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On the Fourier transform for operators on homogeneous Banach spaces

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1 Introduction

Let $\boldsymbol{T}=\boldsymbol{R}/2\pi\boldsymbol{Z}$ be the circle group. We may identify functions on $\boldsymbol{T}$ with 2π-periodic functions on $\boldsymbol{R}$. For $t\in\boldsymbol{T}$, let R_t denote the translation operator on functions on $\boldsymbol{T}$ given by $(R_tf)(s)=f(s-t)$. A homogeneous Banach space on $\boldsymbol{T}$(see [4]) is a linear subspace B of $L^1(\boldsymbol{T})$ having a norm $\|\cdot\|_B\geq\|\cdot\|_1$ under which it is a Banach space, and having the following properties:

(a) If $f\in B$ and $t\in\boldsymbol{T}$, then $R_tf\in B$ and $\|R_tf\|_B=\|f\|_B$.

(b) For any $f\in B$ and any $t\in\boldsymbol{T}$, $\lim_{s\to t}\|R_sf-R_tf\|_B=0$.

The space $L^p(\boldsymbol{T})$, $1\leq p<\infty$, and the space $C^n(\boldsymbol{T})$, $n\geq 0$, are typical examples of homogeneous Banach spaces on $\boldsymbol{T}$.

Throughout this paper, B will be a homogeneous Banach space on $\boldsymbol{T}$ and $\mathscr{L}(B)$ will denote the Banach algebra of all bounded linear operators on B with the usual operator norm $\|\cdot\|$. We call an operator $\boldsymbol{T}$ in $\mathscr{L}(B)$ almost invariant if $\lim_{t\to 0}\|TR_t-R_tT\|=0$. The set of almost invariant operators in $\mathscr{L}(B)$ will be denoted by $\mathscr{L}_{\#}(B)$.

Following DeLeeuw [1, 2], we define the Fourier transform of $T\in\mathscr{L}(B)$ to be the $\mathscr{L}(B)$-valued function $\hat{T}$ defined on $\boldsymbol{Z}$ by

$$\hat{T}(n)f=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-int}R_{-t}TR_tf\ dt \qquad (n\in\boldsymbol{Z}, f\in B)$$

and we call the formal series $T\sim\sum_{n=-\infty}^{\infty}\hat{T}(n)$ the Fourier series of T. As usual, we denote by $\sigma_n(T)$ the nth $C-1$ sum and by $S_n(T)$ the nth partial sum of the Fourier series of T. The basic result concerning the Fourier series is

PROPOSITION 1.1. ([1]) (a) *If* $T\in\mathscr{L}_{\#}(B)$, *then* $\|\sigma_n(T)-T\|\to 0$ *as* $n\to\infty$.

(b) *If* $T\in\mathscr{L}(B)$, *then for all* $f\in B$, $\|\sigma_n(T)f-Tf\|_B\to 0$ *as* $n\to\infty$.

As a corollary, we have an analogue of the Riemann-Lebesgue lemma.

COROLLARY 1.2. (a) *If* $T\in\mathscr{L}_{\#}(B)$, *then* $\|\hat{T}(n)\|\to 0$ *as* $|n|\to\infty$.
(b) *If* $T\in\mathscr{L}(B)$, *then for all* $f\in B$, $\|\hat{T}(n)f\|_B\to 0$ *as* $|n|\to\infty$.

This paper is devoted to a study of the Fourier transform for two classes of operators. In Section 2, we introduce a class of operators $Lip_\alpha(B)$ which is dense in $\mathscr{L}_{\#}(B)$ in the operator norm for $0<\alpha\leq 1$. We give some examples and establish several of their algebraic properties. We also show that $\|\hat{T}(n)\|=O(|n|^{-\alpha})$ for $T\in Lip_\alpha(B)$. Section 3 is concerned with the rates of convergence of $\sigma_n(T)$ and $S_n(T)$ for $T\in Lip_\alpha(B)$. By introducing the Fejér and Dirichlet kernels, we are able to prove the following results:

$$\|\sigma_n(T)-T\|=\begin{cases}O(n^{-\alpha}) & \text{if } 0<\alpha<1\\ O(n^{-1}\log n) & \text{if } \alpha=1\end{cases}$$

$$\|S_n(T)-T\|=O(n^{-\alpha}\log n) \qquad \text{if } 0<\alpha\leq 1.$$

In particular, the Fourier series of T converges to T in the operator norm. Finally, in Section 4, we restrict ourselves to the case where $B=L^2(\boldsymbol{T})$ and study the Fourier transform of positive operators on $L^2(\boldsymbol{T})$. We prove that T is a positive operator if and only if the sequence $\{\hat{T}(n)\}$ is positive-definite.

