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Regularly varying correlation functions and KMO-Langevin equations

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Abstract. We study a variant of Okabe's first KMO-Langevin equation. After establishing unique existence of a stationary solution, we precisely describe the long-time behavior of the correlation function R of the solution. In particular, the behavior such as $R(t) \sim ct^{-1}$ as $t \to \infty$ is characterized by using Π-variation. Correlation functions regularly varying with index $p \in [-1, 0)$ are characterized in terms of outer functions.

Key words: first KMO-Langevin equation, stationary process, reflection positivity, correlation function, outer function, regular variation, Π-variation, stationary random distribution.

1. Introduction

In [O4], Okabe introduced the linear stochastic delay equation

$$\dot{X}(t) = -\beta X(t) - \int_{-\infty}^{t} \gamma(t-s)\dot{X}(s)ds + \alpha \dot{B}(t). \tag{1.1}$$

This equation is called a *first KMO-Langevin equation*. Here, α and β are positive numbers, $\dot{B}$ is a Gaussian white noise, and the kernel function $\gamma : (0, \infty) \to [0, \infty)$ has a representation of the form

$$\gamma(t) = \int_{0}^{\infty} e^{-t\lambda} d\rho(\lambda) \qquad (t > 0), \tag{1.2}$$

where ρ is a Borel measure on $(0, \infty)$ such that

$$\int_{0}^{\infty} \frac{1}{\lambda + 1} d\rho(\lambda) < \infty. \tag{1.3}$$

The key feature of equation (1.1) is that it describes the time evolution of a stationary Gaussian process X with *reflection positivity*: the correlation function R of X, which is defined by $R(t) := E[X(t)X(0)]$, takes the form

$$R(t) = \int_{0}^{\infty} e^{-|t|\lambda} d\sigma(\lambda) \quad (t \in \mathbb{R}), \tag{1.4}$$

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where σ is a bounded Borel measure on $(0,\infty)$. If the triple (α,β,ρ) in (1.1) with (1.2) varies satisfying $\alpha>0$, $\beta>0$ and (1.3), then the measure σ ranges over all non-zero bounded Borel measures on $(0,\infty)$ such that $m_{-1}(\sigma)<\infty$, $m_1(\sigma)<\infty$. Here we write $m_k(\sigma)$ for the k-th moment of σ:

$$m_k(\sigma) := \int_0^\infty \lambda^k d\sigma(\lambda) \quad (k \in \mathbb{Z}).$$

Let D be the *diffusion coefficient* of X:

$$D := \int_0^\infty R(t)dt.$$

Since $m_{-1}(\sigma)$ is equal to D, we have $D<\infty$ for the equation (1.1). For details, see [O4].

For equation (1.1), the long-time behavior of R was considered by [O6], and subsequently by [I1] and [OI]. Thus, for $q\in(0,\infty)$ and l slowly varying at infinity,

$$R(t) \sim t^{-(1+q)} l(t)\alpha^2\beta^{-3}q \quad (t\to\infty) \tag{1.5}$$

if and only if

$$\gamma(t) \sim t^{-q} l(t) \quad (t\to\infty). \tag{1.6}$$

Functions $1/(1+|t|)^{1+q}$ $(0<q<\infty)$ are examples of such R.

Now choosing σ suitably in (1.4), we obtain non-negative definite functions R such that, for $0<p\le 1$ and l slowly varying at infinity,

$$R(t) \sim t^{-p} l(t) \quad (t\to\infty). \tag{1.7}$$

The prototype of such functions is $R(t)=1/(1+|t|)^p$, $0<p\le 1$. No stationary Gaussian process X satisfying (1.7) with $0<p<1$ can be described by equation (1.1) because $D=\infty$ for such X. The case $p=1$ is delicate, and requires separate treatments. Early in [O3], Okabe showed that $[\alpha,\beta,\gamma]$-*Langevin equations*, which had been introduced by [O1], also describe a class of reflection positive, stationary Gaussian processes with $D=\infty$; they describe the class $m_2(\sigma)<\infty$. Thus the question arises of extending the class of first KMO-Langevin equations (1.1) of [O4] to include the case $D=\infty$, and also of characterizing the asymptotic behavior (1.7) with $0<p\le 1$ in terms of the quantities in the equations. As already suggested by [O2] and [I2], the desired extension of [O4] turns out to be given by admitting the value $\beta=0$ in equation (1.1).

Concerning [O2] and [I2], we comment briefly on Okabe's theory of *second KMO-Langevin equations*. They are linear stochastic delay equations similar to (1.1) but, instead of $\dot{B}$, they have a *Kubo noise*. The need to consider Kubo noise comes from a physical requirement — the *fluctuation-dissipation theorem*. In [O4], Okabe introduced second KMO-Langevin equations for reflection positive, stationary Gaussian processes with $D < \infty$, motivated by the equation describing a non-Markovian effect, known as the *Alder-Wainwright effect* ([AW], [OoKU]). In [O5], he also developed the theory without assuming reflection positivity. The relevant part of [O2] — an analogue of [O5, Theorem 5.1] for $\beta = 0$ — suggested an extension of the class of second KMO-Langevin equations of [O5], to include the case $\lim_{\epsilon\to 0+} |\int_0^\infty e^{-\epsilon t} R(t)dt| = \infty$. The results of [I2] may be regarded as realizing this possibility at the cost of restricting the class of X to reflection positive ones. The point of [I2] is the construction of Kubo noise with desired causality condition.

Now we state the contents of the present paper. As suggested above, we consider the following variant of Okabe's first KMO-Langevin equation (1.1):

$$\dot{X}(t) = -\int_{-\infty}^{t} \gamma(t-s)\dot{X}(s)ds + \alpha\dot{B}(t), \tag{1.8}$$

where $\alpha > 0$, $\dot{B}$ is a Gaussian white noise, and $\gamma : (0,\infty) \to (0,\infty)$ is a function of the form (1.2). We also call equation (1.8) a first KMO-Langevin equation.

For equation (1.8), the measure ρ in (1.2) is assumed to be in a subset C of the class (1.3). The subset C is defined in a rather indirect way but a simple criterion for ρ to be in C is given in terms of the asymptotic behavior of γ. For example, for ρ such that $\gamma(t) = t^{-p}$, ρ is in C if and only if $0 < p < 1/2$. We remark that, for (1.1), we may take $\gamma(t) = t^{-p}$ with $0 < p < 1$.

For $\alpha > 0$, $\rho \in C$ and Gaussian white noise $\dot{B}$, under an appropriate causality condition, there exists a unique stationary random distribution X satisfying (1.8). The solution X is a reflection positive, stationary Gaussian process such that $m_{-1}(\sigma) = \infty$, $m_1(\sigma) < \infty$; in particular, $D = \infty$. Conversely, any such X is a solution of (1.8) for some $\alpha > 0$, $\rho \in C$, and $\dot{B}$; in fact, as $\dot{B}$, we may take the derivative of the canonical Brownian motion of X.

The arguments to prove the results above are more or less similar to those of [O1], [O3], [O4] and [I2], whence owe the basic ideas to Okabe. However the method to prove uniqueness of solutions deserves comment here. In [O4, Theorem 4.2], the uniqueness of a solution of the equation (1.1) was shown in a rather small class — the class of reflection positive, stationary Gaussian processes such that $m_{-1}(\sigma) < \infty$, $m_1(\sigma) < \infty$. In this paper, we develop a projection method which enables us to prove uniqueness of solutions in a larger class — the class of stationary random distributions. This applies to both (1.1) and (1.8).

Our main interest is in the characterization of the asymptotic behavior (1.7) with $0 < p \leq 1$ in terms of α and γ in the equation (1.8). This is the counterpart of the results of [O6] and [I1] stated above but the situation of the present paper requires harder analysis. The difficulty is in the characterization of (1.7) with $0 < p \leq 1$ in terms of outer functions. The arguments of [O6] and [I1] do not apply any more. To overcome this hurdle, we use the relation between the asymptotic behavior of spectral densities and that of outer functions. This relation has its own interest, apart from the application to equation (1.8). We also need an essentially new technique — Π-variation — to deal with the boundary case $p = 1$. See [I3] for the usefulness of Π-variation in the study of stationary processes.

