Hokkaido Mathematical Journal Vol. 23 (1994) p. 383-398

Notes on spatial representations of graphs

Toshiki ENDO and Takashi OTSUKI
(Received June 24, 1993)

§ 1. Introduction

Throughout this paper, every graph is assumed to be finite, simple, and connected. We refer to [2] for basic terminology and notation of graph theory. We regard a graph as a topological space in the natural way. The image of a tame embedding of a graph G into the 3-dimensional Euclidean space R^3 is called a *spatial representation* of G. We will denote a spatial representation of a graph G always by $\overline{G}$.

In [5], Kobayashi defined several kinds of spatial representations of graphs and discussed their properties from the knot-theoretic point of view (see [3] for knot theory).

DEFINITION 1. Let G be a graph and let $\mathscr{C}=\{C_1, C_2, \cdots, C_n\}$ be a set of cycles in G that represents a basis of $H_1(G;Z)$. A spatial representation $\overline{G}$ of G is *locally unknotted* (with respect to $\mathscr{C}$) if there exist internally disjoint disks $D_1, D_2, \cdots, D_n$ in R^3 such that $D_i \cap \overline{G}=\partial D_i=\overline{C_i}$ for $i=1, 2, \cdots, n$.

Kobayashi showed that any complete graph has a locally unknotted spatial representation, but he mentioned nothing about other class of graphs. In Section 2, we will show that any graph has a locally unknotted spatial representation.

DEFINITION 2. A spatial representation $\overline{G}$ of a graph G is *globally unknotted* if $\overline{C}$ is a trivial knot for every cycle C in G.

For a spatial representation of the complete graph, Kobayashi established several connections between local unknottedness and global unknottedness. In Section 3, we will present another relationship between them.

For $n \geq 0$, a subspace of R^3 consisting of a line (called a *binder*) and n distinct half-planes (called *sheets*) with the line as their common boundary is called a *book with n sheets*. A graph containing a Hamilton path is called *pseudo-Hamiltonian*. For the class of pseudo-Hamiltonian graphs, Kobayashi defined the folllowing spatial representation.

DEFINITION 3. Let G be a pseudo-Hamiltonian graph and let P be a Hamilton path in G. A spatial representation $\bar{G}$ is called a *book presentation* with n sheets (with respect to P) if $\bar{G}$ satisfies that P is embedded into the binder, any edge of $E(G)-E(P)$ is embedded into exactly one sheet, and at least one edge of $E(G)-E(P)$ is embedded into any sheet.

As an example, consider the complete graph with seven vertices. Fig. 1 illustrates a book presentation with 4 sheets. Notice that $\bar{e}$ can be embedded into any other sheet.

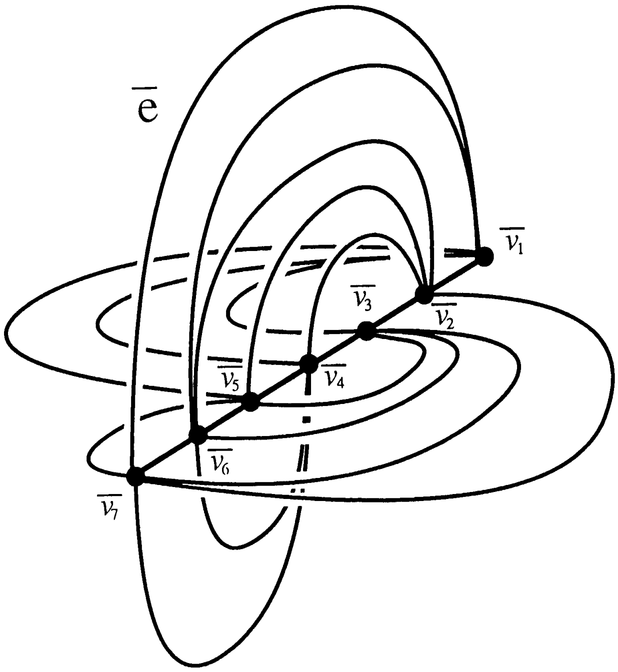

Fig. 1

Let G be a pseudo-Hamiltonian graph and let P be a Hamilton path in G. It is easy to show that there exists a book presentation of G with respect to P. The minimum number of sheets requiring for a book presentation of G is called the *sheetnumber* of G, where minimum is taken over all Hamilton paths in G. If the number of sheets of a book presentation of G is equal to the sheetnumber of G, then the book presentation is called a *minimal book presentation* of G. Let B_n be a book with n sheets $S_1, S_2, \cdots, S_n$ and let L be the binder of B_n. A self-homeomorphism h of B_n is called a *sheet translation* if the restriction of h to L is the identity map and there is a permutation σ of $\{1, 2, \cdots, n\}$ such that $h(S_i)=S_{\sigma(i)}$ for $i=1, 2, \cdots, n$. Kobayashi conjectured that a minimal book presentation of the complete graph is unique up to sheet translations and ambient isotopies of R^3. In Section 4, we will characterize the book presentation of the complete graph and verify the conjecture.

§ 2. Locally unknotted spatial representations.

In this section, we shall show the following.

THEOREM 1. *Any graph G has a locally unknotted spatial representation.*

In order to prove Theorem 1, we need to prepare two lemmas. A cycle C of a graph G is called a *dominating cycle* (D-*cycle*) if every edge of G is incident with at least one vertex of C. A graph containing a D-cycle is called D-*cyclic*.

