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Asymptotic behaviors of radial solutions to semilinear wave equations in odd space dimensions

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0. Introduction

This paper is concerned with semilinear wave equations of the form

$$u_{tt}-u_{rr}-\frac{n-1}{r}u_{r}=F(u, u_{t}, u_{r}) \quad \text{in} \quad \boldsymbol{R}^{2}, \tag{0.1}$$

where $u=u(r, t)$ is a real-valued function and $n=2m+3$ with m a non-negative integer. For a large class of the nonlinear term F we will show that "small" solutions of (0.1) exist and are asymptotic to the solutions of the linear wave eqaution

$$u_{tt}-u_{rr}-\frac{n-1}{r}u_{r}=0 \quad \text{in} \quad \boldsymbol{R}^{2}, \tag{0.2}$$

namely, there exist solutions u_{-}, u_{+} of (0.2) and $u(t)-u_{\pm}(t)\rightarrow 0$ as $t\rightarrow\pm\infty$ in the sense of the energy norm.

As is well known, the equation (0.1) is the radially symmetric version of a special case of

$$u_{tt}-\Delta u=F_{0}(u, Du, D_{x}Du) \quad \text{in} \quad \boldsymbol{R}^{n}\times\boldsymbol{R}, \tag{0.3}$$

where $D=(D_{x}, D_{t})$, $D_{x}=(\partial/\partial x_{1}, \ldots, \partial/\partial x_{n})$ and $D_{t}=\partial/\partial t$. The existence of global small solutions of the Cauchy problem for (0.3) has been shown by Christodoulou [3], Li Ta-tsien and Chen Yun-mei [11], and Li Ta-tsien and Yu Xin [12], provided the nonlinear term F_{0} and the initial data prescribed on $t=0$ are "nice". Moreover the asymptotic behaviors as $t\rightarrow\pm\infty$ for solutions of (0.3), which guarantee the existence of the scattering operator, have been researched by Strauss [19], Mochizuki and Motai [13], [14], Pecher [15], Tsutaya [22] and Kubota and Mochizuki [10] in the case where F_{0} is independent of Du, $D_{x}Du$, *i.e.*, $F_{0}=F_{1}(u)$, and by Klainerman [7], Shatah [16] and Klainerman and Ponce [8] in the case where F_{0} does not explicitly depend on u, *i.e.*, $F_{0}=F_{2}(Du, D_{x}Du)$.

In the present paper we study the asymptotic behaviors of radial solutions to (0.3), which guarantee the existence of the scattering opeartor, in the case where F_0 is independent of D_xDu, *i.e.*, $F_0=F_3(u, Du)$.

Although in this paper we restrict ourselves to the case of odd space dimensions, one can similarly deal with the case of even space dimensions. The details will be published elsewhere.

1. Statements of main results

First consider the Cauchy problem for the linear wave equation (0. 2):

$$\begin{aligned} &u_{tt}-u_{rr}-\frac{n-1}{r}u_r=0 \quad \text{in} \quad \boldsymbol{R}^2, \\ &u(r,0)=f(r),\ u_t(r,0)=g(r) \quad \text{for} \quad r\in\boldsymbol{R}, \end{aligned} \tag{1.1}$$

where $n=2m+3$ and m is a nonnegative integer. We assume the pair $\{f, g\}$ of the initial data belongs to the following function space E_0: The set of $\{f, g\}$ such that f, g are even functions of $\boldsymbol{R}$, $f\in C^{m+3}(\boldsymbol{R})$, $g\in C^{m+2}(\boldsymbol{R})$ and $\|\{f, g\}\|_0<\infty$. Here the norm is defined by

$$\begin{aligned} \|\{f, g\}\|_0 = &\sup_{-\infty<r<\infty}|f(r)|\langle r\rangle^{m+1+\kappa} \\ &+\sum_{j=0}^{m+2}\sup_{-\infty<r<\infty}(|f^{(j+1)}(r)|+|g^{(j)}(r)|)\langle r\rangle^{m+2+\kappa}, \end{aligned} \tag{1.2}$$

where $\langle r\rangle=\sqrt{1+r^2}$, $g^{(j)}(r)=\left(\frac{d}{dr}\right)^j g(r)$ and κ is a positive parameter. Note that the function $\langle|x|\rangle^{-m-1-\kappa}$ does not belong to $L^2(\boldsymbol{R}^n)$ if $0<\kappa\leqq 1/2$.

We also introduce another function space X in which we will look for solutions of the nonlinear wave equation of (0. 1):

$$\begin{aligned} X=\{u(r, t)\in C^1(\boldsymbol{R}^2)\,;\ &r^ju(r, t)\in C^{j+2,0}(\boldsymbol{R}^2),\ r^ju_t(r, t)\in C^{j+1,0}(\boldsymbol{R}^2) \\ &\text{for } 0\leqq j\leqq m+1 \quad\text{and}\quad \|u\|<\infty\}, \end{aligned} \tag{1.3}$$

where

$$\begin{aligned} \|u\|=&\sum_{|\alpha|\leq 1}\sup_{(r, t)\in\boldsymbol{R}^2}|D^\alpha_{r,t}u(r, t)|\langle r\rangle^m\langle|r|+|t|\rangle\langle|r|-|t|\rangle^\kappa \\ &+\sum_{j=0}^{m+1}\sup_{(r, t)\in\boldsymbol{R}^2}(|r^jD_r^{j+2}u(r, t)|+|r^jD_r^{j+1}D_tu(r, t)|) \\ &\times\langle r\rangle^{m-j}\langle|r|+|t|\rangle\langle|r|-|t|\rangle^\kappa \end{aligned} \tag{1.4}$$

with the same parameter κ as in (1. 2), and $C^{k,0}(\boldsymbol{R}^2)$ stands for the set of continuous functions $u(r, t)$ on $\boldsymbol{R}^2$ such that $D_r^ju(r, t)$ are continuous on $\boldsymbol{R}^2$ for $0\leqq j\leqq k$. We also write $X_d=\{u\in X\,;\ \|u\|\leqq d\}$.

In (1. 4) if $j\geqq 1$, we interpret $r^jD_r^{j+2}u(r, t)$ for $r=0$ as a linear combination of $D_r^{k+2}(r^ku(r, t))$ for $k=0, 1, \ldots, j$, because

$$r^j D_r^{j+2} u(r, t) = \sum_{k=0}^{j} C_k D_r^{k+2}(r^k u(r, t)) \quad \text{for} \quad r \neq 0,$$

so that $r^j D_r^{j+2} u(r, t)$ is continuous on $\boldsymbol{R}^2$.

Now, for the Cauchy problem (1.1) we have

Theorem 1.1 *Let* $\{f, g\} \in E_0$. *Then* (1.1) *admits a unique solution* $u(r, t)$ *which belongs to* $X \cap C^2(\boldsymbol{R}^2)$ *and is even in* r. *Moreover*

$$r^j u(r, t) \in C^{j+2}(\boldsymbol{R}^2) \quad \text{if} \quad 1 \leqq j \leqq m+1 \tag{1.5}$$

and for $(r, t) \in \boldsymbol{R}^2$ *we have*

$$|u(r, t)| \leqq C \|\{f, g\}\|_0 \langle r \rangle^{-m} \langle |r| + |t| \rangle^{-1} \langle |r| - |t| \rangle^{-\kappa}, \tag{1.6}$$

$$|D_{r,t}^{\alpha} u(r, t)| \leqq C \|\{f, g\}\|_0 \langle r \rangle^{-m-1} \langle |r| - |t| \rangle^{-\kappa-1} \quad \text{if} \quad 1 \leqq |\alpha| \leqq 2 \tag{1.7}$$

and

$$\begin{aligned} &|r^{|\alpha|-2} D_{r,t}^{\alpha} u(r, t)| \\ &\quad \leqq C \|\{f, g\}\|_0 \langle r \rangle^{-m+|\alpha|-3} \langle |r| - |t| \rangle^{-\kappa-1} \quad \text{if} \quad 3 \leqq |\alpha| \leqq m+3, \end{aligned} \tag{1.8}$$

where C *is a constant depending only on* n *and* κ.

Remark. Employing the methods in Kubo [9], we find that the conclusions of Theorem 1.1 are still valid for a wider class of initial data, say, E_0': The set of $\{f, g\}$ such that f, g are continuous even functions on $\boldsymbol{R}$, $r^{m+1} f(r) \in C^{m+3}(\boldsymbol{R})$, $r^{m+1} g(r) \in C^{m+2}(\boldsymbol{R})$ and $\|\{f, g\}\|_0' < \infty$, where

$$\begin{aligned} \|\{f, g\}\|_0' = &\sup_{-\infty < r < \infty} |f(r)| \langle r \rangle^{m+1+\kappa} \\ &+ \sum_{j=0}^{m+2} \sup_{-\infty < r < \infty} \{|D_r^{j+1}(r^{m+1} f(r))| + |D_r^j (r^{m+1} g(r))|\} \langle r \rangle^{\kappa+1}. \end{aligned}$$

Next consider the nonlinear wave equation (0.1). Let $p_0(m)$ be the positive root of the quadratic equation

$$\begin{aligned} \Phi(m, p) &\equiv (m+1)p^2 - (m+2)p - 1 \\ &= \frac{n-1}{2} p^2 - \frac{n+1}{2} p - 1 = 0. \end{aligned} \tag{1.9}$$

Then we impose the following three conditions $(H)_1$, $(H)_2$ and $(H)_3$ on the nonlinear term F in (0.1):

$(H)_1$ $\quad F(\lambda) = F(\lambda_1, \lambda_2, \lambda_3) \in C^{m+2}(\boldsymbol{R}^3)$

and there exist positive numbers p and A satisfying

$$p_0(m) < p \leqq m+3 \tag{1.10}$$

and

$$|D^\alpha F(\lambda)| \leqq A|\lambda|^{p-|\alpha|} \quad \text{if} \quad |\alpha|<p,\ |\alpha| \leqq m+2 \ \text{and} \ |\lambda| \leqq 1. \tag{1.11}$$

$(H)_2$ $F(\lambda_1, \lambda_2, \lambda_3)$ is even in λ_3.

$(H)_3$ $D^{m+2}F$ is locally Hölder continuous, namely, for $|\alpha|=m+2$ and $|\lambda|$, $|\lambda'| \leqq 1$ we have

$$|D^\alpha F(\lambda)-D^\alpha F(\lambda')| \leqq B|\lambda-\lambda'|^\delta, \tag{1.12}$$

where B, δ are positive numbers depending only on F, n such that $\delta=p-(m+2)$ if $m+2<p\leqq m+3$ and δ is an arbitrary positive number satisfying $0<\delta\leqq 1$ if $p\leqq m+2$.

Examples. If $m\geqq 1$, then

$$F(\lambda)=\sum_{j=1}^{3} a_j\lambda_j^2$$

satisfies $(H)_1$, $(H)_2$ and $(H)_3$, where a_j are constants. If $m=0$, then

$$F(\lambda)=\sum_{j=1}^{3} b_j|\lambda_j|^{p_j}$$

satisfies $(H)_1$, $(H)_2$ and $(H)_3$ provided $p_j>1+\sqrt{2}$, where b_j are constants. Indeed, we take $p=\min\{3, p_1, p_2, p_3\}$.

We also impose the following condition on the positive parameter κ in (1.2) and (1.4):

$$\frac{2}{p-1}-m-1<\kappa\leqq q, \tag{1.13}$$

where $q=q(m, p)$ is a positive number defined by

$$q=\frac{1+\Phi(m, p)}{p}=(m+1)p-(m+2) \tag{1.14}$$

with the $\Phi(m, p)$ in (1.9).

We are now in a position to state the main theorem in this paper.

Theorem 1.2 *Let $u_-(r, t)$ be the solution of the Cauchy problem* (1.1) *with $\{f, g\}\in E_0$. Assume conditions $(H)_1$, $(H)_2$, $(H)_3$ and* (1.13) *hold. Then there are positive constants ε_0, d having the following property, where ε_0 depends only on F, n and κ, and d only on F and n: If $\|\{f, g\}\|_0\leqq\varepsilon\leqq\varepsilon_0$, then there exists uniquely a solution $u(r, t)$ of the nonlinear wave equation* (0.1) *which belongs to $X_d\cap C^2(\boldsymbol{R}^2)$, is even in r and has the following asymptotic behavior*

$$\begin{aligned}|D_r(u(r, t)-u_-(r, t))|+|D_t(u(r, t)-u_-(r, t))|\\ \leqq C_1\|u\|^p\langle r\rangle^{-m-1}\langle|r|-|t|\rangle^{-1}\langle|r|-t\rangle^{-\kappa}\\ \textit{for}\quad (r, t)\in\boldsymbol{R}^2,\end{aligned} \tag{1.15}$$

where C_1 *is a constant depending only on* F *and* n. *Moreover we have*

$$\|u\| \leqq 2\|u_-\| \leqq C_0\varepsilon, \tag{1.16}$$

$$\|u(t)-u_-(t)\|_e \leqq C_2\|u\|^p\langle t\rangle^{-\kappa} \quad \text{for} \quad t\leqq 0, \tag{1.17}$$

where

$$\|v(t)\|_e = \left\{\int_0^\infty (|D_r v(r, t)|^2 + |D_t v(r, t)|^2) r^{n-1} dr\right\}^{1/2},$$

$$\begin{aligned} |u(r, t)-u_-(r, t)| \leqq C_3\|u\|^p\langle r\rangle^{-m}\langle |r|+|t|\rangle^{-1} \\ \times\langle |r|-t\rangle^{-\kappa} \quad \text{for} \quad (r, t)\in \boldsymbol{R}^2 \end{aligned} \tag{1.18}$$

and

$$\begin{aligned} &|r^{\alpha+\beta-2}D_t^\beta D_r^\alpha(u(r, t)-u_-(r, t))| \\ &\leqq C_4\|u\|^p\langle r\rangle^{-m+\alpha+\beta-3}\langle |r|-|t|\rangle^{-1}\langle |r|-t\rangle^{-\kappa} \\ &\qquad \text{for } (r, t)\in \boldsymbol{R}^2,\ 2\leqq \alpha+\beta\leqq m+3 \text{ and } 0\leqq\beta\leqq 2, \end{aligned} \tag{1.19}$$

where C_2, C_3 *and* C_4 *are constants depending only on* F, n *and* C_0 *is a constant depending only on* n *and* κ.

Furthermore there exists uniquely a solution $u_+(r, t)$ *of the linear wave equation* (0.2) *which belongs to* $X\cap C^2(\boldsymbol{R}^2)$ *and has the asymptotic behaviors* (1.15), (1.17), (1.18) *and* (1.19) *with* $u_-(r, t)$, $\langle |r|-t\rangle^{-\kappa}$ *and "for* $t\leqq 0$*" replaced by* $u_+(r, t)$, $\langle |r|+t\rangle^{-\kappa}$ *and "for* $t\geqq 0$*", respectively. Besides we have*

$$\begin{aligned} &\|u_+(0)\|_e^2-\|u_-(0)\|_e^2 \\ &\quad = 2\int_{-\infty}^\infty dt \int_0^\infty F(u, u_t, u_r)(r, t)u_t(r, t)r^{n-1}dr. \end{aligned} \tag{1.20}$$

Remarks. 1) Let E be the closure of E_0 with respect to the energy norm $\|u_-(0)\|_e$. Then the "scattering operator": $\{f, g\}\mapsto\{u_+(r, 0), (D_t u_+)(r, 0)\}$ is shown to be exist on a dense set of a neighborhood of 0 in E.

2) Condition (1.2) on the decay rate of initial data can be somewhat relaxed, provided the number p in $(H)_1$ is large, say, $p>(m+3)/(m+1)$. For the details see Appendix below.

3) To prove the uniqueness of such a solution of (0.1) as in Theorem 1.2, we will show that the solution satisfies a certain integral equation (see Lemma 5.5 in § 5) and that a solution of the integral equation is unique in X_d (see Proposition 5.4).

4) For the last factor in (1.15), (1.18) and (1.19), we note that $\langle |r|-t\rangle^{-\kappa}=\langle |r|+|t|\rangle^{-\kappa}$ for $t\leqq 0$.

5) In [15] where $m=0$, Pecher takes the parameter in (1.4) as $\varkappa=q(0, p)=p-2$.

