfies the following:*

$$(\ln(3 + T^*(\varepsilon)))^{1/p} (\ln(\ln(3 + T^*(\varepsilon))))^{1/(p-1)} \varepsilon^{p-1} \ge A. \tag{2.2}$$

REMARK 2.2. If the left-hand side of (2.2) does not include the factor $(\ln(\ln(3 + T^*(\varepsilon))))^{1/(p-1)}$ of slower growth, the estimate is sharp, since the dominant factor $(\ln(3 + T^*(\varepsilon)))^{1/p}$ with the growth rate $A\varepsilon^{-(p-1)}$ gives precisely (1.4). In this respect (2.2) is almost optimal.

The plan in the present paper is as follows. In Section 3, we first describe the decay estimates for the homogeneous wave equation (see [2],

[7], [12, 13, 14]). Next, we derive the basic estimates needed for the proof of Theorem 2.1. Finally, in Section 4 we prove Theorem 2.1. We denote a constant in the estimates by C, which will change from step to step.

3 Decay estimates and basic estimates

We consider the linear problem

$$\begin{cases} \partial_t^2 u - \Delta u = w(x, t), & (x, t) \in R^n \times (0, \infty), \\ u(x, 0) = \varepsilon f(x),\ \partial_t u(x, 0) = \varepsilon g(x), & x \in R^n (n=2, 3). \end{cases} \tag{3.1}$$

We see that the solution $u(x, t)$ of (3.1) is given by

$$u(x, t) = u^0(x, t) + Lw(x, t), \tag{3.2}$$

where $u^0(x, t)$ is the solution of $\partial_t^2 u^0 - \Delta u^0 = 0$ with initial data $\varepsilon f(x)$, $\varepsilon g(x)$, and

$$Lw(x, t) = \begin{cases} \dfrac{1}{4\pi} \displaystyle\int_0^t (t-\tau) \int_{|\omega|=1} w(x + (t-\tau)\omega, \tau)\, d\omega d\tau & (n=3), \\ \dfrac{1}{2\pi} \displaystyle\int_0^t \int_0^{t-s} \frac{\rho}{\sqrt{(t-s)^2 - \rho^2}} \int_{|\omega|=1} w(x + \rho\omega, s)\, d\omega d\rho ds & (n=2) \end{cases} \tag{3.3}$$

is the solution of $\partial_t^2 u - \Delta u = w$ with zero data, where $d\omega$ is the surface measure on the unit sphere $|\omega| = 1$. Note that $w \geq 0$ implies $Lw \geq 0$.

The following lemma is proved in [2], [7] and [12, 13, 14].

LEMMA 3.1. *Assume that $f(x)$, $g(x)$ satisfy $(H2)$. Then, the solution u^0 of $\partial_t^2 u^0 - \Delta u^0 = 0$ with initial data $\varepsilon f(x)$, $\varepsilon g(x)$ satisfies the following*:

(i) *if* $n=3$,

$$\sum_{|\alpha| \leq 2} |D_x^\alpha u^0(x, t)| \leq \begin{cases} \dfrac{C_k \varepsilon}{(1+t+r)(1+|t-r|)^{k-1}} & (k>1), \\ \dfrac{C_k \varepsilon}{1+t+r} \left(1 + \ln \dfrac{1+t+r}{1+|t-r|}\right) & (k=1), \\ \dfrac{C_k \varepsilon}{(1+t+r)^k} & (0<k<1), \end{cases} \tag{3.4}$$

(ii) *if* $n=2$,

$$\sum_{|\alpha|\le 2}|D_x^\alpha u^0(x,t)|\le\begin{cases}\dfrac{C_k\varepsilon}{\sqrt{(1+t+r)(1+|t-r|)}} & (k>1),\\ \dfrac{C_k\varepsilon\ln(2+|t-r|)}{\sqrt{(1+t+r)(1+|t-r|)}} & (k=1),\\ \dfrac{C_k\varepsilon}{\sqrt{1+t+r}(1+|t-r|)^{k-1/2}} & \left(\dfrac{1}{2}<k<1\right),\\ \dfrac{C_k\varepsilon}{\sqrt{1+t+r}}\left(1+\ln\dfrac{1+t+r}{1+|t-r|}\right) & \left(k=\dfrac{1}{2}\right),\\ \dfrac{C_k\varepsilon}{(1+t+r)^k} & \left(0<k<\dfrac{1}{2}\right),\end{cases} \tag{3.5}$$

where $r=|x|$, *and a constant* C_k *depends only on* k.

We next introduce the function

$$Z_n(|x|,t)=(1+t+|x|)^m(1+|t-|x||)^{1/p}(\ln(3+|t-|x||))^{1/p}(\ln(\ln(3+|t-|x||)))^{1/(p-1)}, \tag{3.6}$$

where

$$m=\begin{cases}1 & (n=3),\\ \dfrac{1}{2} & (n=2),\end{cases}$$

and define the norm for functions $u(x,t)$ which are continuous on $R^n\times[0,T)$

$$\|u\|_n=\sup_{\substack{x\in R^n\\ 0\le t<T}} Z_n(|x|,t)|u(x,t)|. \tag{3.7}$$

The following lemma provides the basic estimate for the existence proof.