We write $\mathcal{R}_0$ for the class of functions slowly varying at infinity: the class of positive measurable f, defined on some neighborhood of infinity, such that, for any $\lambda > 0$,

$$\lim_{x\to\infty} f(\lambda x)/f(x) = 1.$$

For $l \in \mathcal{R}_0$ such that $\int^\infty l(s)ds/s < \infty$, we write

$$\bar{l}(t) = \int_t^\infty l(s)ds/s.$$

Then $\bar{l}$ is also slowly varying. For $l \in \mathcal{R}_0$ such that $\int^\infty l(s)ds/s = \infty$, we set

$$\tilde{l}(t) = \int_M^t l(s)ds/s \qquad (t \geq M),$$

where we choose M so large that l is locally integrable on $[M, \infty)$. Then $\tilde{l}$ is also slowly varying. For $l \in \mathcal{R}_0$, the class Π_l is the set of measurable f

satisfying, for all $\lambda > 0$,

$$\lim_{x\to\infty}\{f(\lambda x) - f(x)\}/l(x) = c\log\lambda,$$

for some constant c called the *l-index* of f. See Bingham, Goldie and Teugels [BGT] for details. We write $B(\cdot,\cdot)$ for the beta function.

Now we are ready to state our main results.

Theorem 1.1 *Let X be the solution of* (3.6) *with* (3.7), *and let R be the correlation function of X.*

(1) *Let $l \in \mathcal{R}_0$ and $0 < q < 1/2$. Then $\gamma(t) \sim t^{-q}l(t)$ as $t \to \infty$ if and only if*

$$R(t) \sim t^{-(1-2q)}l(t)^{-2}B(q, 1-2q)\pi^{-2}\alpha^2\sin^2(q\pi) \quad (t \to \infty).$$

(2) *Let $l \in \mathcal{R}_0$ such that $\int^\infty l(s)^{-2}ds/s < \infty$. Then $\gamma(t) \sim t^{-1/2}l(t)$ as $t \to \infty$ if and only if $R \in \Pi_{l_1}$ with index -1, where $l_1(t) = l(t)^{-2}\alpha^2\pi^{-2}$.*

(3) *Let $l \in \mathcal{R}_0$ such that $\int^\infty l(s)ds/s < \infty$. Then $\gamma \in \Pi_l$ with index -1 if and only if $R(t) \sim t^{-1}l(t)\bar{l}(t)^{-3}\alpha^2$ as $t \to \infty$.*

For the meaning of 'solution', see §3, in particular, Theorem 3.4. We only remark that equation (3.6) is the precise form of (1.1), and that (3.7) is a causality condition associated with it. We can also state Theorem 1.1 (3) in the following way:

Theorem 1.1 (3)′ *Let $l \in \mathcal{R}_0$ such that $\int^\infty l(s)ds/s = \infty$. We write $l_1(t) = l(t)\{2\tilde{l}(t)\}^{-3/2}$. Then $R(t) \sim t^{-1}l(t)\alpha^2$ as $t \to \infty$ if and only if $\gamma \in \Pi_{l_1}$ with index -1. In particular, $R(t) \sim t^{-1}\alpha^2$ as $t \to \infty$ if and only if $\gamma \in \Pi_l$ with index -1, where $l(t) = (2\log t)^{-3/2}$.*

The last assertion illustrates the usefulness of Π-variation.

We remark that the asymptotic behavior such as (1.7) with $p = 1$ can also occur for equation (1.1). Naturally the question arises of obtaining the analogue of Theorem 1.1 (3) for equation (1.1), to supplement the results of [O6] and [I1].

Theorem 1.2 *Let X be the solution of* (3.9), *and let R be the correlation function of X. Let $l \in \mathcal{R}_0$ such that $\int^\infty l(s)ds/s < \infty$. Then $\gamma \in \Pi_l$ with index -1 if and only if $R(t) \sim t^{-1}l(t)\alpha^2\beta^{-3}$ as $t \to \infty$.*

The equation (3.9) is the precise form of (1.1) in the sense of the present paper. One may take as X the solution of (1.1) in the sense of [O4], too.

In §2, we develop a projection method. Basic facts on equation (1.8) are given in §3; we also give a refinement of [O4, Theorem 4.2] concerning the uniqueness of a solution of (1.1). Sections 4, 5 and 6 are mainly devoted to the proof of Theorem 1.1. In §4, we characterize (1.7) with $0 < p < 1$ in terms of outer functions. In §5, we consider the same problem for the boundary cases $p = 0, 1$. We complete the proof of Theorem 1.1 in §6.

2. Projection method

We denote by H the Hibert space of $\mathbb{C}$-valued random variables, defined on a probability space $(\Omega, \mathcal{F}, P)$, with zero expectation and finite variance: $(f, g) = E[f\overline{g}]$, $\|f\| = (f, f)^{1/2}$. By $\mathcal{D}(\mathbb{R})$ we denote the space of all $\phi \in C^{\infty}(\mathbb{R})$ with compact support, endowed with the usual topology. A *random distribution* is a linear and continous map from $\mathcal{D}(\mathbb{R})$ to H. A random ditribution X is *stationary* if $(X(\tau_h\phi), X(\tau_h\psi)) = (X(\phi), X(\psi))$ for all $\phi, \psi \in \mathcal{D}(\mathbb{R})$ and $h \in \mathbb{R}$, where $\tau_h\phi(t) = \phi(t + h)$. We write $\mathcal{S}$ for the class of stationary random distributions. A $\mathbb{C}$-valued, mean-continuous, stochastic process $X = (X(t) : t \in \mathbb{R})$ with zero expectation and finite variance is simply called a *process*. For $X \in \mathcal{S}$ we denote by μ_X the *spectral measure* of X: $(X(\phi), X(\psi)) = \int_{-\infty}^{\infty} \hat{\phi}(\xi)\overline{\hat{\psi}(\xi)}d\mu_X(\xi)$, where $\hat{\phi}$ is the Fourier transform of ϕ: $\hat{\phi}(\xi) = \int_{-\infty}^{\infty} e^{-it\xi}\phi(t)dt$. Let $\mathcal{S}_k$ be the class of $X \in \mathcal{S}$ such that $\int_{-\infty}^{\infty}(1 + \xi^2)^{-k}d\mu_X(\xi) < \infty$. Then we have $\mathcal{S} = \bigcup_{k=0}^{\infty} \mathcal{S}_k$. The class $\mathcal{S}_0$ coincides with the class of stationary processes. Any $X \in \mathcal{S}$ has the following *spectral representation*: $X(\phi) = \int_{-\infty}^{\infty} \hat{\phi}(\xi)dZ_X(\xi)$, where Z_X is the associated random measure. We write DX for the derivative of a random distribution X: $DX(\phi) = -X(\dot{\phi})$. We refer to Ito [It] and Yaglom [Y] for details.

Let X and Y be random distributions. Then X is said to be *stationarily correlated* with Y if $(X(\tau_h\phi), Y(\tau_h\psi)) = (X(\phi), Y(\psi))$ for all $\phi, \psi \in \mathcal{D}(\mathbb{R})$ and $h \in \mathbb{R}$; this is equivalent to $(X(t+s), Y(s)) = (X(t), Y(0))$ for all $t, s \in \mathbb{R}$ if X and Y are both processes. We denote by $M(Y)$ the closed linear hull of $\{Y(\phi) : \phi \in \mathcal{D}(\mathbb{R})\}$ in H. Then we have $M(Y) = \{\int_{-\infty}^{\infty} g(\xi)dZ_Y(\xi) : g \in L^2(\mu_Y)\}$. We define a random distribution $P_Y X$ by $P_Y X(\phi) = p_Y(X(\phi))$, where p_Y is the orthogonal projection of H to $M(Y)$. Clearly $P_Y X$ is equivalent to the process $(p_Y(X(t)) : t \in \mathbb{R})$ if X is a process. Note that the operator P_Y commutes with D: $P_Y DX = DP_Y X$.