6) Consider the following Cauchy problem

$$\begin{aligned} &u_{tt}-\Delta u=|u|^p \quad \text{in} \quad \boldsymbol{R}^n\times[0,\infty),\ n\geqq 2,\ p>1,\\ &u(x,0)=0,\ u_t(x,0)=g(x) \quad \text{for} \quad x\in\boldsymbol{R}^n. \end{aligned} \tag{1.21}$$

Let $\tilde{p}_0(n)$ be the positive root of (1.9). Then it is necessary for (1.21) to admit global (in time) solutions that $p>\tilde{p}_0(n)$ if $n=2$, 3, and $p\geqq\tilde{p}_0(n)$ if $n\geqq 4$. (For the details see John [6], Glassey [5], Sideris [18] and Schaeffer [17]). Therefore the first half of condition (1. 10), *i. e.*, $p>p_0(m)$ is almost necessary to obtain Theorem 1.2.

7) Let $p>\tilde{p}_0(n)$. Then it is necessary for (1.21) to admit global radial solutions that the parameter $\varkappa$ in (1.2) satisfies $\varkappa+m+1\geqq 2/(p-1)$, where $\varkappa$ is a real number and $m=(n-2)/2$ if n is even. (For the details see Asakura [2], Agemi and Takamura [1], Tsutaya [21] and Takamura [20]). Therefore the first half of condition (1.13) is almost necessary to obtain Theorem 1.2.

The plan of this paper is as follows. In the next section we summarize some known results concerning the fundamental solution for the Cauchy problem (1.1). In § 3 we prove Theorem 1.1, by employing the results in § 2. In § 4 we establish a priori estimates for the nonlinear term in (0.1). In § 5 we prove Theorem 1.2, by employing the results in §§ 2 through 4. Finally in the Appendix we relax condition (1.2) on the decay rate of the initial data in (1.1), when $p>(m+3)/(m+1)$.

2. Preliminaries

In this section we shall summarize some known results concerning the fundamental solution for the Cauchy problem (1.1). We start with

Lemma 2.1 *Let $H(\rho)\in C^2(\boldsymbol{R})$ and set*

$$u(r,t)=\int_{-1}^{1}H(t+r\sigma)(1-\sigma^2)^m d\sigma, \tag{2.1}$$

where $m=(n-3)/2$. Then $u(r,t)$ belongs to $C^2(\boldsymbol{R}^2)$ and is even in r. Moreover $u(r,t)$ satisfies the linear wave equation (0.2), where $r^{-1}u_r(r,t)$ is interpreted as

$$\frac{1}{r}u_r(r,t)=\int_0^1 u_{rr}(r\lambda,t)d\lambda.$$

For the proof see for example Courant and Hilbert [4], p. 700 or Kubo [9], § 2. For the Cauchy problem (1.1) we have

Lemma 2. 2 *Let $f(r)$, $g(r)\in C^0(\boldsymbol{R})$ be functions satisfying $r^{m+1}f(r)\in C^{m+3}(\boldsymbol{R})$ and $r^{m+1}g(r)\in C^{m+2}(\boldsymbol{R})$. Then $r^k f(r)\in C^{k+2}(\boldsymbol{R})$ and $r^k g(r)\in C^{k+1}(\boldsymbol{R})$ for $0\leqq k\leqq m$. Moreover set.*

$$H_g(r) = (2m!)^{-1}\left(\frac{\partial}{\partial r^2}\right)^m (r^{2m+1}g(r)). \tag{2. 2}$$

Then $H_f(r)\in C^3(\boldsymbol{R})$ and $H_g(r)\in C^2(\boldsymbol{R})$. Furthermore put

$$\begin{aligned} u(r, t) = &\int_{-1}^{1} H_g(t+r\sigma)(1-\sigma^2)^m d\sigma \\ &\qquad + D_t \int_{-1}^{1} H_f(t+r\sigma)(1-\sigma^2)^m d\sigma. \end{aligned} \tag{2. 3}$$

If f, g are even functions, we have

$$u(r, 0) = f(r), \ \ u_t(r, 0) = g(r). \tag{2. 4}$$

Besides if $f=0$ then $u(r, t)$ is odd in t.

For the proof see Kubo [9], § 2. From this lemma we have easily

Corollary 2. 3 *Let $g(r)\in C^0(\boldsymbol{R})$ be an even function such that $r^{m+1}g(r)\in C^{m+2}(\boldsymbol{R})$. Let $H_g(r)$ be defined by (2. 2). Then*

$$\int_{-1}^{1} H_g(r\sigma)(1-\sigma^2)^m d\sigma = 0$$

and

$$\int_{-1}^{1} H_g'(r\sigma)(1-\sigma^2)^m d\sigma = g(r).$$

For the inhomogeneous wave equation

$$u_{tt} - u_{rr} - \frac{n-1}{r} u_r = G(r, t) \quad \text{in} \quad \boldsymbol{R}^2, \tag{2. 5}$$

we have

Lemma 2. 4 *Let $G(r, t)\in C^0(\boldsymbol{R}^2)$ and $r^{m+1}G(r, t)\in C^{m+2,0}(\boldsymbol{R}^2)$. Set*

$$H(r, t) = (2m\,!)^{-1}\left(\frac{\partial}{\partial r^2}\right)^m (r^{2m+1}\, G(r, t))$$

and

$$w(r, t) = \int_s^t d\tau \int_{-1}^{1} H(t-\tau+r\sigma, \ \tau)(1-\sigma^2)^m d\sigma,$$

where s is an arbitrary real number. Then $H(r, t)\in C^{2,0}(\boldsymbol{R}^2)$ and $w(r, t)$

is even in r. Moreover suppose $G(r, t)$ is even in r. Then

$$D_t w(r, t)=\int_s^t d\tau \int_{-1}^1 D_t H(t-\tau+r\sigma, \tau)(1-\sigma^2)^m d\sigma. \tag{2.6}$$

Furthermore $w(r, t)$ belongs to $C^2(\boldsymbol{R})$ and satisfies (2.5) with the zero initial data $w(r, s)=0$ and $w_t(r, s)=0$.

For the proof see e. g. [9], the proof of Proposition 3. 4. The following lemma shows that the function $u(r, t)$ given by (2. 1) does gain regularity by multiplying r.

Lemma 2. 5 *Let $u(r, t)$ be defined by (2. 1), where $H(\rho)\in C^1(\boldsymbol{R})$. Let α, β be nonnegative integers such that $0\leqq\alpha+\beta\leqq m$. Then*

$$(D_t r)^\beta (D_r r)^\alpha u(r, t)=\int_{-1}^1 H(t+r\sigma)\psi_{\alpha,\beta}(\sigma)d\sigma, \tag{2.7}$$

where

$$\psi_{\alpha,\beta}(\sigma)=\left(-\frac{d}{d\sigma}\right)^\beta\left(-\sigma\frac{d}{d\sigma}\right)^\alpha(1-\sigma^2)^m$$

and $(D_r r)u=D_r(ru)$. Moreover if $\alpha+\beta=m$ we have

$$\begin{aligned}(D_t r)^{\beta+1}(D_r r)^\alpha u(r, t)=&\int_{-1}^1 H(t+r\sigma)\left(-\frac{d}{d\sigma}\right)\psi_{\alpha,\beta}(\sigma)d\sigma\\ &+H(t+r)\psi_{\alpha,\beta}(1)-H(t-r)\psi_{\alpha,\beta}(-1)\end{aligned} \tag{2.8$_1$}$$

and

$$\begin{aligned}(D_r r)^{m+1}u(r, t)=&\int_{-1}^1 H(t+r\sigma)\left(-\sigma\frac{d}{d\sigma}\right)\psi_{m,0}(\sigma)d\sigma\\ &+H(t+r)\psi_{m,0}(1)+H(t-r)\psi_{m,0}(-1).\end{aligned} \tag{2.8$_2$}$$

For the proof see [9], the proof of Proposition 2. 5.

Corollary 2. 6 *Let $H(\rho, \tau)\in C^{l,0}(\boldsymbol{R}^2)$, where l is a positive integer. Set*

$$v_k(r, t, \tau)=r^k\int_{-1}^1 H(t+r\sigma, \tau)(1-\sigma^2)^m d\sigma,$$

where k is an integer such that $0\leqq k\leqq m+1$. Then

$$D_t^\beta D_r^\alpha v_k(r, t, \tau)\in C^0(\boldsymbol{R}^3) \tag{2.9}$$

for nonnegative integers α, β such that $0\leqq\alpha+\beta\leqq k+l$.

Proof. It suffices to prove (2. 9) when $k\leqq\alpha+\beta\leqq k+l$. Since

$$D_t^\beta D_r^\alpha(r^{\alpha+\beta}v_0(r, t, \tau))=\sum_{j=0}^\alpha C_j(D_t r)^\beta(D_r r)^j v_0$$

with some constants C_j such that $C_\alpha=1$, by Lemma 2.5 we get

$$D_t^\beta D_r^\alpha(r^{\alpha+\beta}v_0(r,t,\tau))=\int_{-1}^{1}H(t+r\sigma,\tau)\tilde{\psi}_{\alpha,\beta}(\sigma)d\sigma + C_{\alpha,\beta}H(t+r,\tau)+C'_{\alpha,\beta}H(t-r,\tau)$$

for $0\leqq\alpha+\beta\leqq m+1$, where $\tilde{\psi}_{\alpha,\beta}(\sigma)$ is a polynomial and $C_{\alpha,\beta}$, $C'_{\alpha,\beta}$ are constants. Therefore we obtain (2.9) for α, β such that $k\leqq\alpha+\beta\leqq k+\ell$. The proof is complete.

We will need also another representation for the solution of (1.1).

Lemma 2.7 *Let $g(r)$, $H_g(r)$ and $u(r,t)$ be as in Lemma 2.2 with $f=0$. Assume g is an even function. Then*

$$u(r,t)=\int_{|t-r|}^{t+r}g(\rho)K(\rho,r,t)d\rho \quad \textit{if} \quad r>0,\ t\geqq0, \tag{2.10}$$

where $K(\rho,r,t)=(2r)^{-1}\rho$ if $m=0$, and

$$K(\rho,r,t)=\frac{(-1)^m}{2m!}\left(\frac{\rho}{r}\right)^{2m+1}\left(\frac{\partial}{\partial\rho}\frac{1}{2\rho}\right)^m\phi^m(\rho,r,t) \tag{2.11}$$

with $\phi(\rho,r,t)=r^2-(t-\rho)^2$ if $m\geqq1$. Moreover we have

$$K(-\rho,r,t)=-K(\rho,r,t) \tag{2.12}$$

and

$$K(\rho,-r,t)=-K(\rho,r,t). \tag{2.13}$$

For the proof see [9], the proof of Lemma 2.3. For the above function $K(\rho,r,t)$ and the polynomial $\phi^m(\rho,r,t)$, we will need the follwing three lemmas which have been obtained in [9], section 4. In what follows we denote constants independent of ρ, r and t by C_1, C_2 and so on. Besides, $\alpha=(\alpha_1,\alpha_2)$ and $\beta=(\beta_1,\beta_2)$ stand for multi-indices of nonnegative integers such that $|\alpha|+|\beta|\leqq 2m+4$.

Lemma 2.8 *Let $r\geqq0$, $t\geqq0$ and $|t-r|\leqq\rho\leqq t+r$. Then*

$$|D_{r,t}^\alpha\phi^m(\rho,r,t)|\leqq C_1r^{2m-|\alpha|} \quad \textit{for} \quad 0\leqq|\alpha|\leqq2m+4, \tag{2.14}$$

where $C_1=0$ unless $0\leqq|\alpha|\leqq2m$, and

$$|D_{r,t}^\alpha\phi^m(\rho,r,t)|\leqq C_2r^m\rho^{m-|\alpha|} \quad \textit{for} \quad 0\leqq|\alpha|\leqq m. \tag{2.15}$$

Lemma 2.9 *Let $r\geqq0$ and $t\geqq0$. Then*

$$|D_{r,t}^\alpha(D_{r,t}^\beta\phi^m(\rho,r,t)|_{\rho=t\pm r})| \leqq C_3r^{2m-|\alpha|-|\beta|} \quad \textit{for} \quad 0\leqq|\alpha|+|\beta|\leqq2m+3, \tag{2.16}$$

where $C_3=0$ *unless* $m\leqq|\alpha|+|\beta|\leqq 2m$. *Moreover suppose* $r\geqq t$. *Then*

$$|D_{r,t}^{\alpha}(D_{r,t}^{\beta}\phi^m(\rho, r, t)|_{\rho=r-t})| \leqq C_4 r^m(r-t)^{m-|\alpha|-|\beta|} \quad for \quad 0\leqq|\alpha|+|\beta|\leqq m \tag{2.17$_1$}$$

and

$$|D_{r,t}^{\alpha}(D_{r,t}^{\beta}\phi^m(\rho, r, t)|_{\rho=r-t})| \leqq C_5 r^{2m-|\alpha|-|\beta|} \quad for \quad m+1\leqq|\alpha|+|\beta|\leqq 2m+3, \tag{2.17$_2$}$$

where $C_5=0$ *if* $|\alpha|+|\beta|\geqq 2m+1$.

Lemma 2.10 *Let* $r\geqq 1$, $t\geqq 0$ *and* $|t-r|\leqq\rho\leqq t+r$. *Then*

$$|D_{r,t}^{\alpha}K(\rho, r, t)|\leqq C_6 r^{-m-1}\chi_{\alpha}(\rho) \quad for \quad 0\leqq|\alpha|\leqq m+4, \tag{2.18}$$

where

$$\chi_{\alpha}(\rho)=\begin{cases}\langle\rho\rangle^{m+1} & if\ |\alpha|=0,\\ \langle\rho\rangle^{m}+r^{-1}\langle\rho\rangle^{m+1} & if\ |\alpha|=1,\\ \langle\rho\rangle^{m-1}+r^{-1}\langle\rho\rangle^{m}+r^{-2}\langle\rho\rangle^{m+1} & if\ |\alpha|\geqq 2.\end{cases} \tag{2.19}$$

Moreover we have

$$|D_{r,t}^{\alpha}(D_{r,t}^{\beta}K(\rho, r, t)|_{\rho=t\pm r})| \leqq C_7 r^{-m-1}\chi_{\alpha+\beta}(t\pm r) \quad for \quad 0\leqq|\alpha|+|\beta|\leqq m+3. \tag{2.20}$$

Furthermore suppose $r\geqq t$. *Then*

$$|D_{r,t}^{\alpha}(D_{r,t}^{\beta}K(\rho, r, t)|_{\rho=r-t})| \leqq C_8 r^{-m-1}\chi_{\alpha+\beta}(r-t) \quad for \quad 0\leqq|\alpha|+\beta|\leqq m+3. \tag{2.21}$$

3. The linear wave equation

The purpose of this section is to prove Theorem 1.1. We start with

Lemma 3.1 *Let* $f(r)$, $g(r)\in C^0(\boldsymbol{R})$ *be even functions. Suppose* $r^{m+1}f(r)\in C^{m+3}(\boldsymbol{R})$ *and* $r^{m+1}g(r)\in C^{m+2}(\boldsymbol{R})$. *Then there exists a unique solution* $u(r, t)\in C^2(\boldsymbol{R}^2)$ *of the Cauchy problem* (1.1). *Moreover the solution* $u(r, t)$ *is given by* (2.3), *even in* r *and satisfies*

$$r^k u(r, t)\in C^{k+2}(\boldsymbol{R}^2) \quad for \quad 0\leqq k\leqq m+1. \tag{3.1}$$

Proof. Let $u(r, t)$ be given by (2.3) with (2.2). Then it follows from Lemmas 2.1 and 2.2 that $u(r, t)$ belongs to $C^2(\boldsymbol{R}^2)$, is even in r and satisfies (1.1). Moreover by Lemma 2.2 and Corollary 2.6 with $H(\rho, \tau)=H_f'(\rho)+H_g(\rho)$ we obtain (3.1). Since the C^2-solution of (1.1) is unique, we complete the proof.

Now, to prove Theorem 1.1 we have only to establish the estimates (1.6), (1.7) and (1,8), because of Lemma 3.1.

In what follows we suppose $\{f, g\} \in E_0$ and set $\varepsilon = \|\{f, g\}\|_0$.

First we shall deal with the case where $|r|$ is small.

