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On a conjecture of J. M. Lee

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Abstract

We deal with the Lee conjecture (compact strictly pseudoconvex CR manifolds whose CR structure has a vanishing first Chern class admit a global pseudo-Einstein structure[1]). We solve in affirmative the Lee conjecture for compact strictly pseudoconvex CR manifolds with a regular (in the sense of R. Palais, [Pal]) contact vector. The regularity assumption leads (via the Boothby-Wang theorem ([Boo-Wan]) and B. O'Neill's fundamental equations of a submersion ([Nei])) to zero pseudohermitian torsion (and we may apply a result of [Lee2]).

Moreover we construct a family $\mathbf{H}_n(s)$, $0<s<1$, of compact strictly pseudoconvex CR manifolds, so that each $\mathbf{H}_n(s)$ satisfies the Lee conjecture. We endow $\mathbf{H}_n(s)$ with the contact form (4); our construction is reminiscent of W. C. Boothby's Hermitian metric (cf. [Boo]) on a complex Hopf manifold.

1 Introduction.

Let $(M, T_{1,0}(M), \theta)$ be a pseudohermitian manifold of CR dimension n. Then M is termed *pseudo-Einstein* if the pseudohermitian Ricci tensor of θ is proportional to the Levi form, cf. [Lee2]. One may formulate the following natural problem: *given a nondegenerate CR manifold M find a pseudohermitian structure θ so that (M, θ) is pseudo-Einstein.* The solution of the local problem (i.e. find a pseudo-Einstein structure on some neighborhood of each point of M) is intimately related to the question of imbeddability. Indeed, if M admits a CR imbedding into $\mathbf{C}^{n+1}$ then M admits a pseudo-Einstein structure (cf. [Lee2], Corollary B, p. 158). On the other hand, by a result of L. Boutet de Monvel, [Bou], a compact strictly pseudoconvex CR manifold can always be imbedded locally in $\mathbf{C}^{n+1}$. Also local imbeddability holds in the noncompact case if $n>2$ by results of M. Kuranishi, [Kur], and T. Akahori, [Aka]. Thus, if M is strictly pseudoconvex then M is locally pseudo-Einstein provided that

[1] A CR analogue of the Calabi conjecture.

either M is compact or $\dim_{\mathbf{R}} M \geq 7$.

As to the solution of the global problem, J. M. Lee has found (cf. [Lee2]) the following obstruction: if M is a compact strictly pseudoconvex pseudo-Einstein manifold then $c_1(T_{1,0}(M))=0$. Here $c_1(T_{1,0}(M)) \in H^2(M; \mathbf{R})$ is the first Chern class of the CR structure. He also conjectured that any compact strictly pseudoconvex CR manifold M with $c_1(T_{1,0}(M))=0$ admits a global pseudo-Einstein structure.

The conjecture (referred hereafter as the Lee conjecture) is known to hold when M has transverse symmetry (i.e. M admits a 1-parameter group of CR automorphisms transverse to $T_{1,0}(M)$). By a result of S. Webster, [Web] if M has transverse symmetry then M admits a contact form θ with vanishing pseudohermitian torsion τ(and then, by [Lee2], p. 176, there is $u \in C^\infty(M)$ so that $\exp(2u)\theta$ is pseudo-Einstein).

In the defense of the Lee conjecture we construct an example of a compact strictly pseudoconvex CR manifold, which is globally pseudo-Einstein and has non-vanishing pseudohermitian torsion. This is obtained as a quotient of the Heisenberg group $\mathbf{H}_n$ by a discrete group of CR automorphisms (and is a CR analogue of the construction of H. Hopf, [Hop], endowing $S^{2n-1} \times S^1$ with a complex structure).

2 Quotients of $\mathbf{H}_n$ by properly discontinuous groups of CR automorphisms.

Let $\delta_s : \mathbf{H}_n - \{0\} \to \mathbf{H}_n - \{0\}$, $s>0$, be the parabolic dilations of the Heisenberg group (i.e. $\delta_s(z, t)=(sz, s^2 t)$, $z \in \mathbf{C}^n$, $t \in \mathbf{R}$, $(z, t) \neq 0$). If $m \in \mathbf{Z}$, $m>0$, set $\delta_s^m = \delta_s \circ \ldots \circ \delta_s$($m$ factors). Also $\delta_s^{-m} = \delta_{1/s}^m$. Consider the discrete group $G_s = \{\delta_s^m : m \in \mathbf{Z}\}$. We establish the following:

THEOREM 1. *Let $0<s<1$ and $n>1$. Then G_s acts freely on $\mathbf{H}_n - \{0\}$ as a properly discontinuous group of CR automorphisms of $\mathbf{H}_n - \{0\}$. The quotient space $\mathbf{H}_n(s) = (\mathbf{H}_n - \{0\})/G_s$ is a compact strictly pseudoconvex CR manifold of CR dimension n.*

PROOF. Clearly $\delta_s^m x = x$ for some $x \in \mathbf{H}_n - \{0\}$ yields $m=0$. Thus the action of G_s on $\mathbf{H}_n - \{0\}$ is free.

Let $|x| = (|z|^4 + t^2)^{1/4}$, $x=(z, y)$, be the Heisenberg norm on $\mathbf{H}_n$. Let $x_0 \in \mathbf{H}_n - \{0\}$ and set $U_\epsilon(x_0) = \{x \in \mathbf{H}_n - \{0\} : |x - x_0| < \epsilon\}$, $\epsilon > 0$. Let $\|x\|$ be the Euclidean norm on $\mathbf{H}_n \approx \mathbf{R}^{2n+1}$. Cf. G. B. Folland & E. M. Stein, [Fol-Ste], p. 449, for any $x \in \mathbf{H}_n$ with $|x| \leq 1$ one has $\|x\| \leq |x| \leq \|x\|^{1/2}$. Thus the sets $U_\epsilon(x)$, $x \in \mathbf{H}_n - \{0\}$, $0 < \epsilon < 1$, form a fundamental system of neighborhoods in $\mathbf{H}_n - \{0\}$.

