Hokkaido Mathematical Journal Vol. 24(1995) p. 53-62

Multiplicity results for nonlinear wave equations with nonlinearities crossing eigenvalues

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(Received March 22, 1994)

Introduction

In this paper we investigate the existence of solutions $u(x, t)$ for a nonlinear perturbation $f(x, u)$ of the 1-dimensional wave operator $u_{tt}-u_{xx}$ under Dirichlet boundary condition on the interval $-\frac{\pi}{2}<x<\frac{\pi}{2}$ and π-periodic condition on the variable t,

$$\begin{aligned} &u_{tt}-u_{xx}+f(x, u) = W(x) \quad \text{in} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\times R, \\ &u\left(\pm\frac{\pi}{2}, t\right)=0, \\ &u(x, t+\pi) = u(x, t) = u(x, -t) = u(-x, t). \end{aligned} \tag{0.1}$$

When a string with a nonuniform density vibrates up and down, the upward restoring coefficient and the downward restoring coefficient of it are not uniform. Hence it happens a nonlinear perturbation $f(x, u)$ in the wave of a string and we have a nonlinear wave equation (0.1).

In [CJK], it was shown that when the nonlinear perturbation $f(x, u)$ is piecewise linear one $bu^{+}-au^{-}$ with $-5<a<-1$, $3<b<7$, the equation

$$\begin{aligned} &u_{tt}-u_{xx}+bu^{+}-au^{-} = s\phi_{00} \quad \text{in} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\times R, \\ &u\left(\pm\frac{\pi}{2}, t\right)=0, \\ &u(x, t+\pi) = u(x, t) = u(x, -t) = u(-x, t), \end{aligned} \tag{0.2}$$

where $s>0$ and ϕ_{00} is the positive eigenfunction of the wave operator, has at least four solutions. This was proved by topological methods.

In [CJ], the authors applied the variational reduction method to show that if $s>0$ and $-1<a<3<b<7$ with $\frac{1}{\sqrt{a+1}}+\frac{1}{\sqrt{b+7}}>1$, then equation

[1] Research supported in part by GARC-KOSEF and by Inha University Research Foundation

(0. 2) has at least three solutions.

In this paper, we improve the earlier result of [CJK] in a way. In [CJK], the authors have been concerned only with the piecewise linear perturbation, but here we do not restrict the nonlinear perturvation only to the piecewise linear case.

Our main idea of this paper is the following. We use the Contraction Mapping Theorem to reduce the problem from an infinite dimensional one in $L^2(\Omega)$ to a 2-dimensional one. Next we convert the two dimensional problem into degree theoretic statements in the space $L^2(\Omega)$, and then show that these results can be perturbed to give the result for the non-linear equation with large coefficient s of ϕ_{00}.

Let L be the 1-dimensional wave operator, in R^2,

$$Lu = u_{tt} - u_{xx}.$$

Then the eigenvalue problem

$$\begin{aligned} &Lu = \lambda u \quad \text{in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times R, \\ &u\left(\pm\frac{\pi}{2}, t\right) = 0, \\ &u(x, t) = u(-x, t) = u(x, -t) = u(x, t+\pi) \end{aligned}$$

has infinitely many eigenvalues

$$\lambda_{mn} = (2n+1)^2 - 4m^2 \qquad (m, n=0, 1, 2, \cdots).$$

Hence the eigenvalues in the interval $(-15, 9)$ are given by

$$\lambda_{32} = -11 < \lambda_{21} = -7 < \lambda_{10} = -3 < \lambda_{00} = 1 < \lambda_{11} = 5.$$

The eigenfunction ϕ_{mn} corresponding to λ_{mn} is given by

$$\phi_{mn} = \cos 2mt \cos(2n+1)x \quad (m, n=0, 1, 2, \cdots).$$

Let Q be the square $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and H the Hilbert space defined by

$$H = \{u \in L^2(Q) | u \text{ is even in } x \text{ and } t\}.$$

Then the set $\{\phi_{mn} | m, n=0, 1, 2, \cdots\}$ is an orthogonal set in H. We will look for π-periodic solutions of (0. 1) when the nonlinear perturbation $f(x, u)$ crosses the first positive eigenvalue and the first negative one. The existence of π-periodic solutions of (0. 1) will be shown by topological methods (cf. [LM1], [MRW], [LM]).