Lemma 3.2 *Let* $|r| \leqq 1$. *Then* (1.6), (1.7) *and* (1.8) *hold.*

Proof. From (2.2) and (2.3) we have

$$u(r, t) = \int_{-1}^{1} H(t + r\sigma)(1 - \sigma^2)^m d\sigma, \tag{3.2}$$

where

$$\begin{aligned} H(\rho) &= H_f'(\rho) + H_g(\rho) \\ &= a_0 f(\rho) + \sum_{j=0}^{m} \rho^{j+1}(a_{j+1} f^{(j+1)}(\rho) + b_j g^{(j)}(\rho)) \end{aligned} \tag{3.3}$$

with some constants a_k, b_j. Moreover (1.2) yields

$$|H^{(j)}(\rho)| \leqq C_j\, \varepsilon \langle \rho \rangle^{-\kappa-1} \quad \text{for} \quad j = 0, 1, 2. \tag{3.4}$$

First we shall derive

$$|D_{r,t}^{\alpha} u(r, t)| \leqq C\varepsilon \langle t \rangle^{-\kappa-1} \quad \text{for} \quad 0 \leqq |\alpha| \leqq 2, \tag{3.5}$$

which implies (1.6) and (1.7) for $|r| \leqq 1$, since $\langle t \rangle^{-1} \leqq 2\langle |r| + |t| \rangle^{-1}$ for $|r| \leqq 1$. Let $0 \leqq |\alpha| \leqq 2$. Then (3.2) and (3.4) imply

$$|D_{r,t}^{\alpha} u(r, t)| \leqq C\varepsilon \int_{-1}^{1} \langle t + r\sigma \rangle^{-\kappa-1} d\sigma.$$

Since $\langle t + r\sigma \rangle^{-1} \leqq 2\langle t \rangle^{-1}$ for $|r\sigma| \leqq 1$, we obtain (3.5).

Next we shall prove

$$|r^{|\alpha|-2} D_{r,t}^{\alpha} u(r, t)| \leqq C\varepsilon \langle t \rangle^{-\kappa-1} \quad \text{for} \quad 3 \leqq |\alpha| \leqq m+3. \tag{3.6}$$

Let $\alpha = (\alpha_1, \alpha_2)$ and $3 \leqq |\alpha| \leqq m+3$. Write as

$$\begin{aligned} & r^{|\alpha|-2} D_t^{\alpha_2} D_r^{\alpha_1} u(r, t) \\ &= \sum_{|\gamma|=2} \sum_{|\beta| \leqq |\alpha|-2} C_{\beta,\gamma} D_{r,t}^{\gamma} (D_t r)^{\beta_2} (D_r r)^{\beta_1} u(r, t), \end{aligned}$$

where $C_{\beta,\gamma}$ are constants. Since it follows from (3.2) and Lemma 2.5 that

$$\begin{aligned} (D_t r)^{\beta_2} (D_r r)^{\beta_1} u(r, t) = & \int_{-1}^{1} H(t + r\sigma) \tilde{\psi}_{\beta}(\sigma) d\sigma \\ & + C_{\beta} H(t + r) + C_{\beta}' H(t - r), \end{aligned}$$

where $\tilde{\psi}_\beta(\sigma)$ is a polynomial and C_β, C'_β are constants, we obtain (3.6) according to (3.4), as before. The proof is complete.

The rest of this section will be devoted to prove (1.6), (1.7) and (1.8) for $|r|\geqq 1$. Since $u(-r, t)=u(r, t)$, one can assume $r\geqq 1$. We may also assume $t\geqq 0$, since the first term or the second one on the right hand side of (2.3) is odd or even in t, respectively.

We shall first deal with the case where t/r is large.

Lemma 3.3 *Let* $t\geqq 3r$ *and* $r\geqq 1$. *Then* (1.6), (1.7) *and* (1.8) *hold.*

Proof. We claim that the function $H(\rho)$ given by (3.3) can be represented as

$$H(\rho)=\sum_{j=0}^{m} D_\rho^j F_j(\rho), \tag{3.7}$$

where F_j belongs to $C^{j+2}(\boldsymbol{R})$ and satisfies

$$|D_\rho^\alpha F_j(\rho)|\leqq C_{j,\alpha}\,\varepsilon\langle\rho\rangle^{-m-1-\kappa+j} \quad \text{for} \quad 0\leqq\alpha\leqq j+2,\, 0\leqq j\leqq m. \tag{3.8}$$

Indeed, from (3.3) we have

$$H(\rho)=a_0 f(\rho)+\sum_{j=0}^{m} a'_{j+1} D_\rho^j(\rho^{j+1}f'(\rho))+\sum_{j=0}^{m} b'_j D_\rho^j(\rho^{j+1}g(\rho))$$

with other constants a'_κ and b'_j. Hence, setting

$$F_j(\rho)=\rho^{j+1}(a'_{j+1}f'(\rho)+b'_j g(\rho)) \quad \text{for} \quad 1\leqq j\leqq m$$

and

$$F_0(\rho)=a_0 f(\rho)+\rho(a'_1 f'(\rho)+b'_0 g(\rho)),$$

we see that $F_j\in C^{j+2}(\boldsymbol{R})$ and (3.8) holds, because $\{f, g\}\in E_0$.

Next, changing a variable in (3.2) by $\rho=t+r\sigma$, we have

$$u(r, t)=r^{-2m-1}\int_{t-r}^{t+r} H(\rho)\phi^m(\rho, r, t)d\rho,$$

where $\phi(\rho, r, t)=r^2-(t-\rho)^2$ is the same function as in (2.11). Note that

$$D_{r,t}^\alpha \phi^m(\rho, r, t)=0 \quad \text{for} \quad \rho=t\pm r,\ 0\leqq|\alpha|\leqq m-1.$$

Therefore, using (3.7) and integrating by parts, we obtain

$$u(r, t)=r^{-2m-1}\sum_{j=0}^{m}\int_{t-r}^{t+r} F_j(\rho)D_t^j\phi^m(\rho, r, t)d\rho. \tag{3.9}$$

We are now in a position to prove (1.6), (1.7) and (1.8) for $t\geqq 3r$. It follows from (2.14), (3.8) and (3.9) that

$$|u(r,t)| \leqq C\varepsilon \sum_{j=0}^{m} r^{-j-1} \int_{t-r}^{t+r} \langle\rho\rangle^{-m-1-\kappa+j} d\rho.$$

Since $t-r \geqq (t+r)/2$ for $t \geqq 3r$, and

$$\int_{t-r}^{t+r} \langle\rho\rangle^{-m-1-\kappa+j} d\rho \leqq 2r\langle t-r\rangle^{-m-1-\kappa+j},$$

we have

$$\begin{aligned}|u(r,t)| &\leqq C\varepsilon r^{-m} \sum_{j=0}^{m} r^{m-j} \langle t+r\rangle^{-m+j-\kappa-1}\\ &\leqq (m+1)C\varepsilon r^{-m}\langle t+r\rangle^{-\kappa-1},\end{aligned}$$

which implies (1. 6). Next suppose $1 \leqq |\alpha| \leqq m+3$. Then, using (2. 16) also, we get

$$\begin{aligned}|D_{r,t}^{\alpha} u(r,t)| \leqq C\varepsilon \sum_{j=0}^{m} \{ & r^{-j-|\alpha|-1} \int_{t-r}^{t+r} \langle\rho\rangle^{-m-1-\kappa+j} d\rho \\ & + r^{-m-1}\langle t\pm r\rangle^{-m-1-\kappa+j} \},\end{aligned}$$

which implies (1. 7) and (1. 8). Thus we prove the lemma.

Next we shall treat the case where t/r is bounded.

Lemma 3. 4 *Let* $0 \leqq t \leqq 3r$ *and* $r \geqq 1$. *Then* (1. 6), (1. 7) *and* (1. 8) *hold.*

Proof. By virtue of (2. 3) and Lemma 2. 7 we get

$$u(r,t) = \int_{|t-r|}^{t+r} g(\rho) K(\rho,r,t) d\rho + D_t \int_{|t-r|}^{t+r} f(\rho) K(\rho,r,t) d\rho,$$

where $K(\rho, r, t)$ is the function given by (2. 11). Moreover we have from (1. 2)

$$|f(\rho)| \leqq C\varepsilon\langle\rho\rangle^{-m-1-\kappa}$$

and

$$|f^{(j+1)}(\rho)| + |g^{(j)}(\rho)| \leqq C\varepsilon\langle\rho\rangle^{-m-2-\kappa} \quad \text{for} \quad 0 \leqq j \leqq m+2.$$

Therefore by Lemma 2. 10 we obtain

$$\begin{aligned}|u(r,t)| &\leqq C\varepsilon r^{-m-1} \Big(\int_{|t-r|}^{t+r} \langle\rho\rangle^{-\kappa-1} d\rho + \langle t-r\rangle^{-\kappa} \Big)\\ &\leqq C\varepsilon r^{-m-1}\langle t-r\rangle^{-\kappa},\end{aligned}$$

since $r^{-1}\langle\rho\rangle$ is bounded for $0 \leqq \rho \leqq 4r$, $r \geqq 1$. Therefore (1. 6) follows, because $r \geqq (t+r)/4$ for $0 \leqq t \leqq 3r$. Similarly we obtain (1. 7) and (1. 8). The proof is complete.

Proof of Theorem 1.1. All conclusions of the theorem follows immediately from Lemmas 3.1, 3.2, 3.3 and 3.4. Thus we prove Theorem 1.1.

4. The nonlinear wave equation

As will be seen, a solution of (0.1), which belongs to $X \cap C^2(\boldsymbol{R}^2)$, is even in r and has the asymptotic behavior (1.15), satisfies the following integral equation

$$u(r, t)=u_-(r, t)+L(u)(r, t), \tag{4.1}$$

where u_- is the solution of (1.1),

$$L(u)(r, t)=\int_{-\infty}^{t} d\tau \int_{-1}^{1} H(t-\tau+r\sigma, \tau)(1-\sigma^2)^m d\sigma, \tag{4.2}$$

$$H(\rho, \tau)=(2m!)^{-1}\left(\frac{\partial}{\partial \rho^2}\right)^m (\rho^{2m+1} G(\rho, \tau)) \tag{4.3}$$

and

$$G(\rho, \tau)=F(u(\rho, \tau),\ u_t(\rho, \tau),\ u_r(\rho, \tau)) \tag{4.4}$$

with F the function in (0.1). (See Lemma 5.5 in the next section).

The purpose of this section is to establish basic a priori estimates for the integral operator L. Throughout the present section, by L, H and G we mean the operator or functions defined by (4.2), (4.3) and (4.4), respectively. By C we also denote various constants depending only on F and n, unless stated otherwise.

We shall start with estimating G.

Lemma 4.1 *Assume condition* $(H)_1$ *holds. Let* $u(r, t) \in X$. *Then*

$$\rho^k G(\rho, \tau) \in C^{k+1,0}(\boldsymbol{R}^2) \quad for \quad 0 \leqq k \leqq m+1. \tag{4.5}$$

Moreover suppose $\|u\| \leqq 1$. *Then*

$$|D_\rho^j G(\rho, \tau)| \leqq C\|u\|^p \langle \rho \rangle^{-mp} \langle |\rho|+|\tau| \rangle^{-p} \langle |\rho|-|\tau| \rangle^{-p\kappa} \quad for \quad |\rho| \geqq 1,\ 0 \leqq j \leqq m+2 \tag{4.6}$$

and

$$|\rho^k D_\rho^j G(\rho, \tau)| \leqq C\|u\|^p \langle \tau \rangle^{-p-p\kappa} \quad for \quad |\rho| \leqq 1, 0 \leqq j \leqq k+1 \quad and \quad 0 \leqq k \leqq m+1, \tag{4.7}$$

where p *is the number in* $(H)_1$.

For the proof see e.g. Kubo [9], the proof of Lemma 3.2. Next we shall estimate $H(\rho, \tau)$.

Lemma 4.2 *Assume condition* $(H)_1$ *holds. Let* $u(r, t)\in X$. *Then* $H(\rho, \tau)\in C^{2,0}(\boldsymbol{R}^2)$. *Moreover, setting*

$$w_k(r, t, \tau)=r^k\int_{-1}^{1}H(t-\tau+r\sigma, \tau)(1-\sigma^2)^m d\sigma, \tag{4.8}$$

where k *is an integer such that* $0\leqq k\leqq m+1$, *we have*

$$D_{r,t}^{\alpha}w_k(r, t, \tau)\in C^0(\boldsymbol{R}^3) \quad for \quad 0\leqq|\alpha|\leqq k+2. \tag{4.9}$$

Furthermore suppose $\|u\|\leqq 1$. *Then*

$$|D_\rho^j H(\rho, \tau)|\leqq C\|u\|^p\langle\rho\rangle^{p-q-1}\langle|\rho|+|\tau|\rangle^{-p}\langle|\rho|-|\tau|\rangle^{-p\kappa} \quad for \quad j=0, 1, 2 \tag{4.10}$$

and

$$|D_{r,t}^{\alpha}w_k(r, t, \tau)|\leqq C\|u\|^p\langle r\rangle^k\langle\tau\rangle^{-\min\{p,q+1\}} \quad for \quad 0\leqq|\alpha|\leqq k+2, \tag{4.11}$$

where q *is the positive number defined by* (1.14).

Proof. From (4.3) we have

$$H(\rho, \tau)=\sum_{j=0}^{m}C_j D_\rho^j(\rho^{j+1}G(\rho, \tau)) \tag{4.12}$$

with some constants C_j. Therefore (4.5) yields $H(\rho, \tau)\in C^{2,0}(\boldsymbol{R}^2)$ and hence by Corollary 2.6 we get (4.9). Next let $\|u\|\leqq 1$. If $|\rho|\leqq 1$, the estimate (4.10) follows from (4.7) and (4.12), since $\langle\tau\rangle^{-1}\leqq 2\langle|\rho|+|\tau|\rangle^{-1}$ for $|\rho|\leqq 1$. If $|\rho|\geqq 1$, by (4.6) we have

$$|D_\rho^j H(\rho, \tau)|\leqq C\|u\|^p\langle\rho\rangle^{m+1-mp}\langle|\rho|+|\tau|\rangle^{-p}\langle|\rho|-|\tau|\rangle^{-p\kappa},$$

which implies (4.10), because $mp=q-p+m+2$ according to (1.14). Finally, (4.11) is a direct consequence of (4.8) and (4.10) provided $0\leqq|\alpha|\leqq 2$. If $3\leqq|\alpha|\leqq k+2$, we refer to the proof of Corollary 2.6. Then we obtain (4.11), as above. The proof is complete.

For the operator L we have

Lemma 4.3 *Assume condition* $(H)_1$ *holds. Let* $u(r, t)\in X$. *Then* $L(u)(r, t)$ *is even in* r *and satisfies*

$$r^k L(u)(r, t)\in C^{k+2,0}(\boldsymbol{R}^2) \quad for \quad 0\leqq k\leqq m+1. \tag{4.13}$$

Moreover assume $G(\rho, \tau)$ *is even in* ρ. *Then*

$$D_t L(u)(r, t)=\int_{-\infty}^{t}D_t w_0(r, t, \tau)d\tau, \tag{4.14}$$

where w_0 *is given by* (4. 8). *Furthermore* $L(u)$ *belongs to* $C^2(\boldsymbol{R}^2)$ *and satisfies the inhomogeneous wave equation* (2. 5). *Besides we have*

$$r^k D_t L(u)(r,t) \in C^{k+1,0}(\boldsymbol{R}^2) \quad \text{for} \quad 0 \leqq k \leqq m+1. \tag{4.15}$$

Proof. Define $w_k(r, t, \tau)$ by (4. 8). Then (4. 2) implies

$$L(u)(r, t) = \int_{-\infty}^{t} w_0(r, t, \tau) d\tau. \tag{4.16}$$

Hence (4. 13) follows from (4. 9) and (4. 11), because $p>1$ and $q+1>1$. Besides, $w_0(r, t, \tau)$ is even in r hence so is $L(u)(r, t)$.

Next suppose $G(\rho, \tau)$ is even in ρ. Then $H(\rho, \tau)$ is odd in ρ hence $w_0(r, t, \tau)=0$ for $\tau=t$. Therefore by (4. 11) and (4. 16) we get (4. 14), because $H(\rho, \tau) \in C^{2,0}(\boldsymbol{R}^2)$ according to the preceding lemma. Moreover by (4. 9) we have $L(u) \in C^2(\boldsymbol{R}^2)$. Furthermore

$$\begin{aligned}
&\left(D_t^2 - D_r^2 - \frac{n-1}{r} D_r\right) L(u)(r, t) \\
&\quad = D_t w_0(r, t, \tau)|_{\tau=t} + \int_{-\infty}^{t} \left(D_t^2 - D_r^2 - \frac{n-1}{r} D_r\right) w_0(r, t, \tau) d\tau \\
&\quad = D_t w_0(r, t, \tau)|_{\tau=t}
\end{aligned}$$

where the last equality follows from Lemma 2. 1. Since Corollary 2. 3 yields

$$D_t w_0(r, t, \tau) = G(r, t) \quad \text{for} \quad \tau = t, \tag{4.17}$$

we thus see that $L(u)$ satisfies (2. 5). Now, (4. 15) follows from (4. 14) and Lemma 4. 2. The proof is complete.

