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A class of singular integral operators with rough kernel on product domains*

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Abstract. The L^{2}-boundedness for a class of singular integral operators with rough kernel on product domain is discussed in terms of block decompositions. The main result is an improvment of corresponding one on L^{2}-boundedness due to J. Duoandikoetxea.

1. Introduction

It is well known that the singular integral operators on product domain $R^{n}\times R^{m}$ defined by

$$Tf(x, y)=p.v. \int_{R^{n}\times R^{m}}K(\xi, \eta)f(x-\xi, y-\eta)d\xi d\eta$$

are bounded on $L^{p}(R^{n}\times R^{m})$, $1<p<\infty$, provided

$$K(x, y)=\Omega(x/|x|, y/|y|)|x|^{-n}|y|^{-m},$$

Ω is homogeneous of degree zero, $\int_{S^{n-1}}\Omega(u, v)du=\int_{S^{m-1}}\Omega(u, v)dv=0$, and some regularity conditions on Ω are assumed (see [2]). The L^{p}-boundedness of T with the rough condition $\Omega\in L^{q}(S^{n-1}\times S^{m-1})$ instead of regularity is obtained in [1]. In this paper, we shall use the method of block decomposition for functions to improve the result of L^{2}-boundedness above. It should be pointed out that the method of block decomposition for functions is originated by M. H. Taibleson and G. Weiss in the study of the convergence of the Fourier series (see [6]). Latter on, many applications of the block decomposition to Harmonic analysis were discovered (see [5]). For example, a sort of method related to block decompositions is applied to study the L^{p}-boundedness of singular integral operators with rough kernel in [3]-[4]. Thus, this paper can also be regarded as generalization of the one-parameter results in [3]-[4].

Let us begin with the definition of q-block on $S^{n-1}\times S^{m-1}$.

Definition 1 A function $b(u, v)$ on $S^{n-1}\times S^{m-1}$ is called a q-block,

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$1<q\leq\infty$, if it satisfies the following two conditions:

(a) $\mathrm{Supp} b \subset Q$, where Q is a interval on $S^{n-1}\times S^{m-1}$, i.e. $Q=\{x'\in S^{n-1}: d(x', x_0')<\alpha\}\times\{y'\in S^{m-1}: d(y', y_0')<\beta\}$ with $x_0'\in S^{n-1}$ and $y_0'\in S^{m-1}$.

(b) $\|b\|_q \leq |Q|^{-1/q'}$, $1/q+1/q'=1$.

Now, we may define a class of function space generated by blocks.

Definition 2 Let ϕ be a nonnegative function on R^+. The space $B_q^{\phi}(S^{n-1}\times S^{m-1})$ generated by q-blocks is defined by

$$B_q^{\phi}(S^{n-1}\times S^{m-1})=\{f\in L^1(S^{n-1}\times S^{m-1}): f(u, v)=\sum_{k=1}^{\infty} C_k b_k(u, v),$$
$$\text{each } b_k \text{ is a } q\text{-block, and } M_q^{\phi}(\{C_k\})<\infty\},$$

where $M_q^{\phi}(\{C_k\}) = \sum_{k=1}^{\infty}|C_k|(1+\phi(|Q_k|))$, and Q_k is the supporting interval of b_k.

In particular, taking

$$\phi(t) = \phi_{\mu,\nu}(t) = \begin{cases} \int_t^1 u^{-1-\mu}\log^{\nu}\frac{1}{u}du, & 0<t<1; \\ 0, \quad t\geq 1, \end{cases}$$

where $\mu\geq 0$, $\nu\in R$ the corresponding spaces are denoted by $B_q^{\mu,\nu}(S^{n-1}\times S^{m-1})$, and $M_q^{\phi}(C_k)$ are denoted by $M_q^{\mu,\nu}(C_k)$.