Our projection method is based on the following theorem.

Theorem 2.1 *Let $Y \in \mathcal{S}$, and let $X \in \mathcal{S}$ such that $X \in \mathcal{S}_k$, $k \geq 0$. Assume that X is stationarily correlated with Y. Then there exists $g \in L^2((1+\xi^2)^{-k}\mu_Y)$ such that $P_Y X(\phi) = \int_{-\infty}^{\infty} \hat{\phi}(\xi) g(\xi) dZ_Y(\xi)$ for $\phi \in \mathcal{D}(\mathbb{R})$. In particular, $P_Y X \in \mathcal{S}$.*

Proof. For simplicity we abbreviate $P_Y X$ as X'. First we assume $X \in \mathcal{S}_0$ and $Y \in \mathcal{S}_0$. Then there exists $g \in L^2(\mu_Y)$ such that $X'(0) = \int_{-\infty}^{\infty} g(\xi) dZ_Y(\xi)$. Let $(U_t : t \in \mathbb{R})$ be the one-parameter group of unitary operators on $M(Y)$ generated by $U_t Y(s) = Y(t+s)$. Then for $t, s \in \mathbb{R}$,

$$\begin{aligned} (U_t X'(0), Y(s)) &= (X'(0), U_{-t} Y(s)) \\ &= (X'(0), Y(s-t)) = (X'(t), Y(s)), \end{aligned}$$

so that $U_t X'(0) = X'(t)$. Therefore $X'(t) = \int_{-\infty}^{\infty} e^{-it\xi} g(\xi) dZ_Y(\xi)$, as desired.

Next we prove the theorem for general X and Y. For $M > 0$, we write $e_M(t) = \exp(t)$ for $t \in [-M, 0]$, and $= 0$ otherwise. Let e_M^n be the n-times convolution of e_M with itself. For $l \in \mathbb{N} \cup \{0\}$ such that $Y \in \mathcal{S}_l$, we write $Y_l(\phi) = \lim_{M\to\infty} Y(e_M^l * \phi)$. Then the limit exists, and is equal to $\int_{-\infty}^{\infty} \hat{\phi}(\xi)(1-i\xi)^{-l} dZ_Y(\xi)$; in particular, Y_l is a stationary process with spectral measure $(1+\xi^2)^{-l}\mu_Y$. Clearly $(1+D)^l Y_l = Y$, so that $M(Y_l) = M(Y)$. We also write $X_k(\phi) = \lim_{M\to\infty} X(e_M^k * \phi)$. Then similarly we have $X' = P_Y(1+D)^k X_k = (1+D)^k P_Y X_k$. It is easy to show that X_k is stationarily correlated with Y_l. Then by the case of $\mathcal{S}_0$ there exists $f \in L^2((1+\xi^2)^{-l}\mu_Y)$ such that $P_Y X_k(\phi) = \int_{-\infty}^{\infty} \hat{\phi}(\xi) f(\xi)(1-i\xi)^{-l} dZ_Y(\xi)$, whence by operating $(1+D)^k$ we obtain $X'(\phi) = \int_{-\infty}^{\infty} \hat{\phi}(\xi) f(\xi)(1-i\xi)^{k-l} dZ_Y(\xi)$; the theorem follows with $g(\xi) = f(\xi)(1-i\xi)^{k-l}$. □

Remark 2.2. The author learned Theorem 2.1, in the case of $\mathcal{S}_0$, from Hida, Maruyama and Nisio [HiMN].

Now we turn to the convolution which appears in (1.8). We set

$$M = \{\rho : \rho \text{ is a Borel measure on } (0,\infty) \text{ satisfying (1.3)}\}.$$

For $\rho \in M$, we write

$$K_\rho(t) := \chi_{(0,\infty)}(t) \int_0^\infty e^{-t\lambda} d\rho(\lambda) \qquad (t \in \mathbb{R}),$$

$$F_\rho(\zeta) := \int_0^\infty \frac{1}{\lambda - i\zeta} d\rho(\lambda) \qquad (\mathrm{Im}\,\zeta \geq 0).$$

Proposition 2.3 *Let $\rho \in M$ and let $X \in \mathcal{S}$. Then for $\phi \in \mathcal{D}(\mathbb{R})$,*

$$\underset{M\to\infty}{\text{l.i.m.}} \int_0^M K_\rho(s) DX(\tau_s\phi)ds = -\int_{-\infty}^\infty i\xi F_\rho(\xi)\hat{\phi}(\xi)dZ_X(\xi).$$

The integral on the left hand side is an H-valued Bochner integral; in fact we have

$$\left\| \int_0^M K_\rho(s) DX(\tau_s\phi)ds \right\| \leq \|\chi_{(0,M)}K_\rho\|_1 \cdot \|DX(\phi)\|.$$

The proof of Proposition 2.3 is almost the same as that of [I2, Proposition 5.1]. We only note that the integral on the right converges by the estimate

$$\begin{aligned} |\xi F_\rho(\xi)| &\leq \int_0^1 d\rho(\lambda) + |\xi| \int_1^\infty \frac{1}{\lambda} d\rho(\lambda) \\ &\leq (1+\xi^2)^{1/2} \int_0^\infty \frac{2}{1+\lambda} d\rho(\lambda). \end{aligned}$$

For $\rho \in M$ and $X \in \mathcal{S}$ we define $K_\rho * DX \in \mathcal{S}$ by

$$(K_\rho * DX)(\phi) = \underset{M\to\infty}{\text{l.i.m.}} \int_0^M K_\rho(s) DX(\tau_s\phi)ds \quad (\phi \in \mathcal{D}(\mathbb{R})).$$

Lemma 2.4 *If $\rho \in M$ and $X \in \mathcal{S}$, then X is stationarily correlated with $DX + K_\rho * DX$.*

Proof. By Proposition 2.3, we have $Y(\phi) = \int_{-\infty}^\infty \hat{\phi}(\xi)\{-i\xi - i\xi F_\rho(\xi)\}dZ_X(\xi)$; the lemma follows immediately. □

Lemma 2.5 *If $\rho \in M$ and $X \in \mathcal{S}$, then X is a solution of $DX + K_\rho * DX = 0$ if and only if $X = a$ with some $a \in H$.*

Proof. For $\phi \in \mathcal{D}(\mathbb{R})$,

$$\begin{aligned} \|(DX + K_\rho * DX)(\phi)\|^2 &= \int_{-\infty}^\infty |\hat{\phi}(\xi)\xi\{1 + F_\rho(\xi)\}|^2 d\mu_X(\xi) \\ &\geq \int_{-\infty}^\infty |\hat{\phi}(\xi)|^2 \xi^2 d\mu_X(\xi), \end{aligned}$$

where we used the estimate $\mathrm{Re}\{1 + F_\rho(\xi)\} \geq 1$. Thus $DX + K_\rho * DX = 0$

if and only if $\operatorname{supp}\mu_X = \{0\}$. Hence the lemma follows. □

3. First KMO-Langevin equations

By B we denote a one-dimensional Brownian motion $(B(t) : t \in \mathbb{R})$ such that $B(0) = 0$, which we simply call a *Brownian motion*. Then $DB \in \mathcal{S}$. For $t \in \mathbb{R}$ and $Y \in \mathcal{S}$ we denote by $M_t(Y)$ the closed linear hull of $\{Y(\phi) : \phi \in \mathcal{D}(\mathbb{R}),\ \operatorname{supp}\phi \subset (-\infty, t]\}$ in H. Let $X = (X(t) : t \in \mathbb{R})$ be a real stationary process: X is a real, mean-continuous, weakly stationary process with zero expectation. We write R for the correlation function of X: $R(t) := E[X(t)X(0)]$. Suppose that X is *purely non-deterministic*: $\bigcap_{t\in\mathbb{R}} M_t(X) = \{0\}$. Then X has a *spectral density* Δ of Hardy class: $R(t) = \int_{-\infty}^{\infty} e^{-it\xi}\Delta(\xi)d\xi$, $(1+\xi^2)^{-1}\log\Delta(\xi) \in L^1(\mathbb{R})$. We write h for the *outer function* of X:

$$h(\zeta) := \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{1+\xi\zeta}{\xi-\zeta}\cdot\frac{\log\Delta(\xi)}{1+\xi^2}d\xi\right\} \qquad (\operatorname{Im}\zeta > 0).$$

