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Lorentz spaces as L_1-modules and multipliers

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Let w be a weight function on a locally compact group G, so that the weighted L_1-space $L_1(w)$ forms a Banach algebra under convolution. Suppose that G acts on a locally compact space Ω, and that B is a Banach space consisting of Radon measures on Ω which is also a left Banach $L_1(w)$-module. Under certain conditions on B, we shall characterize those bounded linear operators $T: L_1(w) \rightarrow B$ which satisfy $T(f*g)=f*T(g)$. We shall also show that there are numerous examples of Lorentz spaces which form left Banach $L_1(w)$-modules with respect to appropriate weight functions.

Introduction.

Let G be a locally compact group, and let w be a weight function on G (for the definition, see (2.3)). Thus the weighted L_1-space $L_1(w)$ forms a Banach algebra under convolution. Suppose that Ω is a locally compact (Hausdorff) space and that G acts on Ω as a transformation group. Thus, for appropriate paris (μ, ν) of Radon measures μ on G and ν on Ω, the convolution product $\mu*\nu$ can be defined in a natural fashion (2.1)(b). We are interested in those Banach spaces consisting of Radon measures on Ω which form left Banach $L_1(w)$-modules. For such Banach modules B, we wish to characterize those bounded linear operators T: $L_1(w) \rightarrow B$ which satisfy $T(f*g)=f*T(g)$ for all $f, g \in L_1(w)$. Such attempts have been made by a number of authors, including S.L. Gulick, T.-S. Liu and A.C.M. van Rooji [3] and [4]; M. Rieffel [12] ; and C.V. Comisky [2]. Our approach to the problem has naturally led us to the concept of *weak tilde-closures* of such spaces.

The present paper consists of four sections (plus this introduction). §1 studies normed spaces comprised of Radon measures on the locally compact space Ω. We shall introduce the concept of weak tilde-closures of such normed spaces. In the study of translation invariant linear functionals on translation invariant Banach spaces B on G, the "weak" compactness of the closed unit ball $(B)_1$ plays a vital role (cf. S. Saeki [16]). Our result, (1.5), gives a necessary and sufficient condition in order that $(B)_1$ be "weakly" compact.

In §2, we shall consider some (weighted) convolution algebras $L_1(w)$. For appropriate normed spaces B of Radon measures on Ω, the convolution $f*\mu$ is always defined for $f\in L_1(w)$ and $\mu\in B$. It turns out that whenever B forms a left normed $L_1(w)$-module, then the weak tilde-closure $B^{\sim}$ of B forms a left Banach $L_1(w)$-module (2.8).

§3 is essentially an appendix, and gives brief accounts of Lorentz spaces $L_{p,q}$. Naturally most of the results are stated only with sketchy proofs or even without any proofs. The classical paper [7] of R. Hunt gives a nice treatment of the Lorentz space theory. Appendix A of the second author's 1991 Ph.D. dissertation [18] provides detailed studies on the subject. Although §3 contains some sharper results than those in Hunt [7], the reader who is familiar with Lorentz spaces may completely skip §3 except for some notation.

The final section, §4, shows that there are "many" Lorentz spaces $L_{p,q}$ on Ω which form a left Banach $L_1(w)$-module for some weight function w on G. We hope that these results will justify our introduction of weighted L_1-spaces in §2.

§1. Weak tilde-closure.

Throughout the paper, Ω denotes a locally compact Hausdorff space, $C(\Omega)$ the space of all bounded continuous (complex-valued) functions on Ω, and $C_c(\Omega)=\{f\in C(\Omega):\mathrm{supp}(f)$ is compact$\}$, where $\mathrm{supp}(f)$ is the closure of $\{f\neq 0\}$. For each compact subset K of Ω, let $C_K(\Omega)$ be the space of all $f\in C_c(\Omega)$ with $\mathrm{supp}(f)\subset K$. Thus $C_K(\Omega)$ forms a Banach space with respect to the uniform norm $\|\cdot\|_u$.

Now let $C_c'(\Omega)$ denote the space of all linear functionals on $C_c(\Omega)$. The value of $\mu\in C_c'(\Omega)$ at $\phi\in C_c(\Omega)$ is denoted by $\mu(\phi)$ or $\langle\phi,\mu\rangle$. An element μ of $C_c'(\Omega)$ is *continuous* if for each compact set $K\subset\Omega$, there exists a finite positive constant $\gamma=\gamma(\mu,K)$ such that

$$|\langle\phi,\mu\rangle|\leq\gamma\|\phi\|_u\quad\forall\phi\in C_K(\Omega).$$

A continuous linear functional on $C_c(\Omega)$ is often called a *Radon measure* on Ω [13]. Let $C_c^*(\Omega)$ denote the space of all such functionals. For each subset B of $C_c'(\Omega)$, we write $\sigma(B, C_c)$ to denote the weak topology of B induced by $C_c(\Omega)$. Thus a net (μ_α) in B converges in $\sigma(B, C_c)$ to $\mu\in B$ if and only if for each $\phi\in C_c(\Omega)$, we have $\langle\phi,\mu_\alpha\rangle\to\langle\phi,\mu\rangle$. *Unless other-*

wise mentioned, $C_c^*(\Omega)$ will always be equipped with $\sigma(C_c^*, C_c)$. Thus $C_c^*(\Omega)$ forms a locally convex topological (Hausdorff) vector space.

For each normed space B and $r>0$, let $(B)_r=\{\mu\in B: \|\mu\|_B\leq r\}$. As in [16], we call B a *normed space on* Ω if B is a subspace of $C_c^*(\Omega)$ and if the imbedding $\mu\mapsto\mu: B\to C_c^*(\Omega)$ is continuous. A Banach space on Ω is a Banach space which is also a normed space on Ω.

(1.1) DEFINITION. Given a normed space B on Ω, we define the weak tilde-closure $B^\sim$ of B in $C_c'(\Omega)$ as the collection of all $\mu\in C_c'(\Omega)$ for which there exists a norm-bounded net (μ_α) in B such that $\mu_\alpha\mapsto\mu$ in $\sigma(C_c', C_c)$. For $\mu\in B^\sim$, let

$$\|\mu\|_{B^\sim}=\inf\{\sup_\alpha\|\mu_\alpha\|_B\},$$

where the infimum is taken over all nets (μ_α) in B as above.

(1.2) REMARKS: (a) $B^\sim$ is a vector space and $\|\cdot\|_{B^\sim}$ is a seminorm on it. Moreover, $B\subset B^\sim$ and $\|\mu\|_{B^\sim}\leq\|\mu\|_B$ for each $\mu\in B$.

(b) Our definition of weak tilde-closure was motivated by the definition of the tilde-algebra associated with a Banach function algebra (cf. Katznelson-McGehee [8] and Varopoulos [19]).

(1.3) PROPOSITION. *Let B be a normed space and let $T: B\to C_c^*(\Omega)$ be a linear mapping. Then T is continuous if and only if for each compact set $K\subset\Omega$, there exists a finite constant $\eta=\eta(K)$ such that*

$$(*)\qquad |\langle\phi, T\mu\rangle|\leq\eta\|\phi\|_u\cdot\|\mu\|_B \quad \forall\phi\in C_K(\Omega) \text{ and } \mu\in B.$$

PROOF: Suppose T is continuous, and let

$$(1)\qquad F(\phi, \mu)=\langle\phi, T\mu\rangle \quad \forall\phi\in C_c(\Omega) \text{ and } \mu\in B.$$

Then, for each $\phi\in C_c(\Omega)$, $F(\phi, \cdot)$ is a continuous linear functional on B. So there exists a finite constant $\alpha(\phi)$ such that

$$(2)\qquad |F(\phi, \mu)|\leq\alpha(\phi)\|\mu\|_B \quad \forall\mu\in B.$$

Now let $K\subset\Omega$ be an arbitrary compact set. Then $C_K(\Omega)$ is a Banach space, and for each $\mu\in B$, $F(\cdot, \mu)$ is a bounded linear functional on $C_K(\Omega)$ since $T(B)\subset C_c^*(\Omega)$. It follows from (2) and the Banach-Steinhaus Theorem that the functional norms of $F(\cdot, \mu)_{C_K(\Omega)}$, $\mu\in(B)_1$, are uniformly bounded. That is, there exists a finite constant $\eta=\eta(K)$ such that

$$(3)\qquad |F(\phi, \mu)|\leq\eta\|\phi\|_u \quad \forall\phi\in C_K(\Omega) \text{ and } \mu\in(B)_1.$$

In light of (1), (3) is equivalent to $(*)$.

The converse is an immediate consequence of the definition of $\sigma(C_c^*, C_c)$.

(1.4) THEOREM. *Let B be a normed space on Ω. Then*
(a) *$B^\sim$ equipped with $\|\cdot\|_{B^\sim}$ is a Banach space on Ω.*
(b) *$(B^\sim)_1$ agrees with the closure of $(B)_1$ in $C_c'(\Omega)$ and is $\sigma(C_c^*, C_c)$-compact.*
(c) *$(B^\sim)^\sim = B^\sim$ isometrically.*

PROOF: Let $K \subset \Omega$ be compact. By (1.3) and the definition of a normed space on Ω, there exists a finite constant η_K such that

(1) $$|\langle\phi, \mu\rangle| \leq \eta_K \|\phi\|_u \cdot \|\mu\|_B \quad \forall \phi \in C_K(\Omega) \text{ and } \mu \in B.$$

It follows from the definition of $\|\cdot\|_{B^\sim}$ that (1) holds with B replaced by $B^\sim$. That is,

(2) $$|\langle\phi, \mu\rangle| \leq \eta_K \|\phi\|_u \cdot \|\mu\|_{B^\sim} \quad \forall \phi \in C_K(\Omega) \text{ and } \mu \in B^\sim.$$

Since K was an arbitrary compact subset of Ω, we infer from (2) and Remark (1.2)(a) that $B^\sim \subset C_c^*(\Omega)$ and $\|\cdot\|_{B^\sim}$ is a norm on $B^\sim$. Moreover, (2) shows that $\mu \mapsto \mu : B^\sim \to C_c^*(\Omega)$ is continuous, and so $B^\sim$ is a normed space on Ω.

Now let $\mu \in (B^\sim)_1$ be given. For each $\varepsilon > 0$, the definition of $\|\cdot\|_{B^\sim}$ shows that $(1+\varepsilon)^{-1}\mu \in ((B)_1)^-$, where the closure is taken in $C_c'(\Omega)$ with respect to $\sigma(C_c', C_c)$. (Notice also that $((B)_1)^-$ agrees with the $\sigma(C_c^*, C_c)$-closure of $(B)_1$ in $C_c^*(\Omega)$ since $((B)_1)^- \subset B^\sim \subset C_c^*(\Omega)$.) Since $(1+\varepsilon)^{-1}\mu \to \mu$ in $\sigma(C_c', C_c)$ as $\varepsilon \downarrow 0$, it follows that μ itself is in $((B)_1)^-$. As $\|\mu\|_{B^\sim} \leq \|\mu\|_B$ for every $\mu \in B$, we conclude that $(B^\sim)_1 = ((B)_1)^-$. In particular, $(B^\sim)^\sim = B^\sim$ isometrically.

Now let (μ_α) be a net in $(B^\sim)_1$. By (2) and Tychonov's Theorem, (μ_α) has a subnet which converges in $\sigma(C_c', C_c)$ to some $\mu \in C_c'(\Omega)$. Since $(B^\sim)_1$ is closed in $C_c'(\Omega)$, it follows that $\mu \in (B^\sim)_1$. This shows that $(B^\sim)_1$ is $\sigma(C_c^*, C_c)$-compact.

It remains to show the completeness of $B^\sim$. Let $(\mu_n)_1^\infty$ be a Cauchy sequence in $B^\sim$. Then $(\mu_n)_1^\infty$ is norm-bounded. Since $(B^\sim)_1$ is $\sigma(C_c^*, C_c)$-compact, it follows that $(\mu_n)_1^\infty$ has a $\sigma(C_c^*, C_c)$-cluster point $\mu \in B^\sim$. But then, for each $m \in \mathbf{N}$, $\mu - \mu_m$ is a $\sigma(C_c^*, C_c)$-cluster point of the sequence $(\mu_n - \mu_m)_{n=1}^\infty$. As $(B^\sim)^\sim = B^\sim$ isometrically, it follows that

$$\begin{aligned}\|\mu - \mu_m\|_{B^\sim} &= \|\mu - \mu_m\|_{(B^\sim)^\sim} \\ &\leq \lim_n \|\mu_n - \mu_m\|_{B^\sim}.\end{aligned}$$

Since $(\mu_n)_1^\infty$ is a Cauchy sequence in $B^\sim$, we conclude that $\|\mu-\mu_m\|_{B^\sim}\to 0$ as $m\to\infty$. Hence $B^\sim$ is a Banach space, as desired.

(1.5) COROLLARY. *For a normed space B on Ω, $B^\sim=B$ isometrically if and only if the closed unit ball of B is $\sigma(C_c^*, C_c)$-compact.*

PROOF: If $B^\sim=B$ isometrically, then $(B)_1=(B^\sim)_1$, and so (1.4) shows that $(B)_1$ is $\sigma(C_c^*, C_c)$-compact. Conversely, suppose $(B)_1$ is $\sigma(C_c^*, C_c)$-compact. Then $(B)_1$ is closed in $C_c'(\Omega)$. Hence $(B^\sim)_1=(B)_1$ by the last theorem. Therefore $B^\sim=B$ isometrically.

(1.6) REMARKS: Fix $1<p<\infty$, and let $B=L_p(\mathbf{R})=L_p(\mathbf{R},\lambda)$, where λ is Lebesgue measure on the real line $\mathbf{R}$. We identify B with a subspace of $C_c^*(\mathbf{R})$ via

$$\langle\phi, f\rangle=\int_{-\infty}^{\infty}\phi f\,dx \quad \forall\phi\in C_c(\Omega) \text{ and } f\in B.$$

Now choose and fix any linear functional F on B which annihilates all of $C_c(\mathbf{R})$, and define $\|f\|_B=\|f\|_p+|F(f)|$ for $f\in B$. Then B equipped with $\|\cdot\|_B$ forms a normed space on $\mathbf{R}$. Moreover,

(a) $B^\sim=B$ and $\|f\|_{B^\sim}=\|f\|_p$ for each $f\in B^\sim$;

(b) $\|\cdot\|_B$ and $\|\cdot\|_{B^\sim}$ are equivalent norms on B if and only if F is a bounded linear functional on $L_p(\mathbf{R})$;

(*c*) $B^\sim=B$ isometrically if and only if $F=0$.