We mention that the problem of convergence of Fourier series for operators in the von Neumann-Schatten p-class has been studied by Huang in [**3**].

2 Operators of class $Lip_\alpha(B)$

DEFINITION 2.1. For a homogeneous Banach space B on $\boldsymbol{T}$, we denote by $\mathrm{Lip}_\alpha(B)$ the set of operators in $\mathscr{L}(B)$ which satisfy the Lipschitz condition of order α; that is, an operator $T\in\mathrm{Lip}_\alpha(B)$ if there is a constant M so that $\|T_t-T\|\leq M|t|^\alpha$ for all $t\in\boldsymbol{T}$, where $T_t=R_{-t}TR_t$.

REMARKS (a) Only the case $0<\alpha\leq 1$ is interesting. For, if $\alpha>1$, then T_t is differentiable in norm with derivative identically zero, so that T_t is constant. It turns out that T commutes with translation.

(b) It may be useful to notice that if for each $f\in B$, there is a constant M_f, depending only on f, so that $\|(T_t-T)f\|_B\leq M_f|t|^\alpha$ for all $t\in\boldsymbol{T}$, then $T\in Lip_\alpha(B)$. This follows immediately from the uniform boundedness principle.

We now give some concrete examples:

$T\in Lip_\alpha(B)$.

We next establish several algebraic properties for operators in $Lip_\alpha(B)$.

PROPOSITION 2.2. (a) *$Lip_\alpha(B)$ forms a subalgebra of $\mathscr{L}(B)$.*

(b) *If $T\in Lip_\alpha(B)$ and if z is in the resolvent set of T, then $(T-z)^{-1}\in Lip_\alpha(B)$.*

(c) *If $T\in Lip_\alpha(B)$ and if μ is a finite Borel measure on $\boldsymbol{T}$, then the convolution $\mu*T$ defined by $(\mu*T)f=\int_{-\pi}^{\pi}T_{-t}f\,d\mu(t)$ $(f\in B)$ belongs to $Lip_\alpha(B)$.*

PROOF. (a) Let S, $T\in Lip_\alpha(B)$. Then there is a constant M so that $\|S_t-S\|\le M|t|^\alpha$ and $\|T_t-T\|\le M|t|^\alpha$ for all $t\in\boldsymbol{T}$. Since $(ST)_t=S_tT_t$, we have, by the triangle inequality, $\|(ST)_t-ST\|\le\|S_t\|\|T_t-T\|+\|S_t-S\|\|T\|\le M|t|^\alpha(\|S\|+\|T\|)$ and so $ST\in Lip_\alpha(B)$. By a similar argument, we can prove that $cS+T\in Lip_\alpha(B)$ for all $c\in\boldsymbol{C}$.

(b) Since $[(T-z)^{-1}]_t=[(T-z)_t]^{-1}$, it follows that

$$\begin{aligned}[(T-z)^{-1}]_t-(T-z)^{-1}&=[(T-z)_t]^{-1}[(T-z)-(T-z)_t](T-z)^{-1}\\&=[(T-z)^{-1}]_t(T-T_t)(T-z)^{-1}.\end{aligned}$$

Thus, $\|[(T-z)^{-1}]_t-(T-z)^{-1}\|\le\|(T-z)^{-1}\|^2\|T-T_t\|$, which shows that $(T-z)^{-1}\in Lip_\alpha(B)$ if $T\in Lip_\alpha(B)$.