To show that G_s is properly discontinuous, given $x_0 \in \mathbf{H}_n - \{0\}$ one needs to choose $\epsilon > 0$ such that:

$$\delta_s^m(U_\epsilon(x_0)) \cap U_\epsilon(x_0) = \emptyset \tag{1}$$

for any $m \in \mathbf{Z}$, $m \neq 0$. Cf. [Fol-Ste], Lemma 8.9., p. 449, there exists $\gamma \geq 1$ so that $|x+y| \leq \gamma(|x|+|y|)$ for any x, $y \in \mathbf{H}_n$. Consequently:

$$|x| - \gamma|y| \leq \gamma|x-y| \tag{2}$$

for any x, $y \in \mathbf{H}_n$. Let:

$$\xi_m = |\delta_s^m(x_0) - x_0|$$

for $m \in \mathbf{Z}$. As G_s acts freely on $\mathbf{H}_n - \{0\}$, it follows that $\xi_m \geq 0$, and $\xi_m \Longleftrightarrow m=0$. Next, as $0<s<1$, one obtains:

$$0 \leq m_1 < m_2 \Longrightarrow \xi_{m_1} < \xi_{m_2},\ \xi_{-m_1} < \xi_{-m_2}.$$

Therefore:

$$\xi_m \geq \min(\xi_1, \xi_{-1}) = \xi_1$$

for any $m \in \mathbf{Z}$, $m \neq 0$. Set $N = 2\gamma + 1$. Choose $0 < \epsilon < \frac{1}{N}\xi_1$. Let $x \in U_\epsilon(x_0)$. Then:

$$|\delta_s^m(x) - \delta_s^m(x_0)| = s^m|x - x_0| < s^m \epsilon < \epsilon$$

shows that:

$$\delta_s^m(U_\epsilon(x_0)) \subseteq U_\epsilon(\delta_s^m(x_0)). \tag{3}$$

Using (2)-(3) we have the estimates:

$$\begin{aligned}\gamma|x_0 - \delta_s^m(x)| &= \gamma|x_0 - \delta_s^m(x_0) - (\delta_s^m(x) - \delta_s^m(x_0))| \geq \\ &\geq |x_0 - \delta_s^m(x_0)| - \gamma|\delta_s^m(x) - \delta_s^m(x_0)| > \\ &> \xi_m - \gamma\epsilon \geq \xi_1 - \gamma\epsilon > N\epsilon - \gamma\epsilon = (\gamma+1)\epsilon\end{aligned}$$

so that:

$$|x_0 - \delta_s^m(x)| > \frac{\gamma+1}{\gamma}\epsilon > \epsilon.$$

This shows that $\delta_s^m(x) \notin U_\epsilon(x_0)$, for any $x \in U_\epsilon(x_0)$, $m \in \mathbf{Z}$, $m \neq 0$, so that (1) holds.

Let $\pi : \mathbf{H}_n - \{0\} \rightarrow \mathbf{H}_n(s)$ be the natural map. Let:

$$\Sigma^{2n} = \{x \in \mathbf{H}_n : |x| = 1\}.$$

Then Σ^{2n} is a compact real hypersurface in $\mathbf{H}_n$. The map $\mathbf{H}_n(s)\to\Sigma^{2n}\times S^1$ defined by:

$$\pi(x)\mapsto\left(\frac{z}{|x|},\ \frac{t}{|x|^2},\ \exp\left(\frac{2\pi i\log|x|}{\log s}\right)\right)$$

is a diffeomorphism, $x=(z,t)\in\mathbf{H}_n-\{0\}$. Thus $\mathbf{H}_n(s)$ is compact. As π is a local diffeomorphism $\mathbf{H}_n(s)$ inherits a structure of CR hypersurface of CR dimension n. Let $(U, z^1, \ldots, z^n, t)$ be a local coordinate system on $\mathbf{H}_n(s)$, $z^\alpha=x^\alpha+iy^\alpha$. Set:

$$\theta=|x|^{-2}\{dt+2\sum_{\alpha=1}^{n}(x^\alpha dy^\alpha-y^\alpha dx^\alpha)\} \tag{4}$$

on U. The right hand member of (4) is G_s-invariant and thus defines a global 1-form on $\mathbf{H}_n(s)$. Let $\{\theta^\alpha\}$ be dual to T_α, where $T_\alpha=\frac{\partial}{\partial z^\alpha}+i\bar{z}^\alpha\frac{\partial}{\partial t}$ on U. The Levi form associated with (4) is given by:

$$L_\theta=|x|^{-2}\delta_{\alpha\beta}\theta^\alpha\wedge\theta^{\bar{\beta}}$$

on U. Thus θ is strictly pseudoconvex. Our Theorem 1. is completely proved.

Let M be a CR manifold and $\mathscr{P}$ the sheaf of CR-pluriharmonic functions (ie. real parts of CR-holomorphic functions) on M. By a result in [Lee2], p. 172, if M in locally realizable then there exists a CR invariant cohomology class $\gamma(M)\in H^1(M, \mathscr{P})$ so that $\gamma(M)=0$ iff M admits a global pseudo-Einstein structure. We need to recall the construction of $\gamma(M)$. The notations we employ are those in [Gol], p. 271-275.

If M is locally realizable, in the neighborhood of each point there exists a pseudo-Einstein structure. Let $\{(U_i, \theta_i)\}_{i\in I}$ be a covering of M with such neighborhoods. On each $U_i\cap U_j\neq\emptyset$ one may write $\theta_j=\exp(2u_{ji})\theta_i$, for some $u_{ji}\in C^\infty(U_i\cap U_j)$. By a result of [Lee2], i.e. Prop. 5.1., p.172, $u_{ji}\in\mathscr{P}(U_i\cap U_j)$ and $u_{ij}+u_{ji}=0$, $u_{ij}+u_{jk}+u_{ki}=0$. Let $N(\mathscr{U})$ be the nerve of $\mathscr{U}=(U_i)_{i\in I}$. Let f map each 1-simplex $\sigma=(U_iU_j)$ of $N(\mathscr{U})$ in $u_{ji}\in\mathscr{P}(\cap\sigma)$. Then $f\in Z^1(N(\mathscr{U}),\mathscr{P})$, i.e. f so built is a 1-cocycle with coefficients in $\mathscr{P}$. Let $\gamma(M)\in H^1(M,\mathscr{P})$ be the equivalence class of $[f]\in H^1(N(\mathscr{U}),\mathscr{P})$. It is known (cf. [Lee2], p. 173) that $\gamma(M)$ depends only on the CR structure of M.