1. A piecewise linear perturbation case

In this section, we remind of the process of the proof of the existence of the solutions of the piecewise linear wave operator with constant coefficients, $-5<a<-1=-\lambda_{00}$, $-\lambda_{10}=3<b<7$,

$$Lu+bu^{+}-au^{-}=s\phi_{00} \quad \text{in} \quad H. \tag{1.1}$$

Let V be the two dimensional subspace of H spanned by $\{\phi_{00}, \phi_{10}\}$ and W the orthogonal complement of V in H. Let P be an orthogonal projection H onto V. Then for all $u\in H$, $u=v+w$, where $v=Pu$, $w=(I-P)u$. Therefore equation (1.1) is equivalent to

$$\begin{aligned} &\text{(a)} \quad Lw+(I-P)(b(v+w)^{+}-a(v+w)^{-})=0, \\ &\text{(b)} \quad Lv+P(b(v+w)^{+}-a(v+w)^{-})=s\phi_{00}. \end{aligned} \tag{1.2}$$

We note that for fixed v, (1.2.a) has a unique solution $w=\vartheta(v)$, and that furthermore, $\vartheta(v)$ is Lipschitz continuous in terms of v.

If $v\geq 0$ or $v\leq 0$, then $\vartheta(v)\equiv 0$. Since $v=c_1\phi_{00}+c_2\phi_{10}$, there exists a cone C_1 defined by $c_1\geq 0$, $|c_2|\leq\varepsilon_0 c_1$ so that $v\geq 0$ for all $v\in C_1$ and a cone C_3 defined by $c_1\leq 0$, $|c_2|\leq\varepsilon_0|c_1|$ so that $v\leq 0$ for all $v\in C_3$. In particular, since $\phi_{00}=\cos x$ and $\phi_{10}=\cos x\cos 2t$, in this case, the elementary calculation shows $\varepsilon_0=1$. Also, we define another cone C_2 by $c_2\geq 0$, $|c_1|\leq c_2$ and a cone C_4 by $c_2\leq 0$, $|c_1|\leq|c_2|$. Then the plane spanned by ϕ_{00} and ϕ_{10} consists of four cones C_1, C_2, C_3, C_4.

We do not know $\vartheta(v)$ for all $v\in PH$, but we know $\vartheta(v)\equiv 0$ for $v\in C_1\cup C_3$. We define a map $\Phi: PH\to PH$ by

$$\Phi(v)=Lv+P(b(v+\vartheta(v))^{+}-a(v+\vartheta(v))^{-}).$$

Then there exists $d>0$ such that

$$(\Phi(c_1\phi_{00}+c_2\phi_{10}), \phi_{00})\geq d|c_2|$$

(see [CJK], [Mc]). Hence the map $\Phi: PH\to PH$ takes the value ϕ_{00} once in each of the four different regions $C_i(1\leq i\leq 4)$ of the plane (see [CJK]). We define a map $F: R^2\to R^2$ by

$$F(s_1, s_2)=(t_1, t_2) \quad \text{if} \quad v=s_1\phi_{00}+s_2\phi_{10} \quad \text{and} \quad \Phi(v)=t_1\phi_{00}+t_2\phi_{10}.$$

The next lemma gives an information on the degree of the map in the regions $C_i(1\leq i\leq 4)$.

Lemma 1.1 *Let $p=(1, 0)$. Let r be so large that $r>1$, $r(b+1)>1$ and $rd>1$. Let*

$$D_1 = \{(s_1, s_2) \mid 0 < s_1 < r,\ |s_2| < s_1\},$$
$$D_2 = \{(s_1, s_2) \mid |s_1| \leq r,\ |s_1| < s_2 < r\},$$
$$D_3 = \{(s_1, s_2) \mid -r < s_1 < 0,\ |s_2| < |s_1|\},$$
$$D_4 = \{(s_1, s_2) \mid |s_1| \leq r,\ -r < s_2 < -|s_1|\}.$$