For different values of μ and ν, it is easy to verify the following relations:

$$B_q^{\mu,\nu_2}\subset B_q^{\mu,\nu_1}, \text{ if } \nu_1<\nu_2; \tag{1.1}$$

$$B_q^{\mu_2,\nu_2}\subset B_q^{\mu_1,\nu_1}\subset B_q^{0,\nu_0}, \text{ if } 0<\mu_1<\mu_2, \text{ and } \nu_i\in R \ (i=0, 1, 2); \tag{1.2}$$

$$L^q\subset B_q^{\mu,\nu}\subset B_{q_1}^{\mu,\nu}, \text{ if } \ 1<q_1<q. \tag{1.3}$$

Let $K(x, y)$ be a kernel of the form

$$K(x, y) = \frac{h(|x|, |y|)\,\Omega(x', y')}{|x|^n|y|^m},$$

where $x\in R^n$, $y\in R^m$, $x'=x/|x|$, $y'=y/|y|$, and $h\in L^{\infty}(R^+\times R^+)$. Suppose Ω satisfies the cancellation condition

$$\int_{S^{n-1}}\Omega(x', y')dx'=\int_{S^{m-1}}\Omega(x', y')dy'=0, \tag{1.4}$$

and Ω is homogeneous of degree zero. We are going to study the L^2-boundedness of singular integral operators

$$Tf(x, y) = p.v. \iint_{R^n \times R^m} K(\xi, \eta) f(x-\xi, y-\eta) d\xi d\eta.$$

The main result of this paper is stated as follows.

Theorem *Suppose $\Omega \in B_q^{0,\nu}(S^{n-1} \times S^{m-1})$ with some $q>1$ and $\nu \geq 1$. Then T is bounded on $L^2(R^n \times R^m)$ for $n \geq 2$ and $m \geq 2$, where $h \in L^\infty(R^+ \times R^+)$.*

Remark. Under the hypothesis on Ω in Theorem, whether T is bounded on L^p is still a question.

2. Proof of Theorem

Let

$$M_q^{\mu,\nu}(\Omega) = inf\{M_q^{\mu,\nu}(\{C_k\})\},$$

where infimun is taken over all q-block decompositions of Ω. By (1.1), (1.2) and (1.3), it will suffice to prove Theorem for the case of $1<q<2$ and $\nu=1$. We need the following.

Proposition *Suppose $\Omega \in B_q^{0,1}(S^{n-1} \times S^{m-1})$, $n \geq 2$, and $m \geq 2$. Let*

$$l_1(r_1, y', \xi') = \int_{S^{n-1}} \Omega(x', y') e^{-ir_1 x' \cdot \xi'} dx',$$

$$l_2(r_2, x', \eta') = \int_{S^{m-1}} \Omega(x', y') e^{-ir_2 y' \cdot \eta'} dy',$$

and

$$l(r_1, r_2, \xi', \eta') = \iint_{S^{n-1} \times S^{m-1}} \Omega(x', y') e^{-i(r_1 x' \cdot \xi' + r_2 y' \cdot \eta')} dx' dy'.$$

Then we have

$$\int_{S^{m-1}} \int_1^\infty |l_1(r_1, y', \xi')| \frac{dr_1}{r_1} dy' \leq CM_q^{0,0}(\Omega), \tag{2.1}$$

$$\int_{S^{n-1}} \int_1^\infty |l_2(r_2, x', \eta')| \frac{dr_2}{r_2} dx' \leq CM_q^{0,0}(\Omega), \tag{2.2}$$

and

$$\int_1^\infty \int_1^\infty |l(r_1, r_2, \xi', \eta')| \frac{dr_1 dr_2}{r_1 r_2} \leq CM_q^{0,1}(\Omega), \tag{2.3}$$

where C is independent of ξ' and η'.