We write E for the *canonical representation kernel* of X: $E := \hat{h}$, where $\hat{h}$ is the Fourier transform of $h(\cdot) := \text{l.i.m.}_{\eta\downarrow 0} h(\cdot + i\eta) \in L^2(\mathbb{R})$. We have

$$h(\zeta) = \frac{1}{2\pi}\int_0^{\infty} e^{i\zeta t}E(t)dt \qquad (\operatorname{Im}\zeta > 0), \tag{3.1}$$

$$R(t) = \frac{1}{2\pi}\int_0^{\infty} E(|t|+s)E(s)ds \qquad (t \in \mathbb{R}). \tag{3.2}$$

If X is Gaussian, then there exists a unique B, called the *canonical Brownian motion* of X, such that, for $t \in \mathbb{R}$, $X(t) = (2\pi)^{-1/2}\int_{-\infty}^{t} E(t-s)dB(s)$. It satisfies $M_t(X) = M_t(DB)$ for any $t \in \mathbb{R}$. For details, see [O4, §2] and the reference cited there.

From [I2], we recall some facts which we need in this section. We remark that these facts originated in Okabe's work ([O1], [O3], [O4]). The point of [I2] is the use of an extension of [O7, Theorem A] in the arguments. Now consider the following condition

$$\int_0^{\infty}\int_0^{\infty}\frac{1}{\lambda+\lambda'}d\nu(\lambda)d\nu(\lambda') < \infty. \tag{3.3}$$

We set

$$\Sigma = \{\sigma : \ \sigma \text{ is a non-zero bounded Borel measure on } (0,\infty)\},$$

$$\Sigma^1 = \{\sigma \in \Sigma : \ m_{-1}(\sigma) = \infty\},$$
$$\Sigma^{10} = \{\sigma \in \Sigma : \ m_{-1}(\sigma) = \infty, \ m_1(\sigma) < \infty\},$$
$$N = \{\nu : \ \nu \text{ is a non-zero Borel measure on } (0, \infty) \text{ satisfying } (3.3)\},$$
$$N^1 = \{\nu \in N : \ m_{-1}(\nu) = \infty\},$$
$$N^{10} = \{\nu \in N : \ m_{-1}(\nu) = \infty, \ m_0(\nu) < \infty\}.$$

For $\nu \in N$, we define $\sigma \in \Sigma$ by

$$d\sigma(\lambda) = \frac{1}{2\pi}\left\{\int_0^\infty \frac{1}{\lambda + \lambda'} d\nu(\lambda')\right\} d\nu(\lambda). \tag{3.4}$$

By direct calculations, $m_{-1}(\sigma) = m_{-1}(\nu)^2/(4\pi)$ and $m_1(\sigma) = m_0(\nu)^2/(4\pi)$. Hence by [I2, Theorem 2.5] we obtain the following theorem.

Theorem 3.1 *For $\nu \in N^1$ let $T(\nu) \in \Sigma^1$ be the measure σ defined by (3.4). Then the map $\nu \mapsto T(\nu)$ becomes a bijection from N^1 onto Σ^1. Moreover we have $T(N^{10}) = \Sigma^{10}$.*

For $\sigma \in \Sigma$, we write

$$R_\sigma(t) := \int_0^\infty e^{-|t|\lambda} d\sigma(\lambda) \qquad (t \in \mathbb{R}).$$

Any real stationary process X such that $R = R_\sigma$, $\sigma \in \Sigma$, is purely non-deterministic. By [I2, Theorem 2.6], we obtain the following theorem.

Theorem 3.2 *For $\sigma \in \Sigma^1$, let X be a real stationary process such that $R = R_\sigma$. Then for $\nu = T^{-1}(\sigma) \in N^1$ we have $E = K_\nu$, $h = (2\pi)^{-1} F_\nu$.*

We write

$$M^1 = \{\mu \in M : \ m_{-1}(\mu) = \infty\}.$$

We consider the following relation between ν and (α, ρ):

$$F_\nu(\zeta)\{-i\zeta - i\zeta F_\rho(\zeta)\} = (2\pi)^{1/2}\alpha \quad (\operatorname{Im}\zeta > 0). \tag{3.5}$$

Theorem 3.3 ([I2]). *For any $\nu \in \Sigma^1$, there exists a unique pair $(\alpha, \rho) \in (0, \infty) \times M^1$ satisfying* (3.5). *If we write $L(\nu)$ for the pair (α, ρ), then the map $\nu \mapsto L(\nu)$ becomes a bijection from Σ^1 onto $(0, \infty) \times M^1$.*

Let T and L be as above. Since $N^{10} = \{\nu \in \Sigma^1 : \nu \text{ satisfying } (3.3)\} \subset \Sigma^1$, the image $L(N^{10})$ forms a subset of $(0, \infty) \times M^1$. For $c > 0$, $\nu \in N^{10}$

and $(\alpha, \rho) = L(\nu)$, it follows from (3.5) that $L(c\nu) = (c\alpha, \rho)$. Hence we can define a subset C of M^1 by

$$L(N^{10}) = (0, \infty) \times C.$$

Clearly the restriction of L to N^{10} gives a bijection from N^{10} onto $(0, \infty) \times C$.

We put the following interpretation

$$DX = -K_\rho * DX + \alpha DB \tag{3.6}$$

on the first KMO-Langevin equation (1.8), where $\alpha > 0$, $\rho \in C$, and B is a Brownian motion. We are concerned with a solution $X \in \mathcal{S}$ of (3.6) with causality condition

$$M(X) \subset M(DB). \tag{3.7}$$

If $X \in \mathcal{S}$ is a solution of (3.6), then $M(DB) \subset M(X)$. Hence (3.6) plus (3.7) implies $M(X) = M(DB)$.

For $\nu \in N^{10}$ and Brownian motion B, we define a real stationary Gaussian process $X = (X(t) : t \in \mathbb{R})$ by

$$X(t) = (2\pi)^{-1/2} \int_{-\infty}^{t} K_\nu(t - s) dB(s) \qquad (t \in \mathbb{R}). \tag{3.8}$$

Since $K_\nu \in L^2(\mathbb{R})$, the integral on the right-hand side converges. For $\sigma = T(\nu) \in \Sigma^{10}$, we have $R = R_\sigma$, whence it follows from Theorem 3.2 that $E = K_\nu$ and $h = (2\pi)^{-1} F_\nu$. By the uniqueness, B coincides with the canonical Brownian motion of X. By using the spectral representation $DB(\phi) = \int_{-\infty}^{\infty} \hat{\phi}(\xi) dZ_{DB}(\xi)$, we have $X(\phi) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \hat{\phi}(\xi) F_\nu(\xi) dZ_{DB}(\xi)$.

Now we are ready to show the unique existence of a solution for (3.6) with (3.7).