We shall now state the main resut of this section.

Proposition 4. 4 *Assume conditions* $(H)_1$, $(H)_2$ *and* (1. 13) *hold. Suppose* $u(r, t)$ *is even in* r *and belongs to* $X_1 = \{u \in X\,; \|u\| \leqq 1\}$. *Then* $L(u)(r, t)$ *belongs to* X *and is even in* r. *Moreover for* $(r, t) \in \boldsymbol{R}^2$ *we have*

$$|L(u)(r, t)| \leqq C\|u\|^p \langle r \rangle^{-m} \langle |r| + |t| \rangle^{-1} \langle |r| - t \rangle^{-\kappa} \ \text{if} \ |r| \geqq 1, \tag{4.18}$$

$$|D_t^\beta D_r^\alpha L(u)(r, t)| \leqq C\|u\|^p \langle r \rangle^{-m-1} \langle |r| - |t| \rangle^{-1} \langle |r| - t \rangle^{-\kappa} \quad \text{if} \ |r| \geqq 1,\ 1 \leqq \alpha + \beta \leqq m+3 \ \text{and} \ 0 \leqq \beta \leqq 2, \tag{4.19}$$

$$|D_t^\beta D_r^\alpha L(u)(r, t)| \leqq C\|u\|^p \langle t \rangle^{-\kappa-1} \quad \text{if} \quad |r| \leqq 1,\ 0 \leqq \alpha + \beta \leqq 2 \tag{4.20}$$

and

$$|r^{\alpha+\beta-2}D_t^\beta D_r^\alpha L(u)(r,t)|\leqq C\|u\|^p\langle t\rangle^{-\kappa-1} \quad (4.21)$$
$$\text{if } |r|\leqq 1,\ 3\leqq\alpha+\beta\leqq m+3 \text{ and } 0\leqq\beta\leqq 2.$$

Proof. By $(H)_2$ and (4.4) we see that $G(\rho,\tau)$ is even in ρ. Therefore it follows from Lemma 4.3 that $L(u)(r,t)$ belongs to $C^2(\boldsymbol{R}^2)$ and is even in r. Moreover $r^jL(u)(r,t)\in C^{j+2,0}(\boldsymbol{R}^2)$ and $r^jD_tL(u)(r,t)\in C^{j+1,0}(\boldsymbol{R}^2)$ for $1\leqq j\leqq m+1$. Consequently we find from (1.3) that $L(u)\in X$ if $\|L(u)\|<\infty$, which follows from (4.18) through (4.21). Thus we must only to prove these estimates. In doing so, one can assume $r\geqq 0$, since $L(u)(r,t)$ is even in r. From now on we suppose the hypotheses of the proposition are fulfilled.

We shall first deal with the case where $\beta=0$.

Lemma 4.5 *Let $0\leqq r\leqq 1$. Then (4.20) and (4.21) hold for $\beta=0$.*

The proof will be given later. In what follows we assume $r\geqq 1$ to prove (4.18) and (4.19). Then we adopt another representation for the operator L. Let $w_0(r,t,\tau)$ be given by (4.8). Then from Lemma 2.7 we have

$$w_0(r,t,\tau)=\int_{|\rho_-|}^{\rho_+}G(\rho,\tau)K(\rho,r,t-\tau)d\rho, \quad (4.22)$$

where $\rho_\pm=t-\tau\pm r$ and $K(\rho,r,t)$ is given by (2.11). We also regard $L(u)$ as $r^{-m-1}(r^{m+1}L(u))$.

We start with

Lemma 4.6 *Let $r\geqq 1$ and $0\leqq\alpha\leqq m+3$. Then we have*

$$|D_r^\alpha(r^{m+1}L(u)(r,t))|\leqq C_\alpha\|u\|^p(I_{1,\alpha}+I_2+I_3+I_4). \quad (4.23)$$

Here $I_2=I_3=I_4=0$ if $\alpha=0$, and

$$I_{1,0}=\int_{-\infty}^t d\tau\int_{|\rho_-|}^{\rho_+}\langle\rho\rangle^{p-q-1}\langle\rho+|\tau|\rangle^{-p}\langle\rho-|\tau|\rangle^{-p\kappa}d\rho, \quad (4.24)$$

where $\rho_\pm=t-\tau\pm r$ and q is the positive number given by (1.14),

$$I_{1,\alpha}=r^{-1}I_{1,0}+\int_{-\infty}^t d\tau\int_{|\rho_-|}^{\rho_+}\langle\rho\rangle^{p-q-2}\langle\rho+|\tau|\rangle^{-p}\langle\rho-|\tau|\rangle^{-p\kappa}d\rho \quad (4.25)$$
$$\text{if } 1\leqq\alpha\leqq m+3,$$

$$I_2=\int_{-\infty}^t\langle\rho_+\rangle^{p-q-1}\langle\rho_++|\tau|\rangle^{-p}\langle\rho_+-|\tau|\rangle^{-p\kappa}d\tau, \quad (4.26)$$

$$I_3=\int_{-\infty}^t\langle\rho_-\rangle^{p-q-1}\langle|\rho_-|+|\tau|\rangle^{-p}\langle|\rho_-|-|\tau|\rangle^{-p\kappa}d\tau \quad (4.27)$$

and

$$I_4=\langle r-t\rangle^{-p-p\kappa}. \tag{4.28}$$

Proof. It follows from (2.18), (4.6), (4.7) and (4.22) that

$$|w_0(r,t,\tau)|\leqq C\|u\|^p r^{-m-1}\int_{|\rho_-|}^{\rho_+}\langle\rho\rangle^{m+1-mp}\langle\rho+|\tau|\rangle^{-p}\langle\rho-|\tau|\rangle^{-p\kappa}d\rho.$$

Therefore by (4.16) we get (4.23) for $\alpha=0$, since $m+1-mp=p-q-1$ according to (1.14).

Next suppose $1\leqq\alpha\leqq m+3$. From (4.11) and (4.16) we have

$$D_r^\alpha(r^{m+1}L(r)(r,t))=\int_{-\infty}^t D_r^\alpha(r^{m+1}w_0(r,t,\tau))d\tau. \tag{4.29}$$

If $|t-\tau-r|\geqq 1$, we have $|\rho_-|\geqq 1$ hence by Lemma 2.10, (4.6) and (4.22) we obtain

$$\begin{aligned}|D_r^\alpha(r^{m+1}&w_0(r,t,\tau))|\\ \leqq C_\alpha\|u\|^p\Big\{&\int_{|\rho_-|}^{\rho_+}(r^{-1}\langle\rho\rangle^{p-q-1}+\langle\rho\rangle^{p-q-2})\\ &\qquad\times\langle\rho+|\tau|\rangle^{-p}\langle\rho-|\tau|\rangle^{-p\kappa}d\rho\\ &+\langle\rho_+\rangle^{p-q-1}\langle\rho_++|\tau|\rangle^{-p}\langle\rho_+-|\tau|\rangle^{-p\kappa}\\ &+\langle\rho_-\rangle^{p-q-1}\langle|\rho_-|+|\tau|\rangle^{-p}\langle|\rho_-|-|\tau|\rangle^{-p\kappa}\Big\}.\end{aligned} \tag{4.30}$$

If $0<|\rho_-|\leqq 1$, we write as

$$\begin{aligned}w_0(r,t,\tau)&=\int_{|\rho_-|}^1 G(\rho,\tau)K(\rho,r,t-\tau)d\rho+\int_1^{\rho_+}G(\rho,\tau)K(\rho,r,t-\tau)d\rho\\ &\equiv w_0'(r,t,\tau)+w_0''(r,t,\tau).\end{aligned}$$

For the second term w_0'' we get an estimate analogous to (4.30). Consider the first term w_0'. From (2.11) we have

$$K(\rho,r,t)=r^{-2m-1}\sum_{j=0}^m C_j\rho^{j+1}D_t^j\phi^m(\rho,r,t)$$

with some constants C_j, so that

$$r^{m+1}w_0'(r,t,\tau)=r^{-m}\sum_{j=0}^m C_j\int_{|\rho_-|}^1\rho^{j+1}G(\rho,\tau)D_t^j\phi^m(\rho,r,t-\tau)d\rho.$$

Therefore it follows from Lemmas 2.8, 2.9 and (4.7) that

$$|D_r^\alpha(r^{m+1}w_0'(r,t,\tau))|\leqq C\|u\|^p\langle\tau\rangle^{-p-p\kappa}\leqq C\|u\|^p\langle t-r\rangle^{-p-p\kappa},$$

since $\langle\tau\rangle^{-1}\leqq 2\langle t-r\rangle^{-1}$ for $|t-r-\tau|\leqq 1$. Thus by (4.29) and (4.30) we obtain (4.23). The proof is complete.

From now on we shall estimate the quantities given by (4.24) through (4.27). Notice that condition $p>p_0(m)$ is equivalent to

$$p>1 \quad \text{and} \quad q>\frac{2}{p-1}-m-1, \tag{4.31}$$

since (1.9) and (1.14) imply

$$(p-1)\left(q-\frac{2}{p-1}+m+1\right)=\Phi(m, p).$$

Moreover (1.13) yields

$$0<\kappa\leqq q=(m+1)p-(m+2) \tag{4.32}$$

and

$$p\kappa+q>\kappa+1. \tag{4.33}$$

We will often employ the following two lemmas also.

Lemma 4.7 *Let a, b be real numbers such that $a\geqq 0$, $b\geqq 0$ and $a+b>1$. Then*

$$\int_{-\infty}^{\infty}\langle x\rangle^{-a}\langle x+y\rangle^{-b}dx\leqq C \quad \textit{for} \quad y\in \boldsymbol{R}, \tag{4.34}$$

where C is a constant depending only on $a+b$.

Proof. Since

$$\langle x\rangle^{-a}\langle x+y\rangle^{-b}\leqq\langle x\rangle^{-a-b}+\langle x+y\rangle^{-a-b},$$

we have easily (4.34).

Lemma 4.8 *Suppose* (4.31), (4.32) *and* (4.33) *hold. Then*

$$\int_{-\infty}^{\infty}\langle x\rangle^{-p\kappa}\langle x+y\rangle^{-q}dx\leqq C\langle y\rangle^{-\kappa} \quad \textit{for} \quad y\in \boldsymbol{R}, \tag{4.35}$$

$$\int_{-\infty}^{\infty}\langle y-x\rangle^{-q}\langle y-2x\rangle^{-p\kappa}\,dx\leqq C\langle y\rangle^{-\kappa} \quad \textit{for} \quad y\in \boldsymbol{R} \tag{4.36}$$

and

$$\int_{y}^{\infty}\langle x-y\rangle^{p-q-1}\langle 2x-y\rangle^{-p}dx\leqq C\langle y\rangle^{p\kappa-\kappa-1} \quad \textit{for} \quad y\geqq 0, \tag{4.37}$$

where C is a constant independent of y.

Proof. First consider (4.35). By $I(y)$ we denote the left hand side of the inequality. If $y\geqq 0$, we write

$$I(y)=\int_{-\infty}^{-y/2}\langle x\rangle^{-p\kappa}\langle x+y\rangle^{-q}\,dx+\int_{-y/2}^{\infty}\langle x\rangle^{-p\kappa}\langle x+y\rangle^{-q}dx$$
$$\equiv I_1(y)+I_2(y).$$

Then

$$I_1(y)\leqq\langle\frac{y}{2}\rangle^{-\kappa}\int_{-\infty}^{-y/2}\langle x\rangle^{\kappa-p\kappa}\langle x+y\rangle^{-q}dx.$$

Since $\kappa-p\kappa\leqq 0$, $q\geqq 0$ and $\kappa-p\kappa-q<-1$, by (4. 34) we get (4. 35) for I_1. Next

$$I_2(y)\leqq\langle\frac{y}{2}\rangle^{-\kappa}\int_{-y/2}^{\infty}\langle x\rangle^{-p\kappa}\langle x+y\rangle^{\kappa-q}dx.$$

Since $\kappa-q\leqq 0$, we obtain (4. 35) for $y\geqq 0$, as above. If $y\leqq 0$, changing the variable by $x+y=z$, we have

$$I(y)=\int_{-\infty}^{\infty}\langle z+|y|\rangle^{-p\kappa}\langle z\rangle^{-q}dz.$$

Therefore we get (4. 35) analogously to the preceding case.

Next consider (4. 36). Setting $y-2x=z$, we have

$$\int_{-\infty}^{\infty}\langle y-x\rangle^{-q}\langle y-2x\rangle^{-p\kappa}dx=\frac{1}{2}\int_{-\infty}^{\infty}\langle z\rangle^{-p\kappa}\langle\frac{y+z}{2}\rangle^{-q}dz.$$

Hence (4. 36) follows from (4. 35).

Finally we shall prove (4. 37). Since

$$\int_{y}^{\infty}\langle x-y\rangle^{p-q-1}\langle 2x-y\rangle^{-p}dx\leqq\int_{y}^{\infty}\langle x-y\rangle^{-q-1}dx\quad\text{for}\quad y\geqq 0\quad\text{and}\quad q>0,$$

we get (4. 37) easily if $p\kappa-\kappa-1\geqq 0$. Now, suppose $p\kappa-\kappa-1<0$. Since $2x-y\geqq y$ for $x\geqq y$, we have

$$\langle 2x-y\rangle^{-p}\leqq\langle y\rangle^{p\kappa-\kappa-1}\langle 2x-y\rangle^{-p-p\kappa+\kappa+1}$$

for $x\geqq y\geqq 0$. Noting that $p\kappa+p>\kappa+1$, we get therefore

$$\int_{y}^{\infty}\langle x-y\rangle^{p-q-1}\langle 2x-y\rangle^{-p}dx$$
$$\leqq\langle y\rangle^{p\kappa-\kappa-1}\int_{y}^{\infty}\langle x-y\rangle^{-q-p\kappa+\kappa}dx\quad\text{for}\quad y\geqq 0.$$

By (4. 33) we thus obtain (4. 37). The proof is complete.

We are now in a position to estimate (4. 24) through (4. 27). In what follows we assume (4. 31), (4. 32) and (4. 33) hold.

First we shall deal with (4. 24).

Lemma 4. 9 *Let* $r \geqq 0$. *Then*

$$I_{1,0}(r,\ t) \leqq C\langle r-t\rangle^{-\kappa} \tag{4.38}$$

and

$$I_{1,0}(r,\ t) \leqq Cr\langle r+|t|\rangle^{-\kappa-1} \quad \textit{if} \quad |t| \geqq 3r. \tag{4.39}$$

Proof. We shall divide $I_{1,0}$ into two parts as follows:

$$I_{1,0} = I_{1,0}^{+} + I_{1,0}^{-}. \tag{4.40}$$

where

$$I_{1,0}^{+}(r,\ t) = \int_0^{t_+} d\tau \int_{|\rho_-|}^{\rho_+} \langle\rho\rangle^{p-q-1}\langle\rho+\tau\rangle^{-p}\langle\rho-\tau\rangle^{-p\kappa}d\rho,$$

$$I_{1,0}^{-}(r,\ t) = \int_{-\infty}^{t_-} d\tau \int_{|\rho_-|}^{\rho_+} \langle\rho\rangle^{p-q-1}\langle\rho-\tau\rangle^{-p}\langle\rho+\tau\rangle^{-p\kappa}d\rho$$

and $t_+ = \max\{t, 0\}$, $t_- = \min\{t, 0\}$, so that $I_{1,0}^{+}(r,\ t) = 0$ for $t \leqq 0$. Moreover, introducing characteristic coordinates by

$$\xi = \rho+\tau, \ \eta = \rho-\tau, \tag{4.41}$$

we have

$$I_{1,0}^{+}(r,\ t) = \frac{1}{2}\int_{|t-r|}^{t+r} \langle\xi\rangle^{-p}d\xi \times \int_{r-t}^{\xi} \langle\frac{\xi+\eta}{2}\rangle^{p-q-1}\langle\eta\rangle^{-p\kappa}d\eta \quad \text{for} \quad t>0, \tag{4.42$_+$}$$

$$I_{1,0}^{-}(r,\ t) = \frac{1}{2}\int_{t-r}^{t+r} \langle\xi\rangle^{-p\kappa}d\xi \times \int_{\max\{|t-r|,\xi\}}^{\infty} \langle\frac{\xi+\eta}{2}\rangle^{p-q-1}\langle\eta\rangle^{-p}d\eta. \tag{4.42$_-$}$$

First consider $I_{1,0}^{+}$. Let $t>0$. Since $p>1$, we have

$$I_{1,0}^{+}(r,\ t) \leqq C\int_{|t-r|}^{t+r} \langle\xi\rangle^{-1}d\xi \int_{-\infty}^{\infty} \langle\xi+\eta\rangle^{-q}\langle\eta\rangle^{-p\kappa}d\eta.$$

By (4. 35) we get therefore

$$I_{1,0}^{+}(r,t) \leqq C\int_{|t-r|}^{t+r} \langle\xi\rangle^{-1-\kappa}d\xi.$$

Hence (4.38) and (4.39) follows for $I_{1,0}^{+}$, because $t-r \geqq (t+r)/2$ for $t \geqq 3r \geqq 0$.