(1.7) THEOREM. *Let B be a normed space on Ω. Define*

$$\|\phi\|_{B^*}=\sup\{|\langle\phi,\mu\rangle| : \mu\in(B)_1\} \quad \forall\phi\in C_c(\Omega), \textit{ and}$$
$$\|\mu\|=\sup\{|\langle\phi,\mu\rangle| : \phi\in C_c(\Omega) \textit{ and } \|\phi\|_{B^*}\le 1\} \quad \forall\mu\in C_c^*(\Omega).$$

Then

$$B^\sim=\{\mu\in C_c^*(\Omega) : \|\mu\|<\infty\}$$

and $\|\mu\|_{B^\sim}=\|\mu\|$ for each $\mu\in B^\sim$.

PROOF: Plainly $\|\cdot\|_{B^*}$ is a semi-norm on $C_c(\Omega)$, and $|\langle\phi,\mu\rangle|\le\|\phi\|_{B^*}\cdot\|\mu\|_B$ for $\phi\in C_c(\Omega)$ and $\mu\in B$. Hence

(1) $$|\langle\phi,\mu\rangle|\le\|\phi\|_{B^*}\cdot\|\mu\|_{B^\sim} \quad \forall\phi\in C_c(\Omega) \text{ and } \mu\in B^\sim.$$

Therefore $\mu\in B^\sim$ implies $\|\mu\|\le\|\mu\|_{B^\sim}$.

To complete the proof, it will suffice to show that $\{\mu\in C_c^*(\Omega) : \|\mu\|\le 1\}$ is contained in $(B^\sim)_1$. Suppose to the contrary that this is false. Then we can find $\mu\in C_c^*(\Omega)$ such that $\|\mu\|\le 1$ and $\mu\notin(B^\sim)_1$. But $(B^\sim)_1$ is a $\sigma(C_c^*,$

C_c)-compact convex balanced subset of $C_c^*(\Omega)$ by Theorem (1.4). So the Hahn-Banach convexity theorem provides us with $\phi\in C_c(\Omega)$ such that

(2) $$|\langle\phi, \nu\rangle|\leq 1<|\langle\phi, \mu\rangle| \quad \forall\nu\in(B^{\sim})_1.$$

As $(B)_1\subset(B^{\sim})_1$, the first inequality in (2) shows that $\|\phi\|_{B^*}\leq 1$. It follows from the second inequality in (2) that
$$1<|\langle\phi, \mu\rangle|\leq\|\phi\|_{B^*}\cdot\|\mu\|\leq\|\mu\|,$$
which contradicts our choice of μ.

(1.8) CONVENTION. We shall close this section with some convention about Radon measures, which will be needed in sections 2 and 4.

(a) Let τ be a positive regular Borel measure on Ω. We denote by $L_1^{loc}(\Omega, \tau)$ the space of all (equivalence classes of) τ-measurable functions f on Ω such that $\int_K|f|d\tau<\infty$ for each compact set $K\subset\Omega$. Unless otherwise stated, we shall imbed $L_1^{loc}(\Omega, \tau)$ into $C_c^*(\Omega)$ via $f\to f\tau$:

$$\langle\phi, f\tau\rangle=\int\phi f\,d\tau \quad \forall\phi\in C_c(\Omega) \quad \text{and} \quad f\in L_1^{loc}(\Omega, \tau).$$

(b) Let $\mu\in C_c^*(\Omega)$ be given. For each $\phi\in C_c^+(\Omega)$, define

$$F(\phi)=\sup\{|\langle\psi, \mu\rangle| : \psi\in C_c(\Omega) \quad \text{and} \quad |\psi|\leq\phi\}.$$

Then F extends to a positive linear functional on $C_c(\Omega)$. So the Riesz Representation Theorem yields a unique positive regular Borel measure $|\mu|$ on Ω such that

$$F(\phi)=\int\phi d|\mu| \quad \forall\phi\in C_c(\Omega).$$

Moreover, $L_1^*(\Omega, \tau)=L_\infty(\Omega, \tau)$ by Theorem (12.18) of E. Hewitt and K.A. Ross [5]. So we can find a $|\mu|$-measurable function v on Ω, with $|v|=1$, such that

$$\langle\phi, \mu\rangle=\int\phi v d|\mu| \quad \forall\phi\in C_c(\Omega).$$

Such a v is unique in the obvious sense. For $\phi\in C_c(\Omega)$ and $f\in L_1^{loc}(\Omega, |\mu|)$, we define

$$\langle\phi, f\mu\rangle=\int\phi f d\mu=\int\phi v d|\mu|.$$

(c) $M(\Omega)$ is the space of all bounded regular Borel measures on Ω

equipped with the total variation norm $\|\cdot\|_M$. We shall identify $M(\Omega)$ with the Banach space dual of $C_0(\Omega)$, where $C_0(\Omega)$ is the uniform closure of $C_c(\Omega)$ in $C(\Omega)$.

§2. Banach modules over weighted L_1-spaces.

Let G be a locally compact group with left Haar measure $d\lambda_G(x)=dx$. Given two (Borel) measurable functions f, g on G, the convolution product of f, g is defined by

$$(f*g)(x)=\int f(y)g(y^{-1}x)dy$$

at each $x\in G$ for which $y\mapsto f(y)g(y^{-1}x)$ is λ_G-integrable; see [5]. If $\mu\in C_c^*(G)$, then $\mu*g$ is defined by

$$(\mu*g)(x)=\int g(y^{-1}x)\, d\mu(y)$$

at each $x\in G$ for which $y\mapsto g(y^{-1}x)$ is in $L_1(G,|\mu|)$. If both f and g are $[0,\infty]$-valued and $\mu\geq 0$, we shall also use the above formulas to define $(f*g)(x)$ and $(\mu*g)(x)$ for *all* $x\in G$.

It is straightforward to show that if either supp(f) or supp$(g)$is σ-compact, then $|f|*|g|$ is measurable on G, and $f*g$ is measurable on the set $\{|f|*|g|<\infty\}$. The same result also holds with f replaced by μ. Moreover, if f, g, h are $[0,\infty]$-valued measurable functions on G, and if at least two of them have σ-compact supports, then $(f*g)*h=f*(g*h)$ everywhere. For partial proofs of these facts, see the appendix to Vol. II of Hewitt-Ross [5].

Throughout this section and section 4, G is a locally compact group, and all the Lebesgue spaces $L_p(G)$, $0<p\leq\infty$, as well as $L_1^{loc}(G)$ are taken with respect to λ_G. We assume that G acts on the locally compact space Ω as a topological transformation group. The latter means that the pair (G, Ω) is equipped with a continuous mapping

$$(x,\omega)\mapsto x\cdot\omega : G\times\Omega\to\Omega$$

such that

(T. 1) $e\cdot\omega=\omega \quad \forall\omega\in\Omega$, and
(T. 2) $x\cdot(y\cdot\omega)=(xy)\cdot\omega \quad \forall x, y\in G$ and $\omega\in\Omega$.

For examples of group actions, see §4.

(2. 1) DEFINITIONS. (a) Given a function h on Ω, define

$$({}_xh)(\omega)=h_\omega(x)=h(x\cdot\omega) \quad \forall x\in G \quad \text{and} \quad \omega\in\Omega.$$

Note that if $h\in C_c(\Omega)$, then ${}_xh\in C_c(\Omega)$ for $x\in G$ and $h_\omega\in C(G)$ for $\omega\in\Omega$. Moreover, $h\in C_c(\Omega)$ and $\nu\in C_c^*(\Omega)$ implies that $x\mapsto\langle {}_xh, \nu\rangle$ is continuous and

$$\langle {}_xh, \nu\rangle=\int_\Omega h(x\cdot\omega)\ d\nu(\omega) \quad \forall x\in G.$$

(b) Let $\mu\in C_c^*(G)$ and $\nu\in C_c^*(\Omega)$. Suppose that for each $\phi\in C_c(\Omega)$, $x\mapsto\langle {}_x\phi, \nu\rangle$ is in $L_1(G, |\mu|)$. Then we define the convolution product $\mu*\nu \in C_c'(\Omega)$ by setting

$$\langle\phi, \mu*\nu\rangle=\int_G\langle {}_x\phi, \nu\rangle\ d\mu(x) \quad \forall\phi\in C_c(\Omega).$$

(2.2) REMARKS: (a) Whenever $\mu*\nu$ is defined, $\mu*\nu\in C_c^*(\Omega)$.

To see this, fix any compact subset K of Ω. By applying the closed graph theorem, one checks that

$$\phi\mapsto\langle {}_x\phi, \nu\rangle : C_K(\Omega)\to L_1(G, |\mu|)$$

is continuous. So there exists a finite constant η_K such that

$$|\langle\phi, \mu*\nu\rangle|\leq\int_G|\langle {}_x\phi, \nu\rangle|d|\mu|(x)\leq\eta_K\|\phi\|_u \quad \forall\phi\in C_K(\Omega).$$

Hence $\mu*\nu\in C_c^*(\Omega)$.

(b) If $\Omega=G$ but $(x, \omega)\mapsto x\cdot\omega$ is *not* the group multiplication in G, our definition of $\mu*\nu$ is confusing. In such a case, we shall write $\mu*_\theta\nu$ for $\mu*\nu$, where $\theta(x, \omega)=x\cdot\omega$ for $x\in G$ and $\omega\in\Omega$.

(2.3) DEFINITION. A weight (function) on G is a strictly positive (finite) Borel function w on G which is submultiplicative:

$$w(xy)\leq w(x)w(y) \quad \forall x, y\in G.$$

Let w be such a function.

(a) Let $L_1(w)=L_1(w\lambda_G)=L_1(G, w\lambda_G)$. Thus $L_1(w)=L_1(G)$ for $w=1$.

(b) For $\mu\in C_c^*(G)$, define $\|\mu\|_{M(w)}=\|w\mu\|_M$. Let

$$M(w)=\{\mu\in C_c^*(G) : \|\mu\|_{M(w)}<\infty\}.$$

Our definition of a weight function is weaker than the corresponding definition in H. Reiter [11; p.83], where the reader will find some interesting examples of weight functions. The following two results are essentially well-known.

(2. 4) LEMMA. *Let w be a weight function on G. Then*:

(a) *Both w and $1/w$ are locally bounded on G.*

(b) *For each $a<1$, $\{w>a\}$ contains a neighborhood of $e\in G$.*

(c) *$L_1(w)\subset M(w)$ isometrically in the sence that $\|f\lambda_G\|_{M(w)}=\|fw\|_1$ for $f\in L_1(w)$.*

PROOF: Choose $\eta<\infty$ so that $\{x\in G: w(x)+w(x^{-1})<\eta\}$ contains a compact set E with $\lambda_G(E)>0$. Then $E^{-1}E$ contains a neighborhood of e by the Steinhaus Theorem, and

$$w(x^{-1}y)\leq w(x^{-1})w(y)<\eta^2 \quad \forall x, y\in E.$$

Therefore w is bounded on some nighborhood of e, and hence on each compact subset of G. Since $0<w(e)\leq w(x^{-1})w(x)$ for each $x\in G$, $1/w$ is also bounded on each compact subset of G, which establishes (a).

To prove (b), pick any $a<1$ and any precompact neighborhood U of e. Since $1/w$ is bounded on U by (a), there exists $\gamma>0$ such that $w(x)\geq \gamma$ for all $x\in U$. Choose a natural number n and a neighborhood V of e so that $V^n\subset U$ and $\gamma^{1/n}>a$. Then $x\in V$ implies

$$w(x)\geq[w(x^n)]^{1/n}\geq\gamma^{1/n}>a.$$

Hence $V\subset\{w>a\}$, which confirms (b).

Part (c) is obvious.

(2. 5) LEMMA. *Let w be a weight function on G. Then each of $L_1(w)$ and $M(w)$ forms a Banach algebra under convolution. Moreover, $f\to f\lambda_G$ is an isometric isomorphism of $L_1(w)$ onto a closed two-sided ideal of $M(w)$.*

PROOF: Plainly $M(w)$ is a vector subspace of $C_c^*(G)$ and $\|\cdot\|_{M(w)}$ is a complete norm on it.

Now let μ, $\nu\in M(w)$ be given. To prove that $\mu*\nu$ exists, pick any $\phi\in C_c(G)$. Since $1/w$ is bounded on $\operatorname{supp}\phi$ by the last lemma, we can find $a<\infty$ such that $|\phi|\leq aw$ on G. Then

$$\begin{aligned}|\langle{}_x\phi, \nu\rangle|&\leq\int|\phi(xy)|d|\nu|(y)\\ &\leq a\int w(xy)d|\nu|(y)\\ &\leq aw(x)\int w(y)d|\nu|(y)\\ &=aw(x)\|\nu\|_{M(w)}.\end{aligned}$$

Since $\mu\in M(w)$, it follows that $x\to\langle{}_x\phi, \nu\rangle$ is in $L_1(G, |\mu|)$. As $\phi\in C_c(G)$

was arbitrary, we conclude that $\mu*\nu$ exists.

Now w is strictly positive on G, and so each element of $M(w)$ has σ-compact support. In particular, $\mu*\nu$ has σ-compact support. So it is easy to show that

$$\begin{aligned}\|w(\mu*\nu)\|_M &= \int wd|\mu*\nu| \\ &\leq \int d|\mu|(x)\int w(xy)d|\nu|(y) \\ &\leq \int w(x)d|\mu|(x)\int w(y)d|\nu|(y) \\ &= \|\mu\|_{M(w)}\|\nu\|_{M(w)}.\end{aligned}$$

Hence $\mu*\nu\in M(w)$ and $\|\mu*\nu\|_{M(w)}\leq\|\mu\|_{M(w)}\|\nu\|_{M(w)}$.

Plainly the convolution product on $M(w)$ is bilinear. Also its associativity follows from (2.9) stated below. Hence $M(w)$ is a Banach algebra under convolution. The proof of the second assertion is left to the reader.

(2.6) LEMMA. *Let $\nu\in C_c^*(G)$, $\rho\in C_c^*(\Omega)$, and h a continuous function on Ω. Suppose that $|\nu|*|\rho|$ exists and $h\in L_1(\Omega, |\nu|*|\rho|)$. Then:*

(a) *The function*

$$g(x):=\int_\Omega |h(x\cdot\omega)|d|\rho|(\omega) \quad \forall x\in G$$

is lower semi-continuous and belongs to $L_1(G, |\nu|)$.

(b) *The function*

$$x\rightarrow\int_\Omega h(x\cdot\omega)d\rho(\omega)$$

defined on $\{g<\infty\}$ is Borel measurable and belongs to $L_1(G, |\nu|)$.