LEMMA 3.2. *If* $p=p_0(n)$ $(n=2,3)$, *then there exists a constant* C_p *depending only on* p *such that*

$$\|L|u|^p\|_n\le C_p(\ln(3+T))^{1/p}(\ln(\ln(3+T)))^{1/(p-1)}\|u\|_n^p. \tag{3.8}$$

PROOF. Following [13], we first prove the lemma above in the case $n=3$. If w is spherically symmetric, i.e., $w(x,t)=w(r,t)$, then from (3.3) we can write the solution of $\partial_t^2u-\Delta u=w$ with zero data in the form

$$u(r,t)=\frac{1}{2r}\int_0^t\int_{|r-t+\tau|}^{r+t-\tau}\rho w(\rho,\tau)d\rho d\tau. \tag{3.9}$$

We denote (3.9) by $u=Pw$. We define the function

$$\tilde{u}(r, t)=\sup_{|x|=r}|u(x, t)|.$$

Then, we have

$$|(Lw)(x, t)|\leq(P\tilde{w})(r, t) \tag{3.10}$$

(see [2], p. 1470 and also [5], Lemma II). Thus, it is sufficient to estimate

$$(P\tilde{u}^{p})(r, t)=\frac{1}{2r}\int_{0}^{t}\int_{|r-t+\tau|}^{r+t-\tau}\rho\tilde{u}(\rho, \tau)^{p}d\rho d\tau.$$

We use the following lemma.

LEMMA 3.3. *Let* $k>1$. *Then,*

$$\frac{1}{2r}\int_{|r-t|}^{r+t}\frac{1}{(1+\rho)^{k}}d\rho\leq\frac{\max(1, k-1)\min(r, t)}{(k-1)r(1+t+r)(1+|t-r|)^{k-1}}$$

for all t, $r\geq 0$.

PROOF. See [2]. ■

To prove (3.8), we distinguish two cases, $r\geq t$ and $t\geq r$.

(a) $r\geq t$.

By (3.6) and (3.7), we have

$$(P\tilde{u}^{p})(r, t)\leq\frac{\|u\|_{3}^{p}}{2r}\int_{0}^{t}\int_{r-t+\tau}^{r+t-\tau}\frac{\rho}{(1+\tau+\rho)^{p}(1+\rho-\tau)}$$
$$\times\frac{1}{\ln(3+\rho-\tau)(\ln(\ln(3+\rho-\tau)))^{p/(p-1)}}d\rho d\tau.$$

Changing variables by

$$\alpha=\rho+\tau,\ \beta=\rho-\tau, \tag{3.11}$$

we obtain

$$\begin{aligned}(P\tilde{u}^{p})(r, t) &\leq\frac{\|u\|_{3}^{p}}{4r}\int_{r-t}^{r+t}\frac{1}{(1+\alpha)^{p-1}}\int_{r-t}^{\alpha}\frac{1}{(1+\beta)\ln(3+\beta)(\ln(\ln(3+\beta)))^{p/(p-1)}}d\beta d\alpha\\ &\leq\frac{C\|u\|_{3}^{p}}{r(\ln(\ln(3+r-t)))^{1/(p-1)}}\int_{r-t}^{r+t}\frac{1}{(1+\alpha)^{p-1}}d\alpha\\ &\leq\frac{C\|u\|_{3}^{p}(\ln(3+r))^{1/p}t}{r(1+t+r)(1+r-t)^{p-2}(\ln(3+r-t))^{1/p}(\ln(\ln(3+r-t)))^{1/(p-1)}}.\end{aligned}$$

We have used here Lemma 3.3. Since the function $\varphi(x)=x/(\ln(3+x))^{q}$ $(0<q<1)$ is increasing for $x\geq 0$ and $p-2=1/p$, we obtain

$$
\begin{aligned}
(P\tilde{u}^p)(r,t) &\le \frac{C\|u\|_3^p(\ln(3+t))^{1/p}}{(1+t+r)(1+r-t)^{p-2}(\ln(3+r-t))^{1/p}(\ln(\ln(3+r-t)))^{1/(p-1)}} \\
&\le \frac{C\|u\|_3^p(\ln(3+t))^{1/p}}{(1+t+r)(1+r-t)^{1/p}(\ln(3+r-t))^{1/p}(\ln(\ln(3+r-t)))^{1/(p-1)}}.
\end{aligned}
$$

(b) $t \ge r$.