LEMMA 1. *A D-cyclic graph G has a locally unknotted spatial representation.*

PROOF. Let C be a D-cycle in G. We can construct a locally unknotted spatial representation of G as follows: At first, we choose the unit circle in the xy-plane in R^3 as the representation curve of C and represent the edges corresponding to the chords of C by internally disjoint arcs in the lower half space separated by the xy-plane. Next, we represent all vertices of $V(G)-V(C)$ by distinct points in the z-axis ($z>0$) and represent edges joining a vertex of $V(C)$ and a vertex of $V(G)-V(C)$ by internally disjoint arcs in the upper half space separated by the xy-plane (see Fig. 2). It is not hard to see that the resulting spatial representation is locally unknotted. □

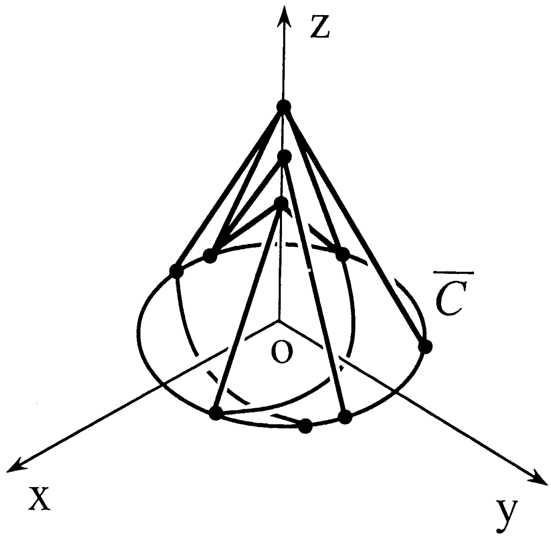

Fig. 2

Remark that, in the proof of Lemma 1, the D-cycle C can be chosen as an element of a basis of $H_1(G;Z)$.

Let G_1 and G_2 be graphs and let $f: K_1 \to K_2$ be an isomorphism from a subgraph K_1 of G_1 to a subgraph K_2 of G_2. The *amalgamation* $G_1 *_f G_2$ is the graph obtained from the union of G_1 and G_2 by identifying the subgraphs K_1 and K_2 according to f.

LEMMA 2. *Let G_i be a graph and let $\mathscr{C}_i$ be a set of cycles in G_i that represents a basis of $H_1(G_i;Z)(i=1,2)$. Let $f: P_1 \to P_2$ be an isomorphism, where P_i is a section of some cycle C_i in $\mathscr{C}_i$ $(i=1,2)$. If G_i has a locallly unknotted spatial representation with respect to $\mathscr{C}_i(i=1,2)$, then the amalgamation $G_1 *_f G_2$ has a locally unknotted spatial representation with respect to $\mathscr{C}_1 \cup \mathscr{C}_2$.*

PROOF. Let $\overline{G_i}$ be a locally unknotted spatial representation of G_i with respect to $\mathscr{C}_i(i=1,2)$. We may assume that $\overline{G_1}$ and $\overline{G_2}$ are splitted in R^3. Let D_i be a disk bounded by $\overline{C_i}$ in R^3 as in the definition of the locally unknotted spatial representation $(i=1,2)$. Then, by using the disk D_i, we can show that there exist disjoint 3-balls B_1 and B_2 in R^3 such that $\overline{G_i} \subset B_i$ and $\overline{G_i} \cap \partial B_i = \overline{C_i}(i=1,2)$. By sewing B_1 and B_2 by some homeomorphism $h: \overline{P_1} \to \overline{P_2}$, we obtain a spatial representation of $G_1 *_f G_2$. It is easy to see that the resulting spatial representation is locally unknotted with respect to $\mathscr{C}_1 \cup \mathscr{C}_2$. □

In order to prove Theorem 1, we shall actually prove a slightly stronger result.

THEOREM 2. *Let G be a graph and let C be a cycle in G. Then there is a basis $\mathscr{C}$ of $H_1(G;Z)$ that contains C, and there exists a locally unknotted spatial representation of G with respect to $\mathscr{C}$.*

PROOF. We may assume that the minimum degree of G is more than or equal to 3, since we consider a basis of $H_1(G;Z)$ with a set of cycles in G. The proof proceeds by induction on the number $|E(G-V(C))|$. Suppose first that $|E(G-V(C))|=0$. Then the result follows from Lemma 1, since C is a D-cycle and hence G is D-cyclic.

Now assume that the theorem holds for any graph G and any cycle C in G with $|E(G-V(C))|<N$, and let G be a graph and C be a cycle in G with $|E(G-V(C))|=N\geq 1$. Let G' be a connected component of $G-V(C)$ such that $E(G')\neq\phi$. Let e be an edge of G' such that its end-vertex u_1 is adjacent to some vertex, say v_1, of C and let e_1 be the edge joining u_1 and v_1. Two cases now arise, depending on whether v_1 is a cut vertex or not.

Case 1. Suppose that v_1 is not a cut vertex. Then there is an edge $e_2=u_2v_2$ such that $u_2\in V(G')$ and $v_2\in V(C)-\{v_1\}$. Let P be a path connecting v_1 and v_2 in C. Then we may assume, without loss of generality, that any vertex of C that is adjacent to a vertex of G' contains in P (see Fig. 3).

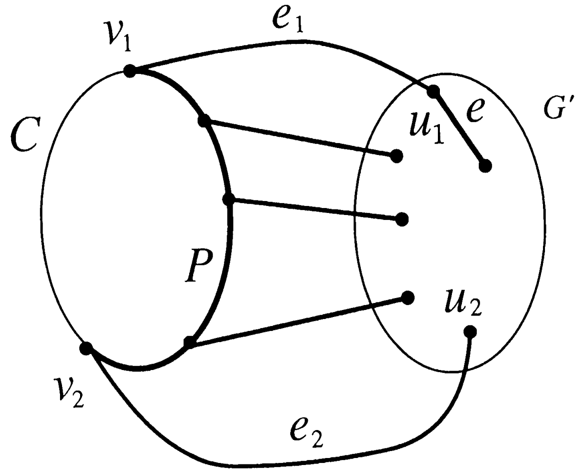

Fig. 3

Let Q be a (u_1, u_2)-path in G' (if $u_1=u_2$, then $Q=u_1$). Let $G_1=G-V(G')$ and let G_2 be the subgraph of G induced by the edges incident to a vertex of G' together with the edges of P. We denote two cycles C and $P\cup e_1\cup e_2\cup Q$ by C_1 and C_2, respectively, then we have $|E(G_i-V(C_i))|<N(i=1, 2)$. Hence, by the induction hypothesis, there exists a locally unknotted spatial representation $\overline{G}_i$ such that C_i is contained in the basis of $H_1(G_i\,;\,Z)(i=1, 2)$. The theorem, therefore, follows from Lemma 2, since G is the amalgamation of G_1 and G_2 at P.