(c) *$\nu*\rho$ exists, $h\in L_1(\Omega, |\nu*\rho|)$, and*

$$\int_\Omega hd(\nu*\rho)=\int_G\left\{\int_\Omega h(x\cdot\omega)d\rho(\omega)\right\}d\nu(x).$$

PROOF: Without loss of generality, we may suppose that $h\geq 0$. For each nonnegative Borel function f on Ω, define

$$f'(x)=\int_\Omega f(x\cdot\omega)d|\rho|(\omega) \quad \forall x\in G.$$

Now let $D=\{\phi\in C_c^+(\Omega):\phi\leq h\}$. Notice that D is directed in the sense that given $\phi_1, \phi_2\in D$, there exists $\phi_3\in D$ such that $\max\{\phi_1, \phi_2\}\leq\phi_3$. Moreover, h is a nonnegative lower semi-continuous (in fact, continuous) function on Ω. It follows from Theorem (9.11) of E. Hewitt and K. Stromberg [6] that for each $x \in G$,

$$(1) \qquad h'(x) = \int_\Omega h(x \cdot \omega) d|\rho|(\omega)$$
$$= \sup\{\phi'(x) : \phi \in D\}.$$

Since each ϕ' with $\phi \in D$ is continuous, we see that h' is lower semi-continuous (and hence Borel measurable).

Now it is clear that $\{\phi' : \phi \in D\}$ is directed. So the theorem cited above ensures that

$$\begin{aligned}\int_G h' d|\nu| &= \sup\left\{\int_G \phi' d|\nu| : \phi \in D\right\} \quad \text{by (1)} \\ &= \sup\left\{\int_G \langle {}_x\phi, |\rho|\rangle d|\nu|(x) : \phi \in D\right\} \\ &= \sup\{\langle \phi, |\nu|*|\rho|\rangle : \phi \in D\} \\ &= \int_\Omega hd(|\nu|*|\rho|) < \infty.\end{aligned}$$

Therefore (a)-(c) hold with (ν, ρ) replaced by $(|\nu|, |\rho|)$.

Finally, there exist positive regular Borel measures ν_k on G $(1 \leq k \leq 4)$, with $\nu_k \leq |\nu|$, such that $\nu = \nu_1 - \nu_2 + i(\nu_3 - \nu_4)$ as linear functionals on $C_c(G)$. Similarly there exist positive regular Borel measures ρ_k on Ω $(1 \leq k \leq 4)$, with $\rho_k \leq \rho$, such that $\rho = \rho_1 - \rho_2 + i(\rho_3 - \rho_4)$ as linear functionals on $C_c(\Omega)$. Since $|\nu|*|\rho|$ exists and $h \in L_1(\Omega, |\mu|*|\nu|)$, it is clear that $\nu_j*\rho_k$ exists and $h \in L_1(\Omega, \nu_j*\rho_k)$ for $j, k \in \{1, 2, 3, 4\}$. Therefore (a)-(c) hold with (ν, ρ) replaced by each (ν_j, ρ_k), which obviously completes the proof.

(2.7) DEFINITIONS. (a) A convolution algebra on G is a normed space A on G such that for each $\mu, \nu \in A$, the convolution $\mu*\nu$ exists in A and $\|\mu*\nu\|_A \leq \|\mu\|_A \|\nu\|_A$.

(b) Let A be a convolution algebra on G. A left normed (resp, Banach) A-module on Ω is a normed (resp. Banach) space B on Ω such that for each $\nu \in A$ and $\rho \in B$, $\nu*\rho$ exists in B and $\|\nu*\rho\|_B \leq \|\nu\|_A \|\rho\|_B$.

(c) A normed space S on Ω has the L-property if $\phi \in C_c(\Omega)$ and $\nu \in S$ implies $\phi\nu \in S$ and $\|\phi\nu\|_S \leq \|\phi\|_u \|\nu\|_S$ (cf. J.E. Taylor [17]). If, in addition, the elements of S with compact support are dense in S, then S has the strong L-property.

Note that, in (a), the associativity of the convolution on A is not postulated.

(2.8) THEOREM. *Let A be a convolution algebra on G which has the L-property, and let B be a normed space on Ω. Suppose that for each* $(\nu,$

$\rho)\in A\times B$, $\nu*\rho$ exists in $B^{\sim}$ and $\|\nu*\rho\|_{B^{\sim}}\leq\|\nu\|_{A}\|\rho\|_{B}$. Then $A^{\sim}$ is a convolution algebra on G, $B^{\sim}$ is a left Banach $A^{\sim}$-module on Ω, and

(*) $\quad(\mu*\nu)*\rho=\mu*(\nu*\rho)\quad\forall\mu,\ \nu\in A^{\sim}$ *and* $\rho\in B^{\sim}$.

PROOF: Given $\rho\in B^{\sim}$, choose a net (ρ_{α}) in B such that

(1) $$\sup_{\alpha}\|\rho_{\alpha}\|_{B}\leq\|\rho\|_{B^{\sim}}\quad\text{and}\quad\rho_{\alpha}\to\rho\text{ in }\sigma(C_{c}^{*},C_{c}).$$

Such a net exists by Theorem (1. 4). We claim that for each $\phi\in C_{c}(\Omega)$ and each compact set $E\subset G$,

(2) $$\langle_{x}\phi,\rho_{\alpha}\rangle\to\langle_{x}\phi,\rho\rangle\quad\text{uniformly in }x\in E.$$

In fact, let $S=\operatorname{supp}\phi$ and $K=\{x^{-1}\cdot\omega:x\in E\text{ and }w\in S\}$. Then K is a compact subset of Ω, and $_{x}\phi\in C_{K}(\Omega)$ for each $x\in E$. Since B is a normed space on Ω, there exists a finite constant η_{K} such that

(3) $$|\langle\psi,\rho'\rangle|\leq\eta_{K}\|\psi\|_{u}\|\rho'\|_{B}\quad\forall\psi\in C_{K}(\Omega)\text{ and }\rho'\in B.$$

Therefore $x,y\in E$ implies

$$\begin{aligned}|\langle_{x}\phi,\rho_{\alpha}\rangle-\langle_{y}\phi,\rho_{\alpha}\rangle|&=|\langle_{x}\phi-{}_{y}\phi,\rho_{\alpha}\rangle|\\&\leq\eta_{K}\|_{x}\phi-{}_{y}\phi\|_{u}\|\rho_{\alpha}\|_{B}\quad\text{by (3)}\\&\leq\eta_{K}\|_{x}\phi-{}_{y}\phi\|_{u}\|\rho\|_{B^{\sim}}\quad\text{by (1)}.\end{aligned}$$

As $\phi\in C_{c}(\Omega)$, this shows that the functions $x\to\langle_{x}\phi,\rho_{\alpha}\rangle$ are equi-continuous at each point of E. Since $\langle_{x}\phi,\rho_{\alpha}\rangle\to\langle_{x}\phi,\rho\rangle$ for each $x\in G$ by (1) and E is compact, (2) follows from the Arzelà-Ascoli Theorem.

Now set

(4) $$\|\phi\|_{B^{*}}=\sup\{|\langle\phi,\rho\rangle|:\rho\in B^{\sim}\text{ and }\|\rho\|_{B^{\sim}}\leq1\}\quad\forall\phi\in C_{c}(\Omega).$$

We claim that $\nu\in A^{\sim}$ and $\rho\in B^{\sim}$ implies

(5) $$\int|\langle_{x}\phi,\rho\rangle|d|\nu|(x)\leq\|\phi\|_{B^{*}}\|\nu\|_{A^{\sim}}\|\rho\|_{B^{\sim}}\quad\forall\phi\in C_{c}(\Omega).$$

Suppose to the contrary that (5) is false for some $\phi\in C_{c}(\Omega)$:

$$\|\phi\|_{B^{*}}\|\nu\|_{A^{\sim}}\|\rho\|_{B^{\sim}}<\int|\langle_{x}\phi,\rho\rangle|d|\nu|(x).$$

Since $x\to\langle_{x}\phi,\rho\rangle$ is continuous, it follows from the regularity of $|\nu|$ that there exists $\psi\in C_{c}(G)$ such that $\|\psi\|_{u}\leq1$ and

$$\|\phi\|_{B^{*}}\|\nu\|_{A^{\sim}}\|\rho\|_{B^{\sim}}<\left|\int\langle_{x}\phi,\rho\rangle\psi(x)d\nu(x)\right|.$$

As $\rho\in B^{\sim}$, Theorem (1. 4) and (2) ensure that there exists $\rho'\in B$ such that $\|\rho'\|_B \le \|\rho\|_{B^\sim}$ and

$$\|\phi\|_{B^*}\|\nu\|_{A^\sim}\|\rho'\|_B < \left|\int \langle {}_x\phi, \rho'\rangle \psi(x) d\nu(x)\right|.$$

Similarly we can find $\nu' \in A$ such that $\|\nu'\|_A \le \|\nu\|_{A^\sim}$ and

$$\text{(6)} \qquad \|\phi\|_{B^*}\|\nu'\|_A\|\rho'\|_B < \left|\int \langle {}_x\phi, p'\rangle \psi(x) d\nu'(x)\right|.$$

Upon utilizing the L-property of A, we obtain

$$\begin{aligned}\|\phi\|_{B^*}\|\nu'\|_A\|\rho'\|_B &< |\langle \phi, (\psi\nu')*\rho'\rangle| \quad \text{by (6)}\\ &\le \|\phi\|_{B^*}\|(\psi\nu')*\rho'\|_{B^\sim} \quad \text{by (4)}\\ &\le \|\phi\|_{B^*}\|\psi\nu'\|_A\|\rho'\|_B \quad \text{by hypothesis}\\ &\le \|\phi\|_{B^*}\|\nu'\|_A\|\rho'\|_B,\end{aligned}$$

which is of course a contradiction. This *reductio ad absurdum* establishes (5).

Recall that $(B^\sim)^\sim = B^\sim$ isometrically by Theorem (1. 4). It follows from (5) and Theorem (1. 7) that $\nu \in A^\sim$ and $\rho \in B^\sim$ implies $\nu*\rho$ exists in $B^\sim$ and

$$\text{(7)} \qquad \|\nu*\rho\|_{B^\sim} \le \|\nu\|_{A^\sim}\|\rho\|_{B^\sim}.$$

If we take $\Omega = G$, $x\cdot\omega = x\omega (x, \omega \in G)$ and $B = A$, then (7) ensures that $A^\sim$ is a convolution algebra on G. Therefore, again by (7), $B^\sim$ is a left Banach $A^\sim$-module on Ω.

Now we claim that $\mu \in A^\sim$ implies $|\mu| \in A^\sim$ and $\||\mu|\|_{A^\sim} \le \|\mu\|_{A^\sim}$. In fact, an easy application of Lusin's Theorem yields a net (ψ_α) in $C_c(G)$ such that $\|\psi_\alpha\|_u \le 1$ for each α and $\psi_\alpha\mu \to |\mu|$ in $\sigma(C_c^*, C_c)$. Since A has the L-property, it is clear that $A^\sim$ has the L-property as well. Hence $\|\psi_\alpha\mu\|_{A^\sim} \le \|\mu\|_{A^\sim}$ for each α, and so $|\mu| \in A^\sim$ and $\||\mu|\|_{A^\sim} \le \|\mu\|_{A^\sim}$ by Theorem (1. 4).

To complete the proof, pick any $\mu, \nu \in A^\sim$ and $\rho \in B^\sim$. Then $|\mu|, |\nu| \in A^\sim$ by the above claim, $|\mu|*|\nu| \in A^\sim$ since $A^\sim$ is a convolution algebra on G, and so $(|\mu|*|\nu|)*\rho$ exists since $B^\sim$ is a left Banach $A^\sim$-module on Ω. If $\phi \in C_c(\Omega)$, it follows that the continuous function $x \to \langle {}_x\phi, \rho\rangle$ belongs to $L_1(G, |\mu|*|\nu|)$; hence

$$\begin{aligned}\langle \phi, (\mu*\nu)*\rho\rangle &= \int_G \langle {}_x\phi, \rho\rangle d(\mu*\nu)(x)\\ &= \int_G\int_G \langle {}_{xy}\phi, \rho\rangle d\nu(y) d\mu(x) \quad \text{by (2. 6)}\\ &= \int_G \langle {}_x\phi, \nu*\rho\rangle d\mu(x) \quad \text{since } {}_{xy}\phi = {}_y({}_x\phi)\\ &= \langle \phi, \mu*(\nu*\rho)\rangle.\end{aligned}$$

As $\phi \in C_c(\Omega)$ was arbitrary, this establishes (*), which completes the proof.

(2.9) REMARKS: By (2.8) with $\Omega = G$, $x \cdot \omega = x\omega (x, \omega \in G)$ and $B = A$, the convolution on a convolution algegra A is associative whenever A has the L-property. In particular, the convolution on $M(w)$ is associative for each weight function w on G.

(2.10) PPOROSITION. *In addition to the hypotheses of* (2.8), *suppose that A has the strong L-property, $\nu \in A$, $\rho \in B^{\sim}$, and (ρ_α) is a norm-bounded net in $B^{\sim}$ such that $\rho_\alpha \to \rho$ in $\sigma(C_c^*, C_c)$. Then $\nu * \rho_\alpha \to \nu * \rho$ in $\sigma(C_c^*, C_c)$.*

PROOF: Upon replacing each ρ_α by $\rho_\alpha - \rho$, we may suppose that $\rho = 0$. Pick any $\phi \in C_c(\Omega)$. We need to show that $\langle \phi, \nu * \rho_\alpha \rangle \to 0$.

Let $\varepsilon > 0$ be given. As A has the strong L-property, we can find $\nu' \in A$ with compact support such that $\|\nu - \nu'\|_A < \varepsilon$. Moreover, the net (ρ_α) is norm-bounded in $B^{\sim}$ and $\rho_\alpha \to 0$ in $\sigma(C_c^*, C_c)$. Hence the functions $x \to \langle {}_x\phi, \rho_\alpha \rangle$ converge to 0 uniformly on each compact subset of G (see the proof of (2.8)). As supp ν' is compact, it follows that

$$\text{(1)} \qquad \langle \phi, \nu' * \rho_\alpha \rangle = \int \langle {}_x\phi, \rho_\alpha \rangle d\nu'(x) \to 0.$$

Also there exists a finite constant η_K such that

$$\text{(2)} \qquad |\langle \phi, \rho' \rangle| \leq \eta_K \|\phi\|_u \|\rho'\|_{B^{\sim}}, \quad \forall \rho' \in B^{\sim},$$

where $K = \text{supp } \phi$. Hence

$$\text{(3)} \qquad \begin{aligned} |\langle \phi, (\nu - \nu') * \rho_\alpha \rangle| &\leq \eta_K \|\phi\|_u \|(\nu - \nu') * \rho_\alpha\|_{B^{\sim}} \quad \text{by (2)} \\ &\leq \eta_K \|\phi\|_u \|\nu - \nu'\|_{A^{\sim}} \|\rho_\alpha\|_{B^{\sim}} \quad \text{by (2.8)} \\ &\leq \eta_K \|\phi\|_u \cdot \varepsilon\gamma, \end{aligned}$$

where $\gamma = \sup_\alpha \|\rho_\alpha\|_{B^{\sim}} < \infty$. Since $\varepsilon > 0$ was arbitrary, we conclude from (1) and (3) that $\langle \phi, \nu * \rho_\alpha \rangle \to 0$, as desired.