Next consider $I_{1,0}^{-}$. From (4.42)$_-$ we have

$$I_{1,0}^{-}(r, t) \leqq C \int_{t-r}^{\infty} \langle \xi \rangle^{-p\kappa} \, d\xi \int_{|t-r|}^{\infty} \langle \xi + \eta \rangle^{-q-1} d\eta.$$
$$\leqq C \int_{-\infty}^{\infty} \langle \xi \rangle^{-p\kappa} \langle \xi + |t-r| \rangle^{-q} d\xi.$$

Hence by (4.35) we obtain (4.38) for $I_{1.0}^{-}$.

Finally we shall prove (4.39) for $I_{1,0}^{-}$. Let $|t| \geqq 3r$. Note that $|t \pm r| \geqq (|t|+r)/2$. If $p\kappa \geqq \kappa+1$, we have from (4.42)$_-$

$$I_{1,0}^{-}(r, t) \leqq C \langle |t|+r \rangle^{-p\kappa} \int_{t-r}^{t+r} d\xi \int_{-\infty}^{\infty} \langle \xi + \eta \rangle^{-q-1} d\eta.$$

Hence (4.39) follows. If $p\kappa < \kappa+1$, we have

$$I_{1,0}^{-}(r, t) \leqq C \int_{t-r}^{t+r} \langle \xi \rangle^{-\kappa-1} d\xi \int_{-\infty}^{\infty} \langle \xi + \eta \rangle^{-q-p\kappa+\kappa} d\eta,$$

since

$$\langle \eta \rangle^{-p} \leqq \langle \xi \rangle^{p\kappa-\kappa-1} \langle \eta \rangle^{-p-p\kappa+\kappa+1} \quad \text{for} \quad \eta \geqq \xi \geqq 0.$$

Therefore by (4.33) we get

$$I_{1,0}^{-}(r, t) \leqq C \langle |t|+r \rangle^{-\kappa-1} \int_{t-r}^{t+r} d\xi,$$

which implies (4.39) for $I_{1,0}^{-}$. The proof is complete.

Next we shall deal with (4.25).

Lemma 4.10 *Let* $r \geqq 1$. *Then*

$$I_{1,a}(r, t) \leqq C \langle r-t \rangle^{-\kappa-1} \quad \textit{for} \quad 1 \leqq a \leqq m+3. \tag{4.45}$$

Proof. Since the first term on the right hand side of (4.25) is dominated by $C\langle r+|t| \rangle^{-1} \langle r-t \rangle^{-\kappa}$ according to the preceding lemma, we have only to estimate the second term. For convenience set

$$J(r, t) = \int_{-\infty}^{t} d\tau \int_{|\rho_-|}^{\rho_+} \langle \rho \rangle^{p-q-2} \langle \rho + |\tau| \rangle^{-p} \langle \rho - |\tau| \rangle^{-p\kappa} d\rho$$

and divide J into two parts, analogously to (4.40):

$$J(r, t) = \int_{-\infty}^{t_-} d\tau + \int_{0}^{t_+} d\tau \equiv J_- + J_+.$$

Then we have estimates similar to (4.42)$_\pm$:

$$J_-(r,t)=\frac{1}{2}\int_{t-r}^{t+r}\langle\xi\rangle^{-p\kappa}d\xi\int_{\max\{|t-r|,\xi\}}^{\infty}\langle\frac{\xi+\eta}{2}\rangle^{p-q-2}\langle\eta\rangle^{-p}d\eta, \qquad (4.46)_-$$

$$J_+(r,t)=\frac{1}{2}\int_{|t-r|}^{t+r}\langle\xi\rangle^{-p}d\xi\int_{r-t}^{\xi}\langle\frac{\xi+\eta}{2}\rangle^{p-q-2}\langle\eta\rangle^{-p\kappa}d\eta, \qquad (4.46)_+$$

where $J_+(r,t)=0$ if $t\leqq 0$.

First consider J_-. We have

$$J_-(r,t)\leqq C\langle t-r\rangle^{-1}\int_{t-r}^{\infty}\langle\xi\rangle^{-p\kappa}d\xi\int_{|t-r|}^{\infty}\langle\xi+\eta\rangle^{-q-1}d\eta$$
$$\leqq C\langle t-r\rangle^{-1}\int_{t-r}^{\infty}\langle\xi\rangle^{-p\kappa}\langle\xi+|t-r|\rangle^{-q}\,d\xi.$$

Therefore by (4.35) we obtain

$$J_-(r,t)\leqq C\langle r-t\rangle^{-1-\kappa}. \qquad (4.47)_-$$

Next we shall prove

$$J_+(r,t)\leqq C\langle r-t\rangle^{-1-\kappa} \quad \text{if} \quad t>0. \qquad (4.47)_+$$

Setting

$$I(\xi)=\langle\xi\rangle^{-p}\int_{r-t}^{\xi}\langle\frac{\xi+\eta}{2}\rangle^{p-q-2}\langle\eta\rangle^{-p\kappa}d\eta,$$

we have from $(4.46)_+$

$$J_+(r,t)\leqq\int_{|t-r|}^{2|t-r|}I(\xi)d\xi+\int_{2|t-r|}^{\infty}I(\xi)d\xi.$$

First consider the second term on the right hand side. If $\xi\geqq 2|t-r|$, we have

$$I(\xi)\leqq C\langle\xi-|t-r|\rangle^{-\kappa-2}\int_{-\infty}^{\infty}\langle\xi+\eta\rangle^{\kappa-q}\langle\eta\rangle^{-p\kappa}d\eta,$$

hence by (4.33) and (4.34) we obtain

$$\int_{2|t-r|}^{\infty}I(\xi)d\xi\leqq C\langle t-r\rangle^{-\kappa-1}. \qquad (4.48)$$

Next consider the first term. Let $|t-r|\leqq\xi\leqq 2|t-r|$. Then

$$I(\xi)\leqq C\langle\xi\rangle^{-p-p\kappa}\int_{-|t-r|}^{-\xi/2}\langle\xi+\eta\rangle^{p-q-2}d\eta$$

$$+C\langle\xi\rangle^{-\kappa-2}\int_{-\xi/2}^{\xi}\langle\xi+\eta\rangle^{\kappa-q}\langle\eta\rangle^{-p\kappa}d\eta$$
$$\equiv I_1(\xi)+I_2(\xi).$$

By Lemma 4.7 we have

$$I_2(\xi)\leqq C\langle\xi\rangle^{-\kappa-2}.$$

Moreover

$$\begin{aligned}I_1(\xi)&\leqq C\langle\xi\rangle^{-1-\kappa}\int_{-|t-r|}^{\infty}\langle\xi+\eta\rangle^{\kappa-q-p\kappa-1}d\eta\\&\leqq C\langle t-r\rangle^{-1-\kappa}\langle\xi-|t-r|\rangle^{\kappa-q-p\kappa}.\end{aligned}$$

Therefore

$$\begin{aligned}\int_{|t-r|}^{2|t-r|}I(\xi)d\xi\leqq C\langle t-r\rangle^{-1-\kappa}&\int_{-\infty}^{\infty}\langle\xi-|t-r|\rangle^{\kappa-q-p\kappa}d\xi\\&+C\int_{|t-r|}^{\infty}\langle\xi\rangle^{-\kappa-2}d\xi.\end{aligned}$$

Hence by (4.33) we obtain

$$\int_{|t-r|}^{2|t-r|}I(\xi)d\xi\leqq C\langle t-r\rangle^{-\kappa-1}.$$

This and (4.48) imply $(4.47)_+$. Now (4.45) follows from $(4.47)_\pm$. Thus we prove Lemma 4.10.

Thirdly we shall treat (4.26).

Lemma 4.11 *Let* $r\geqq 0$. *Then*

$$I_2(r,t)\leqq C\langle r+t\rangle^{-1}\langle r-t\rangle^{-\kappa}. \tag{4.49}$$

Proof. We divide (4.26) into two parts as follows:

$$\begin{aligned}I_2(r,t)=&\int_{-\infty}^{t_-}\langle t+r-\tau\rangle^{p-q-1}\langle t+r-2\tau\rangle^{-p}\langle t+r\rangle^{-p\kappa}d\tau\\&+\int_0^{t_+}\langle t+r-\tau\rangle^{p-q-1}\langle t+r\rangle^{-p}\langle t-r-2\tau\rangle^{-p\kappa}d\tau\\\equiv& I_2^-(r,t)+I_2^+(r,t),\end{aligned}\tag{4.50}$$

where $t_-=\min\{t,0\}$, $t_+=\max\{t,0\}$, so that $I_2^+(r,t)=0$ if $t\leqq 0$.

First consider the second term I_2^+. Let $t>0$. Since $p>1$ and $t+r\geqq t+r-\tau\geqq 0$ for $0\leqq\tau\leqq t$, we have

$$I_2^+(r,t)\leqq\langle t+r\rangle^{-1}\int_0^t\langle t+r-\tau\rangle^{-q}\langle t+r-2\tau\rangle^{-p\kappa}d\tau.$$

Hence by (4.36) we get

$$I_2^+(r,t) \leqq C\langle t+r\rangle^{-1-\kappa},$$

which implies (4. 49) for I_2^+.

Next consider the first term I_2^-. If $t \geqq 0$, we have

$$I_2^-(r,t) \leqq \langle t+r\rangle^{-p\kappa} \int_{-\infty}^{0} \langle t+r-\tau\rangle^{-q-1} d\tau$$
$$\leqq C\langle t+r\rangle^{-p\kappa-q}.$$

Hence by (4. 33) we get

$$I_2^-(r,t) \leqq C\langle t+r\rangle^{-\kappa-1},$$

which implies (4. 49) for I_2^- and $t \geqq 0$.

Now let $t<0$. Then

$$I_2^-(r,t) \leqq \langle r-|t|\rangle^{-p\kappa} \int_{|t|}^{\infty} \langle \tau-|t|+r\rangle^{p-q-1} \langle 2\tau-|t|+r\rangle^{-p} d\tau.$$

First suppose $|t| \geqq 3r$. We have

$$I_2^-(r,t) \leqq \langle |t|-r\rangle^{-p\kappa} \int_{|t|-r}^{\infty} \langle \tau-|t|+r\rangle^{p-q-1} \langle 2\tau-|t|+r\rangle^{-p}\, d\tau.$$

Hence by (4. 37) with $y=|t|-r$ we get

$$I_2^-(r,t) \leqq C\langle |t|-r\rangle^{-\kappa-1},$$

which implies (4. 49), since $|t|-r \geqq (|t|+r)/2$ for $|t| \geqq 3r$.

Finally suppose $0 \leqq |t| \leqq 3r$. We have

$$I_2^-(r,t) \leqq \langle r-|t|\rangle^{-p\kappa} \int_{|t|}^{\infty} \langle \tau-|t|+r\rangle^{-q-1} d\tau$$
$$\leqq C\langle r-|t|\rangle^{-p\kappa} \langle r\rangle^{-q}.$$

Since $r \geqq (r+|t|)/4$ for $|t| \leqq 3r$ and $0<\kappa \leqq q$, we get

$$I_2^-(r,t) \leqq C\langle r+|t|\rangle^{-\kappa} \langle r-|t|\rangle^{\kappa-q-p\kappa}.$$

Therefore by (4. 33) we obtain (4. 49). The proof is complete.

Fourthly we shall deal with (4. 27).

Lemma 4. 12 *Let* $r \geqq 0$. *Then*

$$I_3(r,t) \leqq C\langle r-t\rangle^{-\kappa-1}. \tag{4. 51}$$

Proof. We shall divide (4. 27) into two parts, as follows:

$$I_3(r,t) = I_{3,1}(r,t) + I_{3,2}(r,t), \tag{4. 52}$$

where

$$I_{3,1}(r,t)=\int_{-\infty}^{t-r}\langle t-r-\tau\rangle^{p-q-1}\langle t-r-\tau+|\tau|\rangle^{-p}$$
$$\times\langle t-r-\tau-|\tau|\rangle^{-p\kappa}\,d\tau \tag{4.53}$$

and

$$I_{3,2}(r,t)=\int_{t-r}^{t}\langle \tau+r-t\rangle^{p-q-1}\langle \tau+r-t+|\tau|\rangle^{-p}$$
$$\times\langle \tau+r-t-|\tau|\rangle^{-p\kappa}d\tau. \tag{4.54}$$

First consider (4.53). If $-\infty<t\leqq r$, we have

$$I_{3,1}(r,t)=\int_{-\infty}^{t-r}\langle t-r-\tau\rangle^{p-q-1}\langle t-r-2\tau\rangle^{-p}\langle t-r\rangle^{-p\kappa}d\tau$$
$$=\langle r-t\rangle^{-p\kappa}\int_{r-t}^{\infty}\langle \tau-r+t\rangle^{p-q-1}\langle 2\tau-r+t\rangle^{-p}d\tau.$$

Hence by (4.37) with $y=r-t$ we get (4.51) for $I_{3,1}$. If $t>r\geqq 0$, we have

$$I_{3,1}(r,t)=\int_{-\infty}^{0}\langle t-r-\tau\rangle^{p-q-1}\langle t-r-2\tau\rangle^{-p}\langle t-r\rangle^{-p\kappa}d\tau$$
$$+\int_{0}^{t-r}\langle t-r-\tau\rangle^{p-q-1}\langle t-r\rangle^{-p}\langle t-r-2\tau\rangle^{-p\kappa}d\tau$$
$$\equiv I'_{3,1}+I''_{3,1}.$$

For the first term we have

$$I'_{3,1}(r,t)\leqq\langle t-r\rangle^{-p\kappa}\int_{-\infty}^{0}\langle t-r-\tau\rangle^{-q-1}d\tau$$
$$\leqq C\langle t-r\rangle^{-p\kappa-q}.$$

Hence by (4.33) we obtain (4.51) for $I'_{3,1}$. For the second term we have

$$I''_{3,1}(r,t)\leqq\langle t-r\rangle^{-1}\int_{0}^{t-r}\langle t-r-\tau\rangle^{-q}\langle t-r-2\tau\rangle^{-p\kappa}d\tau.$$

Therefore by (4.36) we get (4.51) for $I''_{3,1}$. Thus (4.51) holds for $I_{3,1}$.

Next consider (4.54). If $t\geqq r\geqq 0$, we have

$$I_{3,2}(r,t)\leqq\langle t-r\rangle^{-p\kappa}\int_{t-r}^{\infty}\langle \tau-t+r\rangle^{p-q-1}\langle 2\tau-t+r\rangle^{-p}d\tau.$$

Hence by (4.37) we get (4.51) for $I_{3,2}$. Now suppose $-\infty<t<r$. Then we divide (4.54) into two parts, as follows:

$$I_{3,2}(r,t)=\int_{t-r}^{t_-}\langle \tau+r-t\rangle^{p-q-1}\langle r-t\rangle^{-p}\langle 2\tau+r-t\rangle^{-p\kappa}d\tau$$
$$+\int_{t_-}^{t}\langle \tau+r-t\rangle^{p-q-1}\langle 2\tau+r-t\rangle^{-p}\langle r-t\rangle^{-p\kappa}d\tau$$
$$\equiv I'_{3,2}(r,t)+I''_{3,2}(r,t),$$

where $t_-=\min\{t, 0\}$, so that $I''_{3,2}(r, t)=0$ for $t\leqq 0$. For the first term we have

$$I'_{3,2}(r, t)\leqq\langle r-t\rangle^{-1}\int_{t-r}^{0}\langle\tau+r-t\rangle^{-q}\langle 2\tau+r-t\rangle^{-p\kappa}d\tau.$$

Hence by (4. 36) we get (4. 51) for $I'_{3,2}$. For the second term, assuming $t>0$, we have

$$\begin{aligned}I''_{3,2}(r, t)&\leqq\langle r-t\rangle^{-p\kappa}\int_{0}^{t}\langle\tau+r-t\rangle^{-q-1}d\tau\\&\leqq C\langle r-t\rangle^{-p\kappa-q}.\end{aligned}$$

Therefore by (4. 33) we get (4. 51) for $I''_{3,2}$. Thus (4. 51) holds for $I_{3,2}$. In view of (4. 52) we complete the proof.