Let:

$$\mathscr{H}_s=T_{1,0}(\mathbf{H}_n(s))$$

and:

$$\gamma_s=\gamma(\mathbf{H}_n(s))$$

for simplicity. Let $\{(U_i, z_i^\alpha, t_i)\}_{i\in I}$ be an atlas on $\mathbf{H}_n(s)$ so that for any $i, j \in I$ with $U_i\cap U_j\neq\emptyset$ the coordinate transformation reads:

$$z_j^\alpha=s^{m_{ji}}z_i^\alpha,\ t_j=s^{2m_{ji}}t_i \tag{5}$$

for some $m_{ji}\in\mathbf{Z}$. Define:

$$\theta_i=dt_i+2\sum_{\alpha=1}^{n}(x_i^\alpha dy_i^\alpha-y_i^\alpha dx_i^\alpha)$$

on U_i, $i\in I$. Each (U_i, θ_i) is a strictly pseudoconvex CR manifold with vanishing Ricci tensor (in particular each θ_i is pseudo-Einstein). Let $\gamma_s\in H^1(\mathbf{H}_n(s), \mathscr{P})$ be the corresponding CR invariant cohomology class. As a consequence of (5) one has:

$$\theta_j=\exp(2m_{ji}\log s)\theta_i$$

on $U_i\cap U_j$. Let $c=(2m_{ij}\log s)\in Z^1(N(\mathscr{U}), \mathbf{R})$ be the corresponding cocycle. If $i: \mathscr{C}^1(N(\mathscr{U}), \mathbf{R})\to\mathscr{C}^1(N(\mathscr{U}), \mathscr{P})$ is the natural cochain map then γ_s is the image of $[c]$ via $i_*: H^1(M, \mathbf{R})\to H^1(M, \mathscr{P})$. We are going to show that (4) is globally pseudo-Einstein so that (cf. Prop. 5.2 of [Lee2], p. 172) $\gamma_s=0$. Yet $c\neq0$ (as $\operatorname{Ker}(i_*)\neq0$). Indeed $[c]$ corresponds (under the isomorphism $H^1_{DR}(\mathbf{H}_n(s))\approx H^1(\mathbf{H}_n(s), \mathbf{R})$) to the De Rham cohomology class $[\omega]$ of the 1-form $\omega=d\log|x|^{-1}$ (which is not exact)[2]. Also, by Prop. D of [Lee2], p. 159, $\gamma_s=0$ yields $c_1(\mathscr{H}_s)=0$. We may show that actually all Chern classes of $\mathscr{H}_s$ vanish (by constructing a flat connection D in $\mathscr{H}_s$). We do this in the following more general setting.

Let $(M, T_{1,0}(M), \theta)$ be a nondegenerate CR manifold. Let $u\in C^\infty(M)$ be a real valued smooth function on M. Let $\{T_\alpha\}$ be a frame in $T_{1,0}(M)$ defined on some open set $U\subseteq M$. Let $\hat{\theta}=e^{2u}\theta$, $\hat{\theta}^\alpha=\theta^\alpha+2iu^\alpha\theta$ and $\hat{T}=e^{-2u}\{T-2iu^\beta T_\beta+2iu^{\bar\beta}T_{\bar\beta}\}$, where $u^\alpha=h^{\alpha\bar\sigma}u_{\bar\sigma}$, $u_{\bar\sigma}=T_{\bar\sigma}(u)$ and $u^{\bar\alpha}=(u^\alpha)^-$. Note that, with these choices, one has $\hat{T}\rfloor\hat{\theta}=1$, $\hat{T}\rfloor d\hat{\theta}=0$ and $\hat{T}\rfloor\hat{\theta}^\alpha=0$. By (A. 0) one has $G_{\hat\theta}=e^{2u}G_\theta$ so that $\hat{h}_{\alpha\bar\beta}=e^{2u}h_{\alpha\bar\beta}$, where $\hat{h}_{\alpha\bar\beta}=L_{\hat\theta}(T_\alpha, T_{\bar\beta})$. We shall need the following:

PROPOSITION 1. *Let $(M, T_{1,0}(M), \theta, T)$ be a nondegenerate CR manifold. Then, under a transformation $\hat{\theta}=e^{2u}\theta$, the Christoffel symbols of the Webster connections of $(T_{1,0}(M), \theta)$ and $(T_{1,0}(M), \hat{\theta})$ are related by:*

$$\begin{aligned}\hat{\Gamma}^\sigma_{\beta\alpha}&=\Gamma^\sigma_{\beta\alpha}+2u_\beta\delta^\sigma_\alpha+2u_\alpha\delta^\sigma_\beta\\ \hat{\Gamma}^\sigma_{\bar\beta\alpha}&=\Gamma^\sigma_{\bar\beta\alpha}-2u^\sigma h_{\bar\beta\alpha}\end{aligned} \tag{6}$$

[2] Note that $d\log|x|^{-1}$ is G_s-invariant, so that ω is globally defined.

$$e^{2u}\hat{\Gamma}^{\sigma}_{0\alpha}=\Gamma^{\sigma}_{0\alpha}+2u_0\delta^{\sigma}_{\alpha}-4iu_{\alpha}u^{\sigma}+$$
$$+2iu_{\alpha,}{}^{\sigma}+2i\Gamma^{\sigma}_{\bar{\mu}\alpha}u^{\bar{\mu}}-2i\Gamma^{\sigma}_{\mu\alpha}u^{\mu}.$$

Consequently, the connection forms ω^{σ}_{α}, $\hat{\omega}^{\sigma}_{\alpha}$ *are related by*[3]:

$$\hat{\omega}^{\sigma}_{\alpha}=\omega^{\sigma}_{\alpha}+2(u_{\alpha}\theta^{\sigma}-u^{\sigma}\theta_{\alpha})+\delta^{\sigma}_{\alpha}(u_{\beta}\theta^{\beta}-u^{\beta}\theta_{\beta})+$$
$$+i(u_{\alpha,}{}^{\sigma}+u^{\sigma}{}_{,\alpha}+4u_{\alpha}u^{\sigma}+4\delta^{\sigma}_{\alpha}u_{\beta}u^{\beta})\theta+\delta^{\sigma}_{\alpha}du \tag{7}$$

where $u_{\alpha,}{}^{\sigma}=u_{\alpha,\bar{\beta}}h^{\sigma\bar{\beta}}$, $\theta_{\alpha}=h_{\alpha\bar{\beta}}\theta^{\bar{\beta}}$, *etc.*

PROOF. The first two identities in (6) are a straightforward consequence of (A. 3)-(A. 4). To prove the last identity in (6) note that (A. 5) may be also written:

$$\Gamma^{\rho}_{0\alpha}h_{\rho\bar{\sigma}}=T(h_{\alpha\bar{\sigma}})+g\theta([T_{\bar{\sigma}}, T], T_{\alpha}).$$

The desired formula follows from:

$$e^{2u}\pi_{-}[T_{\bar{\sigma}}, \hat{T}]=\pi_{-}[T_{\bar{\sigma}}, T]+2i[T_{\bar{\sigma}}, T_{\bar{\mu}}]u^{\bar{\mu}}+$$
$$+2i\{u^{\bar{\rho}}{}_{,\bar{\sigma}}-2u_{\bar{\sigma}}u^{\bar{\rho}}+\Gamma^{\bar{\rho}}_{\mu\bar{\sigma}}u^{\mu}-\Gamma^{\bar{\rho}}_{\bar{\mu}\bar{\sigma}}u^{\bar{\mu}}\}T_{\bar{\rho}}$$

and:

$$e^{2u}\hat{\Gamma}^{\sigma}_{0\alpha}+2iu^{\beta}\hat{\Gamma}^{\sigma}_{\beta\alpha}-2iu^{\bar{\beta}}\hat{\Gamma}^{\sigma}_{\bar{\beta}\alpha}=$$
$$=\Gamma^{\sigma}_{0\alpha}+2u_0\delta^{\sigma}_{\alpha}+2iu_{\alpha,}{}^{\sigma}+4iu_{\beta}u^{\beta}\delta^{\sigma}_{\alpha}+4iu_{\alpha}u^{\sigma}$$

where $u_0=T(u)$.