Case 2. Suppose that v_1 is a cut vertex. If e_1 is not a cut edge, then a similar argument can be used to show that G has a locally unknotted spatial representation. Hence we assume that e_1 is a cut edge. Let G^* be the connected component of $G-e_1$ containing v_1, and let C' be a cycle in G'(C' always exists because the minimum degree of G is more than or equal to 3). Clearly it holds that $|E(G'-V(C'))|<N$ and $|E(G^*-V(C))|<N$. Thus there exist locally unknotted spatial representations $\overline{G'}$ and $\overline{G^*}$. Then the spatial representation obtained from the disjoint union of $\overline{G'}$ and $\overline{G^*}$ by attaching an arc joining $\bar{u}_1$ and $\bar{u}_2$ is a spatial representation of G. Obviously this spatial representation is locally unknotted and C is an element of a basis of $H_1(G\,;\,Z)$. $\square$

We shall conclude this section with one remark. From Theorem 2, we can choose an arbitrary cycle as an element of a basis of $H_1(G\,;Z)$.

But it is not true that, for a given set $\mathscr{C}=\{C_1, C_2, \ldots, C_n\}$ of cycles in G that represents a basis of $H_1(G; Z)$, there exists a locally unknotted spatial representation of G with respect to $\mathscr{C}$. In Fig. 4, it is impossible to construct a locally unknotted spatial representation of G such that C_1, C_2 and C_3 are contained in the basis of $H_1(G; Z)$.

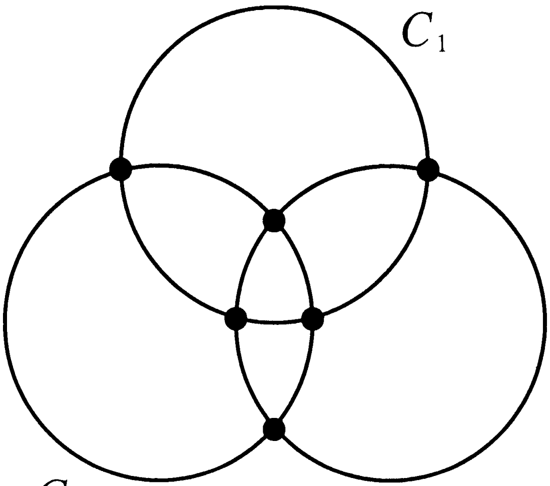

Fig. 4

§ 3. Local unknottedness and global unknottedness.

In this section, we concern only with the class of complete graphs. The complete graph with n vertices is denoted by K_n.

DEFINITION 4. A spatial representation $\overline{K_n}$ of the complete graph K_n is *locally unknotted with respect to a triangle basis* if there is a basis $\mathscr{C}$ of $H_1(K_n; Z)$ consisting of 3-cycles in K_n and $\overline{K_n}$ is locally unknotted with respect to $\mathscr{C}$.

In [5], Kobayashi showed that any complete graph has a locally unknotted spatial representation with respect to a triangle basis. Kobayashi asked if, for K_5 or K_6, there exists a locally unknotted spatial representation with respect to a triangle basis that is not globally unknotted. Remark that, by the result of Conway and Gordon [4], there exists always such a spatial representation of $K_n (n \geq 7)$.

PROPOSITION 1. *If a spatial representation of* K_5 *is locally unknotted with respect to a triangle basis, then it is globally unknotted.*

PROPOSITION 2. *There exists a spatial representation of K_6 that is locally unknotted with respect to a triangle basis and is not globally unknotted.*

PROOF OF PROPOSITION 1. Let $\overline{K_5}$ be a locally unknotted spatial representation of K_5 with respect to a triangle basis. We assume that there exists a cycle C in K_5 such that $\bar{C}$ represents a nontrivial knot in R^3. We choose C with the minimum length. Note that, in the following argument, the arithmetic on the indices of the vertices of C is done modulo the length of C.

Suppose first that $C=v_1v_2v_3v_4v_5v_1$ is a Hamilton cycle in K_5. Then the 3-cycle $\overline{v_iv_{i+1}v_{i+2}v_i}$ does not bound a disk in $R^3(i=1, 2, \cdots, 5)$ since a knotted 4-cycle does not exist. Thus the number of 3-cycles in K_5 that can not bound disks in R^3 is at least 5. Since the number of 3-cycles in K_5 is 10 and the number of 3-cycles that are needed to bound disks in R^3 is 6, $\overline{K_5}$ cannot be locally unknotted with respect to a triangle basis. This is a contradiction.

Suppose next that $C=v_1v_2v_3v_4v_1$ is a 4-cycle. Then, by the minimulity of the length of C, each 3-cycle $\overline{v_iv_{i+1}v_{i+2}v_i}(i=1, 2, 3, 4)$ can not bound a disk in R^3. This implies that the remaining all 3-cycles in $\overline{K_5}$ must bound internally disjoint disks. Suppose that there exist internally disjoint disks D_1, D_2, D_3, D_4 in R^3 such that $D_i\cap\overline{K_5}=\partial D_i=\overline{v_5v_iv_{i+1}v_5}(i=1, 2, 3, 4)$. Then $\bar{C}$ bounds the disk $D_1\cup D_2\cup D_3\cup D_4$ in R^3, that is a contradiction.