Proof. Let $\Omega(x', y') = \sum_{k=1}^\infty C_k b_k(x', y')$ and

$$M_q^{0,1}(\{C_k\}) = \sum_{k=1}^{\infty}|C_k|\{1+(\log^+|Q_k|^{-1})^2\} < \infty.$$

Without loss of generality, we may assume $\xi'=(1,0,\cdots,0)\in S^{n-1}$ and $\eta'=(1,0,\cdots,0)\in S^{m-1}$. Let $x'=(s,x_2',\cdots,x_n')\in S^{n-1}$ and $y'=(t,y_2',\cdots,y_m')\in S^{m-1}$. Then

$$\int_{S^{n-1}} b_k(x',y')e^{-ir_1x'\cdot y'}\,dx' = \int_{R^1} e^{-ir_1s}F_k(s,y')ds = \widehat{F}_k(r_1,y'),$$

where

$$F_k(s,y') = (1-s^2)^{(n-3)/2}\chi_{\{|s|<1\}}(s)\int_{S^{n-2}} b_k(s,\sqrt{1-s^2}\,u'\,;\,y')du'.$$

We consider two cases respectively: $|Q_k|<1$ and $|Q_k|\geq 1$. When $|Q_k|<1$, we have

$$\begin{aligned}\int_{S^{m-1}}\int_1^{\infty}|\widehat{F}_k(r_1,y')|\frac{dr_1}{r_1}dy' &= \int_{S^{m-1}}\int_1^{|Q_k|^{-q'}}|\widehat{F}_k(r_1,y')|\frac{dr_1}{r_1}dy'\\ &\quad+\int_{S^{m-1}}\int_{|Q_k|^{-q'}}^{\infty}|\widehat{F}_k(r_1,y')|\frac{dr_1}{r_1}dy'\\ &:=I_1+I_2.\end{aligned}$$

It is easy to see that

$$\begin{aligned}I_1 &\leq \int_1^{|Q_k|^{-q'}}\iint_{S^{n-1}\times S^{m-1}}|b_k(x',y')|dx'dy'\frac{dr_1}{r_1}\\ &\leq C\int_1^{|Q_k|^{-q'}}\frac{dr_1}{r_1}\\ &\leq C\log(|Q_k|^{-1}).\end{aligned}$$

Setting $\lambda'=(q')^2$ and $1/\lambda+1/\lambda'=1$, then $1<\lambda<q$ and $\lambda'>2$. Applying the Hausdorff-Young inequality and Hölder's inequality to I_2, we have

$$\begin{aligned}I_2 &\leq \left(\int_{|Q_k|^{-q'}}^{\infty}\frac{dr_1}{r_1^{\lambda}}\right)^{1/\lambda}\int_{S^{m-1}}\|\widehat{F}_k(r_1,y')\|_{L^{\lambda'}(R,dr_1)}dy'\\ &\leq C|Q_k|^{1/q'}\int_{S^{m-1}}\|F_k(s,y')\|_{L^{\lambda}(R,ds)}dy'\\ &\leq C|Q_k|^{1/q'}\int_{S^{m-1}}\left\{\int_{-1}^{1}(1-s^2)^{\lambda(n-3)/2}\left|\int_{S^{n-2}} b_k(s,\sqrt{1-s^2}\,u'\,;\,y')du'\right|^{\lambda}ds\right\}^{1/\lambda}dy'.\end{aligned}$$

Using Hölder's inequality again for $q/(q-\lambda)$ and q/λ, we obtain

$$\begin{aligned}I_2 \leq C|Q_k|^{1/q'}\int_{S^{m-1}}&\{\int_{-1}^{1}(1-s^2)^{q(n-3)/2}ds\}^{1/qq'}\\ &\cdot\left\{\int_{-1}^{1}(1-s^2)^{(n-3)/2}\left|\int_{S^{n-2}} b_k(s,\sqrt{1-s^2}\,u'\,;\,y')du'\right|^{q}ds\right\}^{1/q}dy'\end{aligned}$$

$$\leq C|Q_k|^{1/q'}\{\iint_{S^{n-1}\times S^{m-1}}|b_k(x',y')|^q dx'dy'\}^{1/q}\leq C.$$

Thus, if $|Q_k|<1$, then we have

$$\int_{S^{m-1}}\int_1^\infty|\hat{F}_k(r_1,y')|\frac{dr_1}{r_1}dy'\leq C[1+\log(|Q_k|^{-1})].$$