Let $(M, T_{1,0}(M), \theta)$ be a nondegenerate CR manifold admitting a real closed (globally defined) 1-form ω. Let $B=\omega^{\#}$, where # denotes raising of indices with respect to g_{θ}. Next, set $B^{1,0}=\pi_{+}B$. Locally, if:

$$\omega=\omega_{\alpha}\theta^{\alpha}+\omega_{\bar{\alpha}}\theta^{\bar{\alpha}}+\omega_0\theta$$

where $\omega_{\bar{\alpha}}=(\omega_{\alpha})^{-}$, then:

$$B^{1,0}=h^{\alpha\bar{\beta}}\omega_{\bar{\beta}}T_{\alpha}.$$

By the Poincaré lemma, there exists an open covering $\{U_i\}_{i\in I}$ of M and a family $\{u_i\}_{i\in I}$ of $\mathbf{R}$-valued functions $u_i\in C^{\infty}(U_i)$ so that $\omega|_{U_i}=du_i$, $i\in I$. Set $\theta_i=\exp(2u_i)\theta|_{U_i}$. By applying (6) to $u=u_i$ it follows that the Webster connections of the nondegenerate CR hypersurfaces (U_i, θ_i), $i\in I$, glue up to a (globally defined) linear connection D on M expressed by:

$$D_Z W=\nabla_Z W+2\{\omega(Z)W+\omega(W)Z\}$$
$$D_{\bar{Z}} W=\nabla_{\bar{Z}} W-2L_{\theta}(\bar{Z}, W)B^{1,0}$$

[3] The formula (7) has been obtained by J. M. Lee, cf. [Lee1]. Yet there is an error in (5.7) of [Lee1], p.421 (the term $\delta^{\sigma}_{\alpha}du$ is missing there).

$$
\begin{aligned}
&D_T W=\nabla_T W+2i\nabla_W B^{1,0}+4i\omega(W)B^{1,0}+4i\|B^{1,0}\|^2 W \\
&D_Z T_\omega=2\omega(Z)T_\omega \\
&D_{T_\omega}T_\omega=2\omega(T)T_\omega
\end{aligned}
\tag{8}
$$

for any $Z,\ W\in T_{1,0}(M)$. Here ∇ denotes the Webster connection of $(M,\ \theta)$ and $T_\omega=T-2iB^{1,0}+2iB^{0,1}$. Note that T_ω is transversal to $H(M)$ (so that the formulae (8) define D everywhere on $T(M)$). In analogy with I. Vaisman, [Vai], we call D the *Weyl connection* of (M, θ, ω).

THEOREM 2. *Let* $0<s<1$ *and* $n>1$. *Then i) all Chern classes of* $\mathscr{H}_s$ *vanish, and ii) the contact form* (4) *is pseudo-Einstein and has nonvanishing pseudohermitian torsion.*

PROOF. Let $M=\mathbf{H}_n(s)$ with the C^∞ atlas $\{U_i, z_i^\alpha, t_i)\}_{i\in I}$ as above. Let $u_i\in C^\infty(U_i)$ be defined by $u_i=\log|x_i|$, $x_i=(z_i, t_i)$. Then (by (5)) we have $u_j-u_i=m_{ji}\log s=\text{const.}$ on $U_i\cap U_j$. Consequently, the local 1-forms du_i glue up to a real (closed) global 1-form ω on $\mathbf{H}_n(s)$. The Webster connections of the local pseudohermitian structures $\{\theta_i\}_{i\in I}$ are flat, so that the Weyl connection D of $(\mathbf{H}_n(s), \theta, \omega)$ (with θ given by (4)) is flat. As $DJ=0$ the Weyl connection is reducible to a (flat) connection in $\mathscr{H}_s$. By the Chern-Weil theorem the characteristic ring of $\mathscr{H}_s$ must vanish.

Let $(M, T_{1,0}(M), \theta)$ be a nondegenerate CR manifold. Set $\hat{\theta}=e^{2u}\theta$, $u\in C^\infty(M)$. As a consequence of Proposition 1 one has:

$$
\hat{A}_{\alpha\beta}=A_{\alpha\beta}+2iu_{\alpha,\beta}-4iu_\alpha u_\beta \tag{9}
$$

(cf. also (2.16) in [Lee2], p. 164). At this point we may prove ii) in Theorem 2. Indeed, we may apply (9) with $u=\log|x|^{-1}$, $A_{\alpha\beta}=0$ and $\omega_\beta^\alpha=0$. If $T_\alpha=\partial/\partial z^\alpha+i\bar{z}^\alpha\partial/\partial t$ then:

$$
\begin{aligned}
&u_\alpha=-\frac{1}{2}|x|^{-4}\bar{z}_\alpha\varphi \\
&T_\alpha(u_\beta)=|x|^{-8}\varphi^2\bar{z}_\alpha\bar{z}_\beta
\end{aligned}
$$

where $\varphi(z, t)=|z|^2+it$. Finally, as $\bar{\varphi}$ is CR-holomorphic, (9) yields $\hat{A}_{\alpha\beta}=2iT_\alpha(u_\beta)-4iu_\alpha u_\beta=i|x|^{-8}\bar{z}_\alpha\bar{z}_\beta\varphi^2$ so that (4) has nonvanishing pseudohermitian torsion.

Let $(M, T_{1,0}(M), \theta)$ be a nondegenerate CR manifold of CR dimension n and $\hat{\theta}=e^{2u}\theta$. Then the pseudohermitian Ricci tensors $R_{\alpha\bar{\beta}}$, $\hat{R}_{\alpha\bar{\beta}}$ of θ, $\hat{\theta}$ are related by:

$$
\begin{aligned}
\hat{R}_{\alpha\bar{\beta}}=&R_{\alpha\bar{\beta}}-(n+2)(u_{\alpha,\bar{\beta}}+u_{\bar{\beta},\alpha})- \\
&-(u_{\rho,}{}^{\rho}+u_{\bar{\rho},}{}^{\bar{\rho}}+4(n+1)u_\rho u^\rho)h_{\alpha\bar{\beta}}
\end{aligned}
\tag{10}
$$

(cf. e.g. (2.17) in [Lee2], p. 164). If $M=\mathbf{H}_n(s)$ and θ is given by (4) then we may apply (10) with $R_{\alpha\bar{\beta}}=0$, $u=\log|x|^{-1}$, $h_{\alpha\bar{\beta}}=\delta_{\alpha\beta}$ and $\omega_\beta^\alpha=0$. Then:

$$u_{\rho,}{}^{\rho}=-\frac{n}{2}|x|^{-4}\varphi$$

$$u_\rho u^\rho=\frac{1}{4}|x|^{-4}|z|^2$$

$$u_{\alpha,\bar{\beta}}=-\frac{1}{2}|x|^{-4}\varphi\delta_{\alpha\beta}$$

so that (10) yields:

$$\hat{R}_{\alpha\bar{\beta}}=(n+1)|x|^{-2}|z|^2\hat{h}_{\alpha\bar{\beta}}$$

and (4) is pseudo-Einstein. Our Theorem 2 is completely proved.