Suppose finally that $C=v_1v_2v_3v_1$ is a 3-cycle. Thus $\overline{v_1v_2v_3v_1}$ cannot bound a disk in R^3, and all $\overline{v_iv_1v_2v_i}$, $\overline{v_iv_2v_3v_i}$, $\overline{v_iv_3v_1v_i}$ cannot bound disks in R^3 for $i=4, 5$. Since the number of 3-cycles that are needed to bound disks in R^3 is 6, we can assume that $\overline{v_4v_2v_3v_4}$ and $\overline{v_4v_3v_1v_4}$ bound disks D_1 and D_2, respectively, and hence $\overline{v_4v_1v_2v_4}$ can not bound a disk in R^3. Then the cycle $\overline{v_5v_1v_2v_5}$ must bound a disk D_3 in R^3, since it is the only 3-cycle containing $\overline{v_1v_2}$. Then both $\overline{v_1v_3v_5v_1}$ and $\overline{v_2v_3v_5v_2}$ cannot bound disks in R^3 because $\bar{C}$ is knotted. Suppose that neither $\overline{v_1v_3v_5v_1}$ nor $\overline{v_2v_3v_5v_2}$ bound a disk in R^3. Then the remaining 3-cycles must bound disks in R^3. In this case, $\overline{v_1v_2v_4v_1}$ can bound a disk in R^3, since there exist D_3 and disks bounded by $\overline{v_1v_4v_5v_1}$ and $\overline{v_2v_4v_5v_2}$, that is a contradiction. Hence, from the symmetry of K_5, we may assume that only $\overline{v_2v_3v_5v_2}$ bounds a disk D_4 in R^3. Suppose that both $\overline{v_2v_4v_5v_2}$ and $\overline{v_3v_4v_5v_3}$ bound disks in R^3. Then the union of these disks and D_2 and D_4 forms a 2-sphere, so at least one of $\overline{v_2v_4v_5v_2}$ and $\overline{v_3v_4v_5v_3}$ can not bound a disk in R^3. Suppose that $\overline{v_2v_4v_5v_2}$ bounds a disk in R^3, then $\overline{v_1v_4v_5v_1}$ cannot bound a disk in R^3, since the 4-cycle $\overline{v_1v_5v_2v_4v_1}$ is ambient isotopic to $\bar{C}$ by the existence of D_1, D_2, and D_3. Suppose that $\overline{v_3v_4v_5v_3}$ bounds a disk in R^3, then $\overline{v_1v_4v_5v_1}$ cannot bound a disk in R^3, since the 4-cycle $\overline{v_1v_5v_3v_4v_1}$ is ambient isotopic to $\bar{C}$ by the existence of D_1, D_3, and D_4. Anyway at least two of $\overline{v_2v_4v_5v_2}$, $\overline{v_3v_4v_5v_3}$ and $\overline{v_1v_4v_5v_1}$ cannot bound disks in R^3. Eventually $\overline{K_5}$ has 5 or more 3-cycles that can not bound disks in R^3. This is a contradiction. □

PROOF OF PROPOSITION 2. The spatial representation $\overline{K_6}$ illustrated in Fig. 5 satisfies the condition of the proposition. Let $\mathscr{C}=\{v_1v_2v_3v_1,$ $v_2v_3v_4v_2,$ $v_3v_4v_5v_3,$ $v_4v_5v_6v_4,$ $v_5v_6v_1v_5,$ $v_6v_1v_2v_6,$ $v_2v_3v_6v_2,$ $v_4v_5v_2v_4,$ $v_6v_1v_4v_6,$ $v_1v_3v_5v_1\}$. Then $\overline{K_6}$ is locally unknotted with respect to $\mathscr{C}$ and contains a trefoil knot $\overline{v_1v_2v_5v_6v_3v_4v_1}$. □

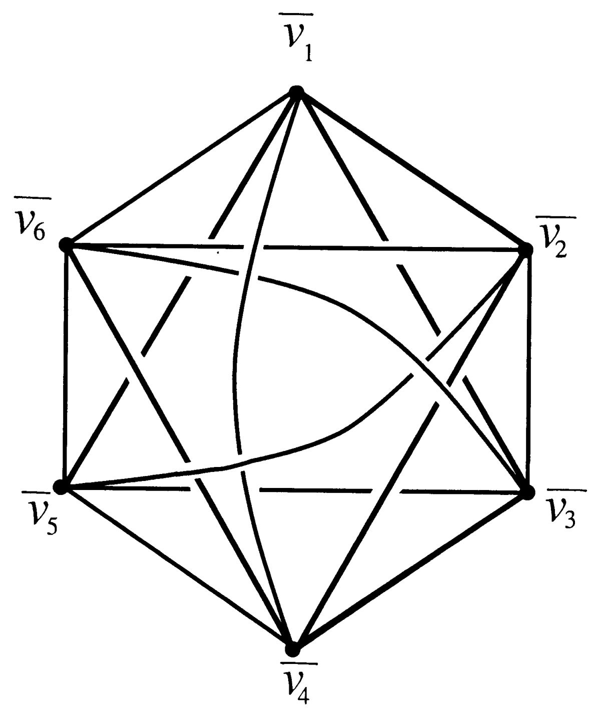

Fig. 5

§ 4. The book presentation of the complete graph.

In this section, we shall characterize the minimal book presentation of the complete graph and prove the following.

THEOREM 3. *Any two minimal book presentations of the complete graph K_n are ambient isotopic in R^3 up to sheet translations.*

If $n=1$, 2, or 3, then the theorem is trivial, since the number of sheets is at most one. Thus we assume from now on that $n \geq 4$. We begin with one familiar result about the book presentation of K_n (see [1] [5], for example).

THEOREM 4. *The sheetnumber of* $K_n (n \geq 4)$ *is equal to* $\left[\frac{n}{2}\right]$.