(2.11) DEFINITIONS. Let A be a convolution abgebra on G, and let B be a left Banach A-module on Ω. A multiplier (or module-homomorphism) from A into B is a bounded linear mapping $T: A \to B$ such that

$$T(\mu * \nu) = \mu * (T\nu) \quad \forall \mu, \nu \in A.$$

The space of all such T's is denoted by $\mathscr{M}(A, B)$. For $\rho \in C_c^*(\Omega)$, we write $\rho \in \mathscr{M}(A, B)$ to mean that the expression $T_\rho \nu = \nu * \rho (\nu \in A)$ defines an element T_ρ of $\mathscr{M}(A, B)$.

Note that $\mathscr{M}(A, B)$ forms a Banach space with respect to the operator norm.

(2.12) THEOREM. *Let A be a convolution Banach algebra on G with the strong L-property, and let B be a left Banach A-module on Ω. Suppose that A has a bounded right approximate identity (ν_α) such that $\nu_\alpha * \rho \to \rho$ in $\sigma(C_c^*, C_c)$ for each $\rho \in B^\sim$. Then*:

(a) *Each $T \in \mathscr{M}(A, B)$ has the form $T = T_\rho$ for a unique $\rho \in B^\sim$.*

(b) *If $A * B^\sim \subset B$, then $\rho \to T_\rho$ is a Banach space isomorphism of $B^\sim$ onto $\mathscr{M}(A, B)$. If, in addition, $\lim_\alpha \|\nu_\alpha\|_A = 1$ and $\|\cdot\|_{B^\sim} = \|\cdot\|_B$ on B, then the isomorphism is isometric.*

PROOF: Pick any $\nu \in C_c^*(G)$ with compact support. For $\phi \in C_c(\Omega)$, set

$$(\phi \cdot \nu)(\omega) = \int \phi(x \cdot \omega) d\nu(x) \quad \forall \omega \in \Omega.$$

It is clear that $\phi \cdot \nu \in C_c(\Omega)$. Moreover, $\nu * \rho$ exists for each $\rho \in C_c^*(\Omega)$, and

$$(1) \qquad \langle \phi \cdot \nu, \rho \rangle = \langle \phi, \nu * \rho \rangle \quad \forall \phi \in C_c(\Omega) \quad \text{and} \quad \rho \in C_c^*(\Omega)$$

by Fubini's Theorem.

Now A has the L-property, so $B^\sim$ is a left Banach $A^\sim$-module (hence A-module) on Ω by Theorem (2.8). Let $T \in \mathscr{M}(A, B)$ be given. Since (ν_α) is a bounded net in A, $(T\nu_\alpha)$ is a bounded net in B. It follows from Theorem (1.4) that $(T\nu_\alpha)$ has a $\sigma(C_c^*, C_c)$-cluster point $\rho \in B^\sim$ such that $\|\rho\|_{B^\sim} \leq \gamma \|T\|$, where $\gamma = \liminf_\alpha \|\nu_\alpha\|_A$. Passing to a subnet, we may suppose that $T\nu_\alpha \to \rho$ in $\sigma(C_c^*, C_c)$. To prove $T = T_\rho$, pick any $\mu \in A$ with compact support. Then $\phi \in C_c(\Omega)$ implies

$$\begin{aligned}
\langle \phi, T\mu \rangle &= \lim_\alpha \langle \phi, T(\mu * \nu_\alpha) \rangle \\
&= \lim_\alpha \langle \phi, \mu * T\nu_\alpha \rangle \\
&= \lim_\alpha \langle \phi \cdot \mu, T\nu_\alpha \rangle \quad \text{by (1)} \\
&= \langle \phi \cdot \mu, \rho \rangle = \langle \phi, \mu * \rho \rangle \quad \text{by (1)}.
\end{aligned}$$

Hence $T\mu = \mu * \rho$ whenever $\mu \in A$ has compact support. In general, choose a sequence (μ_n) in A such that each μ_n has compact support and $\|\mu_n - \mu\|_A \to 0$; such a sequence exists since A has the strong L-property. Then

$$\begin{aligned}
\|T\mu - \mu * \rho\|_{B^\sim} &\leq \|T(\mu - \mu_n)\|_{B^\sim} + \|(\mu_n - \mu) * \rho\|_{B^\sim} \\
&\leq \|T\| \cdot \|\mu - \mu_n\|_A + \|\mu_n - \mu\|_A \|\rho\|_{B^\sim} \\
&\to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Hence $T\mu=\mu*\rho$ for each $\mu\in A$. We have thus proved that

(2) $\|\rho\|_{B^{\sim}}\leq\gamma\|T\|$ and $T=T_{\rho}$.

Moreover, $T\nu_{\alpha}=\nu_{\alpha}*\rho\rightarrow\rho$ in $\sigma(C_{c}^{*}, C_{c})$ by hypothesis, so the element $\rho\in B^{\sim}$ satisfying $T=T_{\rho}$ is unique, which establishes (a).

To prove (b), suppose that $A*B^{\sim}\subset B$. Choose and fix any $\rho\in B^{\sim}$, and let $T_{\rho}\mu=\mu*\rho$ for each $\mu\in A$. Then T_{ρ} is linear and $T_{\rho}(\mu*\nu)=\mu*T_{\rho}\nu$ for each $\mu, \nu\in A$ by Theorem (2.8). To prove $T_{\rho}\in\mathscr{M}(A, B)$, we need to check the continuity of T_{ρ}. Suppose (ν_{n}) is a sequence in A such that $\|\nu_{n}\|_{A}\rightarrow 0$. Then $\|T_{\rho}\nu_{n}\|_{B^{\sim}}=\|\nu_{n}*\rho\|_{B^{\sim}}\leq\|\nu_{n}\|_{A}\|\rho\|_{B^{\sim}}\rightarrow 0$. Since $\|\cdot\|_{B^{\sim}}\leq\|\cdot\|_{B}$ on B, it follows from the Closed Graph Theorem that $T_{\rho}: A\rightarrow B$ is bounded. Hence $T_{\rho}\in\mathscr{M}(A, B)$. One more application of the Closed Graph Theorem shows that

(3) $\rho\rightarrow T_{\rho}: B^{\sim}\rightarrow\mathscr{M}(A, B)$

is bounded. From this and (2), we infer that the mapping in (3) is a surjective Banach space isomorphism.

Now suppose further that $\gamma=1$ and $\|\cdot\|_{B^{\sim}}=\|\cdot\|_{B}$ on B. Then $\rho\in B^{\sim}$ and $\nu\in A$ implies

$$\begin{aligned}\|T_{\rho}\nu\|_{B}&=\|\nu*\rho\|_{B^{\sim}}\\&\leq\|\nu\|_{A}\|\rho\|_{B^{\sim}}\quad\text{by (2.8).}\end{aligned}$$

Hence $\|T_{\rho}\|\leq\|\rho\|_{B^{\sim}}$, which combined with (2) completes the proof.

(2.13) REMARKS: It is possible to reduce the last theorem from Theorem 4 of C.V. Comisky [2]. However, his proof is based upon some results of M. Rieffel [12] involving projective tensor products, and first of all, we can easily modify our proof to obtain a quick and direct proof of his theorem.

(2.14) DEFINITION. A weight function w on G is moderate if given a neighborhood V of $e\in G$ and $a>0$, $V\cap\{w<1+a\}$ has positive Haar measure.

(2.15) COROLLARY. *Let w be a weight function on G, and let B be a left Banach $L_{1}(w)$-module on Ω.*

(i) *If the closed unit ball of B is $\sigma(C_{c}^{*}, C_{c})$-compact, then $\rho\rightarrow T_{\rho}$ is a Banach space isomorphism of B onto $\mathscr{M}(L_{1}(w), B)$. If, in addition, w is moderate, then the isomorphism is isometric.*

(ii) *If B_{0} is a closed subspace of $B^{\sim}$ such that $L_{1}(w)*B^{\sim}\subset B_{0}$, then $\rho\rightarrow T_{\rho}$ is a Banach space isomorphism of $B^{\sim}$ onto $\mathscr{M}(L_{1}(w), B_{0})$. If, in* *addition, w is moderate, then the isomorphism is isometric.*

PROOF: We shall first construct a bounded approximate identity for $L_1(w)$.

By Lemma (2.4), w is locally bounded. So we can find $a<\infty$ such that for each neighborhood V of $e\in G$, $V\cap\{w\leq a\}$ has nonzero Haar measure. Choose and fix any compact neighborhood V_0 of e. With each neighborhood $V\subset V_0$ of e, associate any nonnegative simple Borel function g_V on G such that

(1) $$\int g_V dx=1 \quad \text{and} \quad \{g_V>0\}\subset V\cap\{w\leq a\}.$$

This defines a net (g_V) in $L_1^+(w)$, where the V's are directed downward by set-inclusion. Note that

(2) $$\|g_V w\|_1=\int g_V w dx\leq a$$

by (1).

If $f\in L_1(w)$, then

(3) $$\|(f*g_V)w\|_1\leq\|fw\|_1\|g_V w\|_1\leq a\|fw\|_1$$

by Lemma (2.5) and (2). Moreover, $\psi\in C_c(G)$ implies

$$\text{supp }(\psi*g_V)\subset(\text{supp }\psi)V_0$$

and $\|\psi * g_V-\psi\|_u\to 0$ by (1). Since w is locally bounded, it follows that $\|(\psi*g_V-\psi)w\|_1\to 0$. Moreover, (3) shows that the operators $f\to f*g_V: L_1(w)\to L_1(w)$ have norms $\leq a$. As $C_c(G)$ is dense in $L_1(w)$, we conclude that

(4) $$\|(f*g_V-f)w\|_1\to 0 \quad \forall f\in L_1(w).$$

In other words, the g_V's form a bounded right approximate identity with norm $\leq$a. Also it is easy to check that

(5) $$g_V*\rho\to\rho \quad \text{in} \quad \sigma(C_c^*, C_c) \quad \forall\rho\in C_c^*(\Omega).$$

Now consider the case where w is moderate. Then the net (g_V) can be chosen to satisfy $\|g_V w\|_1\to 1$. In fact, if G is discrete, this is obvious. So suppose that G is nondiscrete. Then, since w is moderate, we can replace a in (1) by $1+\lambda_G(V)$ to get $\|g_V w\|_1\leq 1+\lambda_G(V)$; hence $\|g_V w\|_1\to 1$.

We are now ready to prove (i) and (ii). If $(B)_1$ is $\sigma(C_c^*, C_c)$-compact, then $B^{\sim}=B$ isometrically by Corollary (1.5). Hence (i) follows from Theorem (2.12) with $A=L_1(w)$.

To prove (ii), let B_0 be a closed subspace of $B^\sim$ such that $L_1(w)*B^\sim \subset B_0$. We of course equip B_0 with $\|\cdot\|_{B^\sim}$. Since $(B^\sim)^\sim = B^\sim$ isometrically, it is obvious that

$$\rho \in B_0^\sim \Longrightarrow \rho \in B^\sim \quad \text{and} \quad \|\rho\|_{B^\sim} \leq \|\rho\|_{B_0^\sim}. \tag{6}$$

To show $B_0^\sim = B^\sim$, let a and $g_V \in L_1^+(w)$ be as in the above construction. Then $\rho \in B^\sim$ implies

$$\|g_V*\rho\|_{B_0} = \|g_V*\rho\|_{B^\sim} \leq a\|\rho\|_{B^\sim}$$

by Theorem (2.8) and (2), and $g_V*\rho \to \rho$ in $\sigma(C_c^*, C_c)$ by (5). Therefore

$$\rho \in B^\sim \Longrightarrow \rho \in B_0^\sim \quad \text{and} \quad \|\rho\|_{B_0^\sim} \leq a\|\rho\|_{B^\sim}. \tag{7}$$

From (6) and (7), we infer that $B^\sim = B_0^\sim$ as sets and $\|\rho\|_{B^\sim} \leq \|\rho\|_{B_0^\sim} \leq a\|\rho\|_{B^\sim}$ for each $\rho \in B^\sim$. It is now obvious that if w is moderate, then $B^\sim = B_0^\sim$ isometrically. Hence (ii) also follows Theorem (2.12), which completes the proof.

§ 3. Lorentz spaces $L_{p,q}$.

This section gives brief accounts of Lorentz spaces. For the proofs of some of those results which are stated without proof, see Hunt [7] and Yap [21].

Let (X, μ) be a measure space, and let f be a measurable function on X. For each $0 \leq s < \infty$, let

$$\mu_f(s) = \mu(|f| > s) = \mu(\{|f| > s\}).$$

The function μ_f is called the *distribution function* of f (with respect to μ). The *decreasing rearrangement function* of f is the function f^* on $(0, \infty)$ defined by

$$f^*(t) = \inf\{s \geq 0 : \mu_f(s) \leq t\},$$

where $\inf \emptyset = +\infty$. Thus both μ_f and f^* are $[0, \infty]$-valued decreasing "right-continuous" functions on $(0, \infty)$. Let

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*\, ds \quad \forall t \in (0, \infty).$$

For $p, q \in (0, \infty)$, we define

$$\|f\|_{p,q}^* = \|f\|_{p,q,\mu}^* = \left(\frac{q}{p}\int_0^\infty [f^*(t)]^q t^{\frac{q}{p}-1} dt\right)^{\frac{1}{q}},$$

$$\|f\|_{p,q} = \|f\|_{p,q,\mu} = \left(\frac{q}{p}\int_0^\infty [f^{**}(t)]^q t^{\frac{q}{p}-1} dt\right)^{\frac{1}{q}}.$$

If $0<p<q=\infty$, we also define

$$\|f\|_{p,\infty}^{*}=\sup_{t>0} t^{\frac{1}{p}} f^{*}(t) \quad \text{and} \quad \|f\|_{p,\infty}=\sup_{t>0} t^{\frac{1}{p}} f^{**}(t).$$

Finally, let $\|f\|_{\infty,\infty}^{*}=\|f\|_{\infty,\infty}=\|f\|_{\infty}$.

For $0<p<\infty$ and $0<q\leq\infty$, the Lorentz space $L_{p,q}(X, \mu)$ is defined to be the vector space of all (equivalence classes of) measurable functions f on X with $\|f\|_{p,q}^{*}<\infty$. For $p=q=\infty$, we let $L_{p,q}(X, \mu)=L_{\infty}(X, \mu)$.

(3.1) REMARKS: Let f, g be two measurable functions on $X, 0<p<\infty$, and $0<q\leq\infty$.