REMARK 1. Let $\mathbf{R}^*\approx\{(0,t):t\in\mathbf{R}-\{0\}\}\subset\mathbf{H}_n-\{0\}$. The pseudohermitian Ricci curvature of the contact form (4) vanishes on $\pi(\mathbf{R}^*)$ so that Prop. 6.4. in [Lee2], p. 175 does not apply.

3 Regular strictly pseudoconvex CR manifolds.

Let M be a m-dimensional differentiable manifold. A local chart (U,φ) on M is cubical (of breadth $2a$ centered at $x\in M$) if $\varphi(x)=(0,\dots,0)$ and $\varphi(U)=\{(t^1,\dots,t^m)\in\mathbf{R}^m:|t^j|<a,\ 1\leq j\leq m\}$. Let $(U,\varphi),\varphi=(x^1,\dots,x^m)$, be a cubical local chart on M. Let $1\leq p\leq m$ and $t=(t^{p+1},\dots,t^m)\in\mathbf{R}^{m-p}$ so that $|t^{p+j}|<a$, $1\leq j\leq m-p$. The p-dimensional slice Σ_t of (U,φ) is given by $\Sigma_t=\{y\in U:x^{p+j}(y)=t^{p+j},\ 1\leq j\leq m-p\}$.

Let $(M, T_{1,0}(M), \theta, T)$ be a nondegenerate CR manifold of CR dimension n. Then T is regular if M admits a C^∞ atlas $\{(U,x^i)\}$ so that the intersection with U of any maximal integral curve of T is a 1-dimensional slice of (U,x^i). Let $\langle T\rangle$ be the distribution spanned by T, i.e. $\langle T\rangle_x=\mathbf{R}T(x)$, $x\in M$. If T is regular then, by Theorem VIII in [Pal], p. 19, the quotient space $M/\langle T\rangle$ (i.e. the space of all maximal integral curves of T) admits a natural manifold structure with respect to which the canonical projection $\pi:M\to M/\langle T\rangle$ is differentiable (cf. also Theorem X, [Pal], p. 20). We may state the following:

THEOREM 3. *Let $(M, T_{1,0}(M), \theta, T)$ be a compact strictly pseudoconvex CR manifold. If T is regular then M admits a global pseudo-Einstein structure.*

To prove Theorem 3. we need to recall the essentials of the Boothby-Wang theorem (cf. [Boo-Wan]). As T is regular, its maximal integral curves are closed subsets of M (cf. Theorem VII, [Pal], p. 18). But M is compact so that each maximal integral curve is homeomorphic to S^1. Let λ be the period of T, i.e. $\lambda(x)=\inf\{t>0:\varphi_t(x)=x\}$, $x\in M$, where $\{\varphi_t\}_{t\in\mathbf{R}}$ is the 1-parameter group generated by T. We may assume that $\lambda=1$ (otherwise, as $\lambda=\text{const.}>0$ (by an argument in [Tan]) we may replace T by $\frac{1}{\lambda}T$). Then, by the Boothby-Wang theorem, T generates a free and effective action of S^1 on M. Next M becomes the total space of a principal bundle $S^1\to M\xrightarrow{\pi}B$, where $B=M/\langle T\rangle$. Any principal bundle is in particular a submersion (and we may apply results in [Nei]).

Let g_θ be the Webster metric. Let $\frac{d}{dt}$ be the generator of the Lie algebra $\mathrm{L}(S^1)\approx\mathbf{R}$. Then $\theta\otimes\frac{d}{dt}$ is a connection 1-form in $S^1\to M\longrightarrow \mathrm{B}$. Set:

$$h_\theta(X,\ Y)_u=g_\theta(X^H,\ Y^H)_x$$

where $x\in\pi^{-1}(u)$, $u\in B$ and X, $Y\in T_u(B)$. Here X^H denotes the horizontal lift (cf. [Kob-Nom], vol. I, p. 64) of X with respect to $\theta\otimes\frac{d}{dt}$. The definition of $h_\theta(X,\ Y)_u$ does not depend upon the choice of x in $\pi^{-1}(u)$. It follows that $\pi: M\to B$ is a Riemannian submersion from (M, g_θ) onto (B, h_θ). Let P, Q be the fundamental tensors of π (cf. [Nei], p. 460) that is:

$$P_XY=h\tilde{\nabla}_{vX}vY+v\tilde{\nabla}_{vX}hY \tag{11}$$
$$Q_XY=h\tilde{\nabla}_{hX}hY+v\tilde{\nabla}_{hX}vY \tag{12}$$

for any X, $Y\in T(M)$. Here $\tilde{\nabla}$ denotes the Levi-Civita connection of (M, g_θ). Moreover $h=\pi_H$ and $vX=\theta(X)T$ are the canonical projections associated with (A. 2). Let us substitute from (A. 6) into (12). As $JT=0$, $\tau T=0$, $\nabla T=0$ and $H(M)$ is parallel with respect to ∇, our (12) becomes:

$$\begin{aligned} &Q_XY=\left\{\frac{1}{2}\Omega_\theta(X,\ Y)-A(X,\ Y)\right\}T\\ &Q_XT=\tau(X)+\frac{1}{2}JX\\ &Q_TX=0,\ \ Q_TT=0\end{aligned} \tag{13}$$

for any X, $Y\in H(M)$. By Theorem 6, τ is self-adjoint, while by a result of B. O'Neill (cf. [Nei], p. 460) Q is skew-symmetric on horizontal vectors. Clearly the Levi distribution $H(M)$ coincides with the horizontal distribution of the Riemannian submersion $\pi: M\to B$. Then the first of the formulae (13) yields $A=0$ and thus (cf. [Lee2], p. 176) there is $u\in C^\infty(M)$ so that $\exp(2u)\theta$ is globally pseudo-Einstein. The proof of Theorem 3 is complete.