For the discussion of the book presentation, it is more helpful to consider an equivalent formulation. Let $\overline{K_n}$ be a book presentation of K_n, and let $v_1, v_2, \cdots, v_n$ be the ordering of vertices along the binder. Let C be the Hamilton cycle $v_1 v_2 \cdots v_n v_1$. Consider a projection of $\overline{K_n}$ onto a plane α in R^3 satisfying that C is represented as a circle in α and the edges of $E(K_n) - E(C)$ as chords of the circle. Then the edges in each sheet are regarded as collections of noncrossing chords. After ambient isotopic modification, we can assume that crossings of chords are transversal double points only. The image of such a projection together with an over/under information at every double point is called a *circular diagram* of $\overline{K_n}$ by C. As an example, Fig. 6 illustrates a circular diagram of the book presentation of K_7 in Fig. 1. In the circular diagram, the binder and sheets are considered by C and internally disjoint disks bounded by C, respectively. In what follows, for a cycle $C = v_1 v_2 \cdots v_n v_1$, we shall read subscripts modulo n, and denote the section of C from v_i to v_j by $C[v_i, v_j]$.

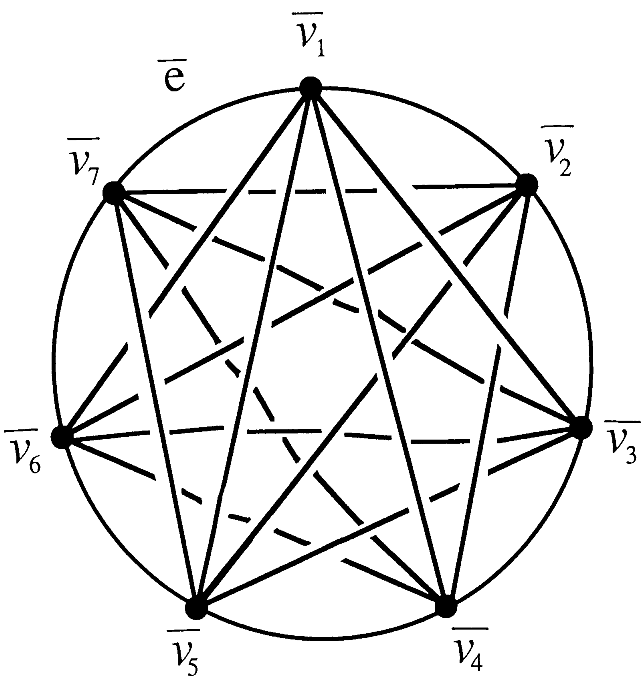

Fig. 6

For convenience, we focus first on the case when the number of the vertices is even. Consider any minimal book presentation of K_{2m}. From Theorem 4, the number of sheets is m. Let $v_1, v_2, \cdots, v_{2m}$ be the ordering of the vertices of K_{2m} along the binder and let C be the Hamilton cycle $v_1v_2\cdots v_{2m}v_1$. We consider this book presentation with the circular diagram by C. We say that a chord is *i-chord* if the distance between its endvertices in C is i. Clearly it holds that $2 \leq i \leq m$ and that the number of i-chords is equal to m if $i=m$, or $2m$ if $2 \leq i \leq m-1$. The following is a simple observation.

LEMMA 3. *For a circular diagram of a minimal book presentation of K_{2m}, there are exactly one m-chord and exactly two i-chords ($i=2, 3, \cdots, m-1$) in every sheet.*

PROOF. Since each m-chord is a diagonal of C, no two such chords can be embedded in the same sheet (this is the reason why m sheets are neccessary for a book presentation of K_{2m}). Thus there is exactly one m-chord in every sheet. For the second claim, notice that the number of chords is $m(2m-3)$. On the other hand, the number of sheets that we are allowed to use is m and the number of chords that we can accomodate in one sheet is at most $2m-3$. Hence the number of chords embedded in each sheet is exactly $2m-3$. Since each sheet contains one m-chord, the lemma follows. □

Let $S_1, S_2, \cdots, S_n$ be the sheets of the book in which K_{2m} is embedded, and suppose that the m-chord v_iv_{i+m} is embedded in S_i for $i=1, 2, \cdots, m$ (this is always possible because we are allowed to use a sheet transration). We shall call such a book presentation *canonical*. We turn attention to (m−1)-chords, if any. From Lemma 3, either v_2v_{m+1} or v_1v_m must be embedded in S_1. Suppose that v_2v_{m+1} is embedded in S_1. Then, from Lemma 3, either v_2v_{m+1} or v_3v_{m+2} must be embedded in S_2. Since v_2v_{m+1} is embedded in S_1, v_3v_{m+2} must be embedded in S_2. By proceeding this argument repeatedly, we can conclude that both $v_{i+1}v_{i+m}$ and $v_{i+m+1}v_{i+2m-1}$ are embedded in S_i for $i=1, 2, \cdots, m$. If we had chosen first v_1v_m as the $(m-1)$-chord in S_1, then it would be hold that both v_iv_{i+m-1} and $v_{i+m}v_{i-1+2m}$ are embedded in S_i for $i=1, 2, \cdots, m$.