We have

$$
\begin{aligned}
(P\tilde{u}^p)(r,t) \le \frac{\|u\|_3^p}{2r}\int_0^t\int_{|r-t+\tau|}^{r+t-\tau} & \frac{\rho}{(1+\tau+\rho)^p(1+|\tau-\rho|)} \\
&\times\frac{1}{\ln(3+|\rho-\tau|)(\ln(\ln(3+|\rho-\tau|)))^{p/(p-1)}}d\rho d\tau.
\end{aligned}
$$

By (3.11),

$$
\begin{aligned}
&(P\tilde{u}^p)(r,t) \\
&\le \frac{\|u\|_3^p}{4r}\int_{t-r}^{t+r}\int_{r-t}^{\alpha}\frac{1}{(1+\alpha)^{p-1}(1+|\beta|)\ln(3+|\beta|)(\ln(\ln(3+|\beta|)))^{p/(p-1)}}d\beta d\alpha \\
&= \frac{\|u\|_3^p}{4r}\int_{t-r}^{t+r}\frac{1}{(1+\alpha)^{p-1}}\int_{r-t}^{t-r}\frac{1}{(1+|\beta|)\ln(3+|\beta|)(\ln(\ln(3+|\beta|)))^{p/(p-1)}}d\beta d\alpha \\
&\quad + \frac{\|u\|_3^p}{4r}\int_{t-r}^{t+r}\frac{1}{(1+\alpha)^{p-1}}\int_{t-r}^{\alpha}\frac{1}{(1+\beta)\ln(3+\beta)(\ln(\ln(3+\beta)))^{p/(p-1)}}d\beta d\alpha \\
&= \frac{\|u\|_3^p}{2r}\int_{t-r}^{t+r}\frac{1}{(1+\alpha)^{p-1}}\int_0^{t-r}\frac{1}{(1+\beta)\ln(3+\beta)(\ln(\ln(3+\beta)))^{p/(p-1)}}d\beta d\alpha \\
&\quad + \frac{\|u\|_3^p}{4r}\int_{t-r}^{t+r}\frac{1}{(1+\alpha)^{p-1}}\int_{t-r}^{\alpha}\frac{1}{(1+\beta)\ln(3+\beta)(\ln(\ln(3+\beta)))^{p/(p-1)}}d\beta d\alpha \\
&\le \frac{C\|u\|_3^p}{r}\int_{t-r}^{t+r}\frac{1}{(1+\alpha)^{p-1}}d\alpha.
\end{aligned}
$$

We use Lemma 3.3 to obtain

$$
\begin{aligned}
&(P\tilde{u}^p)(r,t) \\
&\qquad \le \frac{C\|u\|_3^p(\ln(3+t))^{1/p}(\ln(\ln(3+t)))^{1/(p-1)}}{(1+t+r)(1+t-r)^{1/p}(\ln(3+t-r))^{1/p}(\ln(\ln(3+t-r)))^{1/(p-1)}}.
\end{aligned}
$$

Next, the proof of Lemma 3.2 for the case $n=2$ proceeds in the same way as that of Lemma 4.1 in [14] except that we need to use the following inequality instead of Lemma 4.3 in [14].

$$
\int_0^{\alpha}\frac{1}{\sqrt{a+t-\beta}\,(1+\beta)\ln(3+\beta)(\ln(\ln(3+\beta)))^k}d\beta \le \frac{C_k}{\sqrt{t+a}} \tag{3.12}
$$
$$
\text{for } k>1 \text{ and } 0\le\alpha\le t+a,
$$

where C_k is a constant depending only on k. Slight modification of the proof of Lemma 4.3 in [14] leads to (3.12).

This completes the proof of Lemma 3.2. ■

4 Proof of Theorem 2.1

Let X_n be the linear space defined by

$$X_n=\{u(x,t): D_x^\alpha u(x,t)\in C(R^n\times[0,T)),\ \|D_x^\alpha u\|_n<\infty \text{ for } |\alpha|\le 2\},\quad n=2,3.$$

We can verify easily that X_n is complete with respect to the norm

$$\|u\|_{X_n}=\sum_{|\alpha|\le 2}\|D_x^\alpha u\|_n.$$

We define the sequence of functions $\{u_n\}$ by

$$u_0=u^0,\quad u_{n+1}=u^0+LF(u_n).$$

Here we note that

$$k>\frac{2}{p-1}=\begin{cases}1+\dfrac{1}{p} & (n=3),\\ \dfrac{1}{2}+\dfrac{1}{p} & (n=2),\end{cases}$$
$$>\begin{cases}1 & (n=3),\\ \dfrac{1}{2} & (n=2).\end{cases}$$

Then, Lemma 3.1 yields

$$\|u^0\|_{X_n}\le C_k\varepsilon.$$

Hence, $u^0\in X_n$.

We now assume that ε is so small that

$$2^p p\lambda C_p(\ln(3+T))^{1/p}(\ln(\ln(3+T)))^{1/(p-1)}(C_k\varepsilon)^{p-1}\le 1 \text{ and } C_k\varepsilon\le\frac{1}{2},\tag{4.1}$$

where C_p is a constant given in Lemma 3.2. Then, we have

$$2^p p\lambda C_p(\ln(3+T))^{1/p}(\ln(\ln(3+T)))^{1/(p-1)}\|u^0\|_n^{p-1}\le 1 \text{ and } \|u^0\|_n\le\frac{1}{2}.\tag{4.2}$$

Therefore, as in [5], [2], we see that if u^0 satisfies (4.2), there exists a unique local solution of (2.1). The lower bound estimate (2.2) follows immediately from (4.1).

This completes the proof of Theorem 2.1. ■

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Department of Mathematics
Hokkaido University
Sapporo 060, Japan