Theorem 3.4 *For $\alpha > 0$, $\rho \in C$ and Brownian motion B, there exists a unique solution $X \in \mathcal{S}$ of equation* (3.6) *with* (3.7). *In fact X is the real stationary Gaussian process given by* (3.8) *with $\nu = L^{-1}(\alpha, \rho) \in N^{10}$.*

Proof. Suppose that $X \in \mathcal{S}$ is a solution of (3.6) with (3.7). Then $P_{DB} X = X$ and, by Lemma 2.4, X is stationarily correlated with DB. Thus, by Theorem 2.1, $X(\phi) = \int_{-\infty}^{\infty} \hat{\phi}(\xi) g(\xi) dZ_{DB}(\xi)$ with some $g \in L^2((1+\xi^2)^{-k} d\xi)$, $k \in \mathbb{N} \cup \{0\}$. By Proposition 2.3,

$$0 = \|DX(\phi) + (K_\rho * DX)(\phi) - \alpha DB(\phi)\|^2$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{\phi}(\xi)|^2 \cdot |\{-i\xi - i\xi F_\rho(\xi)\}g(\xi) - \alpha|^2 d\xi,$$

so that $g(\xi) = \alpha\{-i\xi - i\xi F_\rho(\xi)\}^{-1} = (2\pi)^{-1/2}F_\nu(\xi)$ with $\nu = L^{-1}(\alpha, \rho)$. Hence X is equal to the stationary Gaussian process given by (3.8). □

Remark 3.5. Let $\alpha > 0$ and $\rho \in M$. If $m_{-1}(\rho) < \infty$, then there is no solution $X \in \mathcal{S}$ of (3.6) with (3.7). If $\rho \in M^1 \setminus C$, then there is a unique solution $X \in \mathcal{S}$ of (3.6) with (3.7) but it is not a process. Therefore there exists a stationary process X which satisfies (3.6) and (3.7) if and only if $\rho \in C$. Since these facts are not used in this paper, we omit the proof.

The definition of C is rather indirect. So we give the following simple criteria for $\rho \in M$ to be in C.

Theorem 3.6 *Let $l \in \mathcal{R}_0$, $\rho \in M$ and $\gamma = K_\rho$.*
(1) *If $\int^\infty l(s)ds/s < \infty$ and $\gamma \in \Pi_l$ with index -1, then $\rho \in C$.*
(2) *If $\gamma(t) \sim t^{-q}l(t)$ as $t \to \infty$ for $q \in (0, 1/2)$, then $\rho \in C$.*
(3) *Suppose that $\gamma(t) \sim t^{-1/2}l(t)$ as $t \to \infty$. Then $\rho \in C$ holds if and only if $\int^\infty l(s)^{-2}ds/s < \infty$.*
(4) *If $\gamma(t) \sim t^{-q}l(t)$ as $t \to \infty$ for $q \in (1/2, \infty)$, then $\rho \notin C$.*

The proof of Theorem 3.6 will be given in section 6.

Example 3.7. For $0 < q < 1$ and $\rho = \lambda^{q-1}d\lambda/\Gamma(q) \in M$, we have $K_\rho(t) = t^{-q}$ for $t > 0$. Hence, by Theorem 3.6, $\rho \in C$ if and only if $0 < q < 1/2$.

The next theorem follows easily from Theorem 3.4.

Theorem 3.8 *Let X be the solution of Theorem 3.4. We define $\nu \in N^{10}$ and $\sigma \in \Sigma^{10}$ by $\nu = L^{-1}(\alpha, \rho)$ and $\sigma = T(\nu)$, respectively. Then*
(1) *B is the canonical Brownian motion of X;*
(2) *$R = R_\sigma$, $h = \dfrac{1}{2\pi}F_\nu$, $E = K_\nu$; in particular $\int_0^\infty R(t)dt = \int_0^\infty E(t)dt = \infty$;*
(3) $h(iy) = \dfrac{\alpha}{(2\pi)^{1/2}} \cdot \dfrac{1}{y + y\int_0^\infty e^{-yt}K_\rho(t)dt}$ *for $y > 0$.*

If we are given X first, then we have the following theorem.

Theorem 3.9 *For $\sigma \in \Sigma^{10}$, let X be a stationary Gaussian process such that $R = R_\sigma$. Then for $(\alpha, \rho) = L(T^{-1}(\sigma)) \in (0, \infty) \times C$ and the canonical Brownian motion B of X, X is a solution of* (3.6) *with* (3.7).

Proof. By Theorem 3.2, X is in the form (3.8) with $\nu = T^{-1}(\sigma)$. Therefore the theorem follows immediately from Theorem 3.4. $\square$

If we omit the condition (3.7), then the uniqueness of a solution does not hold. More precisely we have the following theorem.

Theorem 3.10 *Let X be the solution of Theorem* 3.4. *Then $Y \in \mathcal{S}$ is a solution of* (3.6) *if and only if $Y = X + a$, where a is an arbitrary element of H perpendicular to $M(DB)$.*

Proof. Suppose that $Y \in \mathcal{S}$ is a solution of (3.6). We set $Y_1 = P_{DB}Y$. Then, by Lemma 2.4 and Theorem 2.1, we have $Y_1 \in \mathcal{S}$. Now $P_{DB}(K_\rho * DY) = K_\rho * DY_1$ because

$$\begin{aligned} &\{P_{DB}(K_\rho * DY)\}(\phi) \\ &\quad = p_{DB}\left(\operatorname*{l.i.m.}_{M\to\infty} \int_0^M K_\rho(s) DY(\tau_s\phi) ds \right) \\ &\quad = \operatorname*{l.i.m.}_{M\to\infty} \int_0^M K_\rho(s) DY_1(\tau_s\phi) ds = (K_\rho * DY_1)(\phi). \end{aligned}$$

Therefore Y_1 is also a solution of (3.6), and so by Theorem 3.4 is equal to X. We set $Y_2 = Y - Y_1$. Then it is easy to show that $Y_2 \in \mathcal{S}$. Since Y and Y_1 are both solutions of (3.6), Y_2 satisfies $DY_2 + K_\rho * DY_2 = 0$. Therefore, by Lemma 2.5, $Y_2 = a$ with $a \perp M(DB)$. $\square$

In the theorem above, $X + a$ is purely non-deterministic if and only if $a = 0$. Hence we have the following corollary.

Corollary 3.11 *A purely non-deterministic solution $X \in \mathcal{S}$ of* (3.6) *is unique, and is equal to the solution of* (3.6) *with* (3.7).

Now we turn to the original first KMO-Langevin equation (1.1). In [O4, Theorem 4.2], Okabe showed the unique existence of a solution for (1.1) in the class of reflection positive, stationary Gaussian processes such that $m_{-1}(\sigma) < \infty$, $m_1(\sigma) < \infty$. If we put the following interpretation

$$DX = -\beta X - K_\rho * DX + \alpha DB \tag{3.9}$$

on equation (1.1), then we are naturally led to the question of showing the uniqueness of a solution in a larger class — the class $\mathcal{S}$. Here is a refinement of [O4, Theorem 4.2].

Theorem 3.12 *For $\alpha > 0$, $\beta > 0$, $\rho \in M$ and Brownian motion B, (3.9) has a unique solution $X \in \mathcal{S}$.*

The solution of Theorem 3.12 is equal to the solution of [O4, Theorem 4.2], whence it is a reflection positive, stationary Gaussian process such that $m_{-1}(\sigma) < \infty$, $m_1(\sigma) < \infty$. We remark that, in contrast with (3.6), the uniqueness of a solution for equation (3.9) follows without the causality condition (3.7). The proof of Theorem 3.12 is almost the same as that of Theorem 3.10. We only note that the following lemma plays the same role as Lemma 2.5.

Lemma 3.13 *If $\beta > 0$, $\rho \in M$ and $X \in \mathcal{S}$, then X is a solution of $DX + \beta X + K_\rho * DX = 0$ if and only if $X = 0$.*

4. Characterization by outer functions

We start the proof of Theorem 1.1. The goal of this section is the following theorem, which characterizes correlation functions satisfying (1.7) with $0 < p < 1$ in terms of outer functions.