If $\deg(F, D_k, p)$ *denotes the Brouwer degree of* F *with respect to* D_k *and* p *for* $1 \leq k \leq 4$, *then* $\deg(F, D_k, p)$ *is defined for* $1 \leq k \leq 4$ *and*

$$\deg(F, D_k, p) = (-1)^{k+1}.$$

Proof. First we consider the Brouwer degree of F with respect to D_1. If $(s_1, s_2) \in \bar{D}_1$ and $v = s_1\phi_{00} + s_2\phi_{10}$, then $\vartheta(v) = 0$. Since $v \geq 0$ in D_1, we have

$$\begin{aligned}\Phi(v) &= Lv + P(b(v+\vartheta(v))^+ - a(v+\vartheta(v))^-)\\ &= L(s_1\phi_{00} + s_2\phi_{10}) + P(b(s_1\phi_{00} + s_2\phi_{10}))\\ &= \lambda_{00}s_1\phi_{00} + \lambda_{10}s_2\phi_{10} + b(s_1\phi_{00} + s_2\phi_{10})\\ &= (\lambda_{00} + b)s_1\phi_{00} + (\lambda_{10} + b)s_2\phi_{10}.\end{aligned}$$

So we have, for $(s_1, s_2) \in \bar{D}_1$,

$$F(s_1, s_2) = ((b + \lambda_{00})s_1,\ (b + \lambda_{10})s_2).$$

Since $1 < r(b+1)$, the equation $F(s_1, s_2) = p$ has a unique solution $(s_1, s_2) = ((b+\lambda_{00})^{-1}, 0)$. Since the determinant of the linear diagonal map B is positive, we have

$$\deg(F, D_1, p) = 1.$$

In the case of $(s_1, s_2) \in \bar{D}_3$, we have the diagonal map

$$F(s_1, s_2) = ((a + \lambda_{00})s_1,\ (a + \lambda_{10})s_2)$$

and the determinant is also positive near the unique solution in this region given by $((a+\lambda_{00})^{-1},\ 0)$. Hence we have

$$\deg(F, D_3, p) = 1.$$

Now we consider the Brouwer degree of F with respect to D_2. The boundary of D_2 consists of three line segments;

(i) a ray in the first quadrant R_1, $s_1 > 0$ and $s_2 = s_1$,
(ii) a ray in the second quadrant R_2, $s_1 < 0$ and $s_2 = -s_1$,
(iii) a line segment L of $s_2 = r$, paralled to the s_1 axis.

The image of R_1 under F will be a straight line segment in the first quadrant, the image of R_2 will be a straight line segment in the fourth quadrant and the image of L will be to the right of line $s_1 = 1$, by virtue of the requirement $rd > 1$.

Now we consider the linear map $u \to Bu$, where B is given by

$$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The image of R_1 under B, BR_1 will be a straight line in the first quadrant. So if $0 \leq \lambda \leq 1$, we have

$$\lambda Bs + (1-\lambda)F(s) \neq p, \quad s = (s_1, s_2) \in R_1.$$

The image of the ray R_2 under B is in the fourth quadrant and again we have, $0 \leq \lambda \leq 1$,

$$\lambda Bs + (1-\lambda)F(s) \neq p, \quad s = (s_1, s_2) \in R_2.$$

If $s \in L$, then $s_2 = r > 1$ and

$$Bs \in \{(s_1, s_2) | s_1 > 1\},$$

and hence $\lambda Bs + (1-\lambda)F(s) \neq p$ for $s \in L$.

By the usual homotopy argument,

$$\deg(F, D_2, p) = \deg(B, D_2, p).$$

But we know that $Bs - p$ has exactly one zero in D_2, and the sign of the determinant of B is -1. Thus

$$\deg(F, D_2, p) = -1.$$

Similarly, we have

$$\deg(F, D_4, p) = -1. \qquad \square$$

Using the definition of the degree of a mapping on an arbitrary finite dimensional space we obtain the following lemma.

Lemma 1.2 *If for* $1 \leq k \leq 4$,

$$U_k = \{v \in V | v = s_1\phi_{00} + s_2\phi_{10}, \ (s_1, s_2) \in D_k\}$$

and $T: V \to V$ *is defined by*

$$Tv = PL^{-1}(b(v+\vartheta(v))^+ - a(v+\vartheta(v))^-),$$

then

$$\deg\left(I+T, U_k, \frac{\phi_{00}}{\lambda_{10}}\right) = (-1)^{k+1}.$$

Next we can deduce, from our knowledge of the two dimensional degree, a result on the degree of the associated map on infinite dimensional space.

We let

$$Nu = L^{-1}(bu^{+} - au^{-}).$$

Note that N is a compact operator from H to H. Now we have the following theorem.