(a) *If* $|f|\leq|g|$ a.e., then $\mu_f\leq\mu_g$ and so $f^{*}\leq g^{*}$.

(b) $(|f|^{p})^{*}=(f^{*})^{p}$ and $(\alpha f)^{*}=|\alpha|f^{*}$ for $\alpha\in\mathbf{C}$.

(c) For $t, s>0$, we have $f^{*}(t)\leq s$ if and only if $\mu_f(s)\leq t$. Hence $\mu_f=\lambda_{f^{*}}$, where λ is the Lebesgue measure restricted to $(0, \infty)$. Moreover, one checks that

$$\|f\|_{p,\infty}^{*}=\sup_{t>0} t^{\frac{1}{p}} f^{*}(t)=\sup_{s>0} s[\mu_f(s)]^{\frac{1}{p}}.$$

(d) If $f\in L_{p,q}(X, \mu)$, then $\|f\|_{p,\infty}^{*}\leq\|f\|_{p,q}^{*}$; in particular, $\mu_f(s)<\infty$ for each $s>0$ and $L_{p,q}\subset L_{p,\infty}$.

(e) $\|f\|_{p,p}^{*}=\|f\|_{p}$ and $L_{p,p}(X, \mu)=L_{p}(X, \mu)$.

(f) It is not difficult to show that for $1\leq q\leq p<\infty$, $\|\cdot\|_{p,q}^{*}$ is a complete norm on $L_{p,q}(X, \mu)$. However, if $p<q$, then $\|\cdot\|_{p,q}^{*}$ is not a norm on $L_{p,q}(X, \mu)$ in general.

(g) For $p\leq 1$, our definition of $\|f\|_{p,q}$ is different from the usual one [7].

(3.2) THEOREM. *Let $1<p<\infty$ and $1\leq q\leq\infty$. Then*

$$\|f\|_{p,q}^{*}\leq\|f\|_{p,q}\leq\frac{p}{p-1}\|f\|_{p,q}^{*}$$

for each measurable function f on X. Moreover, $\|\cdot\|_{p,q}$ is a complete norm on $L_{p,q}(X, \mu)$.

(3.3) LEMMA. *Let f be a measurable function on X, and let $E\subset X$ be a measumable set. Then*

(i) $\int_E |f|\,d\mu\leq\int_0^{\mu(E)} f^{*}\,ds.$

Moreover, if

(ii) $s>0$ *and* $\mu(E\cap\{|f|>s\})<\mu(E)\Longrightarrow\mu(\{|f|>s\}\backslash E)=0,$

then equality obtains in (i). *Conversely, if equality obtains in* (i) *and both sides are finite, then* (ii) *holds.*

PROOF: We may suppose that $f \geq 0$. Since $\mu_f = \lambda_{f^*}$ by (3.1)(c), it follows that

$$(1) \qquad \int f\, d\mu = \int_0^\infty \mu_f(s)\, ds = \int_0^\infty \lambda_{f^*}(s)\, ds = \int_0^\infty f^*\, dt.$$

Also it is easy to check that $f^*(t)=0$ if and only if $t > \mu_f(0)$. It follows from (1) applied to $\chi_E f$ that

$$(2) \qquad \int_E f\, d\mu = \int_0^\infty (\chi_E f)^*\, dt = \int_0^{\mu(E)} (\chi_E f)^*\, dt \leq \int_0^{\mu(E)} f^*\, dt,$$

which establishes (i).

By applying (3.1)(c), we can show that (ii) holds if and only if $(\chi_E f)^* = f^*$ on $(0, \mu(E))$. This fact combined with (2) completes the proof.

(3.4) The Hardy-Littlewood Inequality. Let $f_1, \ldots, f_n$ be measurable functions on X, and let $E \subset X$ be measurable. Then

$$\int_E |f_1 \ldots f_n|\, d\mu \leq \int_0^{\mu(E)} f_1^* \ldots f_1^*\, dt.$$

PROOF: We may suppose that $f_1, \ldots, f_n \geq 0$. Note that for $n=1$, the desired inequality is nothing but (3,3)(i). So suppose that $n \geq 2$ and the result is true with n replaced by $n-1$.

Set $f = f_1$, $g = f_2 \ldots f_n$, and $\phi = f_2^* \ldots f_n^*$. Then the inductive hypothesis ensures shat

$$\int f_1 \ldots f_n\, d\mu = \int \int_0^{f(x)} ds\ g(x)\ d\mu(x) = \int_0^\infty \int_{\{f>s\}} g\, d\mu\, ds \leq \int_0^\infty \int_0^{\mu_f(s)} \phi(t)\, dt\ ds = \int_0^\infty \int_{\{f^*>s\}} \phi(t)\, dt\ ds \quad \text{by (3.1)(c)}$$

$$=\int_0^\infty f^*(t)\phi(t)\,dt.$$

Finally, replace f_1 by $\chi_E f_1$ to get the desired inequality.

(3.5) THEOREM. *Suppose $p, q\in[1,\infty)$ and $q=1$ if $p=1$. If f, g are measurable functions on X, then*

$$\int|fg|\,d\mu\leq\int_0^\infty f^*g^*\,dt$$
$$\leq C_{p,q}\|f\|^*_{p,q}\|g\|^*_{p',q'},$$

where $C_{p,q}=(p/q)^{\frac{1}{q}}(p'/q')^{\frac{1}{q'}}$ if $p, q>1$ and $C_{p,q}=p$ otherwise.

PROOF: The first inequality is the special case of (3.4) with $n=2$ and $E=X$. In the case $p, q>1$, we apply Hölder's Inequality to the functions $f^*(t)t^{\frac{1}{p}}$, $g^*(t)t^{\frac{1}{p'}}$ and to the measure $t^{-1}dt$ to obtain the second inequality. The other case is almost obvious.

(3.6) REMARKS AND DEFINITIONS: Suppose $p, q\in[1,\infty)$ and $q=1$ if $p=1$. If $g\in L_{p',q'}(X,\mu)$, then Theorem (3.5) enables us to define a linear functional ψ_g on $L_{p,q}(X,\mu)$ by

(i) $\psi_g(f)=\int fg\,d\mu \quad \forall f\in L_{p,q}(X,\mu)$.

Let

(ii) $\|\psi_g\|'_{p,q}=\sup\{|\psi_g(f)|:\|f\|^*_{p,q}\leq 1\}$.

Thus

(iii) $\|\psi_g\|'_{p,q}\leq C_{p,q}\|g\|^*_{p',q'}$,

where $C_{p,q}$ is as in (3.5).

(3.7) THEOREM. *If $1<p<\infty$, then $L^*_{p,1}(X,\mu)\simeq L_{p',\infty}(X,\mu)$. Moreover,*

$$\|g\|^*_{p',\infty}\leq\|\psi_g\|'_{p',q}\leq p\|g\|^*_{p',\infty}\quad \forall g\in L_{p',\infty}.$$

(3.8) THEOREM. If $1<p, q<\infty$, then $L^*_{p,q}(X,\mu)\simeq L_{p',q'}(X,\mu)$. Moreover,

(a) $\|g\|^*_{p',q'}\leq(q/p)^{\frac{1}{q}}\|\psi_g\|'_{p,q}$ *for* $1<p\leq q<\infty$, *and*

(b) $\|g\|^*_{p',q'}\leq p'(q'/p')^{\frac{1}{q'}}\|\psi_g\|'_{p,q}$ *for* $1<q\leq p<\infty$.

If (X,μ) has no atoms with finite measure, we have better estimates; namely,

(a') $\|g\|^*_{p',q'}=C^{-1}_{p,q}\|\psi_g\|'_{p,q}$ *for* $1<p\leq q<\infty$, *and*

(b') $\|g\|^*_{p',q'}\leq p'C^{-1}_{p,q}\|\psi_g\|'_{p,q}$ *for* $1<q\leq p<\infty$.

The proof of (3.8) is somewhat lengthy and can be found in Appendix A of [18]. (The corresponding result in Hunt [7] is not as precise as ours. Also his proof appears to be incomplete.)

(3.9) THEOREM. *Let $1<p<\infty$, $1\leq q\leq\infty$, and let (f_α) be a net in $L_{p,q}(X, \mu)$ such that $\|f_\alpha\|_{p,q}\leq C$ for all α's and some $C<\infty$. Then there exits a subnet (f_β) of (f_α) and $f\in L_{p,q}(X, \mu)$ such that $\|f\|_{p,q}\leq C$ and*

$$(*)\qquad \lim_{\beta}\int f_\beta g d\mu=\int fgd\mu \quad \forall g\in S(p', q'),$$

where $S(p', q')$ is the norm-closure of $(L_{p',q'}\cap L_{p',2})(X, \mu)$ in $L_{p',q'}(X, \mu)$.

PROOF: We shall consider the three cases ($1<q<\infty$, $q=1$, and $q=\infty$) separately.

Case 1. Suppose $1<q<\infty$. Then $L_{p,q}(X, \mu)\simeq L^*_{p',q'}(X, \mu)$ by Theorem (3.8). Since (f_α) is a norm-bounded net in $L_{p,q}(X, \mu)$, it follows that (f_α) has a subnet (f_β) which satisfies $(*)$ for some $f\in L_{p,q}(X, \mu)$. To prove $\|f\|_{p,q}\leq C$, choose a bounded a linear function ψ on $L_{p,q}(X, \mu)$ such that

$$(1)\qquad \|\psi\|\leq 1 \quad\text{and}\quad \psi(f)=\|f\|_{p,q}.$$

By (3, 8), ψ is induced by some $g\in L_{p',q'}(X, \mu)$; that is

$$(2)\qquad \psi(h)=\int ghd\mu \ \forall h\in L_{p,r}(X, \mu).$$

Moreover, $S_{p',q'}=L_{p',q'}(X, \mu)$ since $1\leq q'<\infty$. So

$$\begin{aligned}\|f\|_{p,q}&=\int fgd\mu \quad \text{by (1) and (2)}\\ &=\lim_{\alpha}\int f_\alpha gd\mu \quad \text{by } (*)\\ &=\lim_{\alpha}\psi(f_\alpha) \quad \text{by (2)}\\ &\leq\liminf_{\alpha}\|f_\alpha\|_{p,q} \quad \text{by (1)}\\ &\leq C.\end{aligned}$$

Case 2. Suppose $q=1$. Then $\|f_\alpha\|_{p,2}\leq\|f_\alpha\|_{p,1}\leq C$ for all α's. It follows from Case 1 that there exists a subnet (f_β) of (f_α) and $f\in L_{p,2}(X, \mu)$ which satisfy $(*)$ with $q=2$. It is not difficult to show that $f\in L_{p,1}(X, \mu)$ and $(*)$ holds for $q=1$. Since $L^*_{p,1}\simeq L_{p',\infty}$ by (3.7), the proof that $\|f\|_{p,1}\leq C$ is obvious from the corresponding proof in Case 1.

Case 3. Suppose $q=\infty$. As in Case 1, $S_{p',q'}=L_{p',1}(X, \mu)$ and (f_α) has a subnet (f_β) which satisfies $(*)$ for some $f\in L_{p,\infty}(X, \mu)$. To prove $\|f\|_{p,\infty}\leq$

C, we may suppose that (X, μ) is a σ-finite measure space having no atoms; simply replace X by $\{f \neq 0\}$ and then (X, μ) by $(X \times [0, 1], \mu \times \lambda)$, where λ is the Lebesgue measure on $[0, 1]$.

We claim that give $t > 0$, there exists a measurable set $E \subset X$ such that

$$(3) \qquad \int_E |f| d\mu = \int_0^t f^* ds \quad \text{and} \quad \mu(E) \leq t.$$

In fact, if $t \geq \mu_f(0)$, then take $E = \{f \neq 0\}$. If $t < \mu_f(0)$, then we can find $E \subset X$ such that $\mu(E) = t$ and $\{|f| > f^*(t)\} \subset E \subset \{(|f| \geq f^*(t)\}$. In either case, (3) holds by (3. 3).

Upon setting $g = \chi_E \cdot \operatorname{sgn} \bar{f}$, we have

$$\begin{aligned} t^{1/p} f^{**}(t) &= t^{-1/p'} \int_0^t f^* ds \\ &= t^{-1/p'} \int f g d\mu \quad \text{by (3)} \\ &= t^{-1/p'} \lim_\beta \int f_\beta g d\mu \quad \text{by } (*) \\ &\leq t^{-1/P'} \liminf_\beta \int_E |f_\beta| d\mu \\ &\leq t^{-1/p'} \liminf_\beta \int_0^* f_\beta^* ds \quad \text{by (3. 3) and (3)} \\ &= \liminf_\beta t^{1/p} f_\beta^{**}(t). \end{aligned}$$

Hence $\|f\|_{p,\infty} = \sup_t t^{1/p} f^{**}(t) \leq C$.

(3.10) REMARKS: Let $1 < p < \infty$.

(a) As in (3. 9), let $S(p', \infty)$ be the norm-closure of $L_{p',\infty} \cap L_{p',2}$ in $L_{p',\infty}$. For $f \in L_{p,1}$, let $\|f\|'_{p,1}$ denote the norm of the functional $g \to \int f g d\mu$ on $S(p', \infty)$. By applying Theorem (3. 7), it is readily seen that $\|\cdot\|'_{p,1}$ is a norm on $L_{p,1}$ which is equivalent to $\|\cdot\|_{p,1}$. Moreover, our proof of (3. 9) shows that the closed unit ball of $L_{p,1}$ with respect to $\|\cdot\|'_{p,1}$ is weak-$*$ compact, where "weak-$*$" refers to the weak-$*$ topology of $S^*(p', \infty)$. It follows from the Krein-Šmulian Theorem [15; p.108] that $L_{p,1} \simeq S^*(p', \infty)$ in the obvious sense. (A more direct proof is possible.)

(b) If $1 < q < \infty$, then $\|\cdot\|_{p,q}$ in (3. 9) may be replaced by any norm on $L_{p,q}$ which is eqivalent to $\|\cdot\|_{p,q}$; in particular, by $\|\cdot\|^*_{p,q}$ if $q \leq p$. This is also true for $q = 1$ if the new norm $\|\cdot\|$ on $L_{p,1}$ satisfies $\|f\| \leq \|g\|$ whenever $|f| \leq |g|$. Similar comments apply to (3. 11) stated below, as well.