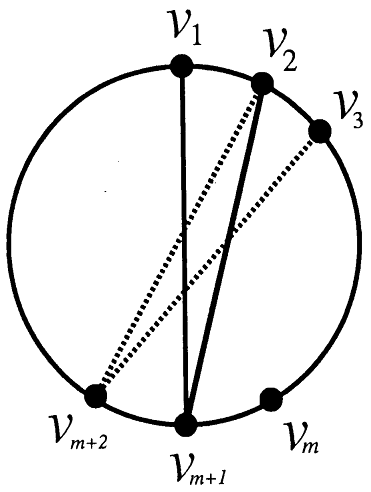

Fig. 7

Next turn attention to $(m-2)$-chords, if any. From Lemma 3, either v_3v_{m+1} or v_2v_m must be embedded in S_1. Suppose that v_3v_{m+1} is embedded in S_1, then v_4v_{m+2} must be embedded in S_2, since either v_4v_{m+2} or v_3v_{m+1} must be embedded in S_2. By proceeding this argument repeatedly, we can conclude that both $v_{i+2}v_{i+m}$ and $v_{i+m+2}v_{i+2+m}$ are embedded in S_i for $i=1, 2, \cdots, m$. If we had chosen first v_2v_m as a $(m-2)$-chord in S_1, then it would be hold that both v_iv_{i+m-1} and $v_{i+m+1}v_{i+2m-1}$ are embedded in S_i for $i=1, 2, \cdots, m$. Hence, by repeating this argument, we can deduce the following.

PROPOSITION 3. *Let $\overline{K_{2m}}$ be a canonical book presentation of K_{2m} and let $v_1, v_2, \cdots, v_{2m}$ be the ordering of the vertices of K_{2m} along the binder and let C be the Hamilton cycle $v_1v_2\cdots v_{2m}v_1$. Then each sheet of the circular diagram of $\overline{K_{2m}}$ by C has a symmetry under the rotation by angle $\frac{2\pi}{m}$ around the center of C.* □

As an example, Fig. 8 illustrates four sheets of a canonical book presentation of K_8.

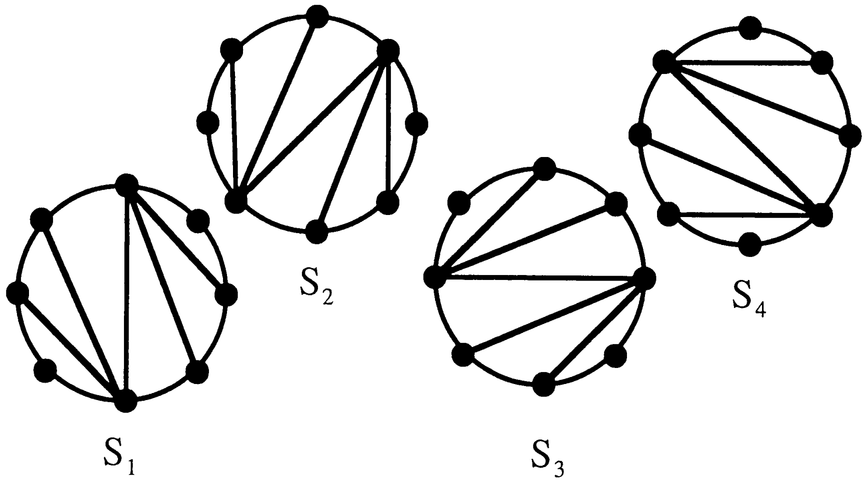

Fig. 8

From Proposition 3, a canonical book presentation $\overline{K_{2m}}$ can be described by the condition of the chords embedded in S_1, more precisely by the condition of the chords embedded in the region, say the *right-side* of S_1, bounded by v_1v_{m+1} and $C[v_1, v_{m+1}]$ in S_1. We shall express $\overline{K_{2m}}$ with a *word* consisting of two letters a and b as follows: We start initially with the $(m-1)$-chord in the right-side of S_1. If v_2v_{m+1} is embedded in it, then associate with a letter a, and if v_1v_m is embedded in it, then associate with a letter b. In general, for the $(i+1)$-chord v_jv_k in the right-side of S_1, if the i-chord $v_{j+1}v_k$ is embedded in it, then associate with a letter a, and if v_jv_{k-1} is embedded in it, then associate with a letter b. As a result we obtain $m-2$ letters. Finally list up these letters from left to right in order. The resulting sequence of letters is the *word* corresponding to $\overline{K_{2m}}$. As an example, the word corresponding to the book presentation in Fig. 8 is bb.

LEMMA 4. *Any two canonical book presentations of K_{2m} are ambient isotopic in R^3.*

PROOF. Let $\overline{K_{2m}}$ be a canonical book presentation of K_{2m}, and let $w_1w_2\cdots w_{m-2}$ be the word corresponding to $\overline{K_{2m}}$. Let $\bar{K}$ be the canonical book presentation such that the word corresponding to $\bar{K}$ is $\underbrace{aa\cdots a}_{m-2}$. We shall show that $\overline{K_{2m}}$ can be transformed into $\bar{K}$ by an ambient isotopy of R^3 (this ambient isotopy fixes the binder of the book).

Assume first that $w_{m-2}=b$. Let v_kv_{k+2} be the 2-chord embedded in the right-side of S_1. Then it holds from Proposition 3 that 2-chords $v_{i+k-1}v_{i+k+1}$ and $v_{i+k-1+m}v_{i+k+1+m}$ are both embedded in S_i for $i=1, 2, \cdots, m$.

Moreover it follows from the difinition of word that 3-chords $v_{i+k-1}v_{i+k+2}$ and $v_{i+k-1+m}v_{i+k+2+m}$ are both embedded in S_i for $i=1, 2, \cdots, m$. Now we move $v_{k+1}v_{k+3}$ from S_2 into S_1. Then v_kv_{k+2} cannot be embedded in S_1, so we move it from S_1 into S_m. By repeating this move, we obtain the book presentation with $w_{m-2}=a$, after moving $v_{k+2}v_{k+4}$ from S_3 into S_2.