Theorem 1.3 *Let U_k, $1 \le k \le 4$, and T be as in the preceding lemma. If $r_2 > 0$ is sufficiently large and for $1 \le k \le 4$,*

$$Y_k = \{u \in L^2(\Omega) | Pu \in U_k,\ \|(I-P)u\| < r_2\},$$

then the Leray-Schauder degree $d\left(I+N,\ Y_k,\ \frac{\phi_{00}}{\lambda_{00}}\right)$ is defined and

$$d\left(I+N,\ Y_k,\ \frac{\phi_{00}}{\lambda_{00}}\right) = d\left(I+T,\ U_k,\ \frac{\phi_{00}}{\lambda_{00}}\right)$$
$$= (-1)^{k+1}.$$

Proof. First we observe that there exists $r_1 > 0$ such that if $v \in \bar{U}_k$, $1 \le k \le 4$ and $w = (1-s)(I-P)N(v+w)$, $0 \le s \le 1$, then $\|w\| < r_1$. In fact, the map $w \to (1-s)(I-P)N(v+w)$ is a contraction on $(I-P)H$, for $s (0 \le s \le 1)$.

Now we choose $r_2 > r_1$ and define $h_1 :\ Y_k \times [0,1] \to L^2$, for some fixed k, by

$$h_1(u, s) = (I-P)N(v+w) + PN(v+w+s(\vartheta(v)-w)),$$

where $v = Pu$, $w = (I-P)u$. Then we obtain

$$u + h_1(u, s) \ne \frac{\phi_{00}}{\lambda_{00}} \quad \text{for} \quad (u, s) \in \partial Y_k \times [0, 1].$$

To prove the above, we consider two possibilities in $u \in \partial Y_k$. One is that $u = v + w$ with $v \in \partial U_k$, $\|w\| < r_2$, $s \in [0,1]$, and

$$u + h_1(u, s) = \frac{\phi_{00}}{\lambda_{00}}.$$

This equation is equivalent to

$$w + (I-P)N(v+w) = 0,$$

$$v + PN(v+w+s(\vartheta(v)-w)) = \frac{\phi_{00}}{\lambda_{00}}.$$

The first of these implies $w=\vartheta(v)$, and the second implies

$$v+PN(v+\vartheta(v)) = v+N(v) = \frac{\phi_{00}}{\lambda_{00}},$$

which contradicts the fact that $v\in\partial U_k$.

Now suppose $v\in U_k$, $w\in(I-P)H$, $\|w\|=r_2$. If $0\leq s\leq 1$ and

$$u+h_1(u, s) = \frac{\phi_{00}}{\lambda_{00}},$$

then

$$w+(I-P)N(v+w) = 0,$$

hence $w=\vartheta(v)$ and $\|w\|\leq r_1<r_2$, which is a contradiction. This shows that

$$u+h_1(u, s) \neq \frac{\phi_{00}}{\lambda_{00}}$$

for all $(u, s)\in\partial Y_k\times[0, 1]$, and it follows by homotopy invariance of degree that

$$d\left(I+N,\ Y_k, \frac{\phi_{00}}{\lambda_{00}}\right) = d\left(I+h_1(\cdot, 1),\ Y_k, \frac{\phi_{00}}{\lambda_{00}}\right).$$

Now let $h_2\colon\ Y_k\times[0, 1] \to L^2(\Omega)$ be defined by

$$h_2(u, s) = (1-s)(I-P)N(u)+PN(v+\vartheta(v)),\ v = Pu.$$

If $v\in\partial U_k$, $w\in(I-P)H$, $0\leq s\leq 1$, $u=v+w$, and $u+h_2(u, s)=\frac{\phi_{00}}{\lambda_{00}}$, then we have

$$v+T(v) = v+PN(v+\vartheta(v)) = P(u+h_2(u, s)) = \frac{\phi_{00}}{\lambda_{00}},$$

which contradicts the fact that there is no solution if $v\in\partial U_k$. If $u=v+w$, $v\in U_k$, $w=(I-P)H$, $\|w\|=r_2$, $0\leq s\leq 1$ and $u+h_2(u, s)=\frac{\phi_{00}}{\lambda_{00}}$, then

$$0 = (I-P)(u+h_2(u, s)) = w+(1-s)(I-P)N(v+w),$$

which would imply that $\|w\|<r_2$, which is a contradiction. Therefore we have

$$u+h_2(u, s)\neq\frac{\phi_{00}}{\lambda_{00}},\quad \text{for}\quad (u, s) \in \partial Y_k\times[0, 1].$$