(c) It is easy to verify that $(\mu*T)_t=\mu*T_t$ and that $\|\mu*T\|\le\|\mu\|\|T\|$ for all $T\in\mathscr{L}(B)$, where $\|\mu\|$ is the total variation norm of μ. Thus, $\|(\mu*T)_t-(\mu*T)\|=\|\mu*(T_t-T)\|\le\|\mu\|\|T_t-T\|$. From this, our assertion follows. □

REMARK. Since the Fourier series of any $T\in\mathscr{L}_\#(B)$ is $C-1$ summable to T in the operator norm, and since $\hat{T}(n)\in Lip_1(B)$ for all n, it follows that $Lip_1(B)$ is dense in $\mathscr{L}_\#(B)$ in the operator norm.

As usual, we denote by B^* the dual space of B. For $B=L^p(\boldsymbol{T})$, $1<p<\infty$, B^* is canonically identified with $L^q(\boldsymbol{T})$, where $q=p/(p-1)$.

PROPOSITION 2.3. *Let $B=L^p(\boldsymbol{T})$, $1<p<\infty$. If $T\in Lip_\alpha(B)$, then its adjoint $T^*\in Lip_\alpha(B^*)$.*

PROOF. It is easy to show that $(R_t)^*=R_{-t}$. Hence $(T^*)_t=(R_{-t}TR_t)^*=(T_t)^*$, and so $\|(T^*)_t-T^*\|=\|(T_t-T)^*\|=\|T_t-T\|$. This proves the proposition. □

We conclude this section with a theorem which is sometimes useful in estimating the magnitude of $\|\hat{T}(n)\|$.

EXAMPLE 1. (the Fourier transform) Let $T\in\mathscr{L}(B)$. By a simple computation, we find that $[\hat{T}(n)]_t=R_{-t}\hat{T}(n)R_t=e^{int}\hat{T}(n)$ for all $n\in\boldsymbol{Z}$ and all $t\in\boldsymbol{T}$. Thus, $\|[\hat{T}(n)]_t-\hat{T}(n)\|=|e^{int}-1|\|\hat{T}(n)\|\leq|n||t|\|\hat{T}(n)\|$, which shows that $\hat{T}(n)\in Lip_1(B)$ for all n.

EXAMPLE 2. (multiplication operators) Let $B=L^p(\boldsymbol{T})$, $1\leq p<\infty$, and let $T\in\mathscr{L}(B)$ be defined by $Tf=\varphi f$, where φ is a Lipschitz function of order α on $\boldsymbol{T}$; that is, there is a constant M so that $|\varphi(x+h)-\varphi(x)|\leq M|h|^\alpha$ for all $x, h\in\boldsymbol{T}$. Then $T\in Lip_\alpha(B)$. For, we have $(T_tf)(x)=\varphi(x+t)f(x)$, hence

$$\|(T_t-T)f\|_p^p=\frac{1}{2\pi}\int_{-\pi}^{\pi}|\varphi(x+t)-\varphi(x)|^p|f(x)|^pdx\leq(M|t|^\alpha)^p\|f\|_p^p$$

and so $\|T_t-T\|\leq M|t|^\alpha$.

EXAMPLE 3. (Volterra integral operators) Let $B=L^p(\boldsymbol{T})$, $1\leq p<\infty$, and consider the operator $T\in\mathscr{L}(B)$ defined by $(Tf)(x)=\int_{-\pi}^{x}f(s)ds$ $(-\pi\leq x\leq\pi)$. We claim that $T\in Lip_{1/q}(B)$, where $q=p/(p-1)$ is the conjugate exponent of p. To prove it, we compute

$$[(TR_t-R_tT)f](x)=\int_{-\pi}^{x}f(s-t)ds-\int_{-\pi}^{x-t}f(s)ds=\int_{-\pi-t}^{-\pi}f(s)ds.$$

Thus, using the Hölder inequality, we obtain

$$\|(T_t-T)f\|_p=\|(TR_t-R_tT)f\|_p=\left|\int_{-\pi-t}^{-\pi}f(s)ds\right|\leq|t|^{1/q}\|f\|_p.$$

EXAMPLE 4. (integral operators with periodic kernels) Let $k:\boldsymbol{R}\times\boldsymbol{R}\to\boldsymbol{C}$ be continuous and $(2\pi\times2\pi)$-periodic, and suppose that there is a constant M so that

$$|k(x+t, y+t)-k(x, y)|\leq M|t|^\alpha \quad \text{for all } x, y, t\in\boldsymbol{R}. \tag{1}$$