Assume next that $w_{m-l}=b$ and $w_{m-l+1}=w_{m-l+2}=\cdots=w_{m-2}=a$ for some $l(3\leq l\leq m-1)$. Let v_kv_{k+l} be the l-chord embedded in the right-side of S_1. Then it holds from Proposition 3 that l-chords $v_{i+k-1}v_{i+k+l-1}$ and $v_{i+k-1+m}v_{i+k+l-1+m}$ are both embedded in S_i for $i=1, 2, \cdots, m$. Moreover it follows from the definition of word that $(l+1)$-chords $v_{i+k-1}v_{i+k+l}$ and $v_{i+k-1+m}v_{i+k+l+m}$ are both embedded in S_i for $i=1, 2, \cdots, m$. Now we move $v_{k+1}v_{k+l+1}$, $v_{k+2}v_{k+l+l}$, $\cdots$, $v_{k+l-1}v_{k+l+l}$ from S_2 into S_1. Next, by the similar way as above, we move v_kv_{k+l}, $v_{k+1}v_{k+l}$, $\cdots$, $v_{k+l-2}v_{k+l}$ from S_1 into S_m. By repeating this move, we finally obtain the book presentation with $w_{m-l}=w_{m-l+1}=\cdots=w_{m-2}=a$. Hence the result follows. □

Next we shall handle the case when the number of vertices is odd. The situation is seriously different from the previous case. Note that, from Theorem 4, the sheetnumber of K_{2m+1} is equal to $m+1$. In this case there might exists a sheet containing a space to accomodate edges. As an example, recall the book presentation of K_7 with 4 sheets (see Fig. 1). This book presentation is a canonical book presentation of K_7. Notice that there is a sheet containing two edges only.

Consider any minimal book presentation of K_{2m+1}. Let v_1, v_2, $\cdots$, v_{2m+1} be the ordering of the vertices along the binder and let C be the Hamilton cycle $v_1v_2\cdots v_{2m+1}v_1$. Again we consider this book presentation with the circular diagram by C. We partition the chords into $m-1$ groups consisting of $2m+1$ chords. We say that a chord is *i-chord* if the distance between its endvertices in C is i. Clearly it holds that $2\leq i\leq m$ and that the number of i-chords is $2m+1$ for $i=2, 3, \cdots, m$.

First we consider m-chords. It is easy to see that no three such chords can be embedded in the same sheet (this is the reason why $m+1$ sheets are needed for a book presentation of K_{2m+1}). Thus exactly m sheets contain two adjacent m-chords and the remaining sheet contains only one m-chord. Let S_1, S_2, $\cdots$, S_{m+1} be the sheets of the book in which K_{2m+1} is embedded, and assume that both v_iv_{i+m} and v_iv_{i+m+1} are embedded in S_i for $i=1, 2, \cdots, m$ and that $v_{m+1}v_{2m+1}$ is embedded in S_{m+1}. Let H_i be the region in S_i bounded by $C[v_i, v_{i+m}]$ and the chord v_iv_{i+m} for $i=1, 2, \cdots, 2m+1$, where the arithmetic on the indices of sheets is done by modulo $m+1$. We shall call H_1, H_2, $\cdots$, H_{2m+1} *half-sheets*. Notice that there exist exactly two half-sheets H_i and H_{i+m+1} in S_i for $i=1, 2, \cdots, m$ and that there exists only one half-sheet H_{m+1} in S_{m+1}. We shall call the region in S_{m+1} bounded by the chord $v_{m+1}v_{2m+1}$ and $C[v_{2m+1}, v_{m+1}]$ the *extra-space*. Now suppose that each half-sheet has exactly one i-chord for $i=2, 3, \cdots, m$ (hence the extra-space has no chord). We shall call such a book presentation *canonical*. The followings can be shown by the similar way as the proofs of Proposition 3 and Lemma 4, respectively.

PROPOSITION 4. *Let $\overline{K_{2m+1}}$ be a canonical book presentation of K_{2m+1}. Let $v_1, v_2, \cdots, v_{2m+1}$ be the ordering of the vertices of K_{2m+1} along the binder and let C be the Hamilton cycle $v_1v_2\cdots v_{2m+1}v_1$. Then each half-sheet of the circular diagram of $\overline{K_{2m+1}}$ by C has a symmetry under the rotation by angle $\frac{2\pi}{2m+1}$ around the center of C.* □

LEMMA 5. *Any two canonical book presentations of K_{2m+1} are ambient isotopic in R^3.* □

In order to complete the proof of Theorem 3, we need the following lemma.

LEMMA 6. *Any minimal book presentation of K_{2m+1} can be transformed into some canonical book presentation by ambient isotopies of R^3 and sheet translations.*

PROOF. Let $\overline{K_{2m+1}}$ be a minimal book presentation of K_{2m+1} and let $v_1, v_2, \cdots, v_{2m+1}$ be the ordering of the vertices of K_{2m+1} along the binder and let C be the Hamilton cycle $v_1v_2\cdots v_{2m+1}v_1$. We consider $\overline{K_{2m+1}}$ with the circular diagram by C. By using m-chords, we can define $2m+1$ half-sheets and the extra-space in $m+1$ sheets of the book. We assume that the extra-space contains at least one chord, and show that $\overline{K_{2m+1}}$ can be transformed into the book presentation such that the extra-space contains no chord by sheet translations and ambient isotopies of R^3 (each ambient isotopy fixes the binder of the book). In the following argument, for convinience, the term *half-sheet* means half-sheet or extra-space.

We first examine 3-chords. If the half-sheet containing the 3-chord v_iv_{i+3} contains either v_iv_{i+2} or $v_{i+1}v_{i+3}$ for $i=1, 2, \cdots, 2m+1$, then examine 4-chords. Otherwise, there exists a half-sheet, say H_1, that contains a 3-chord, say v_1v_4, but contains neither v_1v_3 nor v_2v_4. Then, after a sheet translation if necessary, take v_2v_4 from other half-sheet into H_1. Let H_2 be the half-sheet that contained v_2v_4 before. If H_2 contains the 3-chord v_2v_5, then take v_3v_5 from other half-sheet into H_2 after a sheet translation if necessary, and apply this argument to the half-sheet that contained v_3v_5 before. If H_2 does not contain v_2v_5, then consider the half-sheet, say H_3, that contains v_2v_5. If H_3 contains v_3v_5 , we are done. Otherwise take v_3v_5 from other half-sheet into H_3, and repeat this argument. Consequently, we obtain the book presentation of K_{2m+1} such that every half-sheet containing v_iv_{i+3} contains $v_{i+1}v_{i+3}$ for $i=1, 2, \cdots, 2m+1$. If we had chosen first v_1v_3 as the 2-chord in H_1, then we could conclude that every half-sheet containing v_iv_{i+3} contains v_iv_{i+2} for $i=1, 2, \cdots, 2m+1$.