REMARK 2.

i) Let us substitute from (A. 6) into (11). This procedure leads to $P=0$. Consequently the fibres of the submersion $\pi: M\to B$ are totally-geodesic in (M, g_θ).

ii) By a result of G. Gigante, [Gig], p. 151, and by the proof of Theorem 3, any compact strictly pseudoconvex symmetric (in the sense of [Gig], p. 150) CR manifold is a Sasakian manifold.

Let $(M, T_{1,0}(M), \theta)$ be a nondegenerate CR manifold. Let $\mathscr{E}_{CR}$ be the sheaf of local CR-holomorphic functions on M. There is a short exact sequence:

$$0\to\mathbf{R}\xrightarrow{j}\mathscr{E}_{CR}\xrightarrow{\eta}\mathscr{P}\to 0 \tag{14}$$

where $j_U:\mathbf{R}\to\mathscr{E}_{CR}(U)$, $j_U(c)=ic$, and $\eta_U:\mathscr{E}_{CR}(U)\to\mathscr{P}(U)$, $\eta_U(f)=Re(f)$, for any $c\in\mathbf{R}$, $f\in\mathscr{E}_{CR}(U)$. Indeed, let $\sigma_x\in\mathrm{Ker}(\eta_x)$, $x\in M$. That is, there are an open set $U\subset M$, $x\in U$, and a real valued function $v\in C^\infty(U)$ so that $[iv]_x=\sigma_x$ and $\bar\partial_b(v)=0$. Then $\partial_b v=0$ (by complex conjugation) and $dv=T(v)\theta$. Exterior differentiation gives:

$$0=dT(v)\wedge\theta+T(v)d\theta=dT(v)\wedge\theta+iT(v)h_{\alpha\bar\beta}\theta^\alpha\wedge\theta^{\bar\beta}.$$

Let us apply this to the pair $(T_\alpha, T_{\bar\beta})$ so that to yield $0=\frac{i}{2}T(v)h_{\alpha\bar\beta}$. Finally, contraction with $h^{\alpha\bar\beta}$ gives $T(v)=0$, i.e. there are an open set $V\subset U$, $x\in V$, and a constant $c\in\mathbf{R}$ so that $v=c$ on V. Thus $\sigma_x=[ic]_x=j_x(c)$, Q. E. D.

Consider the Bockstein exact sequence:

$$\cdots\to H^1(M,\mathbf{R})\to H^1(M,\mathscr{E}_{CR})\xrightarrow{\eta_*}H^1(M,\mathscr{P})\xrightarrow{b}H^2(M,\mathbf{R})\to\cdots$$

associated with (14). If M is compact and strictly pseudoconvex one may try to show that i) $b(\gamma(M))=c_1(T_{1,0}(M))$ and ii) $Im(\eta_*)=0$ (by Prop. 5.2 in [Lee2], p. 172, this would imply the Lee conjecture). The example $M=\mathbf{H}_n(s)$ kills a hope to solve the Lee conjecture along the line indicated above. Indeed, $r:\mathbf{H}_n-\{0\}\to\Sigma^{2n}$ defined by:

$$r(x)=\delta_{|x|^{-1}}(x)$$

for any $x\in\mathbf{H}_n-\{0\}$, is a deformation retract. Thus, by $\mathbf{H}_n(s)\approx\Sigma^{2n}\times S^1$

and the Künneth formula it follows that $H^2(\mathbf{H}_n(s), \mathbf{R})=H^2(\Sigma^{2n}, \mathbf{R})=H^2(\mathbf{H}_n-\{0\}, \mathbf{R})=H^2(S^{2n}, \mathbf{R})=0$ and the Bockstein sequence yields:

$$Im(\eta^*)=H^1(\mathbf{H}_n(s), \mathscr{P})$$

4 Locally conformal Heisenberg manifolds.

Let M be a C^∞ real $(2n+1)$-dimensional manifold. Then M is said to be locally Heisenberg if it is equipped with a C^∞ atlas $\mathscr{A}$ whose transition functions (coordinate transformations) are local CR diffeomorphisms of the Heisenberg group $\mathbf{H}_n$. The sphere $S^{2n+1}\subset\mathbf{C}^{n+1}$ is locally Heisenberg. Also $\mathbf{H}_n(s)$ (cf. Section 2) is locally Heisenberg, for any $0<s<1$.

Any locally Heisenberg manifold $(M,\mathscr{A})$ is a CR manifold, in a natural way. Indeed, let $x\in M$ and $(V, \psi)\in\mathscr{A}$ so that $x\in V$. Define $H_x(M)=\psi_*^{-1}H_{\psi(x)}(\mathbf{H}_n)$. The definition of $H_x(M)$ does not depend upon the choice of $(V, \psi)\in\mathscr{A}$. Next, define a real operator $J_x: H_x(M)\otimes\mathbf{C}\to H_x(M)\otimes\mathbf{C}$ by setting $J_xT'_\alpha=iT'_\alpha$, where $T'_\alpha=\psi_*^{-1}W_\alpha$, $W_\alpha=\partial/\partial w^\alpha+i\bar{w}^\alpha\partial/\partial s$, $\psi=(w^1, \dots, w^n, s)$. If $(U,\varphi)\in\mathscr{A}$ is an other chart, $U\cap V\neq\emptyset$, then $F=\psi\varphi^{-1}$ is a CR diffeomorphism. Set $F=(F^1, \dots, F^n, f)$. As F is a CR map, the functions F^α and $|F|^2-if$ are CR-holomorphic, where $|F|^2=F^\alpha F_\alpha$. Thus $F_*Z_\alpha=Z_\alpha(F^\sigma)W_\sigma$, where $Z_\alpha=\partial/\partial_\alpha+i\bar{z}^\alpha\partial/\partial t$, $\varphi=(z^1, \dots, z^n, t)$. Finally $JT_\alpha=J\varphi_*^{-1}Z_\alpha=JZ_\alpha(F^\sigma)\psi_*^{-1}W_\sigma=iT_\alpha$, i.e. J is globally defined. Then $(H(M), J)$ gives M a structure of CR manifold of CR dimension n.

A pseudohermitian manifold $(M, \mathscr{H}, \theta)$ of CR dimension n, is said to be locally conformal Heisenberg if for any $x\in M$ there is a local coordinate neighborhood $(U, z^1, \dots, z^n, t)$, $x\in U$, so that:

$$\theta_{|U}=e^{2u}\{dt+i\sum_{\alpha=1}^n(z^\alpha d\bar{z}^\alpha-\bar{z}^\alpha dz^\alpha)\}$$

for some $\mathbf{R}$-valued function $u\in C^\infty(U)$. For instance $(\mathbf{H}_n(s), |x|^{-2}\{dt+i\sum_{\alpha=1}^n(z^\alpha d\bar{z}^\alpha-\bar{z}^\alpha dz^\alpha)\})$ is locally conformal Heisenberg (with $u=\log|x|^{-1}$).

