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On a division ring with discrete valuation

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Throughout this paper A will be a division ring with non-trivial valuation v, and C will be the center of A. A has the completion with respect to the v-topology, which is also a division ring. We will denote it by A^*. For each division subring B of A the closure of B in A^* with respect to the v-topology will be also denoted by B^*. B^* is isomorphic to the completion of B as topological ring.

The aim of this paper is to show that B^* coincides with the double centralizer of B in A^* for each division C-subalgebra B of A, in the case where A is finite over C and v is discrete.

In this paper we will use the same terminology as [4] and [6]. In particular for each division subring B of A we write

$$O(B)=\{x\in B|v(x)\leqq 1\},\ P(B)=\{x\in B|v(x)<1\}.$$

Let v be non-archimedean and B an arbitrary division subring of A. $O(B)$ is a local ring with the maximal ideal $P(B)$. Hence $O(B)/P(B)$ is a division ring, which will be denoted by $E(B)$. Write $E(B)=K$ and $E(A)=E$. Then K is a division subring of E. We will write $f_r(A/B)=[E:K]_r$, $f_l(A/B)=[E:K]_l$ and $e(A/B)=[v(A^\circ):v(B^\circ)]$, where A° and B° are the unit groups of A and B, respectively. In the case where $[E:K]_l=[E:K]_r$, we will write $f(A/B)$ in stead of $f_r(A/B)$ or $f_l(A/B)$. Note that we have $e(A^*/A)=f(A^*/A)=1$ by Proposition 17.4 and Corollary 17.4 b [4]. Furthermore the Domination Principle, that is, $v(x)<v(y)$ implies $v(x+y)=v(y)$, holds also for a division ring with non-Archimedean valuation (See § 17.2 [4]).

The next lemma is well known in the case where A is a commutative field, and holds also in the case where $v|B$ is trivial

LEMMA 1. *Let A, B and v be as above, then we have $e(A/B)f_r(A/B)\leqq[A:B]_r$. If $[A:B]_r<\infty$, both $e(A/B)$ and $f_r(A/B)$ are finite.*

PROOF. Since $v(ab)=v(a)v(b)=v(b)v(a)=v(ba)$ for any a, $b\in A$, and the Domination Principle holds for A, we can follow the same lines as the proof of Theorem 4.5 Chap. 2 [2].

Let v be discrete, and write $O=O(A)$ and $P=P(A)$. Then there exists $z\in P$ such that $P=zO=Oz$. Such z is called a uniformizer at v. Let $v(z)=d$. Then $0<d<1$, and we have $v(A^{\circ})=\{d^n|n\in Z\}$, where Z is the set of rational integers (See § 17.5 [4]). For each $n\in Z$ we can choose an element $z_n\in A$ such that $v(z_n)=d^n$. On the other hand we can obtain a set Γ which contains 0 and consists of the representatives in O of the cosets of P, that is, Γ satisfies the following conditions;

(γ_1) $0\in\Gamma\subset O$.
(γ_2) If a, $b\in\Gamma$ and $a\neq b$, then $a\not\equiv b$ (P).
(γ_3) For each $x\in O$ there exists $c\in\Gamma$ such that $x\equiv c$ (P).

The next two lemmas are also well known in the case of commutative field (See Theorems 4.6 and 4.7 Chap. 2 [2] and Exercise 3 § 17.5 [4]).

LEMMA 2. *Assume that v is discrete and A is complete. Let z, $\{z_n\}$ and Γ be as above. Then each element a of A has a unique Laurent series representation, namely, $a=\sum a_n z_n$, were $a_n\in\Gamma$ and $a_n=0$ for almost all $0>n\in Z$, in particular we have $a=\sum a_n z^n$ with $a_n\in\Gamma$, $n>k$ for some $k\in Z$.*

PROOF. Completely same as the proof of Theorem 4.6 [2].

LEMMA 3. *Let F be a commutative field and A an F-algebra. Assume furthermore that v is discrete, $v|F$ is non-trivial, F is complete and $f(A/F)=f<\infty$. Then A is complete, and we have $e(A/F)f(A/F)=[A:F]<\infty$.*

PROOF. First assume that A is complete. We will show the equality by the same methods as the proof of Theorem 4.7 Chap. 2 [2]. Let z and c be uniformizers at v and $v|F$, respectively, and $v(z)=d$. Then $v(c)=d^e$ for some natural number e, and we have $e=e(A/F)<\infty$. Moreover the set $\{c^i z^j \mid i, j\in Z, 0\leqq j<e\}$ has the same condition as the set $\{z_n\}$ in Lemma 2. On the other hand by our assumption there exist $w_1, w_2, \cdots, w_f$ in $O(A)$ such that $\bar{w}_1, \bar{w}_2, \cdots, \bar{w}_f$ form a basis of $E(A)$ over $E(F)$, and we can construct a set Γ satisfying the conditions (γ_1), (γ_2) and (γ_3) as a subset of the set $\{\sum k_i w_i \mid k_i\in O(F), 1\leqq i\leqq f\}$. Then since A is complete for each $a\in A$ we can write by Lemma 2

$$a=\textstyle\sum_{i,j,k} a_{ijk} w_i c^j z^k,\ a_{ijk}\in O(F),\ i, j, k\in Z,\ 1\leqq i\leqq f,\ 0\leqq k<e.$$