(3.11) THEOREM. *Let $(X, \mathscr{A}, \nu)$ and $(\Omega, \mathscr{B}, \tau)$ be two σ-finite measure spaces, and let $h: X\times\Omega\to[0,\infty]$ be $(\nu\times\tau)$-measurable. Set $(xh)(\omega)=h(x,\omega)$ for $x\in X$ and $\omega\in\Omega$, and define*

$$\text{(i)}\qquad (\nu h)(\omega)=\int h(x,\omega)d\nu(x)$$

for τ-a.a. $\omega\in\Omega$. Then

$$\text{(ii)}\qquad \|\nu h\|_{p,q}\leq\int\|xh\|_{p,q}d\nu(x)\quad(1<p<\infty \text{ and } 1\leq q\leq\infty);$$

$$\text{(iii)}\qquad \|\nu h\|^*_{p,q}\leq\int\|xh\|^*_{p,q}d\nu(x)\quad(1\leq q\leq p<\infty).$$

PROOF: By the Monotone Convergence and Fubini's Theorem, we may suppose that $\nu(X)+\tau(\Omega)<\infty$ and h is a bounded $(\mathscr{A}\times\mathscr{B})$-measurable function.

Let $\mathscr{B}_0$ be the Borel σ-algebra of $(0,\infty)$. We shall show that $(x,t)\to(xh)^*(t)$ is $(\mathscr{A}\times\mathscr{B}_0)$-measurable. To this end, fix any $s>0$ and set

$$\text{(1)}\qquad g(x)=\int\chi_s(h(x,\omega))d\tau(\omega)\quad\forall x\in X,$$

where χ_s is the indicator of (s,∞). Then g is $\mathscr{A}$-measurable by Fubini's Theorem. Moreover,

$$\begin{aligned}\{(x,t)\in X\times(0,\infty):(xh)^*(t)\leq s\}&=\{(x,t)\in X\times(0,\infty):\tau(xh>s)\leq t\}\\&=\{(x,t)\in X\times(0,\infty):g(x)\leq t\}\quad\text{by (1)},\end{aligned}$$

which is a member of $\mathscr{A}\times\mathscr{B}_0$. As $s>0$ was arbitrary, this shows that $(x,t)\to(xh)^*(t)$ is $(\mathscr{A}\times\mathscr{B}_0)$-measurable.

Now let $1<p<\infty$ and $1\leq q\leq\infty$ be given. If $q<\infty$, then $x\to\|xh\|_{p,q}$ is $\mathscr{A}$-measurable by the above remark and by Fubini's Theorem. This is also true for $q=\infty$ since

$$\|xh\|_{p,\infty}=\sup\{t^{1/p}(xh)^{**}(t):t\in\mathbf{Q}\text{ and }t>0\}.$$

Hence the integral in the right-hand-side of (ii) is well-defined.

To prove the inequality in (ii), first consider the case $q<\infty$. By our additional assumptions on ν, τ, and h, it is clear that $\nu h\in L_{p,q}(\Omega,\tau)$. Moreover, $f\in L_{p',q'}(\Omega,\tau)$ implies

$$\begin{aligned}\left|\int(\nu h)f\,d\tau\right|&=\left|\iint f(\omega)(xh)(\omega)d\tau(\omega)d\nu(x)\right|\quad\text{by (i)}\\&\leq\int\|f\|\cdot\|xh\|_{p,q}d\nu(x),\end{aligned}$$

where $\|f\|$ denotes the norm of f as a linear functional on $L_{p,q}(\Omega, \tau)$ with respect to $\|\cdot\|_{p,q}$. Since $L^*_{p,q} \simeq L_{p',q'}$, (ii) follows from the Hahn-Banach Theorem.

For $q=\infty$, we argue as follows. As in the proof of the last theorem, we may suppose (Ω, τ) has no atoms. Given $t>0$, choose a measurable set $E \subset \Omega$, with $\tau(E) \leq t$, such that

$$(2) \qquad \int_E (\nu h) d\tau = \int_0^t (\nu h)^* ds.$$

Then

$$\begin{aligned} t(\nu h)^{**}(t) &= \int_E (\nu h) d\tau \quad \text{by (2)} \\ &= \iint \chi_E(\omega) h(x, \omega) d\tau(\omega) d\nu(x) \\ &\leq \int\int_0^t (xh)^*(s) ds d\nu(x) \quad \text{by (3.3)} \\ &= t \int (xh)^{**}(t) d\nu(x). \end{aligned}$$

Hence (ii) holds for $q=\infty$ as well.

Finally, (iii) for $p=1$ is a direct consequence of Fubini's Theorem. If $p>1$, then the proof of (iii) is essentially identical with the proof of (ii), which completes the proof.

Note that Theorem (3.11)(iii) with $p=q \in [1, \infty]$ is nothing but Minkowski's Inequality for double integrals.

§ 4. Lorentz spaces as L_1-modules and examples.

As before, G is a locally compact group which acts on the locally compact space Ω as a topological transformasion group. For a function f on G and $x \in G$, let $f^{\#}(x) = f(x^{-1})$. The symbol Δ always denotes the modular function of G. Thus

$$\int f(xa) dx = \Delta(a^{-1}) \int f(x) dx \quad \text{and} \quad \int f(x^{-1}) dx = \int f(x) \Delta(x^{-1}) dx$$

whenever f is a nonnegative Haar measurable function on G and $a \in G$. Note that Δ is a continuous homomorphism of G into the multiplicative group $(0, \infty)$. Therefore, for each $\alpha \in \mathbf{R}$, both Δ^α and $\max\{\Delta^\alpha, 1\}$ are moderate weights on G.

Now let τ be an arbitrary positive regular Borel measure on Ω. In order to avoid non-essential problems about measurability, we shall assume that for $0<p<\infty$ and $0<q\leq\infty$, $L_{p,q}(\Omega, \tau)$ consists of (equivalence classes of) Borel measurable functions. However, $L_{\infty}(\Omega, \tau)$ is assumed to consist of τ-measurable functions; hence $L_1^*(\Omega, \tau)=L_{\infty}(\Omega, \tau)$ isometrically in the obvious sense. A function f is said to *belong locally* to a space S of measurable functions on Ω if for each compact set $K\subset\Omega$, there exists $g=g_K\in S$ such that $g=f$ τ-almost everywhere on K.

(4.1) LEMMA. *Let τ, ρ be two positive regular Borel measures on Ω which are mutually absolutely continuous. Suppose that $1<p<\infty$, $1\leq q\leq\infty$, and the Radon-Nikodym derivative $d\rho/d\tau$ of ρ with respect to τ belongs locally to $L_{p',q'}(\Omega, \tau)$ if $q>1$ or to $S(p', \infty)$ if $q=1$, where $S(p', \infty)$ is the norm-closure of $L_{p',2}(\Omega,\tau)$ in $L_{p',\infty}(\Omega, \tau)$. Imbed $L_{p,q}(\Omega, \tau)$ into $C_c^*(\Omega)$ via $f\to f\rho$:*

$$\langle\phi, f\rho\rangle=\int\phi f d\rho \quad \forall\phi\in C_c(\Omega) \quad \text{and} \quad f\in L_{p,q}(\Omega, \tau).$$

Then $[L_{p,q}(\Omega, \tau)]^{\sim}=L_{p,q}(\Omega, \tau)$ isometrically with respect to $\|\cdot\|_{p,q}$ (always) and also with respect to $\|\cdot\|_{p,q}^*$ (if $q\leq p$).

PROOF. By our assumptions on τ and ρ, $f\to f\rho$ maps $L_{p,q}(\Omega, \tau)$ isometrically onto a Banach space on Ω.

Now let $\|\cdot\|$ denote either $\|\cdot\|_{p,q}$ or $\|\cdot\|_{q,p}^*$ (if $q\leq p$). Consider the closed unit ball B of $L_{p,q}(\Omega, \tau)$ with respect to $\|\cdot\|$. By Corollary (1.5), the desired conclusion is equivalent to the $\sigma(C_c^*, C_c)$-compactness of $B\rho:=\{f\rho : f\in B\}$. Pick any net (f_α) in B. By Theorem (3.9) combined with Remark (3.10)(b), we can find $f\in B$ and a subnet (f_β) of (f_α) such that

$$(1)\qquad \lim_{\beta}\int f_\beta g d\tau=\int fg d\tau \ \forall g\in S(p', q'),$$

where $S(p', q')$ is the norm closure of $(L_{p',q'}\cap L_{p,2})(\Omega, \tau)$ in $L_{p',q'}(\Omega, \tau)$.

We claim that $f_\beta\rho\to f\rho$ in $\sigma(C_c^*, C_c)$. To confirm this, pick any $\phi\in C_c(\Omega)$. By our assumptions on τ and ρ, there exists $v\in S(p', q')$ such that $\phi d\rho=\phi v d\tau$. Hence

$$\begin{aligned}\lim_{\beta}\int f_\beta\phi d\rho&=\lim_{\beta}\int f_\beta\phi v d\tau\\&=\int f\phi v d\tau \quad \text{by (1)}\\&=\int f\phi d\rho,\end{aligned}$$

which confirms our claim. As (f_α) was an arbitrary net in B, this establishes the $\sigma(C_c^*, C_c)$-compactness of $B\rho$, which completes the proof.

(4.2) DEFINITIONS. Let τ be a nonzero positive regular Borel measure on Ω. The maximal Jacobian of τ (under the action of G) is the function J_τ on G defined by

$$J_\tau(x)=\sup\{\int\phi(x^{-1}\cdot\omega)d\tau(\omega):\phi\in C_c^+(\Omega)\quad and \quad \int\phi d\tau=1\}.$$

The measure τ is boundedly quasi-invariant under the action of G if $J_\tau(x)<\infty$ for each $x\in G$.

It is easy to check that J_τ is lower semi-continuous,

$$J_\tau(xy)\leq J_\tau(x)J_\tau(y)\quad \text{for all}\quad x,y\in G,\quad \text{and}$$
$$\int h(x^{-1}\cdot\omega)d\tau(\omega)\leq J_\tau(x)\int h(\omega)d\tau(\omega)\quad \forall x\in G$$

whenever h is a nonnegative Borel function on Ω. In particular, if τ is boundedly quasi-invariant, then J_τ is a weight function on G.

(4.3) THEOREM. *Let τ and ρ be two nonzero positive regular Borel measures on Ω which are mutually absolutely continuous and each boundedly quasi-invariant under the action of G. Suppose that $1<p<\infty$, $1\leq q\leq\infty$, and $d\rho/d\tau$ belongs locally to $L_{p',q'}(\Omega,\tau)$. Imbed $L_{p,q}(\Omega,\tau)$ into $C_c^*(\Omega)$ via $f\to f\rho$. Then $L_{p,q}(\Omega,\tau)$ forms a left Banach $M(J_\tau^{1/p}J_\rho^\#)$-module on Ω with respect to $\|\cdot\|_{p,q}^*$ (always) and also with respect to $\|\cdot\|_{p,q}^*$ (if $q\leq p$).*

PROOF: Let h be a nonnegative Borel function on Ω. For $x\in G$ and $\omega\in\Omega$, write $(x^{-1}h)(\omega)=h(x^{-1}\cdot\omega)$. We claim that

$$(1)\qquad (x^{-1}h)^*(t)\leq h^*(t/J_\tau(x))\quad \forall x\in G\quad \text{and}\quad t>0,$$

where the decreasing rearrangements are taken with respect to τ. To confirm this, we first estimate the distribution functin of $x^{-1}h$. If $s\geq 0$ and χ_s is the indicator of (s,∞), then

$$(2)\qquad \begin{aligned}\tau_{x^{-1}h}(s)&=\int\chi_s(h(x^{-1}\cdot\omega))d\tau(\omega)\\ &\leq J_\tau(x)\int\chi_s(h(\omega))d\tau(\omega)\\ &=J_\tau(x)\tau_h(s).\end{aligned}$$

Given $t>0$, set $s=s(t,x)=h^*(t/J_\tau(x))$. Then $\tau_h(s)\leq t/J_\tau(x)$, or $J_\tau(x)\tau_h(s)\leq t$. This implies $\tau_{x^{-1}h}(s)\leq t$ by (2), or equivalently $(x^{-1}h)^*(t)\leq s$, which establishes (1). Note that $x\in G$ and $t>0$ implies

$$(3)\quad \begin{aligned}(x^{-1}h^{**})(t)&=t^{-1}\int_0^t (x^{-1}h)^*(s)ds\\ &\leq t^{-1}\int_0^t h^*(s/J_\tau(x))ds \quad \text{by (1)}\\ &=J_\tau(x)t^{-1}\int_0^{t/J_\tau(x)} h^*(u)du\\ &=h^{**}(t/J_\tau(x)).\end{aligned}$$

Next we claim that $x\in G$ implies

$$(4)\quad \|x^{-1}h\|_{p,q}^*\leq J_\tau^{1/p}(x)\|h\|_{p,q}^*, \quad \text{and}$$

$$(5)\quad \|x^{-1}h\|_{p,q}\leq J_\tau^{1/p}(x)\|h\|_{p,q}.$$

In fact, if $q<\infty$, then

$$\begin{aligned}\|x^{-1}h\|_{p,q}^*&=\left(\frac{q}{p}\int_0^\infty \{(x^{-1}h)^*(t)\}^q t^{q/p-1}dt\right)^{1/q}\\ &\leq\left(\frac{q}{p}\int_0^\infty \{h^*(t/J_\tau(x))\}^q\, t^{q/p}t^{-1}dt\right)^{1/p} \quad \text{by (1)}\\ &=\left(\frac{q}{p}\int_0^\infty \{h^*(t)\}^q\{J_\tau(x)t\}^{q/p}t^{-1}dt\right)^{1/q}\\ &=J_\tau^{1/p}(x)\|h\|_{p,q}^*.\end{aligned}$$

For $q=\infty$, we have

$$\begin{aligned}\|x^{-1}h\|_{p,\infty}^*&=\sup_{t>0} t^{1/p}(x^{-1}h)^*(t)\\ &\leq\sup_{t>0} t^{1/p}h^*(t/J_\tau(x)) \quad \text{by (1)}\\ &=J_\tau^{1/p}(x)\|h\|_{p,\infty}^*.\end{aligned}$$

Thus (4) holds in either case. Similarly (5) follows from (3).

Now let $w=J_\tau^{1/p}J_\rho^\#$, and let $\|\cdot\|$ denote either $\|\cdot\|_{p,q}$ or $\|\cdot\|_{p,q}^*$ (if $q\leq p$). Plainly w is a weight function on G, and so $M(w)$ is a convolution algebra on G by (2.5). To prove that $L_{p,q}(\Omega,\tau)$ forms a left Banach $M(w)$ module on Ω, pick any $\nu\in M^+(w)$ and $h\in L_{p,q}^+(\Omega,\tau)$. Set

$$(6)\quad [(J_\rho^\#\nu)h](\omega)=\int_G h(x^{-1}\cdot\omega)J_\rho^\#(x)d\nu(x) \quad \forall\omega\in\Omega.$$

Then $(J_\rho^\#\nu)h$ is Borel measurable and

$$(7)\quad \begin{aligned}\|(J_\rho^\#\nu)h\|&\leq\int_G \|x^{-1}h\|J_\rho^\#(x)d\nu(x) \quad \text{by (3.11)}\\ &\leq\|h\|\int_G J_\tau^{1/p}(x)J_\rho^\#(x)d\nu(x) \quad \text{by (4) or (5)}\\ &=\|h\|\,\|\nu\|_{M(\omega)}<\infty.\end{aligned}$$

(For more details about the applicability of (3.11), see (4.4) stated below.)