Theorem 4.1 *Let $0 < p < 1$, $l \in \mathcal{R}_0$, X be a real, purely nondeterministic, stationary process. Suppose that the canonical representation kernel E of X is non-increasing on $(0, \infty)$. Then the following are equivalent:*

$$R(t) \sim t^{-p} l(t) \quad (t \to \infty), \tag{4.1}$$

$$h(iy) \sim \{y^{-(1-p)} l(1/y) \Gamma(1-p) \pi^{-1} \sin(\pi p/2)\}^{1/2} \quad (y \to 0+), \tag{4.2}$$

$$E(t) \sim \{t^{-(1+p)} l(t) 2\pi B((1-p)/2, p)^{-1}\}^{-1/2} \quad (t \to \infty). \tag{4.3}$$

We begin by recalling the theorem of Pitman [P], and Soni and Soni [SS]. In our context, this theorem is stated as follows. .

Theorem 4.2 ([P], [SS]). *Let $0 < p < 1$, and $l \in \mathcal{R}_0$. Let X be a real stationary process. Assume that R is non-increasing on $(0, \infty)$, $\lim_{t\to\infty} R(t) = 0$. Then (4.1) is equivalent to*

$$\Delta(\xi) \sim \xi^{-(1-p)} l(1/\xi) \Gamma(1-p) \pi^{-1} \sin(\pi p/2) \quad (\xi \to 0+). \tag{4.4}$$

See also [BGT, Theorem 4.10.3]. Note that, in Theorem 4.2, we do not need to a priori assume the existence of a spectral density Δ. In fact, we have

$$\Delta(\xi) = \frac{1}{\pi} \int_0^{\infty -} R(t) \cos \xi t dt \qquad (\xi \in \mathbb{R} \setminus \{0\}),$$

where $\int_0^{\infty-}$ denotes an improper integral $\lim_{M\to\infty} \int_0^M$. See [I3, Proposition 5.1].

The next proposition will be used in the proof of the implication (4.3) $\Rightarrow$ (4.1).

Proposition 4.3 *Let $q < 1 < p + q$, and $l_1, l_2 \in \mathcal{R}_0$. Let g be locally integrable on $[0, \infty)$, and let f be measurable on $(0, \infty)$. Suppose $f(t) \sim t^{-p} l_1(t)$ as $t \to \infty$, and $g(t) \sim t^{-q} l_2(t)$ as $t \to \infty$. Then $f(t + \cdot)g(\cdot)$ is integrable on $(0, \infty)$ for sufficiently large t, and $U(t) := \int_0^\infty f(t + s)g(s)ds$ satisfies*

$$U(t) \sim t^{-(p+q-1)} l_1(t) l_2(t) B(p + q - 1, 1 - q) \qquad (t \to \infty).$$

Proof. If $f(t) \sim t^{-p} l_1(t)$ and $g(t) \sim t^{-q} l_2(t)$ as $t \to \infty$, then $\int_X^\infty |f(t + s)g(s)|ds < \infty$ for sufficiently large X. Since $p > 0$, we have $\sup_{0<s<\infty} |f(t+s)| < \infty$ for sufficiently large t, and so $\int_0^X |f(t + s)g(s)|ds < \infty$. Thus $U(t)$ exists for any t large enough. By [BGT, Corollary 1.4.2], we may choose M so large that, on $[M, \infty)$, both f and g are positive, and g is locally bounded. If we set $g_1(s) = 1$ on $(0, M)$, and $= g(s)$ on $[M, \infty)$, then

$$U(t) = \int_0^M f(t + s)\{g(s) - 1\}ds + \int_0^\infty f(t + s)g_1(s)ds.$$

If t is large enough, then by the Uniform Convergence Theorem (e.g., [BGT, Theorem 1.5.2]) $|f(t+s)/f(t)| \le 2$ for $s \in [0, M]$. Moreover, $\lim_{t\to\infty} tg(t) = \infty$ since $1 - q > 0$. Thus

$$\lim_{t\to\infty} \frac{1}{tf(t)g(t)} \int_0^M f(t + s)\{g(s) - 1\}ds = 0.$$

Therefore in order to prove the proposition, we may assume that, in $[0, \infty)$, f and g are both positive, and g is locally bounded; this reduction enables us to apply [BGT, Theorem 1.5.2] in the next step.

Choose $\delta(i)$ $(i = 1, 2, 3)$ so that $\max(0, q) < \delta(1) < 1$, $\delta(2) < p$, $\delta(3) < q$, and $1 < \delta(2) + \delta(3)$. Set $F_1(u) = u^p f(u)$, $G_1(u) = u^{\delta(1)} g(u)$, $F_2(u) = u^{\delta(2)} f(u)$, and $G_2(u) = u^{\delta(3)} g(u)$. Then G_1 is locally bounded on $[0, \infty)$.

Now $U(t)/\{tf(t)g(t)\} = C(t) + D(t)$, where

$$C(t) = \int_0^1 \frac{F_1(t(u+1))}{F_1(t)} \cdot \frac{G_1(tu)}{G_1(t)} \cdot \frac{1}{(u+1)^p u^{\delta(1)}} du,$$
$$D(t) = \int_1^\infty \frac{F_2(t(u+1))}{F_2(t)} \cdot \frac{G_2(tu)}{G_2(t)} \cdot \frac{1}{(u+1)^{\delta(2)} u^{\delta(3)}} du.$$

In $C(t)$, by [BGT, Theorem 1.5.2], $F_1(t(u+1))/F_1(t)$ converges to 1 as $t \to \infty$, and $G_1(tu)/G_1(t)$ to $u^{\delta(1)-q}$, both uniformly in $u \in (0,1)$. Therefore $C(t)$ converges to $\int_0^1 (u+1)^{-p} u^{-q} du$ as $t \to \infty$. In the same way, by [BGT, Theorem 1.5.2], $D(t)$ converges to $\int_1^\infty (u+1)^{-p} u^{-q} du$ as $t \to \infty$. Thus we obtain

$$\lim_{t\to\infty} \frac{U(t)}{tf(t)g(t)} = \int_0^\infty \frac{1}{(u+1)^p u^q} du = B(p+q-1, 1-q),$$

hence the theorem follows. □

The next theorem is a key to prove Theorem 4.1.

Theorem 4.4 *Let $l \in \mathcal{R}_0$ and $p \in \mathbb{R}$. Let f be a positive, even and measurable function on $\mathbb{R}$ such that $\{\log f(\xi)\}/(1+\xi^2)$ is integrable on $\mathbb{R}$. Suppose $f(\xi) \sim \xi^p l(1/\xi)$ as $\xi \to 0+$. Then*

$$\exp\left\{\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{y^2+\xi^2} \log f(\xi) d\xi\right\} \sim y^p l(1/y) \qquad (y \to 0+).$$

Proof. If we put

$$L(x) = x^p f(1/x) \qquad (x > 0),$$

then L is slowly varying and $\{\log L(1/|\xi|)\}/(1+\xi^2)$ is integrable on $\mathbb{R}$. The Representation Theorem (e.g., [BGT, Theorem 1.3.1]) yields

$$\log L(x) = \eta(x) + \int_a^x \epsilon(u) du/u \qquad (x \geq a) \tag{4.5}$$

for some $a > 0$, where $\eta(x)$, $\epsilon(x)$ are bounded and measurable on $[a, \infty)$, $\eta(x) \to c \in \mathbb{R}$, $\epsilon(x) \to 0$ as $x \to \infty$. On taking $\eta(x) = \log L(x)$, $\epsilon(x) \equiv 0$ on $(0, a)$, (4.5) holds for $x > 0$; ϵ is bounded on $(0, \infty)$ again, but η may not be so. As for η, we have the estimate

$$\int_{-\infty}^{\infty} \frac{|\eta(1/|\xi|)|}{1+\xi^2} d\xi \leq 2 \sup_{x \geq a} |\eta(x)| \cdot \int_0^{1/a} \frac{1}{1+\xi^2} d\xi$$

$$+ 2\int_{1/a}^{\infty} \frac{|\log L(1/\xi)|}{1+\xi^2} d\xi < \infty.$$

For $y > 0$, making use of the change of variables $t = \xi/y$, $s = yu$ as well as the identity