Let $B=L^p(\boldsymbol{T})$, $1\leq p<\infty$, and let $T\in\mathscr{L}(B)$ be defined by $(Tf)(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}k(x, y)f(y)dy$. Then, using the periodicity of k and f, we find $(T_tf)(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}k(x+t, y+t)f(y)dy$. Thus, by (1), we have

$$\begin{aligned}|[(T_t-T)f](x)|&\leq\frac{1}{2\pi}\int_{-\pi}^{\pi}|k(x+t, y+t)-k(x, y)||f(y)|dy\\&\leq\frac{1}{2\pi}\int_{-\pi}^{\pi}M|t|^\alpha|f(y)|dy=M|t|^\alpha\|f\|_1.\end{aligned}$$

It follows that $\|(T_t-T)f\|_p\leq M|t|^\alpha\|f\|_1$, so by the preceding remark (b), THEOREM 2.4. *If* $T\in\mathscr{L}(B)$, *then* $\|\hat{T}(n)\|\leq\frac{1}{2}\|T_{\pi/n}-T\|$ *for all* $n\neq 0$.

PROOF. Let $f\in B$. Then we have

$$\hat{T}(n)f=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-int}T_tf\ dt=\frac{-1}{2\pi}\int_{-\pi}^{\pi}e^{-in(t+\pi/n)}T_tf\ dt$$
$$=\frac{-1}{2\pi}\int_{-\pi}^{\pi}e^{-int}T_{t-\pi/n}f\ dt$$

so that

$$\hat{T}(n)f=\frac{1}{4\pi}\int_{-\pi}^{\pi}e^{-int}(T_t-T_{t-\pi/n})f\ dt.$$

Using the fact that $\|T_t-T_s\|=\|T_{t-s}-T\|$, we find that

$$\|\hat{T}(n)f\|_B\leq\frac{1}{4\pi}\int_{-\pi}^{\pi}\|T_t-T_{t-\pi/n}\|\|f\|_B\ dt=\frac{1}{4\pi}\int_{-\pi}^{\pi}\|T_{\pi/n}-T\|\|f\|_B\ dt$$
$$=\frac{1}{2}\|T_{\pi/n}-T\|\|f\|_B.$$

This completes the proof. □

COROLLARY 2.5. *If* $T\in Lip_\alpha(B)$, *then* $\|\hat{T}(n)\|=O(|n|^{-\alpha})$.

3 Convergence of Fourier series

Since every operator in $Lip_\alpha(B)$ is almost invariant, the Fourier transform of $T\in Lip_\alpha(B)$ can be defined directly by

$$\hat{T}(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-int}T_t\ dt$$

also, the Fourier series of T is $C-1$ summable to T in the operator norm (see [**1**]). In this section, we wish to study the rates of convergence of Fourier series for operators in $Lip_\alpha(B)$. Recall that

$$\sigma_n(T)=\sum_{j=-n}^{n}\left(1-\frac{|j|}{n+1}\right)\hat{T}(j)\quad\text{and}\quad S_n(T)=\sum_{j=-n}^{n}\hat{T}(j).$$

THEOREM 3.1. *If* $T\in Lip_\alpha(B)$, *then*

$$\|\sigma_n(T)-T\|=\begin{cases}O(n^{-\alpha}) & \text{if } 0<\alpha<1,\\ O(n^{-1}\log n) & \text{if } \alpha=1.\end{cases}\tag{2}$$

PROOF. By introducing the Fejér kernel:

$$K_n(t)=\sum_{j=-n}^{n}\left(1-\frac{|j|}{n+1}\right)e^{ijt}=\frac{1}{n+1}\left\{\sin\frac{(n+1)t}{2}\Big/\sin\frac{t}{2}\right\}^2$$