Any orientable locally Heisenberg manifold is locally conformal Heisenberg, in a natural way. Indeed, let $(M,\mathscr{A})$ be a locally Heisenberg manifold and $\mathscr{H}$ its natural CR structure. By orientability, let $\theta\in\Gamma^\infty(F)$ be a global, nowhere vanishing section, i.e. a pseudohermitian structure on M. Here $F\to M$ is the real line bundle in the Appendix (i.e. $F_x\subset T_x^*(M)$, $x\in M$, and each covector $f\in F_x$ annihilates $H_x(M)$). Let $(U, \varphi)\in\mathscr{A}$, $\varphi=(z^1, \dots, z^n, t)$. Then $\mathscr{H}_{|U}=\text{Span}\{\partial/\partial z^\alpha+i\bar{z}^\alpha\partial/\partial t\}$ so that $dt+i\sum_{\alpha=1}^n(z^\alpha d\bar{z}^\alpha-\bar{z}^\alpha dz^\alpha)\in\Gamma^\infty(F_{|U})$. Thus there is a $\mathbf{R}$-valued function $f\in$

$C^\infty(U)$, nowhere vanishing, so that $\theta_{|U}=f\{dt+i\sum_{\alpha=1}^{n}(z^\alpha d\bar{z}^\alpha-\bar{z}^\alpha dz^\alpha)\}$. We may assume w. l. o. g. that $f>0$ on U(otherwise start with $-\theta$). If $(V, \psi)\in\mathscr{A}$ is an other coordinate neighborhood, $U\cap V\neq\emptyset$, so that $\psi=(w^1,\ldots, w^n, s)$ and $\theta_{|V}=g\{ds+i\sum_{\alpha=1}^{n}(w^\alpha d\bar{w}^\alpha-\bar{w}^\alpha dw^\alpha)\}$, $g\in C^\infty(V)$, then $g>0$ on V; in particular $(M, \mathscr{H}, \theta)$ is strictly pseudoconvex. Indeed, set $F=\psi\varphi^{-1}$; then F is a local CR diffeomorphism of $\mathbf{H}_n$ and $F^*\{ds+i\sum_{\alpha=1}^{n}(w^\alpha d\bar{w}^\alpha-\bar{w}^\alpha dw^\alpha)\}=\lambda\{dt+i\sum_{\alpha=1}^{n}(z^\alpha d\bar{z}^\alpha-\bar{z}^\alpha dz^\alpha)\}$ with $\lambda=\sum_{\alpha,\beta=1}^{n}|U_{\alpha}^{\beta}|^2>0$ (where $U_{\alpha}^{\beta}=Z_\alpha(F^\beta)$). Finally, note that $f=g\lambda$.

Let $(M, \mathscr{H}, \theta)$ be a locally conformal Heisenberg manifold. There is a covering of M with coordinate neighborhoods $\{(U_j, z_j^\alpha, t_j)\}_{j\in J}$ and a family $\{u_j\}_{j\in J}$ of $\mathbf{R}$-valued functions $u_j\in C^\infty(U_j)$ so that $\theta_{|U_j}=e^{2u_j}\{dt_j+i\sum_{\alpha=1}^{n}(z_j^\alpha d\bar{z}_j^\alpha-\bar{z}_j^\alpha dz_j^\alpha)\}$. If, for any $i, j\in J$ with $U_i\cap U_j\neq\emptyset$, there is $c_{ij}\in\mathbf{R}$ so that $u_i-u_j=c_{ij}$ on $U_i\cap U_j$ then $(M, \mathscr{H}, \theta)$ is termed globally conformal Heisenbrg.

Let $(M, \mathscr{H}, \theta)$ be a globally conformal Heisenberg manifold. Set $\omega_{|U_j}=du_j$, $j\in J$. Then ω is a (closed) globally defined 1-form on M, called the *Lee form* of M. For instance $\mathbf{H}_n(s)$ with the contact form (4) is globally conformal Heisenberg with the Lee form $\omega=d\log|x|^{-1}$.

Let M be a real $(2n+1)$-dimensional C^∞ differentiable manifold admitting a C^∞ atlas $\mathscr{A}$ whose transition functions are dilations $\delta_r:(z, t)\mapsto(rz, r^2t)$, $r\neq0$, of $\mathbf{H}_n$. Let us call such $(M, \mathscr{A})$ a *locally dilation* manifold. For example $\mathbf{H}_n(s)$, $0<s<1$, is a locally dilation manifold.

PROPOSITION 2. *Let M be a locally dilation manifold. Then M is globally conformal Heisenberg.*

PROOF. Any dilation of $\mathbf{H}_n$ is a CR diffeomorphism so that a locally dilation manifold is in particular locally Heisenberg. Let $\theta_{|U}=e^{2u}\varphi^*\theta_1$ and $\theta_{|V}=e^{2v}\psi^*\theta_2$ (where $\theta_1=dt+i\sum_{\alpha=1}^{n}(z^\alpha d\bar{z}^\alpha-\bar{z}^\alpha dz^\alpha)$ and $\theta_2=ds+i\sum_{\alpha=1}^{n}(w^\alpha d\bar{w}^\alpha-\bar{w}^\alpha dw^\alpha)$). Then $F^*\theta_2=\lambda\theta_1$ with $\lambda=|r\delta_{\alpha}^{\alpha}|^2=r^2n^2$ so that $u=v+\log n|r|$ (and M follows to be globally conformal Heisenberg), Q. E. D.

THEOREM 4. *Let M be a globally conformal Heisenberg manifold. If M is locally realizable and its Lee form is exact then M admits a global pseudo-Einstein structure.*

PROOF. As $(M, \mathscr{H}, \theta)$ is globally conformal Heisenberg, there is an open cover $\mathscr{U}=\{U_j\}_{j\in J}$ and a family $\{u_j\}_{j\in J}$, $u_j\in C^\infty(U_j)$, and local coordinates $\varphi_j=(z_j^\alpha, t_j): U_j\to\mathbf{H}_n$ so that $\theta_{|U_j}=e^{2u_j}\{dt_j+i\sum_{\alpha=1}^{n}(z_j^\alpha d\bar{z}_j^\alpha-\bar{z}_j^\alpha dz_j^\alpha)\}$ and $u_i-u_j=c_{ij}\in\mathbf{R}$ on $U_i\cap U_j\neq\emptyset$. Let $f=(c_{ij})\in\mathscr{C}^1(N(\mathscr{U}),\mathbf{R})$ be the corresponding cochain. Note that f is a cocycle so that we may consider its cohomology class $[f]\in H^1(M,\mathbf{R})$. Then $[\omega]$ (the De Rham cohomology class of the Lee form) corresponds to $[f]$ under the isomorphism $H^1_{DR}(M)\approx H^1(M,\mathbf{R})$. Let $\mathscr{P}$ be the sheaf of CR-pluriharmonic functions on M and $\gamma(M)\in H^1(M,\mathscr{P})$ the CR invariant cohomology class in Section 2. Let i : $\mathscr{C}^1(N(\mathscr{U}),\mathbf{R})\to\mathscr{C}^1(N(\mathscr{U}),\mathscr{P})$ be the natural cochain map. Since each $dt_j+i\sum_{\alpha=1}^{n}(z_j^\alpha d\bar{z}_j^\alpha-\bar{z}_j^\alpha dz_j^\alpha)$, $j\in J$, is Ricci flat (and in particular pseudo-Einstein) it follows that $i_*: H^1(M,\mathbf{R})\to H^1(M,\mathscr{P})$ maps $[f]$ onto $\gamma(M)$.