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{y^2+\xi^2} \log|\xi| d\xi = \log y,$$

we are led to

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{y^2+\xi^2} \log f(\xi) d\xi - \log f(y) = \mathrm{I}_1(y) + \mathrm{I}_2(y), \tag{4.6}$$

where

$$\begin{aligned} \mathrm{I}_1(y) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{y^2+\xi^2} \eta(1/|\xi|) d\xi - \eta(1/y), \\ \mathrm{I}_2(y) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+t^2} \left\{ \int_1^{1/|t|} \epsilon(s/y) ds/s \right\} dt. \end{aligned}$$

We claim that $\mathrm{I}_1(y)$ and $\mathrm{I}_2(y)$ both converge to 0 as $y \to 0+$. In fact, since $\eta(1/|\xi|)/(1+\xi^2)$ is integrable on $\mathbb{R}$ and $\eta(1/|\xi|) \to c$ as $\xi \to 0$, the well known property of Poisson integral yields $\mathrm{I}_1(y) \to 0$ as $y \to 0+$. As for I_2, the dominated convergence theorem shows $\mathrm{I}_2(y) \to 0$ as $y \to 0+$ because we have

$$\left| (1+t^2)^{-1} \int_1^{1/|t|} \epsilon(s/y) ds/s \right| \le \sup_{x>0} |\epsilon(x)| \cdot \frac{|\log|t||}{1+t^2}.$$

Thus the left-hand side of (4.6) converges to 0 as $y \to 0+$, whence

$$\exp\left\{ \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{y^2+\xi^2} \log f(\xi) d\xi \right\} \sim f(y) \sim y^p l(1/y) \quad (y \to 0+),$$

and the theorem follows. □

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Since E is non-increasing and square integrable on $(0, \infty)$, $E(t)$ is non-negative and tends to zero as $t \to \infty$. So by (3.2) and the monotone convergence theorem, $R(t)$ is also non-increasing and tends to zero as $t \to \infty$. Since Δ is even, by a simple calculation we have

$$h(iy) = \exp\left\{ \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{y}{y^2+\xi^2} \log \Delta(\xi) d\xi \right\} \qquad (y > 0).$$

Therefore, by Theorems 4.2 and 4.4, (4.1) implies (4.2). By (3.1),

$$h(iy) = \frac{1}{2\pi}\int_0^\infty e^{-yt}E(t)dt \qquad (y > 0),$$

so that by Karamata's Tauberian Theorem (e.g., [BGT, Theorem 1.7.6]) (4.2) implies (4.3). Finally that (4.3) implies (4.1) follows immediately from Proposition 4.3. □

5. Boundary cases

In this section, we consider the boundary cases which correspond to $p = 0, 1$ in Theorem 4.1. These are delicate cases in which slowly varying parts become important. The goal of this section is the following two theorems.

Theorem 5.1 *Let $l \in \mathcal{R}_0$ such that $\int^\infty l(s)ds/s < \infty$. Let $\sigma \in \Sigma^1$, and let X be a real stationary process such that $R = R_\sigma$. Then the following are equivalent:*

(1) $R \in \Pi_l$ *with index* -1,
(2) $\Delta(\xi) \sim \xi^{-1}l(1/\xi)2^{-1}$ *as* $\xi \to 0+$,
(3) $h(iy) \sim \{y^{-1}l(1/y)2^{-1}\}^{1/2}$ *as* $y \to 0+$,
(4) $E(t) \sim \{t^{-1}l(t)2\pi\}^{1/2}$ *as* $t \to \infty$.

Theorem 5.2 *Let X be as in Theorem* 5.1. *Let $l \in \mathcal{R}_0$ such that $\int^\infty l(s)ds/s = \infty$. Then the following are equivalent*:

(1) $R(t) \sim t^{-1}l(t)$ *as* $t \to \infty$,
(2) $\Delta(1/\cdot) \in \Pi_l$ *with index* π^{-1},
(3) $h(i/\cdot) \in \Pi_{l_1}$ *with index* 1, *where* $l_1(t) = l(t)\tilde{l}(t)^{-1/2}2^{-1}\pi^{-1/2}$,
(4) $E(t) \sim t^{-1}l(t)\tilde{l}(t)^{-1/2}\pi^{1/2}$ *as* $t \to \infty$.

We may regard Theorem 5.1 (resp. Theorem 5.2) as corresponding to the boundary case $p = 0$ (resp. $p = 1$) of Theorem 4.1 with Theorem 4.2. We remark that the assertion (1) $\Leftrightarrow$ (2) of Theorem 5.2 holds more generally (see [I3]).

Let σ, R, Δ, h and E be as in Theorem 5.1. We set $g(y) = h(iy)$ for $y > 0$, and define $\nu \in N^1$ by $\nu = S^{-1}(\sigma)$. Then we have the following integral representations: $R = R_\sigma$, $E = K_\nu$, $g(y) = (2\pi)^{-1}F_\nu(iy)$ for $y > 0$, and

$$\Delta(\xi) = \frac{1}{\pi}\int_0^\infty \frac{\lambda}{\lambda^2+\xi^2}d\sigma(\lambda) \qquad (\xi \in \mathbb{R}).$$

See §3 as well as [I2]. By these representations, R, Δ, g and E are all of $C^\infty(0,\infty)$ class. In particular,

$$\begin{aligned}\Delta(1/x) &= \Delta(1) + \int_1^x \{-\dot{\Delta}(1/u)u^{-2}\}du \qquad (x \geq 1),\\ g(1/x) &= g(1) + \int_1^x \{-\dot{g}(1/u)u^{-2}\}du \qquad (x \geq 1).\end{aligned}$$

Proof of Theorem 5.1. Suppose (1) holds. Then by the monotone density theorem of de Haan [H] (see also [BGT, Theorem 3.6.8]) $-\dot{R}(t) \sim t^{-1}l(t)$ as $t \to \infty$. By the representation $R = R_\sigma$, we have $-\dot{R}(t) \downarrow 0$ as $t \to \infty$. Moreover $\dot{R}$ is locally integrable on $[0,\infty)$ because for $M > 0$,

$$\begin{aligned}\int_0^M (-\dot{R}(t))dt &= \lim_{\epsilon \to 0+} \int_\epsilon^M (-\dot{R}(t))dt\\ &= \lim_{\epsilon \to 0+} (R(\epsilon) - R(M)) = R(0) - R(M).\end{aligned}$$

Hence integration by parts yields

$$\Delta(\xi) = (\pi\xi)^{-1} \int_0^{\infty-} \{-\dot{R}(t)\} \sin t\xi dt \qquad (\xi > 0).$$

So by [P, Theorem 1] we obtain (2). By Theorem 4.4, (2) implies (3), while, by Karamata's Tauberian Theorem, (3) implies (4). Finally suppose (4) holds. By the representation $E = K_\nu$, we can justify the equality

$$-\dot{R}(t) = (2\pi)^{-1} \int_0^\infty \{-\dot{E}(t+s)\}E(s)ds \qquad (t > 0).$$

For let $t \in (a,\infty)$, $a > 0$ and $s > 0$. Then

$$-\dot{E}(t+s) = \int_0^\infty \lambda e^{-(t+s)\lambda} d\nu(\lambda) \leq \sup_{0<\lambda<\infty} (\lambda e^{-a\lambda}) \cdot E(s),$$

yielding the equality above. By the Monotone Density Theorem, we have $-\dot{E}(t) \sim \{t^{-3}l(t)2^{-1}\pi\}^{1/2}$ as $t \to \infty$. Therefore by Proposition 4.3, $-\dot{R}(t) \sim t^{-1}l(t)$ as $t \to \infty$, hence (1). This completes the proof. □

Proof of Theorem 5.2. Suppose (1) holds. Then by [P, Theorem 7 (iii)],

$$\Delta(1/x) - \frac{1}{\pi}\int_0^x R(t)dt \sim -\frac{\gamma}{\pi} l(x) \qquad (x \to \infty),$$

where γ is Euler's constant. Since the function $\int_0^t R(s)ds$ in t is in Π_l with index 1, we have for any $\lambda \geq 1$,

$$\begin{aligned}&\frac{\Delta(1/\lambda x)-\Delta(1/x)}{l(x)}\\&\quad=\frac{\Delta(1/\lambda x)-\pi^{-1}\int_0^{\lambda x}R(t)dt}{l(x)}+\frac{\int_0^{\lambda x}R(t)dt-\int_0^{x}R(t)dt}{\pi l(x)}\\&\quad-\frac{\Delta(1/x)-\pi^{-1}\int_0^{x}R(t)dt}{l(x)}\to\pi^{-1}\log\lambda\qquad(x\to\infty),\end{aligned}$$

hence (2).