Since $h_1(u,1)=h_2(u,0)$, we infer by homotopy invariance that

$$d\left(I+N,\ Y_k,\ \frac{\phi_{00}}{\lambda_{00}}\right)=d\left(I+h_2(\cdot,1),\ Y_k,\ \frac{\phi_{00}}{\lambda_{00}}\right).$$

Let B be the open ball of radius r_2 in $(I-P)H$. If $u\in\bar{Y}_k$, $v=Pu$, $w=(I-P)u$, then

$$u+h_2(u,1)=v+PN(v+\vartheta(v))+w.$$

Thus we see that the map $u\to u+h_2(u,1)$ is uncoupled an $PH\oplus(I-P)H$ and is the identity on $(I-P)H$. Therefore by the product property of degree,

$$d\left(I+N,\ Y_k,\ \frac{\phi_{00}}{\lambda_{00}}\right)=d\left(I+T,\ U_k,\ \frac{\phi_{00}}{\lambda_{00}}\right)=(-1)^{k+1}.$$

This concludes the proof of the theorem. □

2. A nonlinear perturbation case

In this section we are in a position to produce solutions to the nonlinear problem

$$Lu+f(u)=s\phi_{00}+h(x), \tag{2.1}$$

where we assume for the nonlinear perturbation,

$$f(\zeta)=b\zeta^+-a\zeta^-+f_0(\zeta)\ \text{with}\ \lim_{|\zeta|\to\infty}\frac{f_0(\zeta)}{\zeta}=0$$

and $-5<a<-1$, $3<b<7$. To apply Theorem 1.3, putting $f_1(u)=bu^+-au^-$, we rewrite (2.1) as

$$Lz+f_1(z)+\frac{f_0(sz)}{s}=\phi_{00}+\frac{h(x)}{s}, \tag{2.2}$$

where $z=\dfrac{u}{s}$. In particular, if $f_0=0$ and $h=0$, then equation (2.2) becomes a nonlinear wave equation with a piecewise linear perturbation bu^+-au^-, $-5<a<-1$, $3<b<7$,

$$Lu+bu^+-au^-=\phi_{00}.$$

Let

$$N_s(z)=L^{-1}\left(f_1(z)+\frac{f_0(sz)}{s}-\frac{h}{s}\right)$$

and let

$$N(z) = L^{-1}(f_1(z)).$$

Then

$$\lim_{s\to\infty} \|N(z) - N_s(z)\| = 0$$

uniformly for z in bounded subsets of $L^2(\Omega)$.

Now we have the main theorem.

Theorem 2. 1 *Assume that the nonlinear perturbation satisfies*

$$f(\zeta) = b\zeta^+ - a\zeta^- + f_0(\zeta) \quad \textit{with} \quad \lim_{|\zeta|\to\infty} \frac{f_0(\zeta)}{\zeta} = 0$$

and $-5 < a < -1$, $3 < b < 7$.

Then there exists $s_0 > 0$ *such that if* $s \geq s_0$ *the nonlinear wave equation*

$$Lu + f(u) = s\phi_{00} + h(x)$$

has at least four solutions.

Proof. We have established that

$$z + N(z) \neq \frac{\phi_{00}}{\lambda_{00}} \quad \text{for all} \quad z \in \partial Y_k,\ 1 \leq k \leq 4.$$

Since ∂Y_k is closed and bounded, and N is continuous and compact, there exists $\eta > 0$ such that

$$\left\| z + N(z) - \frac{\phi_{00}}{\lambda_{00}} \right\| \geq \eta \quad \text{if} \quad z \in \partial Y_k.$$

Now choose s_0 so that

$$\|N_s(z) - N(z)\| < \frac{\eta}{2} \quad \text{for all} \quad z \in \partial Y_k,\ 1 \leq k \leq 4.$$

Then

$$\left\| z + N(z) + (1-\lambda)(N_s(z) - N(z)) - \frac{\phi_{00}}{\lambda_{00}} \right\| \geq \frac{\eta}{2}$$

for $0 \leq \lambda \leq 1$, from which we conclude

$$d\left(I + N_s,\ Y_k,\ \frac{\phi_{00}}{\lambda_{00}}\right) = d\left(I + N,\ Y_k,\ \frac{\phi_{00}}{\lambda_{00}}\right) = (-1)^{k+1}, \quad 1 \leq k \leq 4.$$

This proves the theorem, since we have at least one solution in Y_k, $1 \leq k \leq 4$. □

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