(see, e.g., [4]), we can write

$$\sigma_n(T)-T=\frac{1}{2\pi}\int_{-\pi}^{\pi}K_n(t)(T_t-T)\,dt.$$

Using the facts that $0\le K_n(t)\le \min\{n+1, \pi^2/(n+1)t^2\}$ and $K_n(t)=K_n(-t)$, it follows that

$$\begin{aligned}\|\sigma_n(T)-T\| &\le \frac{1}{2\pi}\int_{-\pi}^{\pi}K_n(t)\|T_t-T\|dt\le\frac{M}{2\pi}\int_{-\pi}^{\pi}K_n(t)|t|^{\alpha}\,dt\\ &=\frac{M}{\pi}\left(\int_0^{1/n}+\int_{1/n}^{\pi}\right)K_n(t)t^{\alpha}\,dt\\ &\le\frac{M}{\pi}\left[\int_0^{1/n}(n+1)t^{\alpha}\,dt+\int_{1/n}^{\pi}\frac{\pi^2}{n+1}t^{\alpha-2}\,dt\right]\\ &=\frac{M(n+1)}{\pi(\alpha+1)n^{\alpha+1}}+\frac{M\pi}{n+1}\int_{1/n}^{\pi}t^{\alpha-2}\,dt\\ &=\text{RHS of (2).}\end{aligned}$$

THEOREM 3.2. *If* $T\in Lip_{\alpha}(B)$, $0<\alpha\le 1$, *then* $\|S_n(T)-T\|=O(n^{-\alpha}\log n)$. *In particular, the Fourier series of T converges to T in the operator norm.*

PROOF. By introducing the Dirichlet kernel:

$$D_n(t)=\sum_{j=-n}^{n}e^{ijt}=\sin\left(n+\frac{1}{2}\right)t\Big/\sin\frac{t}{2}$$

(see, e.g., [4]), we can write

$$S_n(T)=\frac{1}{2\pi}\int_{-\pi}^{\pi}D_n(t)T_t\,dt.$$

Let

$$D_n^*(t)\equiv\frac{1}{2}[D_n(t)+D_{n-1}(t)]=\sin(nt)\cot\frac{t}{2} \tag{3}$$

$$S_n^*(T)\equiv\frac{1}{2\pi}\int_{-\pi}^{\pi}D_n^*(t)T_t\,dt=S_n(T)-\frac{1}{2}[\hat{T}(n)+\hat{T}(-n)]. \tag{4}$$

Then, since $(1/2\pi)\int_{-\pi}^{\pi}D_n(t)dt=1$ and D_n is an even function, we have

$$\begin{aligned}S_n^*(T)-T&=\frac{1}{2\pi}\int_{-\pi}^{\pi}D_n^*(t)(T_t-T)dt\\ &=\frac{1}{2\pi}\int_0^{\pi}D_n^*(t)(T_t+T_{-t}-2T)dt.\end{aligned}$$

Thus, if we put $T(t)=\cot(t/2)(T_t+T_{-t}-2T)$ and $\theta=\pi/n$, then by (3),

$$\begin{aligned}4\pi[S_n^*(T)-T]&=2\int_0^\pi T(t)\sin nt\ dt\\&=\int_0^\pi T(t)\sin nt\ dt-\int_{-\theta}^{\pi-\theta} T(t+\theta)\sin nt\ dt\\&\equiv I_1+I_2+I_3+I_4\end{aligned}\tag{5}$$

where

$$\begin{aligned}I_1&=\int_0^\theta T(t)\sin nt\ dt,\\I_2&=-\int_{-\theta}^\theta T(t+\theta)\sin nt\ dt=\int_0^{2\theta} T(t)\sin nt\ dt,\\I_3&=\int_\theta^{\pi-\theta}[T(t)-T(t+\theta)]\sin nt\ dt,\\I_4&=\int_{\pi-\theta}^\pi T(t)\sin nt\ dt.\end{aligned}$$

Since $|\sin nt\ \cot(t/2)|\leq 2n$ and $\|T_t+T_{-t}-2T\|\leq\|T_t-T\|+\|T_{-t}-T\|\leq 2M|t|^\alpha$, it follows that

$$\|I_1\|+\|I_2\|\leq 8nM\int_0^{2\theta} t^\alpha\ dt=O(n^{-\alpha}).$$

For $n\geq 2$ and $\pi-\theta\leq t\leq\pi$, we have $|\cot(t/2)|\leq 1$ so that

$$\|I_4\|\leq\int_{\pi-\theta}^\pi\|T_t+T_{-t}-2T\|dt\leq 4\theta\|T\|=O(n^{-1}).$$