If $v(a)=d^n$, we have $a_{ijk}=0$ for all $j<[n/e]$ by the Domination Principle, where [ ] is the Gauss symbol. Hence $b_{ik}=\sum_j a_{ijk} c^j$ is an element of F, since F is complete. But we have $a=\sum b_{ik}w_i z^k$, since A is an F-algebra. Therefore we have $[A:F]\leqq ef$. By Lemma 1 we have $[A:F]\geqq ef$. Hence we have $[A:F]=ef<\infty$. Next assume that A may not be complete. But A^* is also an F-algebra by Lemma 1.4 [6]. Since $e(A^*/A)=f(A^*/A)=1$ we have

$$e(A^*/F)=e(A^*/A)e(A/F)=e(A/F)=e \text{ and}$$
$$f(A^*/F)=f(A^*/A)f(A/F)=f(A/F)=f.$$

Then by Lemma 1 and the above argument we have $[A:F]\leqq[A^*:F]=ef\leqq[A:F]$. Thus we have $[A:F]=[A^*:F]$ and $A=A^*$.

Now suppose that A is finite dimensional over C. Then for any division C-subalgebra B of A $V_A(B)$ is a finite dimensional division C-subalgebra, where $V_A(B)$ is the centralizer of B in A, and we have $B=V_A(V_A(B))$ (See e. g., Theorem 12.7 [4]). Therefore A is an H-separable extension of every division C-subalgebra B of A by Theorem 1 [5]. Moreover $\{A/B, A^*/B''\}$ have the centralizer property in the sense of [6], and we have $[A:B]=[A^*:B'']$ and $B''\supset B^*$, where $B''=V_{A^*}(V_{A^*}(B))$ by Theorem 1.3 (3), (4) [6]. In particular C^* is contained in the center of A^*.

Now we are ready to have our main theorem.

THEOREM 1. *Let A be a division ring with non-trivial discrete valuation v. Assume A is finite dimensional over its center C. Then we have*

(1) *The closure C^* of C in A^* coincides with the center of A^*, and we have $[A^*:C^*]=[A:C]$.*

(2) *For any division C-subalgebra B of A, we have $B^*=V_{A^*}(V_{A^*}(B))$, $A^*=B^*A=AB^*$, $B^*\cap A=B$ and $[A:B]=[A^*:B^*]=e(A/B)f(A/B)=e(A^*/B^*)f(A^*/B^*)$.*

PROOF. (1). Write $C''=V_{A^*}(V_{A^*}(C))$. C'' coincides with the center of A^* by Proposition 1.4 [6] . Since v is non-archimedean, we have $e(C^*/C)=e(A^*/A)=1$ and $f(A^*/A)=f(C^*/C)=1$. Then since

$$e(A^*/C)=e(A^*/C^*)e(C^*/C)=e(A^*/A)e(A/C),$$

we have $e(A/C)=e(A^*/C^*)$. Similarly we have $f(A^*/C^*)=f(A/C)<\infty$. On the other hand, $v|C$ and $v|C^*$ are non-trivial, otherwise by Lemma 1 we have

$$\aleph_0=|v(A^\circ)|=e(A/C)\leqq e(A/C)f(A/C)\leqq[A:C]<\infty$$

a contradiction. Then by Lemmas 1 and 3 we have

$$[A^*: C^*]=e(A^*/C^*)f(A^*/C^*)=e(A/C)f(A/C)\leqq[A: C].$$

But by Theorem 1.7 (3) [6] we have $[A: C]=[A^*: C'']\leqq[A^*: C^*]$. Hence we have $[A^*: C^*]=[A^*: C'']$ and $C^*=C''$.

(2). Put $B''=V_{A^*}(V_{A^*}(B))$. B'' is complete, and $[A^*: B'']=[A: B]$ by Theorem 1.7 (3) and (4) [6] On the other hand since $e(B^*/B)=f(B^*/B)=1$, we have

$$e(A^*/B^*)f(A^*/B^*)=e(A/B)f(A/B)\leqq[A: B]$$
$$e(B^*/C^*)f(B^*/C^*)=e(B/C)f(B/C)\leqq[B: C].$$

Where the inequalities $\leqq$' s are due to Lemma 1. Then we have

$$\begin{aligned}[A^*: C^*]&=e(A^*/C^*)f(A^*/C^*)\\&=e(A^*/B^*)e(B^*/C^*)f(A^*/B^*)f(B^*/C^*)\\&\leqq[A: B][B: C]=[A: C]=[A^*: C^*].\end{aligned}$$

This means that $e(A/B)f(A/B)=[A: B]$ and $e(B/C)f(B/C)=[B: C]$. The latter equality shows $[B: C]=e(B^*/C^*)f(B^*/C^*)=[B^*: C^*]$. Then $[A^*: B'']=[A: B]=[A^*: B^*]$ and we have $B^*=B''$. The proof of the remaining part is due to Theorem 1.7 [6].

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