Next suppose $\nu \in M(\omega)$ and $h \in L_{p,q}(\Omega, \tau)$. We claim that $\phi \in C_c(\Omega)$ implies

$$(8) \qquad \int_G |\langle {}_x\phi, h\rho\rangle| d|\nu|(x) \leq \int_\Omega |\phi(\omega)| [(J_\rho^\#|\nu|)|h|](\omega) d\rho(\omega).$$

To prove this, we may suppose that $\phi \geq 0$, $h \geq 0$, and $\nu \geq 0$. Let $S = \{x^{-1} \cdot \omega : x \in \operatorname{supp} \nu$ and $\omega \in \operatorname{supp} \phi\}$. Plainly S is a σ-compact subset of Ω. So we may apply Fubini's Theorem to get

$$\begin{aligned}\int_G \langle {}_x\phi, h\rho\rangle d\nu(x) &= \int_G \int_S \phi(x \cdot \omega) h(\omega) d\rho(\omega) d\nu(x) \\ &\leq \int_G J_\rho(x^{-1}) \int_\Omega \chi_S(x^{-1} \cdot \omega) \phi(\omega) h(x^{-1} \cdot \omega) d\rho(\omega) d\nu(x) \\ &= \int_\Omega \phi(\omega) \int_G \chi_S(x^{-1} \cdot \omega) h(x^{-1} \cdot \omega) J_\rho^\#(x) d\nu(x) d\rho(\omega) \\ &\leq \int_\Omega \phi(\omega) [(J_\rho^\# \nu) h](\omega) d\rho(\omega) \quad \text{by (6)},\end{aligned}$$

which confirms (8).

From (7) and (8), we infer that the convolution $\nu * (h\rho)$ exists. Moreover, the set $\{(J_\rho^\#|\nu|)|h| \neq 0\}$ is σ-finite with respect to τ by (7) and so with respect to ρ. Therefore the Radon-Nikodym-Lebesgue Theorem combined with (8) yields a Borel function $\nu * h$ in $L_1^{\mathrm{loc}}(\Omega, \rho)$ such that

$$(9) \qquad |\nu * h| \leq (J_\rho^\#|\nu|)\,|h| \quad \text{and} \quad \nu * (h\rho) = (\nu * h)\rho.$$

It follows from (9) and (7) that $L_{p,q}(\Omega, \tau)$ forms a left Banach $M(w)$-module on Ω, which completes the proof.

(4.4) REMARKS: In the application of (3.11) in the last proof, we have ignored the basic assumption in (3.11), namely, the σ-finiteness of the underlying measures. ($J_\rho^\#\nu$ is σ-finite, but τ is not in general.) Although this difficulty can be easily circumvented for the purpose of proving (4.3), we shall give more details which justify our application of (3.11) in the last proof. Let $1 < p < \infty$, $1 \leq q \leq \infty$, and $\nu \in M^+(w)$, where $w = J_\tau^{1/p} J_\rho^\#$.

(i) If $h \in L_{p,q}^+(\Omega, \tau)$ has σ-compact support, then the justification of (7) is similar to that of (8). In particular, (7) is valid whenever $h \in C_c^+(\Omega)$.

(ii) If D is a directed collection of nonnegative lower semi-continuous functions on Ω and if $g = \sup\{f : f \in D\}$ pointwise, then $\|g\|_{p,q} = \sup\{\|f\|_{p,q} : f \in D\}$.

[Let $h:\Omega\to[0,\infty]$ be Borel measurable and $t>0$. If $s, u>0$, then $h^*(s)>u$ if and only if $\tau_h(u)>s$. Hence

$$\begin{aligned} th^{**}(t)&=\int_0^t h^*(s)ds\\ &=\int_0^t\int_0^{h^*(s)} duds\\ &=\int_0^\infty \min\{t, \tau_h(u)\}du. \end{aligned}$$

Moreover, $\{\tau_f : f\in D\}$ is a directed collection of nonnegative lower semi-continuous functions on $(0,\infty)$, and its pointwise supremum equals τ_g by the regularity of τ. Hence $g^{**}=\sup\{f^{**}: f\in D\}$ pointwise. So the result for $q=\infty$ is obvious. For $q<\infty$, note that $\{f^{**}: f\in D\}$ is a directed collection of nonnegative lower semi-continuous function on $(0,\infty)$, and so

$$\begin{aligned} \|g\|_{p,q}&=\left(\frac{q}{p}\int_0^\infty [g^{**}(t)]^q t^{q/p-1}dt\right)^{1/q}\\ &=\sup\{\|f\|_{p,q}: f\in D\}, \end{aligned}$$

as desired.]

(iii) Let $g:\Omega\to[0,\infty]$ be lower semi-continuous and let $D=\{\phi\in C_c^+(\Omega): \phi\leq g\}$. Then $\{(J_p^\#\nu)\phi : \phi\in D\}$ is a directed collection of nonnegative lower semi-continuous functions on Ω, and its pointwise supremum equals $(J_p^\#\nu)g$. Hence

$$\begin{aligned} \|(J_p^\#\nu)g\|_{p,q}&=\sup\{\|J_p^\#\nu)\phi\|_{p,q}: \phi\in D\} && \text{by (ii)}\\ &\leq\sup(\|\nu\|_{M(w)}\|\phi\|_{p,q}: \phi\in D\} && \text{by (i)}\\ &\leq\|\nu\|_{M(w)}\|g\|_{p,q}. \end{aligned}$$

(iv) Let $h:\Omega\to[0,\infty]$ be Borel measurable and $h=0$ τ-almost everywhere. Then $(J_p^\#\nu)h=0$ τ-almost everywhere.

[Let $\varepsilon>0$ be given. As $\|h\|_{p,p}=0$, it is easy to costruct a nonnegative lower semi-continuous function g on Ω such that $h\leq g$ everywhere and $\|g\|_{p,p}<\varepsilon$. Then

$$\begin{aligned} \|(J_p^\#\nu)h\|_{p,p}&\leq\|(J_p^\#\nu)g\|_{p,p}\\ &\leq\|\nu\|_{M(w)}\|g\|_{p,p} && \text{by (iii)}\\ &\leq\varepsilon\|\nu\|_{M(w)}. \end{aligned}$$

As $\varepsilon>0$ was arbitrary, this shows that $(J_p^\#\nu)h=0$ τ-almost everywhere.]

(v) Let $h\in L_{p,q}^+(\Omega,\tau)$ be given. Then there exists $h'\in L_{p,q}^+(\Omega,\tau)$ such that $h'=h$ τ-almost everywhere and $\{h'\neq 0\}$ is σ-compact. Hence

$$
\begin{aligned}
\|(J^{\#}_{\rho}\nu)h\|_{p,q} &= \|(J^{\#}_{\rho}\nu)h'\|_{p,q} \quad \text{by (iv)}\\
&\leq \|\nu\|_{M(w)}\|h'\|_{p,q} \quad \text{by (i)}\\
&= \|\nu\|_{M(w)}\|h\|_{p,q}.
\end{aligned}
$$

If $q \leq p$, then we may replace $\|\cdot\|_{p,q}$ by $\|\cdot\|^{*}_{p,q}$ in the last three lines, which completes the detailed proof of the first inequality in (7).

(4.5) REMARK: Let ρ, τ, p, q be as in the hypotheses of (4.3), and let $w = J^{1/p}_{\tau} J^{\#}_{\rho}$. Define $M_{\delta}(G)$ to be the space of all measures on G with finite support, and equip $M_{\delta}(G)$ with $\|\cdot\|_{M(w)}$. If $\nu \in M_{\delta}(G)$, then no difficulties arise in the proof of (4.3). So we can readily conclude that $L_{p,q}(\Omega, \tau)$ forms a left Banach $M_{\delta}(G)$-module on Ω. On the other hand, w is lower semi-continuous, so it is easy to check that $[M_{\delta}(G)]^{\sim} = M(w)$ isometrically. Also $[L_{p,q}(\Omega, \tau)]^{\sim} = L_{p,q}(\Omega, \tau)$ isometrically by Lemma(4.1) [under an additional minor condition if $q=1$]. These facts, combined with Theorem (2.8), yield an alternative proof of (4.3) which requires neither (3.10) nor Fubini's Theorem.

(4.6) REMARKS: Let τ be a positive regular Borel measure on Ω. Suppose that τ is quasi-invariant, that is, $\delta_x * \tau$ and τ are mutually absolutely continuous for each $x \in G$ (δ_x is the Dirac measure at x). Thus, for each $x \in G$, there exists a positive τ-measurable function $J(x, \cdot)$ such that

$$\int h(x^{-1}\cdot\omega)\,d\tau(\omega) = \int h(\omega)J(x, \omega)\,d\tau(\omega)$$

whenever h is a nonnegative Borel function on G with σ-compact support. Gulick, Liu and van Rooij [3] prove, among other things, that the collection $\{J(x, \cdot) : x \in G\}$ can be chosen in such a way that $(x, \omega) \to J(x, \omega)$ is $(\lambda_G \times \tau)$-measurable. They also prove that $L_p(\Omega, \tau)$ forms a left Banach $L_1(G)$-module for $1 \leq p \leq \infty$. However, their definition of the convolution product $f * g$ of $f \in L_1(G)$ and $g \in L_p(\Omega, \tau)$ depends not only on f and g but also on p. That is,

$$(f*g)(\omega) = \int f(x)g(x^{-1}\cdot\omega)J^{1/p}(x^{-1}, \omega)\,dx.$$

Thus their definition is essentially different from ours unless $p=1$.

(4.7) THEOREM. *Let τ, ρ, $1<p<\infty$, and $1 \leq q \leq \infty$ be as in the hypotheses of* (4.1). *Suppose that τ and ρ are each boundedly quasi-invariant. Imbed $L_{p,q}(\Omega, \tau)$ into $C^{*}_{c}(\Omega)$ via $f \to f\rho$. If w is a weight function on G such that $J^{1/p}_{\tau} J^{\#}_{\rho} \leq w$, then:*

(a) *$L_{p,q}(\Omega, \tau)$ forms a left Banach $M(w)$-module on Ω;*

(b) *The multiplier space* $\mathscr{M}(L_1(w), L_{p,q}(\Omega, \tau))$ *is isomorphic to* $L_{p,q}(\Omega, \tau)$ *as Banach spaces*;

(c) *If, in addition, w is moderate, then the isomorphism is isometric [with respect to* $\|\cdot\|_{p,q}$ *(always) and also with respect to* $\|\cdot\|^*_{p,q}$ *(if* $q \leq p$*)].*

PROOF: The first conclusion is an immediate consequence of Theorem (4.3). Since $[L_{p,q}(\Omega, \tau)]^{\sim} = L_{p,q}(\Omega, \tau)$ isometrically by Lemma (4.1), the second and third conlusions follow from Theorem (2.12).

(4.8) REMARKS: In (4.3) and (4.7), we have put aside the important two cases, namely, $p=q=1$ and $p=q=\infty$. We shall discuss these cases below.

(a) Plainly $M(\Omega)$ is a left Banach $M(G)$-module on Ω and $M(\Omega)^{\sim} = M(\Omega)$ isometrically. In particular, $M(\Omega)$ forms a left Banach $L_1(G)$-module on Ω. It follows from Theorem (2.12) that for each closed subspace A of $M(\Omega)$ such that $L_1(G)*M(\Omega) \subset A$, we have $\mathscr{M}(L_1(G), A) \simeq M(\Omega)$ isometrically. For $\Omega = G$, $x \cdot y = xy (x, y \in G)$, and $A = L_1(G)$, this result is nothing but Wendel's Theorem [20].

(b) Let τ and ρ be as in the hypotheses of (4.3), and imbed $L_\infty(\Omega, \tau)$ into $C_c^*(\Omega)$ via $h \to h\rho$. Then it is easy to show that $L_\infty(\Omega, \tau)$ forms a left Banach $M(J_\rho^\#)$-module on Ω. Moreover, $[L_\infty(\Omega, \tau)]^{\sim} = L_\infty(\Omega, \tau)$ isometrically; hence $\mathscr{M}(L_1(J_\rho^\#), L_\infty(\Omega, \tau))$ and $L_\infty(\Omega, \tau)$ are isomorphic as Banach spaces by Theorem (2.12).

(4.9) EXAMPLE (The standard Case). Suppose $\Omega = G$, G acts on itself via group multiplication ($x \cdot y = xy$), and w, v are two weight functions on G. Let $d\tau = w d\lambda_G$ and $d\rho = v d\lambda_G$. Then τ and ρ are mutually absolutely continuous, and the Radon-Nikodym derivative $d\rho/d\tau = v/w$, which is locally bounded by (2.4). Moreover, if $h \geq 0$ is a Borel function on G and $x \in G$, then

$$\begin{aligned} \int h(x^{-1} \cdot y) d\tau(y) &= \int h(x^{-1} y) w(y) dy \\ &= \int h(y) w(xy) dy \\ &\leq w(x) \int h d\tau. \end{aligned}$$

This shows that τ is boundely quasi-invariant and $J_\tau \leq w$. Similarly ρ is boundedly quasi-invariant and $J_\rho \leq v$.

Now let $1 < p < \infty$ and $1 \leq q \leq \infty$, and imbed $L_{p,q}(G, \tau)$ into $C_c^*(G)$ via

$f \to f\rho = fv\lambda_G$. Then by (4. 3), $L_{p,q}(G, \tau)$ forms a left Banach $M(w^{1/p} v^{\#})$-module on G (with respect to $\|\cdot\|_{p,q}$ always and also with respect to $\|\cdot\|_{p,q}^{*}$ if $q \leq p$). Moreover, Theorem (4. 7) ensures that

$$\mathscr{M}(L_1(w^{1/p} v^{\#}),\ L_{p,q}(\Omega, \tau)) \simeq L_{p,q}(\Omega, \tau),$$

and that if $w^{1/p} v^{\#}$ is a moderate weight function, then the isomorphism is isometric. For $\nu \in M(w^{1/p} v^{\#})$ and $f \in L_{p,q}(\Omega, \tau)$, a direct calculation shows that the Radon-Nikodym derivative $d[\nu * (f\rho)]/d\rho$ is given by

$$x \to \frac{1}{v(x)} \int f(y^{-1}x) v(y^{-1}x) d\nu(y).$$

These results with $w=v=1$ and for $\|\cdot\|_{p,q}$ are due to Y.K. Chen and H.C. Lai [1]; see also T.S. Quek and L.Y.H. Yap [10].