It remains to estimate $\|I_3\|$. For this, we write $I_3=I_{3,1}+I_{3,2}$, where

$$\begin{aligned}I_{3,1}&=\int_\theta^{\pi-\theta}\left\{T(t)\tan\frac{t}{2}-T(t+\theta)\tan\frac{t+\theta}{2}\right\}\sin nt\ \cot\frac{t}{2}dt\\&=\int_\theta^{\pi-\theta}(T_t-T_{t+\theta}+T_{-t}-T_{-t-\theta})\sin nt\ \cot\frac{t}{2}dt,\\I_{3,2}&=\int_\theta^{\pi-\theta}T(t+\theta)\left\{\tan\frac{t+\theta}{2}-\tan\frac{t}{2}\right\}\sin nt\ \cot\frac{t}{2}dt\\&=\int_\theta^{\pi-\theta}(T_{t+\theta}+T_{-t-\theta}-2T)\sin nt\left\{\sin\frac{\theta}{2}\Big/\sin\frac{t}{2}\sin\frac{t+\theta}{2}\right\}dt.\end{aligned}$$

Since $|\cot(t/2)|\leq 2t^{-1}$ in $(0,\pi)$ and $\|T_t-T_s\|\leq M|t-s|^\alpha$, we have

$$\|I_{3,1}\|\leq 4M\theta^\alpha\int_\theta^{\pi-\theta}t^{-1}\ dt=O(n^{-\alpha}\log n).$$

On the other hand, using the fact that $2x/\pi<\sin x<x$ in $(0,\pi/2)$, we see that for $\theta\leq t\leq\pi-\theta$, $|\sin\frac{\theta}{2}\Big/\sin\frac{t}{2}\sin\frac{t+\theta}{2}|\leq\pi^2\theta/2t^2$. Thus,

$$\|I_{3,2}\|\leq M\pi^2\theta\int_\theta^{\pi-\theta}(t+\theta)^\alpha t^{-2}\,dt\leq 2^\alpha M\pi^2\theta\int_\theta^{\pi-\theta}t^{\alpha-2}\,dt$$
$$=\begin{cases}O(n^{-1}\log n) & \text{if } \alpha=1,\\ O(n^{-\alpha}) & \text{if } 0<\alpha<1.\end{cases}$$

Combining these estimates with (5), we find that $\|S_n^*(T)-T\|=O(n^{-\alpha}\log n)$, so by (4) and Corollary 2.5,

$$\|S_n(T)-T\|\leq\|S_n^*(T)-T\|+\frac{1}{2}\|\hat{T}(n)+\hat{T}(-n)\|=O(n^{-\alpha}\log n).$$

□

REMARK. Theorems 3.1 and 3.2 contain some classical versions on the rates of convergence of Fourier series for Lipschitz functions (see [**4**], p. 22 and [**5**], p. 64): Let φ be a Lipschitz function of order α on $\boldsymbol{T}$. If $B=L^1(\boldsymbol{T})$, and $T\in\mathscr{L}(B)$ is defined by $Tf=\varphi f$, then $T\in Lip_\alpha(B)$, and $\hat{T}(n)$ is the operator of multiplication by $\hat{\varphi}(n)e^{in\cdot}$, where $\hat{\varphi}(n)$ is the nth Fourier coefficient of φ. Set

$$\sigma_n(\varphi, t)=\sum_{j=-n}^{n}\left(1-\frac{|j|}{n+1}\right)\hat{\varphi}(j)e^{ijt}\ \text{ and }\ S_n(\varphi, t)=\sum_{j=-n}^{n}\hat{\varphi}(j)e^{ijt}.$$

Then $\sigma_n(T)-T$ and $S_n(T)-T$ are the operators of multiplication by $\sigma_n(\varphi, t)-\varphi(t)$ and $S_n(\varphi, t)-\varphi(t)$, respectively. Thus, Theorems 3.1 and 3.2 give that

$$\sup_{t\in\boldsymbol{T}}|\sigma_n(\varphi, t)-\varphi(t)|=\begin{cases}O(n^{-\alpha}) & \text{if } 0<\alpha<1,\\ O(n^{-1}\log n) & \text{if } \alpha=1.\end{cases}$$

and

$$\sup_{t\in\boldsymbol{T}}|S_n(\varphi, t)-\varphi(t)|=O(n^{-\alpha}\log n).$$

4 Positive operators on $L^2(\boldsymbol{T})$

An operator $T\in\mathscr{L}(L^2(\boldsymbol{T}))$ is called positive if $\langle Tf, f\rangle\geq 0$ for all $f\in L^2(\boldsymbol{T})$. This section is devoted to studying the Fourier transform for positive operators on $L^2(\boldsymbol{T})$. We begin with a definition.