For the case of $|Q_k|\geq 1$, by a method similar to the estimates of I_2, we obtain

$$\int_{S^{m-1}}\int_1^\infty|\hat{F}_k(r_1,y')|\frac{dr_1}{r_1}dy'\leq C.$$

Since the above constant C is independent of b_k, we have

$$\begin{aligned}&\int_{S^{m-1}}\int_1^\infty|l_1(r_1,y',\xi')|\frac{dr_1}{r_1}dy'\\&\quad\leq\sum_{k=1}^\infty|C_k|\int_{S^{m-1}}\int_1^\infty\left|\int_{S^{n-1}}b_k(x',y')e^{-ir_1x'\cdot\xi'}dx'\right|\frac{dr_1}{r_1}dy'\\&\quad\leq C\sum_{k=1}^\infty|C_k|(1+\log^+\frac{1}{|Q_k|}).\end{aligned}$$

Taking infimum over all q-block decompositions of Ω we get (2.1). The estimates of (2.2) is the same as above. To prove (2.3), we write

$$\begin{aligned}&\iint_{S^{n-1}\times S^{m-1}}b_k(x',y')e^{-i(r_1x'\cdot\xi'+r_2y'\cdot\eta')}dx'dy'\\&\quad=\iint_{R^2}G_k(s,t)e^{-i(r_1s+r_2t)}dsdt=\hat{G}_k(r_1,r_2)\end{aligned}$$

where

$$\begin{aligned}G_k(s,t)=&[(1-s^2)(1-t^2)]^{(n-3)/2}\chi_{\{|s|<1,|t|<1\}}(s,t)\\&\iint_{S^{n-2}\times S^{m-2}}b_k(s,\sqrt{1-s^2}\,u';\ t,\sqrt{1-t^2}\,v')du'dv'.\end{aligned}$$

If $|Q_k|<1$, then

$$\begin{aligned}\int_1^\infty\int_1^\infty|\hat{G}_k(r_1,r_2)|\frac{dr_1dr_2}{r_1r_2}&=\Big(\int_1^{|Q_k|^{-q'}}\int_1^{|Q_k|^{-q'}}+\int_1^{|Q_k|^{-q'}}\int_{|Q_k|^{-q'}}^\infty+\int_{|Q_k|^{-q'}}^\infty\int_1^{|Q_k|^{-q'}}\\&\quad+\int_{|Q_k|^{-q'}}^\infty\int_{|Q_k|^{-q'}}^\infty\Big)|\hat{G}_k(r_1,r_2)|\frac{dr_1dr_2}{r_1r_2}\\&:=I_3+I_4+I_5+I_6.\end{aligned}$$

It is easy to see that

$$I_3 \le \int_1^{|Q_k|^{-q'}} \int_1^{|Q_k|^{-q'}} \iint_{S^{n-1}\times S^{m-1}} |b_k(x', y')| dx'dy' \le C\log^2(1/|Q_k|)$$

and

$$\begin{aligned} I_4 &\le \int_1^{|Q_k|^{-q'}} \int_{S^{m-1}} \int_{|Q_k|^{-q'}}^{\infty} \left| \int_{S^{n-1}} b_k(x', y') e^{-ir_1 s} dx' \right| \frac{dr_1}{r_1} dy' \frac{dr_2}{r_2} \\ &\le C\, I_2 \log(1/|Q_k|) \le C \log(1/|Q_k|). \end{aligned}$$

The estimate of I_5 is the same as that of I_4. Applying the Hausdorff-Young inequality and Hölder's inequality to I_6, we obtain

$$\begin{aligned} I_6 &\le \left(\int_{|Q_k|^{-q'}}^{\infty} \int_{|Q_k|^{-q'}}^{\infty} \frac{dr_1 dr_2}{r_1^{\lambda} r_2^{\lambda}} \right)^{1/\lambda} \|\widehat{G}_k(r_1, r_2)\|_{L^{\lambda'}(R^2, dr_1 dr_2)} \\ &\le C|Q_k|^{2q'(\lambda-1)/\lambda} \|G_k(s, t)\|_{L^{\lambda}(R^2, dsdt)} \\ &\le C|Q_k|^{2/q'} \|b_k\|_{L^q(S^{n-1}\times S^{m-1})} \le C. \end{aligned}$$

When $|Q_k| \ge 1$, it is easy to see that

$$\int_1^{\infty} \int_1^{\infty} |\widehat{G}_k(r_1, r_2)| \frac{dr_1 dr_2}{r_1 r_2} \le C.$$

Clearly, the above estimates yield (2. 3). This finishes the proof of Proposition.