(4. 10) EXAMPLE (The Right-Translation Case). Suppose G acts on itself via right-translation, that is,

$$\theta(x, y) = x \cdot y = yx^{-1} \quad \forall x, y \in G.$$

Let w, v be two weight functions on G, and let $d\tau = w d\lambda_G$ and $d\rho = v d\lambda_G$. If $h \geq 0$ is a Borel function on G and $x \in G$, then

$$\begin{aligned} \int h(x^{-1} \cdot y) d\tau(y) &= \int h(yx) w(y) dy \\ &= \Delta(x^{-1}) \int h(y) w(yx^{-1}) dy \\ &\leq (w\Delta)^{\#}(x) \int h d\tau. \end{aligned}$$

Hence $J_\tau \leq (w\Delta)^{\#}$, and similarly $J_\rho \leq (v\Delta)^{\#}$. Consequently, results similar to those in Example (4. 9) hold with $w^{1/p} v^{\#}$ replaced by

$$[(w\Delta)^{\#}]^{1/p}\, v\, \Delta = (w^{\#})^{1/p} v \Delta^{1/p'}.$$

If $\nu \in M((w^{\#})^{1/p}\, v\, \Delta^{1/p'})$ and $f \in L_{p,q}(\Omega, \tau)$, then $d[\nu *_\theta (f\rho)]/d\rho$ is given by

$$x \to \frac{1}{v(x)} \int f(xy) v(xy) \Delta(y) d\nu(y).$$

(4. 11) EXAMPLE (The Inner-Automorphism Case.) Suppose G acts on itself via inner-automorphism, that is,

$$\theta(x, y) = x \cdot y = xyx^{-1} \quad \forall x, y \in G.$$

Let w, v, τ, ρ be as in the last example. If $h \geq 0$ is good enough and $x \in G$, then

$$\begin{aligned}\int h(x^{-1}\cdot y)d\tau(y) &= \int h(x^{-1}yx)w(y)dy \\ &= \int h(yx)w(xy)dy \\ &= \Delta(x^{-1})\int h(y)w(xyx^{-1})dy \\ &\leq (\Delta^{\#}w^{\#}w)(x)\int h d\tau.\end{aligned}$$

Hence $J_\tau \leq \Delta^{\#}w^{\#}w$ and similarly $J_\rho \leq \Delta^{\#}v^{\#}v$. Therefore results similar to those in Example (4.9) hold with $w^{1/p}v^{\#}$ replaced by

$$(\Delta^{\#}w^{\#}w)^{1/p}(\Delta^{\#}v^{\#}v)^{\#} = (w^{\#}w)^{1/p}(v^{\#}v)\Delta^{1/p'}.$$

If $\nu \in M((w^{\#}w)^{1/p}v^{\#}v\Delta^{1/p'})$ and $f \in L_{p,q}(G, \tau)$, then $d[\nu *_\theta (f\rho)]/d\rho$ is given by

$$x \to \frac{1}{v(x)}\int f(y^{-1}xy)v(y^{-1}xy)\Delta(y)d\nu(y).$$

(4.12) EXAMPLE (The Linear Group Case). Consider the general linear group $GL(n, \mathbf{R})$, which acts on $\mathbf{R}^n$ in the obvious way. Let λ_n be Lebesgue measure on $\mathbf{R}^n$, and let G be any closed subgroup of $GL(n, \mathbf{R})$. Choose and fix any strictly positive functions w, v in $L_1^{\mathrm{loc}}(\mathbf{R}^n, \lambda_n)$ such that v/w is locally bounded. Take $d\tau = w d\lambda_n$ and $d\rho = v d\lambda_n$. Suppose that there exist weight functions w_G and v_G on G such that,

$$w(Ax) \leq w_G(A)w(x) \quad \text{and} \quad v(Ax) \leq v_G(A)v(x)$$

whenever $A \in G$ and $x \in \mathbf{R}^n$. If $h \geq 0$ is a Borel function on $\mathbf{R}^n$ and $A \in G$, then

$$\begin{aligned}\int_{\mathbf{R}^n} h(A^{-1}x)d\tau(x) &= \int_{\mathbf{R}^n} h(A^{-1}x)w(x)d\lambda_n(x) \\ &= |\det A|\int_{\mathbf{R}^n} h(x)w(Ax)d\lambda_n(x) \\ &\leq |\det A| w_G(A)\int_{\mathbf{R}^n} h(x)d\tau(x).\end{aligned}$$

Hence $J_\tau(A) \leq |\det A| w_G(A)$ and similary $J_\rho(A) \leq |\det A| v_G(A)$ for each $A \in G$.

Let $1 < p < \infty$, $1 \leq q \leq \infty$, and define

$$\begin{aligned}w_p(A) &= (|\det A| w_G(A))^{1/p}(|\det A| v_G(A))^{\#} \\ &= w_G^{1/p}(A)v_G^{\#}(A)/|\det A|^{1/p'}\end{aligned}$$

for each $A \in G$. If we imbed $L_{p,q}(\mathbf{R}^n, \tau)$ into $C_c^*(\mathbf{R}^n)$ via $f \to f\rho$, then results similar to those in Example (4.9) hold with $w^{1/p}v^{\#}$ replaced by w_p.

For $\nu \in M(w_p)$ and $f \in L_{p,q}(\mathbf{R}^n, \tau)$, $d[\nu * (f\rho)]/d\rho$ is given by

$$x \to \frac{1}{v(x)} \int_G f(A^{-1}x) v(A^{-1}x) |\det A|^{-1} d\nu(A).$$

(4.13) EXAMPLE. Let G be a locally compact abelian group with dual Γ (cf. W. Rudin [14]). The Fourier transform of $\mu \in M(G)$ is defined by

$$\hat{\mu}(\gamma) = \int_G \gamma(x^{-1}) d\mu(x) \quad \forall \gamma \in \Gamma.$$

Similarly we define the "inverse" Fourier transform of $\rho \in M(\Gamma)$ by setting

$$(\rho^\vee)(x) = \int_\Gamma \gamma(x) d\rho(\gamma) \quad \forall x \in G.$$

Now choose and fix any positive regular Borel measure τ on Γ such that $(\phi\tau)^\vee \in C_0(G)$ whenever $\phi \in C_c(\Gamma)$. Also let $w \geq 1$ be a continuous moderate weight on G. For $1 < p < \infty$ and $1 \leq q \leq \infty$, define

$$M_{p,q} = M_{p,q,w,\tau} = \{\mu \in M(w) : \hat{\mu} \in L_{p,q}(\Gamma, \tau)\}.$$

Then we have:

(a) $M_{p,q}$ is a left Banach $M(w)$-module on G with respect to the norm $\|\mu\|_{M(p,q)} = \|\mu\|_{M(w)} + \|\hat{\mu}\|_{p,q}$ (always) and also with respect to $\|\mu\|^*_{M(p,q)} = \|\mu\|_{M(w)} + \|\hat{\mu}\|^*_{p,q}$ (if $q \leq p$).

(b) $(M_{p,q})^\sim = M_{p,q}$ isometrically.

(c) If A is a closed subspace of $M_{p,q}$ such that $L_1(w) * M_{p,q} \subset A$, then

$$\mathscr{M}(L_1(w), A) \simeq M(p, q) \quad \text{isometrically.}$$

These results for $w=1$ and $\tau = \lambda_\Gamma$ are due to Chen and Lai [1].

PROOF: We shall prove (a)-(c) only for $\|\cdot\|_{M(p,q)}$.

(a) As $w \geq 1$, the definition of $M_{p,q}$ makes sense and it is obvious that $M_{p,q}$ forms a Banach space on G. If $\nu \in M(w)$ and $\mu \in M_{p,q}$, then

$$\begin{aligned}\|\nu * \mu\|_{M(p,q)} &= \|\nu * \mu\|_{M(w)} + \|\hat{\nu} \cdot \hat{\mu}\|_{p,q} \\ &\leq \|\nu\|_{M(w)} \|\mu\|_{M(w)} + \|\hat{\nu}\|_u \|\hat{\mu}\|_{p,q} \\ &\leq \|\nu\|_{M(w)} \|\mu\|_{M(p,q)}.\end{aligned}$$

Hence $M_{p,q}$ is a left $M(w)$-module on G.

(b) By virtue of (1.5), it suffices to show that $(M_{p,q})_1$ is $\sigma(C_c^*, C_c)$-compact. Let (μ_α) be a net in $(M_{p,q})_1$. We need to show that (μ_α) has a $\sigma(C_c^*, C_c)$-cluster point in $(M_{p,q})_1$. We may suppose that both

$$\lim_\alpha \|\mu_\alpha\|_{M(w)} = a \quad \text{and} \quad \lim_\alpha \|\hat{\mu}_\alpha\|_{p,q} = b$$

exist. Note that $a+b\leq 1$.

As $w\geq 1$, (μ_α) is a bounded net in $M(G)$. So, after passing to a subnet, we may suppose that (μ_α) converges weak-* to some $\mu\in M(G)$:

(1) $$\lim_\alpha \int_G f d\mu_\alpha = \int_G f d\mu \ \forall f\in C_0(G).$$

Since w is continuous and $\|\mu_\alpha\|_{M(w)}\to a$, it is obvious that $\|\mu\|_{M(w)}\leq a$.

To prove $\mu\in(M_{p,q})_1$, pick any $\phi\in C_c(\Gamma)$. Then

(2) $$\int_\Gamma \hat{\mu}_\alpha \phi d\tau = \int_G (\phi\tau)^\vee d\mu_\alpha$$

for each α. Since $(\phi\tau)^\vee\in C_0(G)$ by hypothesis, if follows from (1) and (2) that

(3) $$\lim_\alpha \int_\Gamma \hat{\mu}_\alpha \phi d\tau = \int_G (\phi\tau)^\vee d\mu \ \forall\phi\in C_c(\Gamma).$$

On the other hand, $\|\hat{\mu},{}_\alpha\|_{p,q}\to b$. Therefore (3. 9) and (3) show that there exists $h\in L_{p,q}(\Gamma,\tau)$, with $\|h\|_{p,q}\leq b$, such that

(4) $$\lim_\alpha \int_\Gamma \hat{\mu}_\alpha \phi d\tau = \int_\Gamma h\phi d\tau \ \forall\phi\in C_c(\Gamma).$$

Hence $\phi\in C_c(\Gamma)$ implies

$$\begin{aligned}\int_\Gamma \hat{\mu}\phi d\tau &= \int_G (\phi\tau)^\vee d\mu \\ &= \int_\Gamma h\phi d\tau\end{aligned}$$

by (3) and (4). Therefore $\hat{\mu}=h$ locally τ-almost everywhere. However, $\{h\neq 0\}$ is σ-finite with respect to τ since $h\in L_{p,q}(\Gamma,\tau)$, and $\hat{\mu}$ is continuous on Γ. It follows from the regularity of τ that $\hat{\mu}=h$ τ-almost everywhere. Hence $\hat{\mu}\in L_{p,q}(\Gamma,\tau)$ and

$$\begin{aligned}\|\mu\|_{M(p,q)} &= \|\mu\|_{M(w)} + \|h\|_{p,q} \\ &\leq a+b\leq 1,\end{aligned}$$

as desired

(c) Since w is a moderate weight function on G, (c) is a direct consequence of Theorem (2. 12) combined with (b).

(4. 14) REMARKS: In the last examle, we have used the ℓ_1-norm on $\mathbf{R}^2$, $\|(s,t)\|_1=|s|+|t|$. However, it is clear from our proof that in the definition of $\|\cdot\|_{M(p,q)}$, we may instead use any norm on $\mathbf{R}^2$ such that $\|(s,t\|\leq\|(s',t')\|$ whenever $0\leq s\leq s'$ and $0\leq t\leq t'$. Also the natural analogues

of (4. 11) for $p=q=1$ and for $p=q=\infty$ hold, provided that τ and λ_Γ are mutually absolutely continuous.

References

[1] Y.-K. CHEN and H.-C. LAI, Multipliers of Lorentz Spaces, Hokkaido Math. Jour. 4 (1975), 247-260.

[2] C. V. COMISKY, Multipliers of Banach Modules, Indag. Math. 33 (1971), 32-38.

[3] S. L. GULICK, T. S. LIU and A.C.M. van ROOIJ, Group Algebra Modules I and II, Canadian Jour. Math. 19 (1967), 133-150 and 151-173.

[4] __________, Group Algebra Modules III and IV, Trans. Amer. Math. Soc. 152 (1970), 561-580 and 581-596.

[5] E. HEWITT and K. A. ROSS, Abstract Harmonic Analysis I and II, Springer-Verlag, New York-Heidelberg-Berlin, 1963 and 1970.

[6] E. HEWITT and K. STROMBERG, Real and Abstract Analysis, Springer-Verlag, Berlin -Heidelberg, 1965.

[7] R. A. HUNT, On L_{pq} Spaces, Enseignement Math. 2 (1966), 249-276.

[8] Y. KATZNELSON and O. C. MCGEHEE, Measures and Pseudomeasures on Compact Subsets of the Line, Math. Scand. 23 (1968), 57-68.

[9] T. S. LIU and A. C. M. van ROOIJ, Sums and intersections of normed linear spaces, Math. Nacr. 42 (1969), 29-42.

[10] T. S. QUEK and L. Y. H. YAP, A test for membership in Lorentz Spaces and Some Applications, Hokkaido Math. Jour. 17 (1988), 279-288.

[11] H. REITER, Classical Harmonic Analysis and Locally Compact Groups, Oxford University Press, London, 1968.

[12] M. RIEFFEL, Induced Banach Representations of Banach Algebras and Locally Compact Groups, Jour. Func. Analy. 1 (1967), 443-491.

[13] W. ROBERTSON and A.P. ROBERTSON, Topological Vector Spaces, Cambridge University Press, London, 1968.

[14] W. RUDIN, Fourier Analysis on Groups, Interscience Tract No. 12, New York, Wiley, 1962.

[15] _________, Functional Analysis, McGraw-Hill, New York, 1973.

[16] S. SAEKI, Discontinuous Translation Invariant Functionals, Trans. Amer, Math. Soc. 282 (1984), 403-414.

[17] J. L. TAYLOR, Measure Algebras, Regional Conference Series in math. No. 16, Amer. Math. Soc., 1972.

[18] E. L. THOME, Tilde-Closures and $L_1(G)$-modules, Ph. D. Dissertation, Kansas State University, 1991 (Summer).

[19] N. Th. VAROPOULOS, On a problem of A. Beurling, Jour. Func. Analysis 2 (1968), 24-30.

[20] J. WENDEL, Left centralizers and isomorphisms of group algebras, Pacific Jour. Math. 2 (1952), 251-261.

[21] L.Y.H. YAP, Some Remarks on Convolution Operators and L_{pq} Spaces, Duke Math. Jour. 36 (1969), 647-658.

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