DEFINITION 4.1. Let $\{T_n\}_{n=-\infty}^{\infty}$ be a sequence of operators on $L^2(\boldsymbol{T})$. We say that $\{T_n\}$ is positive-definite if for any sequence of functions $\{f_n\}$ in $L^2(\boldsymbol{T})$ having only a finite number of terms different from zero we have $\sum_{n,m}\langle T_{n-m}f_n, f_m\rangle\geq 0$.

PROOF. (a) Since T is positive, we have for all $f \in L^2(\boldsymbol{T})$ that

$$\langle \hat{T}(0)f, f\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \langle T_t f, f\rangle dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \langle T(R_t f), R_t f\rangle dt \geq 0.$$

So, $\hat{T}(0)$ is positive.

(b) We first observe that if $S \in \mathscr{L}(L^2(\boldsymbol{T}))$, then $(S^*)\hat{}(n) = [\hat{S}(-n)]^*$ for all n ([**2**]). Since T is positive, T is self-adjoint so

$$[\hat{T}(-n)]^* = \hat{T}(n) \quad \text{for all } n. \tag{7}$$

By Theorem 4.2, we have $\sum_{i,j} \langle (\hat{T}i-j) f_i, f_j\rangle \geq 0$ for any sequence of functions $\{f_i\}$ having only a finite number of terms different from zero. Thus, if we put

$$f_i = \begin{cases} f & \text{for } i=0 \\ ag & \text{for } i=n \\ bg & \text{for } i=m \\ 0 & \text{otherwise} \end{cases} \qquad (a, b \in \boldsymbol{C}\,;\, f,\ g \in L^2(\boldsymbol{T}))$$

and use (7), a simple computation then gives

$$\begin{aligned} &2\mathrm{Re}\{a\langle \hat{T}(n)g, f\rangle + b\langle \hat{T}(m)g, f\rangle + b\bar{a}\langle \hat{T}(m-n)g, g\rangle\} \\ &\quad + \langle \hat{T}(0)f, f\rangle + (|a|^2+|b|^2)\langle \hat{T}(0)g, g\rangle \geq 0. \end{aligned} \tag{8}$$

To prove (b), we take $b=0$ and choose a with $|a|=1$ so that $a\langle \hat{T}(n)g, f\rangle = -|\langle \hat{T}(n)g, f\rangle|$. Then (8) simplifies to

$$\langle \hat{T}(0)f, f\rangle + \langle \hat{T}(0)g, g\rangle \geq 2|\langle \hat{T}(n)g, f\rangle|.$$

Since this is true for any $f, g \in L^2(\boldsymbol{T})$, we conclude that $\|\hat{T}(n)\| \leq \|\hat{T}(0)\|$.

(c) Let $x \in \boldsymbol{R}$. If we take $b=-a$ and choose a with $|a|=|x|$ so that

$$a\langle [\hat{T}(n) - \hat{T}(m)]g, f\rangle = x|\langle [\hat{T}(n) - \hat{T}(m)]g, f\rangle|,$$

then (8) becomes

$$\begin{aligned} &2\{\langle \hat{T}(0)g, g\rangle - \mathrm{Re}\langle \hat{T}(m-n)g, g\rangle\}x^2 \\ &\quad + 2|\langle [\hat{T}(n) - \hat{T}(m)]g, f\rangle|x + \langle \hat{T}(0)f, f\rangle \geq 0. \end{aligned} \tag{9}$$

As a result, the discriminant of the quadratic polynomial (9) in x cannot be positive. Thus,

$$\begin{aligned} &4|\langle [\hat{T}(n) - \hat{T}(m)]g, f\rangle|^2 \\ &\quad \leq 8\langle \hat{T}(0)f, f\rangle\{\langle \hat{T}(0)g, g\rangle - \mathrm{Re}\langle \hat{T}(m-n)g, g\rangle\} \\ &\quad = 8\langle \hat{T}(0)f, f\rangle\{\mathrm{Re}\langle [\hat{T}(0) - \hat{T}(m-n)]g, g\rangle\} \\ &\quad \leq 8\|\hat{T}(0)\|\|\hat{T}(0) - \hat{T}(m-n)\|\|f\|^2\|g\|^2. \end{aligned}$$