Finally we shall prove Lemma 4. 5 which is a consequence of Lemma 4. 11 or 4. 12.

Proof of Lemma 4. 5. First we shall prove (4. 20) for $\beta=0$. Let $0\leqq r\leqq 1$ and $0\leqq a\leqq 2$. It follows from (4. 2), (4. 8) and (4. 11) that

$$D_r^a L(u)(r, t)=\int_{-\infty}^{t}D_r^a w_0(r, t, \tau)d\tau.$$

Moreover by (4. 10) we have

$$\begin{aligned}|D_r^a w_0(r, t, \tau)|\leqq C\|u\|^p\int_{-1}^{1}&\langle t-\tau+r\sigma\rangle^{p-q-1}\\&\times\langle|t-\tau+r\sigma|+|\tau|\rangle^{-p}\langle|t-\tau+r\sigma|-|\tau|\rangle^{-p\kappa}d\sigma.\end{aligned}$$

Since $t-\tau-1\leqq t-\tau+r\sigma\leqq t-\tau+1$ for $|r\sigma|\leqq 1$, we get therefore

$$|D_r^a L(u)(r, t)|\leqq C\int_{-\infty}^{t}\langle t-\tau\rangle^{p-q-1}\langle t-\tau+|\tau|\rangle^{-p}\langle t-\tau-|\tau|\rangle^{-p\kappa}d\tau.$$

Noting that the integral on the right hand coincides with (4. 26) for $r=0$, by Lemma 4. 11 we obtain (4. 20) for $\beta=0$.

Next we shall prove (4. 21) for $\beta=0$. Let $3\leqq a\leqq m+3$. Note that

$$r^{a-2}D_r^a L(u)(r, t)=\sum_{j=0}^{a-2}C_j D_r^2 D_r^j(r^j L(u)(r, t))$$

with some constants C_j. Hence by (4. 8) and (4. 11) we have

$$r^{a-2}D_r^2 L(u)(r, t)=\sum_{j=0}^{a-2}C_j\int_{-\infty}^{t}D_r^{j+2}w_j(r, t, \tau)d\tau.$$

In view of the proof of Corollary 2. 6 we have therefore

$$|r^{\alpha-2}D_r^\alpha L(u)(r,t)|\leqq C\int_{-\infty}^t\left\{\int_{-1}^1|(D_\rho^2H)(t-\tau+r\sigma,\tau)|d\sigma\right.$$
$$\left.+|(D_\rho^2H)(t-\tau+r,\tau)|+|(D_\rho^2H)(t-\tau-r,\tau)|\right\}d\tau.$$

Thus by (4.10) and Lemma 4.11 we obtain (4.21) for $\beta=0$, as before. The proof is complete.

End of Proof of Proposition 4.4. As we remarked at the openning of the proof, we have only to prove (4.18) through (4.21) for $r\geqq 0$. If $\beta=0$, these estimate follows immediately from Lemmas 4.5, 4.6 and Lemmas 4.9 through 4.12. If $\beta=1$ or $\beta=2$, we shall employ (4.14) instead of (4.16). Then it follows from (4.11) and (4.17) that

$$D_t^2L(u)(r,t)=\int_{-\infty}^t D_t^2w_0(r,t,\tau)d\tau+G(r,t).$$

Therefore, except for $G(r,t)$, we obtain (4.19), (4.20) and (4.21) for $\beta=1,2$ analogously to the case $\beta=0$. Thus it suffices to prove

$$|D_r^\alpha G(r,t)|\leqq C\|u\|^p\langle r\rangle^{-m-1}\langle r+|t|\rangle^{-\kappa-1}$$
$$\text{for } r\geqq 1,\ 0\leqq\alpha\leqq m+1, \tag{4.55}$$

because $p>1$ hence (4.7) implies

$$|r^\alpha D_r^\alpha G(r,t)|\leqq C\|u\|^p\langle t\rangle^{-\kappa-1}\quad\text{for}\quad 0\leqq r\leqq 1, 0\leqq\alpha\leqq m+1.$$

First suppose $|t|\geqq 3r$ and $r\geqq 1$. Then from (4.6) we have for $0\leqq\alpha\leqq m+1$

$$|D_r^\alpha G(r,t)|\leqq C\|u\|^p\langle r\rangle^{-mp}\langle r+|t|\rangle^{-p-p\kappa}$$
$$\leqq C\|u\|^p\langle r\rangle^{-(m+1)p+\kappa+1-p\kappa}\langle r+|t|\rangle^{-1-\kappa}.$$

Moreover (4.32) implies

$$-(m+1)p+\kappa+1\leqq -m-1.$$

Hence we get (4.55). Next suppose $|t|\leqq 3r$ and $r\geqq 1$. Then (4.6) implies

$$|D_r^\alpha G(r,t)|\leqq C\|u\|^p\langle r\rangle^{-mp-p}\leqq C\|u\|^p\langle r+|t|\rangle^{-m-2-q},$$

since $(m+1)p=q+m+2$ and $\langle r\rangle^{-1}\leqq 4\langle r+|t|\rangle^{-1}$ for $|t|\leqq 3r$. Therefore by (4.32) we obtain (4.55). Thus we prove Proposition 4.4.

5. Proof of Theorem 1.2

In this section we shall prove the main theorem (*i.e.*, Theorem 1.2), by employing the results in the previous sections. The plan is as follows. First we show that the integral equation (4.1) is uniquely solvable in the function space X. (See Proposition 5. 4). Then the solution satisfies the nonlinear wave equation (0. 1) according to Lemma 4. 3. In order to prove the uniqueness of such a solution of (0. 1) as in the theorem, we next show that a solution of (0. 1) having appropriate asymptotic behaviors satisfies (4. 1). (See Lemma 5. 5). Finally we prove the statements about the solution u_+ of (0. 2).

First of all we state

Lemma 5. 1 *The function space X defined by* (1. 3) *is a Banach space.*

Proof. Introduce another norm $\|u\|'$ for $u\in X$ by

$$\|u\|'=\sum_{|\alpha|\leq 1}\sup_{(r,t)\in\mathbf{R}^2}|D_{r,t}^{\alpha}u(r,t)|\langle r\rangle^{m}\langle|r|+|t|\rangle\langle|r|-|t|\rangle^{\kappa}$$
$$+\sum_{j=0}^{m+1}\sup_{(r,t)\in\mathbf{R}^2}\{|D_r^{j+2}(r^{m+1}u(r,t))|+|D_r^{j+1}D_t(r^{m+1}u(r,t))|\}$$
$$\times\langle r\rangle^{-1}\langle|r|+|t|\rangle\langle|r|-|t|\rangle^{\kappa}.$$

Then we find that the two norms are equivalent and (1. 3) can be rewritten as

$$X=\{u(r,t)\in C^1(\mathbf{R}^2)\,;\ r^{m+1}u(r,t)\in C^{m+3,0}(\mathbf{R}^2),$$
$$r^{m+1}u_t(r,t)\in C^{m+2,0}(\mathbf{R}^2)\ \text{and}\ \|u\|'<\infty\}.$$

(See Kubo [9], Lemma 3. 1 and its proof).

Now we shall introduce an auxiliary Banach space Y by

$$Y=\{v(r,t)\in C^1(\mathbf{R}^2)\cap C^{m+3,0}(\mathbf{R}^2)\,;$$
$$D_tv(r,t)\in C^{m+2,0}(\mathbf{R}^2)\quad\text{and}\quad\|v\|_Y<\infty\},$$

where

$$\|v\|_Y=\sum_{|\alpha|\leq 1}\sup_{(r,t)\in\mathbf{R}^2}|D_{r,t}^{\alpha}v(r,t)|\langle r\rangle^{-1}\langle|r|+|t|\rangle\langle|r|-|t|\rangle^{\kappa}$$
$$+\sum_{j=0}^{m+1}\sup_{(r,t)\in\mathbf{R}^2}(|D_r^{j+2}v(r,t)|+|D_r^{j+1}D_tv(r,t)|)$$
$$\times\langle r\rangle^{-1}\langle|r|+|t|\rangle\langle|r|-|t|\rangle^{\kappa}.$$

Let Y_m be a closed subspace of Y given by

$$Y_m=\{v(r,t)\in Y\,;\ D_r^jv(0,t)=0\quad\text{and}$$
$$D_r^jD_tv(0,t)=0\quad\text{for}\quad 0\leq j\leq m\}.$$

Then for $v\in Y_m$ we have

$$v(r,t)=(m!)^{-1}r^{m+1}\int_0^1(1-\lambda)^m(D_r^{m+1}v)(r\lambda,t)d\lambda$$

and

$$v_t(r, t) = (m!)^{-1} r^{m+1} \int_0^1 (1-\lambda)^m (D_r^{m+1} D_t v)(r\lambda, t) d\lambda.$$

Therefore, setting

$$u(r, t) = r^{-m-1} v(r, t) \quad \text{for} \quad v \in Y_m,$$

we have $u \in X$, because

$$|D_{r,t}^{\alpha} u(r, t)| \leqq \sup_{|\rho| \leqq 1} |(D_r^{m+1} D_{r,t}^{\alpha} v(\rho, t))| \quad \text{for} \quad |r| \leqq 1,\ |\alpha| \leqq 1$$

and

$$\|u\|' \leqq C \|v\|_Y.$$

Conversely, setting $v(r, t) = r^{m+1} u(r, t)$ for $u \in X$, we have $v(r, t) \in Y_m$, because

$$\|v\|_Y \leqq C \|u\|'.$$

Since Y_m is a Banach space, we conclude that X is also a Banach space. The proof is complete.

Next from Proposition 4.4 we have easily

Lemma 5.2 *Let the hypotheses of Proposition* 4.4 *be fulfilled. Then* $L(u)(r, t)$ *belongs to* X *and is even in* r. *Moreover we have*

$$\|L(u)\| \leqq C_1 \|u\|^p, \tag{5.1}$$

where C_1 *is a constant depending only on* F *and* n.

In order to show that (4.1) is solvable in X, we shall introduce an auxiliary norm $|||u|||$ for $u \in X$ by

$$\begin{aligned}|||u||| = &\sum_{|\alpha| \leqq 1} \sup_{(r,t) \in \mathbf{R}^2} |D_{r,t}^{\alpha} u(r, t)| \langle r \rangle^m \langle |r| + |t| \rangle \langle |r| - |t| \rangle^{\kappa} \\ &+ \sum_{j=0}^{m} \sup_{(r,t) \in \mathbf{R}^2} \{|r^{j+1} D_r^{j+2} u(r, t)| + |r^{j+1} D_r^{j+1} D_t u(r, t)|\} \\ &\qquad \times \langle r \rangle^{m-j-1} \langle |r| + |t| \rangle \langle |r| - t| \rangle^{\kappa}.\end{aligned}$$

Then we see from (1.4) that

$$|||u||| \leqq \|u\| \quad \text{for} \quad u \in X, \tag{5.2}$$

since $|r| \leqq \langle r \rangle$. Moreover we have

Lemma 5.3 *Assume conditions* $(H)_1$, $(H)_2$, $(H)_3$ *and* (1.13) *hold. Let* $u(r, t)$, $v(r, t) \in X_1$ *be even in* r. *Then*

$$|||L(u)-L(v)|||\leqq C_2|||u-v|||(|||u|||^{p-1}+|||v|||^{p-1}) \tag{5.3}$$

and

$$\begin{aligned}\|L(u)-L(v)\|\leqq C_3\|u-v\|(\|u\|^{p-1}+\|v\|^{p-1})& \\ +C_4|||u-v|||^{\delta}(\|u\|^{p-1}+\|v\|^{p-1}),&\end{aligned} \tag{5.4}$$

where δ is the number in (1.12) *and C_2, C_3 and C_4 are positive constants depending only on F and n.*

Proof. By $H(u)$, $G(u)$ we denote the functions defined by (4.3), (4.4) respectively. Then (4.2) implies

$$\begin{aligned}&L(u)(r, t)-L(v)(r, t)\\&\quad=\int_{-\infty}^{t}d\tau\int_{-1}^{1}(H(u)-H(v))(t-\tau+r\sigma, \tau)(1-\sigma^2)^m d\sigma\end{aligned}$$

with

$$(H(u)-H(v))(\rho, \tau)=(2m!)^{-1}\left(\frac{\partial}{\partial\rho^2}\right)^m\{\rho^{2m+1}(G(u)-G(v))(\rho, \tau)\}.$$

Moreover, analogously to (4.6) and (4.7), by $(H)_1$ we get

$$\begin{aligned}&|D_\rho^j(G(u)-G(v))(\rho, \tau)|\\&\quad\leqq C|||u-v|||(|||u|||^{p-1}+|||v|||^{p-1})\langle\rho\rangle^{-mp}\\&\qquad\times\langle|\rho|+|\tau|\rangle^{-p}\langle|\rho|-|\tau|\rangle^{-p\kappa}\quad\text{for}\quad|\rho|\geqq 1,\ 0\leqq j\leqq m+1\end{aligned}$$

and

$$\begin{aligned}|\rho^k D_\rho^j(G(u)-G(v))(\rho, \tau)|\leqq C|||u-v|||(|||u|||^{p-1}+|||v|||^{p-1})\langle\tau\rangle^{-p-p\kappa}&\\ \text{for}\quad|\rho|\leqq 1,\ 0\leqq j\leqq k\quad\text{and}\quad 0\leqq k\leqq m+1.&\end{aligned}$$

(For the details see e.g. Kubo [9], § 3). Therefore, analogously to Proposition 4.4, we obtain (5.3). Furthermore, employing condition $(H)_3$ also, we get

$$\begin{aligned}&|D_\rho^{m+2}(G(u)-G(v))(\rho, \tau)|\\&\quad\leqq C\{\|u-v\|(\|u\|^{p-1}+\|v\|^{p-1})+|||u-v|||^{\delta}(\|u\|^{m+2}+\|v\|^{m+2})\}\\&\qquad\times\langle\rho\rangle^{-mp}\langle|\rho|+|\tau|\rangle^{-p}\langle|\rho|-|\tau|\rangle^{-p\kappa}\quad\text{for}\quad|\rho|\geqq 1\end{aligned}$$

and

$$\begin{aligned}&|\rho^k D_\rho^{k+1}(G(u)-G(v))(\rho, \tau)|\\&\quad\leqq C\{\|u-v\|(\|u\|^{p-1}+\|v\|^{p-1})+|||u-v|||^{\delta}(\|u\|^{m+2}+\|v\|^{m+2})\}\\&\qquad\times\langle\tau\rangle^{-p-p\kappa}\quad\text{for}\quad|\rho|\leqq 1\quad\text{and}\quad 0\leqq k\leqq m+1.\end{aligned}$$

Therefore we obtain (5.4), as before. The proof is complete.

We are now in a position to solve the integral equation (4.1).

Proposition 5.4 *Let the hypostheses of Theorem* 1.2 *be fulfilled. Then there are positive numbers* ε_0 *and* d *having the following property, where* ε_0 *depends only on* F, n *and* χ, *and* d *only on* F *and* n: *If* $\|\{f, g\}\|_0 \leqq \varepsilon \leqq \varepsilon_0$, *there exists uniquely a solution* $u(r, t)$ *of* (4.1) *which belongs to* X_d *and is even in* r. *Moreover* $u \in C^2(\boldsymbol{R}^2)$ *and we have* (1.15) *through* (1.19).