Let us now turn to prove Theorem. We write

$$\widehat{Tf}(\xi, \eta) = m(\xi, \eta) \hat{f}(\xi, \eta),$$

where

$$\begin{aligned} m(\xi, \eta) &= \widehat{K}(\xi, \eta) \\ &= (2\pi)^{-n-m} \int_0^{\infty} \int_0^{\infty} \iint_{S^{n-1}\times S^{m-1}} \Omega(x', y') e^{-i(r_1 x' \cdot \xi' + r_2 y' \cdot \eta')} dx' dy' h\left(\frac{r_1}{|\xi|}, \frac{r_2}{|\eta|} \right) \frac{dr_1 dr_2}{r_1 r_2}. \end{aligned}$$

To prove Theorem, we need only to show

$$\mathrm{Sup}_{\xi\in R^n, \eta\in R^m} |m(\xi, \eta)| < \infty.$$

Using the cancellation condition (1. 4), we have

$$\begin{aligned} m(\xi, \eta) = & \\ & \int_0^1 \int_0^1 \iint_{S^{n-1}\times S^{m-1}} \Omega(x', y')(e^{-ir_1 x' \cdot \xi'} - 1)(e^{-ir_2 y' \cdot \eta'} - 1) dx'dy'\, h\left(\frac{r_1}{|\xi|}, \frac{r_2}{|\eta|} \right) \frac{dr_1 dr_2}{r_1 r_2} \\ & + \int_0^1 \int_1^{\infty} \iint_{S^{n-1}\times S^{m-1}} \Omega(x', y')(e^{-ir_1 x' \cdot \xi'} - 1) e^{-ir_2 y' \cdot \eta'} dx'dy'\, h\left(\frac{r_1}{|\xi|}, \frac{r_2}{|\eta|} \right) \frac{dr_1 dr_2}{r_1 r_2} \end{aligned}$$

$$+\int_1^\infty\int_0^1 \iint_{S^{n-1}\times S^{m-1}} \Omega(x', y')e^{-ir_1x'\cdot\xi'}(e^{-ir_2y'\cdot\eta'}-1)dx'dy'\, h\Big(\frac{r_1}{|\xi|}, \frac{r_2}{|\eta|}\Big)\frac{dr_1dr_2}{r_1r_2}$$
$$+\int_1^\infty\int_1^\infty \iint_{S^{n-1}\times S^{m-1}} \Omega(x', y')e^{-i(r_1x'\cdot\xi'+r_2y'\cdot\eta')}dx'dy'\, h\Big(\frac{r_1}{|\xi|}, \frac{r_2}{|\eta|}\Big)\frac{dr_1dr_2}{r_1r_2}.$$

Thus, it follows from Propositon that

$$\begin{aligned}|m(\xi, \eta)| \le{}& C\|h\|_\infty\int_0^1\int_0^1 \|\Omega\|_{L^1(S^{n-1}\times S^{m-1})}dr_1dr_2\\ &+C\|h\|_\infty\int_0^1\int_{S^{n-1}}\int_1^\infty |l_2(r_2, x', \eta')|\frac{dr_2}{r_2}dx'dr_1\\ &+C\|h\|_\infty\int_0^1\int_{S^{m-1}}\int_1^\infty |l_1(r_1, y', \xi')|\frac{dr_1}{r_1}dy'dr_2\\ &+C\|h\|_\infty\int_1^\infty\int_1^\infty |l(r_1, r_2, \xi', \eta')|\frac{dr_1dr_2}{r_1r_2}\\ \le{}& C[1+M_q^{0,0}(\Omega)+M_q^{0,1}(\Omega)].\end{aligned}$$

Hence, the proof of Theorem is complete.

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