THEOREM 4.2. *T is a positive operator on $L^2(\boldsymbol{T})$ if and only if the sequence $\{\hat{T}(n)\}$ is positive-definite.*

PROOF. Suppose T is positive, and let $\{f_n\}$ be a sequence of functions in $L^2(\boldsymbol{T})$ having only a finite number of terms different from zero. Then we have

$$\begin{aligned}2\pi\sum_{n,m}\langle\hat{T}(n-m)f_n, f_m\rangle &= \sum_{n,m}\int_{-\pi}^{\pi}e^{-i(n-m)t}\langle T_tf_n, f_m\rangle dt\\ &= \int_{-\pi}^{\pi}\langle T_t(\sum_n e^{-int}f_n), \sum_m e^{-imt}f_m\rangle dt\\ &= \int_{-\pi}^{\pi}\langle TR_t(\sum_n e^{-int}f_n),\ R_t(\sum_n e^{-int}f_n)\rangle dt\geq 0\end{aligned}$$

because T is positive. This shows that $\{\hat{T}(n)\}$ is positive-definite.

Conversely, suppose $\{\hat{T}(n)\}$ is positive-definite. For any $f\in L^2(\boldsymbol{T})$, we have

$$\langle\sigma_k(T)f, f\rangle=\sum_{j=-k}^{k}\left(1-\frac{|j|}{k+1}\right)\langle\hat{T}(j)f, f\rangle=\frac{1}{k+1}\sum_{n,m=1}^{k+1}\langle\hat{T}(n-m)f, f\rangle. \tag{6}$$

Thus, if we put

$$f_n=\begin{cases}f & \text{for } 1\leq n\leq k+1\\ 0 & \text{otherwise}\end{cases}$$

then (6) becomes

$$\langle\sigma_k(T)f, f\rangle=\frac{1}{k+1}\sum_{n,m}\langle\hat{T}(n-m)f_n, f_m\rangle.$$

Since $\{\hat{T}(n)\}$ is positive-definite, it follows that $\langle\sigma_k(T)f, f\rangle\geq 0$ for all $k\in\boldsymbol{N}$ and all $f\in L^2(\boldsymbol{T})$. Thus, by Proposition 1.1. (b), we conclude that

$$\langle Tf, f\rangle=\lim_{k\to\infty}\langle\sigma_k(T)f, f\rangle\geq 0.$$

So, T is a positive operator. □

Finally, we give a proposition which indicates some facts concerning the Fourier transform of a positive operator.

PROPOSITION 4.3. *Let T be a positive operator on $L^2(\boldsymbol{T})$. Then*
(a) *$\hat{T}(0)$ is positive.*
(b) *$\|\hat{T}(n)\|\leq\|\hat{T}(0)\|$ for all n.*
(c) *$\|\hat{T}(n)-\hat{T}(m)\|^2\leq 2\|\hat{T}(0)\|\|\hat{T}(0)-\hat{T}(m-n)\|$ for all m, n.*

Since this holds for all $f, g \in L^2(\boldsymbol{T})$, it follows that

$$\|\hat{T}(n)-\hat{T}(m)\|^2 \leq 2\|\hat{T}(0)\|\|\hat{T}(0)-\hat{T}(m-n)\|.$$

□

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References

[1] K. DELEEUW, An Harmonic Analysis for Operators I: Formal Properties, Illinois J. Math. **19** (1975), 593-606.
[2] K. DELEEUW, An Harmonic Analysis for Operators II: Operators on Hilbert Space and Analytic Operators, Illinois J. Math. **21** (1977), 164-175.
[3] M-J. HUANG, Harmonic Analysis for Operators in Homogeneous Classes, Bull. Inst. Math. Acad. Sinica, **21** (1993), 115-130.
[4] Y. KATZNELSON, An Introduction to Harmonic Analysis, Wiley, New York, 1968.
[5] A. ZYGMUND, Trigonometric Series, Vol. I, Cambridge Univ. Press, 1959.

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