Proof. First of all we set

$$d = \min\{1, (4C_2)^{-1/(p-1)}\}, \tag{5.7}$$

where C_2 is the constant in (5.3). From (5.2) and (5.3) we have then

$$|||L(u) - L(v)||| \leqq \frac{1}{2}|||u - v||| \quad \text{for} \quad u, v \in X_d. \tag{5.8}$$

We now define a sequence of functions u_k $(k = 0, 1, 2, \cdots)$ by $u_0 = u_-$ and

$$u_k = u_0 + L(u_{k-1}) \quad \text{for} \quad k \geqq 1.$$

It follows from Theorem 1.1 that $u_0(r, t)$ belongs to X, is even in r and satisfies

$$\|u_0\| \leqq C_0 \varepsilon \quad \text{for any } \varepsilon > 0, \tag{5.9}$$

where C_0 is a constant dependign only on n and χ. Let ε_0 be the maximum of positive numbers ε satisfying the following three consitions

$$2C_0\varepsilon \leqq d \leqq 1, \tag{5.10$_1$}$$

$$2^p C_1 (C_0 \varepsilon)^{p-1} \leqq 1, \tag{5.10$_2$}$$

and

$$2^{p+1} C_3 (C_0 \varepsilon)^{p-1} \leqq 1, \tag{5.10$_3$}$$

where C_0, C_1 and C_3 are the constants in (5.9), (5.1) and (5.4) respectively and d is the number given by (5.7).

In what follows we suppose $0 < \varepsilon \leqq \varepsilon_0$. Then by induction it follows from Lemma 5.2, (5.9), $(5.10)_1$ and $(5.10)_2$ that $u_k \in X_d$ for $k \geqq 0$ and

$$\|u_k\| \leqq 2\|u_0\| \quad \text{for} \quad k \geqq 1. \tag{5.11}$$

Moreover (5.8) implies

$$|||u_{k+1} - u_k||| \leqq \left(\frac{1}{2}\right)^k |||u_1 - u_0||| \quad \text{for} \quad k \geqq 0.$$

Using (5.4) and $(5.10)_3$ also we thus obtain

$$\|u_{k+1}-u_k\| \leqq \frac{1}{2}\|u_k-u_{k-1}\| + C_5\left(\frac{1}{2}\right)^{\delta k} \quad \text{for} \quad k \geqq 1,$$

where

$$C_5 = 2^{m+3+\delta} C_4 |||u_1-u_0|||^{\delta} \|u_0\|^{m+2}.$$

Consequently we have

$$\|u_{k+1}-u_k\| \leqq \left(\frac{1}{2}\right)^k \|u_1-u_0\| + C_5 k\left(\frac{1}{2}\right)^{\delta k}$$

for $k \geqq 1$, since $0<\delta \leqq 1$. Therefore the sequence u_k $(k=0, 1, 2, \cdots)$ converes to a function u in X, because of Lemma 5. 1. Besides it follows from (5. 9), $(5. 10)_1$ and (5. 11) that $u \in X_d$ and

$$\|u\| \leqq 2\|u_-\| \leqq 2C_0\varepsilon. \tag{5.12}$$

Moreover we see from (5. 2) and (5. 8) that u is a unique solution of (4. 1) in X_d. Now (1. 15) through (1. 19) follows immediately form (5. 12) and Proposition 4. 4, since (1. 17) is a direct consequence of (1. 15). Thus we prove Proposition 5. 4.

In order to prove the uniqueness of a solution of the nonlinear wave equation (0. 1) which satisfies the asymptotic behavior (1. 15), we need the following.

Lemma 5. 5 *Assume conditions* $(H)_1$ *and* $(H)_2$ *hold. Let* $u_-(r, t)$ *be the solution of the Cauchy problem* (1. 1) *which is even in* r *and belongs to* $X \cap C^2(\boldsymbol{R}^2)$. *Let* $u(r, t)$ *be a solution of* (0. 1) *which belongs to* $X \cap C^2(\boldsymbol{R}^2)$, *is even in* r *and satisfies the following asymptotic behavior*

$$\begin{aligned} &|D_r(u(r, t)-u_-(r, t))| + |D_t(u(r, t)-u_-(r, t))| \\ &\quad \leqq C\langle r\rangle^{-m-1}\langle t\rangle^{-\mu} \end{aligned} \tag{5.13}$$

for $(r, t) \in \boldsymbol{R}^2$ *such that* $t<0$, *where* C, μ *are constants independent of* r, t, *and* $0<\mu<1$. *Then* $u(r, t)$ *satisties the integral equation* (4. 1).

Proof. Consider the following Cauchy probrem for (0. 2):

$$\begin{aligned} &w_{tt}-w_{rr}-\frac{n-1}{r}w_r=0 \quad \text{in} \quad \boldsymbol{R}\times[s, \infty), \\ &w(r, s)=f(r, s), \ w_t(r, s)=g(r, s) \quad \text{for} \quad r \in \boldsymbol{R}, \end{aligned} \tag{5.14}$$

where s is a fixed negative number and

$$\begin{aligned} &f(r, s)=u(r, s)-u_-(r, s), \\ &g(r, s)=D_t(u(r, t)-u_-(r, t))|_{t=s}. \end{aligned} \tag{5.15}$$

Since $u(r, t)-u_{-}(r, t)$ belongs to X and is even in r, it follows form Lemmas 2.1 and 2.2 that (5.14) admits a unique solution $w(r, t; s)\in C^2(\boldsymbol{R}^2)$. Moreover the solution is given by

$$w(r, t; s)=\int_{-1}^{1}(D_\rho H_f+H_g)(t-s+r\sigma, s)(1-\sigma^2)^m d\sigma \tag{5.16}$$

for each $s<0$, where $H_f(\rho, s)$, $H_g(\rho, s)$ are defined by (2.2) with $f(r)=f(r, s)$, $g(r)=g(r, s)$ respectively. Besides $w(r, t; s)$ is even in r.

Now set

$$\begin{aligned} v(r, t; s)=u_{-}(r, t)+\int_s^t d\tau\int_{-1}^{1}H(t-\tau+r\sigma, \tau)(1-\sigma^2)^m d\sigma \\ +w(r, t; s), \end{aligned} \tag{5.17}$$

where $H(\rho, \tau)$ is given by (4.3) and (4.4). Then, since $G(\rho, \tau)$ is even in ρ according to $(H)_2$, it follows from (4.5) and Lemma 2.4 that $v(r, t; s)$ belongs to $C^2(\boldsymbol{R}^2)$ and satisfies the inhomogeneous wave equation (2.5). Moreover from (5.15) we have $v(r, t; s)=u(r, t)$ and $v_t(r, t; s)=u_t(r, t)$ for $t=s$. Therefore by the uniqueness of solutions to (2.5) we obtain $v(r, t; s)=u(r, t)$ for $(r, t)\in\boldsymbol{R}^2$, since $u(r, t)$ is also a solution of (2.5) with $G(r, t)=F(u(r, t), u_t(r, t), u_r(r, t))$ regarded as a given function. In order to show that $u(r, t)$ is a solution of (4.1) we thus have only to prove

$$\lim_{s\to-\infty} w(r, t; s)=0 \quad \text{for} \quad (r, t)\in\boldsymbol{R}^2, \tag{5.18}$$

because of (5.17).

Let $(r, t)\in\boldsymbol{R}^2$ be fixed. Since each side of (4.1) is even in r and continuous according to (4.13), one can assume $r>0$. We find from the proof of Lemma 3.3 that the function $D_\rho H_f+H_g$ in (5.16) is represented as

$$(D_\rho H_f+H_g)(\rho, s)=\sum_{j=0}^{m}D_\rho^j F_j(\rho, s),$$

where

$$\begin{aligned} &F_0(\rho, s)=a_0 f(\rho, s)+\rho(a_1 D_\rho f(\rho, s)+b_0 g(\rho, s)), \\ &F_j(\rho, s)=\rho^{j+1}(a_{j+1}D_\rho f(\rho, s)+b_j g(\rho, s)) \quad \text{for } 1\leqq j\leqq m, \end{aligned}$$

and a_k, b_j are constants. Moreover, since $u-u_{-}\in X$, we have $F_j(\rho, s)\in C^{j+2,0}(\boldsymbol{R}^2)$ for $0\leqq j\leqq m$ hence $D_\rho H_f+H_g\in C^{2,0}(\boldsymbol{R}^2)$. For convenience set $\rho=t-s+r\sigma$. Then from (5.16) we get

$$w(r,t;s)=\sum_{j=0}^{m}\int_{-1}^{1}\{D_\rho^j F_j(\rho,s)\}(1-\sigma^2)^m d\sigma.$$

Since

$$D_\rho^j F_j(\rho,s)=r^{-j}D_\sigma^j F_j(\rho,s),$$

integrating by parts, we obtain

$$w(r,t;s)=\sum_{j=0}^{m}r^{-j}\int_{-1}^{1}F_j(\rho,s)(-D_\sigma)^j(1-\sigma^2)^m d\sigma.$$

Furthermore (5.13) and (5.15) yield

$$|F_j(\rho,s)|\leqq C\langle s\rangle^{-\mu}\quad\text{for}\quad 0\leqq j\leqq m,$$

because $u-u_-\in X$ implies

$$|u(r,t)-u_-(r,t)|\leqq C\langle t\rangle^{-\mu}.$$

Thus we have

$$|w(r,t;s)|\leqq C\langle s\rangle^{-\mu}\sum_{j=0}^{m}r^{-j}$$

hence (5.18) follows. The proof is complete.

In order to complete the proof of Theorem 1.2 we also need the following

Proposition 5.6 *Let $u(r,t)\in X_1$ be a solution of* (4.1) *which is even in r, where $u_-(r,t)\in C^2(\boldsymbol{R}^2)$ is a solution of* (0.2). *Suppose conditions $(H)_1$, $(H)_2$ and* (1.13) *hold. Then there exists usiquely a solution $u_+(r,t)$, of the linear wave equation* (0. 2) *which belongs to $X\cap C^2(\boldsymbol{R}^2)$ and has the following asymptotic behaviors*

$$|u(r,t)-u_+(r,t)|\leqq C_1\|u\|^p\langle r\rangle^{-m}\langle|r|+|t|\rangle^{-1}\langle|r|+t\rangle^{-\kappa}\quad \textit{for}\quad (r,t)\in\boldsymbol{R}^2, \tag{5.19}$$

$$\begin{aligned}&|D_t^\beta D_r^\alpha(u(r,t)-u_+(r,t))|\\ &\quad\leqq C_2\|u\|^p\langle r\rangle^{-m-1}\langle|r|-|t|\rangle^{-1}\langle|r|+t\rangle^{-\kappa}\\ &\qquad \textit{for}\quad 1\leqq\alpha+\beta\leqq 2,\quad (r,t)\in\boldsymbol{R}^2,\end{aligned} \tag{5.20}$$

$$\begin{aligned}&|r^{\alpha+\beta-2}D_t^\beta D_r^\alpha(u(r,t)-u_+(r,t))|\\ &\quad\leqq C_3\|u\|^p\langle r\rangle^{-m+\alpha+\beta-3}\langle|r|-|t|\rangle^{-1}\langle|r|+t\rangle^{-\kappa}\\ &\qquad \textit{for}\quad 3\leqq\alpha+\beta\leqq m+3,\ 0\leqq\beta\leqq 2\quad \textit{and}\quad (r,t)\in\boldsymbol{R}^2\end{aligned} \tag{5.21}$$

and

$$\|u(t)-u_+(t)\|_e\leqq C_4\langle t\rangle^{-k}\quad \textit{for}\quad t\geqq 0, \tag{5.22}$$

where C_j are constants depending only on n and F.

Proof. Set

$$u_+(r, t)=u_-(r, t)+\int_{-\infty}^{\infty}d\tau\int_{-1}^{1}H(t-\tau+r\sigma, \tau)(1-\sigma^2)^m\, d\sigma,$$

where $H(\rho, \tau)$ is the function given by (4. 3) and (4. 4). Then it follows from Lemmas 4. 2 and 2. 1 that u_+ belongs to $C^2(\boldsymbol{R}^2)$ and satisfies (0. 2). Moreover from (4. 1) we have

$$\begin{aligned} u(r, t)-u_+(r, t)&=-\int_t^{\infty}d\tau\int_{-1}^{1}H(t-\tau+r\sigma, \tau)(1-\sigma^2)^m d\sigma \\ &=\int_{-\infty}^{-t}d\tau\int_{-1}^{1}H(-t-\tau+r\sigma, -\tau)(1-\sigma^2)^m d\sigma, \end{aligned} \tag{5. 23}$$

since $H(\rho, \tau)$ is odd in ρ. Therefore by virtue of Proposition 4. 4 we have $u-u_+\in X$ hence $u_+\in X$. Furthermore from (4. 18) through (4. 21) we obtain (5. 19) through (5. 22), since (5. 20) implies (5. 22).

Next we shall show the uniqueness of such a solution u_+ of (0. 2). Let $v_+(r, t)$ be another solution and set $w(r, t)=u_+(r, t)-v_+(r, t)$. Then from (0. 2) and (5. 20) we have

$$\|w(t)\|_e^2=\|w(0)\|_e^2 \qquad \text{for} \qquad t\in\boldsymbol{R}.$$

Moreover (5. 22) implies that $\|w(t)\|_e$ tends to zero as $t\to\infty$, since

$$\|w(t)\|_e\leqq\|u_+(t)-u(t)\|_e+\|u(t)-v_+(t)\|_e.$$

Therefore we conclude that $w(r, t)$ is constant. Hence $w(r, t)$ vanishes identically according to (5. 19). The proof is complete.

Proof of Theorem 1. 2. Let ε_0 and d be the same numbers as in Proposition 5. 4. Let $\|\{f, g\}\|_0\leqq\varepsilon\leqq\varepsilon_0$. Then there exists uniquely a solution $u(r, t)$ of (4. 1) which is even in r and bebongs to X_d. Moreover it follows from Lemma 4. 3 that u belongs to $C^2(\boldsymbol{R}^2)$ and satisfies (0. 1). Furthermore we have (1. 15) through (1. 19).

Next we shall show the uniqueness of such a solution of (0. 1). Let $u(r, t)$ be a solution of (0. 1) which is even in r, belongs to $X_d\cap C^2(\boldsymbol{R}^2)$ and has the asymptotic behavior (1. 15). Then we see from Lemma 5. 5 that u satisfies (4. 1). Moreover, a solution of (4. 1) is unique in X_d according to (5. 8). Therefore such a solution of (0. 1) is unique.

Finally the statements for the u_+ follows from Proposition 5. 6. It remains only to prove (1. 20). From (0. 1) we have

$$2\int_0^t d\tau\int_0^\infty G(r,\tau)u_t(r,\tau)r^{n-1}dr=\|u(t)\|_e^2-\|u(0)\|_e^2,$$

where $G(r, t)$ is given by (4. 4), because (1. 7), (1. 15) and (1. 19) implies

$$|D_{r,t}^\alpha u(r,t)|\leqq C\langle r\rangle^{-m-1}\langle|r|-|t|\rangle^{-\kappa-1}\quad\text{for}\quad 1\leqq|\alpha|\leqq 2.$$

Moreover by (5. 22) we get

$$\|u(t)\|_e\longrightarrow\|u_+(0)\|_e\qquad\text{as}\qquad t\to\infty.$$

Therefore

$$2\int_0^\infty d\tau\int_0^\infty G(r,\tau)u_t(r,\tau)r^{n-1}dr=\|u_+(0)\|_e^2-\|u(0)\|_e^2.$$

Analogously we have from (1. 17)

$$2\int_{-\infty}^0 d\tau\int_0^\infty G(r,\tau)u_t(r,\tau)r^{n-1}dr=\|u(0)\|_e^2-\|u_-(0)\|_e^2.$$

Hence (1. 20) follows. Thus we prove Theorem 1. 2.

Appendix

The purpose of this appendix is to show that one can relax condition (1. 2) on decay rate of the initial data in (1. 1), provided the number p in $(H)_1$ is large, say, $p>(m+3)/(m+1)$. In what follows we shall indicate only the points different from the previous sections.