Now we assume that every half-sheet containing v_iv_{i+3} also contains $v_{i+1}v_{i+3}$ for $i=1, 2, \cdots, 2m+1$, and examine 4-chords. If the half-sheet containing the 4-chord v_iv_{i+4} contains either v_iv_{i+3} or $v_{i+1}v_{i+4}$ for $i=1, 2, \cdots, 2m+1$, then examine 5-chords. Otherwise, there exists a half-sheet, say H_1 again, that contains a 4-chord, say v_1v_5, but contains neither v_1v_4 nor v_2v_5. Then, after sheet translation if necessary, take v_2v_5 and v_3v_5 from other half-sheet into H_1. Let H_2 be the half-sheet that contained v_2v_5 and v_3v_5 before. If H_2 contains the 4-chord v_2v_6, then take v_3v_6 and v_4v_6 from other half-sheet into H_2 after a sheet translation if necessary, and apply this argument to the half-sheet that contained v_3v_6 and v_4v_6 before. If H_2 does not contain v_2v_6, then consider the half-sheet, say H_3, that contains v_2v_6. If H_3 contains v_3v_6 and v_4v_6, then we are done. Otherwise take v_3v_6 and v_4v_6 from other half-sheet into H_3, and repeat this argument. Consequently, we obtain the book presentation of K_{2m+1} such that every half-sheet containing v_iv_{i+4} contains $v_{i+1}v_{i+4}$ and $v_{i+2}v_{i+4}$ for $i=1, 2, \cdots, 2m+1$. If we had chosen first v_1v_4 as the 3-chord in H_1, then we could conclude that every half-sheet containing v_iv_{i+4} contains v_iv_{i+3}, $v_{i+1}v_{i+3}$ for $i=1, 2, \cdots, 2m+1$.

By repeating above construction, we finally obtain the book presentation of K_{2m+1} such that the extra-space contains no chord, since the extra-space has no m-chord. Now it is easy to transform this book presentation into some canonical one. Let $S_1, S_2, \cdots, S_{m+1}$ be the sheets of the book in which K_{2m+1} is embedded. If both v_iv_{i+m} and v_iv_{i+m+1} are contained in the same sheet for $i=1, 2, \cdots, m$ and hence the remaining sheet contains $v_{m+1}v_{2m+1}$, then we can obtain a canonical book presentation of K_{2m+1} by using a sheet translation. Otherwise, there is a sheet, say S_{m+1}, that contains $v_{m+1}v_{2m+1}$ and contains either v_mv_{2m+1} or v_1v_{m+1}. If S_{m+1} contains v_mv_{2m+1}, then move v_mv_{2m+1} together with all other chords in the same half-sheet from S_{m+1} into the sheet, say S_m, that contains v_mv_{2m}. If S_m does not contain $v_{m-1}v_{2m}$, then we are done because the resulting book presentation can be transformed into some canonical one by the similar way as above. Otherwise, S_m contains $v_{m-1}v_{2m}$. Then move $v_{m-1}v_{2m}$ together with all other chords in the same half-sheet from S_m into the sheet that contains $v_{m-1}v_{2m-1}$. By repeating this move, we obtain a canonical book presentation similarly. In the case when S_{m+1} contains v_1v_m, we can also obtain a canonical book presentation by the similar way as above. $\square$

We are now ready to prove Theorem 3.

PROOF OF THEOREM 3. Let $\overline{B_1}$ and $\overline{B_2}$ be two minimal book presentations of K_n. If $n=1$, 2, or 3, then the proof is clear, since each of them has at most one sheet. Thus it suffices to prove the theorem for the case when $n\geq 4$. Suppose that n is even. By virtue of a sheet translation, we can assume that both $\overline{B_1}$ and $\overline{B_2}$ are canonical. Therefore, from Lemma 4, they are ambient isotopic in R^3. Suppose that n is odd. From Lemma 6, we can transform each of $\overline{B_1}$ and $\overline{B_2}$ into some canonical book presentation by sheet translation and ambient isotopies of R^3. Hence, from Lemma 5, they are ambient isotopic in R^3. $\square$

Acknowledgement. We are grateful to our advisor, Shin'ichi Suzuki, for helpful discussions and encouragement. We also wish to thank Kazuaki Kobayashi, who guided our interest in this topic.

References

[1] BERNHART, F. and KAINEN, P. C.: The book thickness of a graph, J. Combin. Theory Ser. B **27** (1979), 320-331.

[2] BONDY, J. A. and MURTY, U. S. R.: Graph Theory with Applications, North Holland, New York, 1976.

[3] BURDE, G. and ZIESCHANG, H.: Knots, Walter de Gruyter, Berlin/New York, 1985.

[4] CONWAY, J. H. and GORDON, C. McA.: Knots and links in spatial graphs, J. Graph Theory **7** (1985), 445-453.

[5] KOBAYASHI, K.: Standard spatial graph, Hokkaido Math. J. **21** (1992), 117-140.

[6] WHITE, A. T.: Graphs, Groups and Surfaces, Mathematics Studies 8 Reviced edition, North Holland, New York, 1984.

Department of Mathematics
School of Science and Engineering
Waseda University
3-4-1 Ohkubo, Shinjuku,
Tokyo 169-50, Japan

Information Systems Group
Sharp Corporation
492 Minosyo, Yamato-Koriyama,
Nara 639-11, Japan