5 Appendix.

Let M be a C^∞ manifold of real dimension $2n+1$. A CR structure on M is a complex n-dimensional subbundle $T_{1,0}(M)\subset T(M)\otimes\mathbf{C}$ so that i) $T_{1,0}(M)\cap T_{0,1}(M)=(0)$ and ii) $[T_{1,0}(M), T_{1,0}(M)]\subset T_{1,0}(M)$, where $T_{0,1}(M)=\overline{T_{0,1}(M)}$. A pair $(M, T_{1,0}(M))$ is a CR manifold (of CR dimension n). Set $H(M)=Re\{T_{1,0}(M)\oplus T_{0,1}(M)\}$. Then $H(M)$ is a real rank $2n$ vector subbundle of $T(M)$ (the Levi distribution of M). It carries the complex structure $J(Z+\bar{Z})=i(Z-\bar{Z})$, for any $Z\in T_{1,0}(M)$. Let $F_x\subset T_x^*(M)$ consist of all tangent covectors f so that $\mathrm{Ker}(f)\supseteq H_x(M)$, $x\in M$. Assume from now on that M is orientable. Then the real line bundle $F\to M$ admits global nowhere vanishing sections (termed pseudohermitian structures). With a choice $\theta\in\Gamma^\infty(F)$ of pseudohermitian structure on M we associate the Levi form $L_\theta(Z, W)=L_\theta(\bar{Z}, \bar{W})=0$, $L_\theta(Z, \bar{W})=-i(d\theta)(Z, \bar{W})$ and $L_\theta(\bar{Z}, W)=\overline{L_\theta(Z, \bar{W})}$, for any $Z, W\in T_{1,0}(M)$. The CR manifold M is nondegenerate (respectively strictly pseudoconvex) if, for some choice of θ, L_θ is nondegenerate (respectively positive-definite).

Let M be a nondegenerate CR manifold. Its Webster metric g_θ is given by $g_\theta(X, T)=0$, $g_\theta(T, T)=1$ and $g_\theta(X, Y)=G_\theta(X, Y)$ where:

$$G_\theta(X, Y)=\frac{1}{2}\{(d\theta)(X, JY)-(d\theta)(JX, Y)\} \tag{A.0}$$

for any $X, Y\in H(M)$. Here T is the unique globally defined nowhere vanishing tangent vector field on M transverse to $H(M)$ and subject to:

$$T\rfloor\theta=1,\ T\rfloor d\theta=0 \tag{A.1}$$

Note that:

(A.2) $$T(M)=H(M)\oplus\{\mathbf{R}T\}$$

Extend J to an endomorphism $J: T(M)\to T(M)$ by setting $JT=0$. We recall (cf. [Dra]):

THEOREM 5. *Let $(M, T_{1,0}(M), \theta, T)$ be a nondegenerate CR manifold. There is a unique linear connection ∇ on M satisfying the following axioms*:

i) $X\in T(M),\ Y\in H(M)\Longrightarrow \nabla_X Y\in H(M)$,
ii) $\nabla J=0$,
iii) $\nabla g_\theta=0$,
iv) $\pi_+ Tor\ (Z, W)=0$,

for any $Z\in T_{1,0}(M)$, $W\in T(M)\otimes\mathbf{C}$(where Tor is the torsion of ∇ and $\pi_+: T(M)\otimes\mathbf{C}\to T_{1,0}(M)$ the natural projection).

This is the Webster connection of M. With respect to a (local) frame $\{T_\alpha\}$ of $T_{1,0}(M)$ it is given by:

(A.3) $$2\Gamma^\rho_{\beta\alpha}h_{\rho\bar\sigma}=T_\beta(h_{\alpha\bar\sigma})+T_\alpha(h_{\beta\bar\sigma})+ \\ +g_\theta([T_\beta, T_\alpha], T_{\bar\sigma})+g_\theta([T_{\bar\sigma}, T_\beta], T_\alpha)+g_\theta([T_{\bar\sigma}, T_\alpha], T_\beta)$$

(A.4) $$2\Gamma^\rho_{\bar\beta\alpha}h_{\rho\bar\sigma}=T_{\bar\beta}(h_{\alpha\bar\sigma})-T_{\bar\sigma}(h_{\alpha\bar\beta})+ \\ +g_\theta([T_{\bar\beta}, T_\alpha], T_{\bar\sigma})+g_\theta([T_{\bar\sigma}, T_{\bar\beta}], T_\alpha)+g_\theta([T_{\bar\sigma}, T_\alpha], T_{\bar\beta})$$

(A.5) $$2\Gamma^\rho_{0\alpha}h_{\rho\bar\alpha}=T(h_{\alpha\bar\sigma})+ \\ +g_\theta([T, T_\alpha], T_{\bar\sigma})+g_\theta([T_{\bar\sigma}, T], T_\alpha).$$

Let $\tilde\nabla$ be the Levi-Civita connection of (M,g_θ). Then (cf. [Dra]):

(A.6) $$\tilde\nabla=\nabla+\left(\frac{1}{2}\Omega_\theta-A\right)\otimes T+\tau\otimes\theta+\theta\odot J$$

where $\Omega_\theta(X, Y)=g_\theta(X, JY)$, $A(X, Y)=g_\theta(X, \tau Y)$ and $\tau: T(M)\to T(M)$ given by $\tau X=Tor\ (T, X)$ is the pseudohermitian torsion of the Webster connection. Also $\odot$ stands for the symmetric product. Finally, we recall (cf. [Dra]):

THEOREM 6. *Let $(M, T_{1,0}(M), \theta, T)$ be a strictly pseudoconvex CR manifold. Then τ is self-adjoint (with respect to g_θ) and trace-less. Consequently, the Levi distribution is minimal (in (M, g_θ)).*

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