For § 1, we first replace the number $1+\kappa$ in (1. 2) by μ, so that (1. 2) changes into

$$\|\{f,g\}\|_0=\sup_{-\infty<r<\infty}|f(r)|\langle r\rangle^{m+\mu}$$
$$=\sum_{j=0}^{m+2}\sup_{-\infty<r<\infty}(|f^{(j+1)}(r)|+|g^{(j)}(r)|)\langle r\rangle^{m+1+\mu},\tag{1. 2$'$}$$

where $0<\mu<1$. We also replace (1. 4) by

$$\|u|=\sum_{|\alpha|\leqq 1}\sup_{(r,t)\in\boldsymbol{R}^2}|D_{r,t}^\alpha u(r,t)|\langle r\rangle^m\langle|r|+|t|\rangle^\mu\tag{1. 4$'$}$$
$$+\sum_{j=0}^{m+1}\sup_{(r,t)\in\boldsymbol{R}^2}(|r^jD_r^{j+2}u(r,t)|+|r^jD_r^{j+1}D_tu(r,t)|)\langle r\rangle^{m-j}\langle|r|+|t|\rangle^\mu,$$

where μ is the same number as in (1. 2)$'$. Then, in Theorem 1. 1, the estimates (1. 6), (1. 7) and (1. 8) change into

$$|u(r,t)|\leqq C\|\{f,g\}\|_0\langle r\rangle^{-m}\langle|r|+|t|\rangle^{-\mu},\tag{1. 6$'$}$$

$$|D_{r,t}^\alpha u(r,t)|\leqq C\|\{f,g\}\|_0\langle r\rangle^{-m-1}\langle|r|-|t|\rangle^{-\mu}\quad\text{if}\quad 1\leqq|\alpha|\leqq 2\tag{1. 7$'$}$$

and

$$|r^{|\alpha|-2}D^{\alpha}_{r,t}u(r,t)|\leqq C\|\{f,g\}\|_0\langle r\rangle^{-m+|\alpha|-3}\langle|r|-|t|\rangle^{-\mu} \quad \text{if} \quad 3\leqq|\alpha|\leqq m+3. \tag{1.8$'$}$$

Next consider (0. 1). We replace condition (1. 10) by

$$p>\frac{m+3}{m+1}. \tag{1.10$'$}$$

Note that $(m+3)/(m+1)>p_0(m)$. Besides, if $p>m+3$, we change condition (1. 12) for

$$|D^{\alpha}F(\lambda)-D^{\alpha}F(\lambda')|\leqq B|\lambda-\lambda'|(|\lambda|^{p-m-3}+|\lambda'|^{p-m-3}). \tag{1.12$'$}$$

We also replace condition (1. 13) by

$$\frac{2}{p-1}-m<\mu<1, \quad \mu>\frac{1}{p}. \tag{1.13$'$}$$

Then, in Theorem 1. 2, the estimates (1. 15), (1. 17), (1. 18) and (1. 19) change into

$$\begin{aligned}&|D_r(u(r,t)-u_-(r,t))|+|D_t(u(r,t)-u_-(r,t))|\\ &\qquad\leqq C_1\|u\|^p\langle r\rangle^{-m-1}\langle|r|-t\rangle^{-\mu} \quad \text{for} \quad (r,t)\in\boldsymbol{R}^2\end{aligned} \tag{1.15$'$}$$

$$\|u(t)-u_-(t)\|_e\leqq C_2\|u\|^p\langle t\rangle^{-\mu+(1/2)} \quad \text{if} \quad \mu>(1/2),\ t\leqq 0, \tag{1.17$'$}$$

$$|u(r,t)-u_-(r,t)|\leqq C_3\|u\|^p\langle r\rangle^{-m}\langle|r|+|t|\rangle^{-\mu} \quad \text{for} \quad (r,t)\in\boldsymbol{R}^2 \tag{1.18$'$}$$

and

$$\begin{aligned}&|r^{\alpha+\beta-2}D_t^{\beta}D_r^{\alpha}(u(r,t)-u_-(r,t))|\\ &\qquad\leqq C_4\|u\|^p\langle r\rangle^{-m+\alpha+\beta-3}\langle|r|-t\rangle^{-\mu}\\ &\qquad\text{for} \quad (r,t)\in\boldsymbol{R}^2,\ 2\leqq\alpha+\beta\leqq m+3 \quad\text{and}\quad 0\leqq\beta\leqq 2.\end{aligned} \tag{1.19$'$}$$

For the $u_+(r,t)$, we replace the factor $\langle|r|-t\rangle^{-\mu}$ in (1. 15)′ and (1. 19)′ by $\langle|r|+t\rangle^{-\mu}$. Besides, (1. 20) holds if $\mu>1/2$.

For § 3, we replace $\varkappa+1$ by μ in the proofs of Lemmas 3. 2, 3. 3 and 3. 4. Moreover, in the proof of Lemma 3. 4, we have

$$\begin{aligned}|u(r,t)|&\leqq C\varepsilon r^{-m-1}\left(\int_{|t-r|}^{t+r}\langle\rho\rangle^{-\mu}d\rho+\langle t+r\rangle^{1-\mu}\right)\\ &\leqq C\varepsilon r^{-m-1}\langle t+r\rangle^{1-\mu},\end{aligned}$$

since $\mu<1$.

For § 4, we first replace (4. 6), (4. 7) and (4. 10) by

$$|D_\rho^j G(\rho, \tau)| \leqq C\|u\|^p \langle\rho\rangle^{-mp} \langle|\rho|+|\tau|\rangle^{-p\mu} \quad \text{for} \quad |\rho| \geqq 1, \quad 0 \leqq j \leqq m+2, \tag{4.6$'$}$$

$$|\rho^k D_\rho^j G(\rho, \tau)| \leqq C\|u\|^p \langle\tau\rangle^{-p\mu} \quad \text{for} \quad |\rho| \leqq 1, \ 0 \leqq j \leqq k+1 \quad \text{and} \quad 0 \leqq k \leqq m+1 \tag{4.7$'$}$$

and

$$|D_\rho^j H(\rho, \tau)| \leqq C\|u\|^p \langle\rho\rangle^{p-q-1} \langle|\rho|+|\tau|\rangle^{-p\mu} \quad \text{for} \quad j=0,\ 1,\ 2. \tag{4.10$'$}$$

In Lemmas 4. 2 and 4. 3 we assume condition (1. 13)$'$ as well as the hypotheses holds. Then (4. 11) changes into

$$|D_{r,t}^\alpha w_k(r, t, \tau)| \leqq C\|u\|^p \langle r\rangle^k \langle\tau\rangle^{-\min\{p\mu, \mu+1\}} \quad \text{for} \quad 0 \leqq |\alpha| \leqq k+2. \tag{4.11$'$}$$

In Proposition 4. 4, the estimates (4. 18) and (4. 19) change into

$$|L(u)(r, t)| \leqq C\|u\|^p \langle r\rangle^{-m} \langle|r|+|t|\rangle^{-\mu} \quad \text{if} \quad |r| \geqq 1 \tag{4.18$'$}$$

and

$$|D_t^\beta D_r^\alpha L(u)(r, t)| \leqq C\|u\|^p \langle r\rangle^{-m-1} \langle|r|-t\rangle^{-\mu} \quad \text{if} \quad |r| \geqq 1, \quad 1 \leqq \alpha+\beta \leqq m+3 \quad \text{and} \quad 0 \leqq \beta \leqq 2. \tag{4.19$'$}$$

Besides, the factor $\langle t\rangle^{-\kappa-1}$ in (4. 20) and (4. 21) changes into $\langle t\rangle^{-\mu}$.

In Lemma 4. 6, the factors $\langle\rho+|\tau|\rangle^{-p}\langle\rho-|\tau|\rangle^{-p\kappa}$ and $\langle|\rho_\pm|+|\tau|\rangle^{-p}\langle|\rho_\pm|-|\tau|\rangle^{-p\kappa}$ in (4. 24) through (4. 27) change into $\langle\rho+|\tau|\rangle^{-p\mu}$ and $\langle|\rho_\pm|+|\tau|\rangle^{-p\mu}$, respectively. Besides, the $-p-p\kappa$ in (4. 28) does into $-p\mu$. We also replace conditions (4. 32) and (4. 33) by

$$1/p < \mu < 1 \tag{4.32$'$}$$

and

$$p\mu + q > \mu + p. \tag{4.33$'$}$$

Lemmas 4. 7 and 4. 8 are now unnecessary.

In Lemma 4. 9 we delete (4. 38) and replace (4. 39) by

$$I_{1,0}(r, t) = Cr\langle r+|t|\rangle^{-\mu}. \tag{4.39$'$}$$

For the proof we chasge the factors $\langle\rho+\tau\rangle^{-p}\langle\rho-\tau\rangle^{-p\kappa}$ and$\langle\rho-\tau\rangle^{-p}$ $\langle\rho+\tau\rangle^{-p\kappa}$ in (4. 40) for $\langle\rho+\tau\rangle^{-\rho\mu}$ and $\langle\rho-\tau\rangle^{-p\mu}$ respectively, hence $(4.42)_\pm$ change into

$$I_{1,0}^{+}(r, t) = \frac{1}{2}\int_{|t-r|}^{t+r} \langle \xi \rangle^{-p\mu} d\xi \int_{r-t}^{\xi} \langle \frac{\xi+\eta}{2} \rangle^{p-q-1} d\eta \quad \text{for} \quad t \geqq 0 \qquad (4.42)'_{+}$$

and

$$I_{1,0}^{-}(r, t) = \frac{1}{2}\int_{t-r}^{t+r} d\xi \int_{\max\{|t-r|, \xi\}}^{\infty} \langle \frac{\xi+\eta}{2} \rangle^{p-q-1} \langle \eta \rangle^{-p\mu} d\eta. \qquad (4.42)'_{-}$$

For $(4.42)'_{+}$, assuming $t>0$ we have from $(4.33)'$

$$\begin{aligned} I_{1,0}^{+} &\leqq C\int_{|t-r|}^{t+r} \langle \xi \rangle^{-\mu} d\xi \int_{-\infty}^{\infty} \langle \xi+\eta \rangle^{p-q-1-p\mu+\mu} d\eta \\ &\leqq C\int_{|t-r|}^{t+r} \langle \xi \rangle^{-\mu} d\xi. \end{aligned}$$

Hence we obtain $(4.39)'$. For $(4.42)'_{-}$, if either $t \leqq 0$ or $t \geqq 3r$, we have

$$I_{1,0}^{-} \leqq C\langle r+|t| \rangle^{-\mu} \int_{t-r}^{t+r} d\xi \int_{-\infty}^{\infty} \langle \xi+\eta \rangle^{p-q-1-p\mu+\mu} d\eta.$$

If $0 \leqq t \leqq 3r$, then

$$\begin{aligned} I_{1,0}^{-} &\leqq C\int_{t-r}^{t+r} d\xi \int_{|t-r|}^{\infty} \langle \xi+\eta \rangle^{p-q-1-p\mu} d\eta \\ &\leqq C\int_{t-r}^{t+r} \langle \xi+|t-r| \rangle^{p-q-p\mu} d\xi \\ &\leqq C\int_{0}^{2(t+r)} \langle \xi \rangle^{p-q-p\mu} d\xi. \end{aligned}$$

Therefore by $(4.32)'$ and $(4.33)'$ we obtain $(4.39)'$.

In Lemma 4.10 we replace (4.45) by

$$I_{1,\alpha}(r, t) \leqq C\langle r-t \rangle^{-\mu} \quad \text{for} \quad 1 \leqq \alpha \leqq m+3. \qquad (4.45)'$$

For the proof we note that the $J(r, t)$ changes into

$$J(r, t) = \int_{-\infty}^{t} d\tau \int_{|\rho_-|}^{\rho_+} \langle \rho \rangle^{p-q-2} \langle \rho+|\tau| \rangle^{-p\mu} d\rho,$$

where $\rho_{\pm} = t-\tau \pm r$. Moreover we have

$$|\rho_-|+|\tau| \geqq |r-t| \quad \text{for} \quad \tau \leqq t,$$

because

$$t-r-\tau+|\tau| \geqq r-t \quad \text{if} \quad \tau \leqq t-r \leqq 0$$

and

$$\tau+r-t+|\tau| \geqq t-r \quad \text{if} \quad \tau \geqq t-r \geqq 0.$$

Hence by (4. 33)′ we get

$$J(r, t)\leqq\langle r-t\rangle^{-\mu}\int_{-\infty}^{t}d\tau\int_{|\rho_-|}^{\infty}\langle\rho\rangle^{p-q-2-p\mu+\mu}d\rho$$
$$\leqq C\langle r-t\rangle^{-\mu}\int_{-\infty}^{\infty}\langle\rho_-\rangle^{p-q-1-p\mu+\mu}d\tau$$
$$\leqq C\langle r-t\rangle^{-\mu}.$$

In Lemma 4. 11 we replace (4. 49) by

$$I_2(r, t)\leqq C\langle r+|t|\rangle^{-\mu}. \tag{4.49$'$}$$

For the proof we note that (4. 26) changes into

$$I_2=\int_{-\infty}^{t}\langle t+r-\tau\rangle^{p-q-1}\langle t+r-\tau+|\tau|\rangle^{-p\mu}d\tau. \tag{4.26$'$}$$

Moreover

$$t+r-\tau+|\tau|\geqq r+|t| \quad \text{for} \quad \tau\leqq t,$$

since $-\tau\geqq|t|$ if $t<0$. Hence by (4. 26)′ we get

$$I_2\leqq\langle r+|t|\rangle^{-\mu}\int_{-\infty}^{\infty}\langle t+r-\tau\rangle^{p-q-1-p\mu+\mu}d\tau.$$

Therefore (4. 33)′ yields (4. 49)′.

In Lemma 4. 12 we replace (4. 51) by

$$I_3(r, t)\leqq C\langle r-t\rangle^{-\mu}. \tag{4.51$'$}$$

For the proof we notice that (4. 27) change into

$$I_3=\int_{-\infty}^{t}\langle t-r-\tau\rangle^{p-q-1}\langle|t-r-\tau|+|\tau|\rangle^{-p\mu}d\tau. \tag{4.27$'$}$$

Moreover we have

$$|t-r-\tau|+|\tau|\geqq|r-t| \quad \text{for} \quad \tau\leqq t,$$

as in the proof of Lemma 4. 10. Hence by (4. 33)′ we obtain

$$I_3\leqq\langle r-t\rangle^{-\mu}\int_{-\infty}^{\infty}\langle t-r-\tau\rangle^{p-q-1-p\mu+\mu}d\tau$$
$$\leqq C\langle r-t\rangle^{-\mu}.$$

In the proof of Lemma 4. 5 we have from (4. 10)′

$$|D_r^\alpha w_0(r, t, \tau)|\leqq C\|u\|^p\int_{-1}^{1}\langle t-\tau+r\sigma\rangle^{p-q-1}\langle|t-\tau+r\sigma|+|\tau|\rangle^{-p\mu}d\sigma$$

for $0\leqq r\leqq 1$ and $0\leqq\alpha\leqq 2$. Hence (4. 20) with $\beta=0$ and $\varkappa+1=\mu$ follows, as before.

Finally we replace (4.55) by

$$|D_r^\alpha G(r,t)|\leqq C\|u\|^p\langle r\rangle^{-m-1}\langle r+|t|\rangle^{-\mu} \quad \text{for} \quad r\geqq 1 \text{ and } 0\leqq\alpha\leqq m+1, \tag{4.55$'$}$$

which follows immediately from (4.6)′ and (4.33)′, since $p>1$ and

$$-mp-p\mu+\mu=p-q-p\mu+\mu-m-2<-m-2.$$

Thus we prove Proposition 4.4.

For § 5, we replace the factor $\langle|r|+|t|\rangle\langle|r|-|t|\rangle^{\kappa}$ in the auxiliary norm $|||u|||$ by $\langle|r|+|t|\rangle^{\mu}$. Besides, if $p>m+3$, we take $C_4=0$ in (5.4). In the proof of Lemma 5.3 we replace the factor $\langle|\rho|+|\tau|\rangle^{-p}\langle|\rho|-|\tau|\rangle^{-p\kappa}$ and $\langle\tau\rangle^{-p-p\kappa}$ by $\langle|\rho|+|\tau|\rangle^{-p\mu}$ and $\langle\tau\rangle^{-p\mu}$, respectively. Finally, in Proposition 5.6 we replace the factors $\langle|r|+|t|\rangle^{-1}\langle|r|+t\rangle^{-\kappa}$ in (5.19) and $\langle|r|-|t|\rangle^{-1}\langle|r|+t\rangle^{-\kappa}$ in (5.20), (5.21) by $\langle|r|+t\rangle^{-\mu}$. Besides, if $\mu>1/2$, we change the factor $\langle t\rangle^{-\kappa}$ in (5.22) by $\langle t\rangle^{-\mu+(1/2)}$. Thus we obtain an analogue to Theorem 1.2.

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