Next suppose (2) holds. By the representation

$$-\dot{\Delta}(\xi)=\frac{2\xi}{\pi}\int_0^\infty\frac{\lambda}{(\lambda^2+\xi^2)^2}d\sigma(\lambda)\qquad(\xi>0),\tag{5.1}$$

$\log\{-\dot{\Delta}(1/x)x^{-2}\}$ is slowly decreasing on $(0,\infty)$. Therefore by [BGT, Theorem 3.6.10] $-\dot{\Delta}(\xi)\sim\xi^{-1}l(1/\xi)\pi^{-1}$ as $\xi\to 0+$, and so $\Delta(\xi)\sim\tilde{l}(1/\xi)\pi^{-1}$ as $\xi\to 0+$. By (5.1) we have

$$|\dot{\Delta}(\xi)/\Delta(\xi)|\leq 2\xi^{-1}\qquad(\xi>0).\tag{5.2}$$

Hence if we set

$$A(y)=\frac{1}{2\pi}\int_{-\infty}^\infty\frac{y}{y^2+\xi^2}\log\Delta(\xi)d\xi\qquad(y>0),$$

then $-\dot{g}(y)=B(y)\exp A(y)$ for $y>0$, where

$$B(y)=\frac{1}{\pi}\int_0^\infty\frac{u}{1+u^2}\cdot\frac{\{-\dot{\Delta}(yu)\}}{\Delta(yu)}du\qquad(y>0).$$

By Theorem 4.4, $\exp A(y)\sim\tilde{l}(1/y)^{1/2}\pi^{-1/2}$ as $y\to 0+$. Now we set

$$f(x)=-\dot{\Delta}(1/x)/\{x\Delta(1/x)\}\qquad(x>0).$$

Then $f(x)\sim l(x)\tilde{l}(x)^{-1}$ as $x\to\infty$. Moreover, by (5.2), $x^{1/2}f(x)$ is locally bounded in $[0,\infty)$. Therefore by the Uniform Convergence Theorem,

$$\begin{aligned}\frac{\pi yB(y)}{f(1/y)}&=\int_0^1\frac{1}{(1+\xi^2)\xi^{1/2}}\cdot\frac{(\xi/y)^{1/2}f(\xi/y)}{(1/y)^{1/2}f(1/y)}d\xi\\&\quad+\int_1^\infty\frac{\xi^{1/2}}{1+\xi^2}\cdot\frac{(\xi/y)^{-1/2}f(\xi/y)}{(1/y)^{-1/2}f(1/y)}d\xi\\&\quad\to\int_0^\infty\frac{1}{1+\xi^2}d\xi=\frac{\pi}{2}\qquad(y\to 0+),\end{aligned}$$

whence

$$-\dot{g}(1/x)x^{-2} \sim x^{-1}l(x)\widetilde{l}(x)^{-1/2}2^{-1}\pi^{-1/2} \qquad (x \to \infty). \tag{5.3}$$

Now by the representation

$$-\dot{g}(y) = (2\pi)^{-1}\int_0^\infty \frac{1}{(y+\lambda)^2}d\nu(\lambda) \qquad (y > 0),$$

$\log\{-\dot{g}(1/x)x^{-2}\}$ is slowly decreasing on $(0, \infty)$. Therefore by [BGT, Theorem 3.6.10] we get (3).

Conversely, if (3) holds, then (5.3) holds. Therefore

$$\int_0^\infty e^{-yt}tE(t)dt \sim y^{-1}l(1/y)\widetilde{l}(1/y)^{-1/2}\pi^{1/2} \qquad (y \to 0+).$$

Since $\log tE(t)$ is slowly increasing, we obtain (4) by Karamata's Tauberian Theorem.

Finally suppose (4) holds. Then $\int_0^t E(u)du \sim \widetilde{l}(t)^{1/2}2\pi^{1/2}$ as $t \to \infty$. In particular, $\lim_{s\to\infty} E(t+s)\int_0^s E(u)du = 0$ for any $t > 0$. Therefore by integration by parts we have

$$R(t) = \frac{1}{2\pi}\int_0^\infty \left\{\int_0^s E(u)du\right\}\left\{-\dot{E}(t+s)\right\}ds \qquad (t > 0).$$

By the Monotone Density Theorem, $-\dot{E}(t) \sim t^{-2}l(t)\widetilde{l}(t)^{-1/2}\pi^{1/2}$ as $t \to \infty$. Hence (1) by Proposition 4.3. □

Remark 5.3. In Theorem 4.1, we assumed only the monotonicity of E, not the reflection positivity of X. The question thus arises of extending Theorems 5.1 and 5.2 to this more general setting.

6. Proof of Theorem 1.1

We complete the proof of Theorem 1.1. We also prove Theorems 1.2 and 3.6 in this section.

If $f : (0, \infty) \to (0, \infty)$ is measurable, we define its *Laplace-Stieltjes transform*

$$\check{f}(y) := y\int_0^\infty e^{-ty}f(t)dt.$$

If f is non-decreasing, right-continuous, and $f(0+) = 0$, then we have $\check{f}(y) = \int_{[0,\infty)} e^{-ty}df(t)$. In addition to the original Abel-Tauber Theorem of de Haan [H] (see also [BGT, Theorem 3.9.1]), we need the following variant.

Theorem 6.1 (de Haan's Abel-Tauber Theorem; a variant). *Let $c > 0$ and $l \in \mathcal{R}_0$. Let f be a positive, non-increasing and right-continuous function on $(0, \infty)$. We assume that f is locally integrable on $[0, \infty)$. Then $f \in \Pi_l$ with index $-c$ if and only if $\check{f}(1/\cdot) \in \Pi_l$ with index $-c$.*

Though Theorem 6.1 is a special case of the results of Bingham and Teugels [BT], we prove it for the reader's convenience.

Proof of Theorem 6.1. We set $g(t) = 0$ on $(0,1)$ and $= f(1) - f(t)$ on $[1, \infty)$. Then $f \in \Pi_l$ with index $-c$ if and only if $g \in \Pi_l$ with index c. Since g is bounded, non-decreasing, right-continuous and $g(0+) = 0$, by de Haan's Abel-Tauber Theorem $g \in \Pi_l$ with index c if and only if $\check{g}(1/\cdot) \in \Pi_l$ with index c. By integration by parts we have

$$\check{f}(y) = e^{-y} f(1) + x \int_0^1 e^{-ty} f(t) dt - \check{g}(y) \qquad (y > 0).$$

Since $(e^{-1/x} - e^{-1/\lambda x})/l(x) \to 0$ as $x \to \infty$ for any $\lambda \geq 1$, $\check{g}(1/\cdot) \in \Pi_l$ with index c if and only if $\check{f}(1/\cdot) \in \Pi_l$ with index $-c$. Thus the theorem follows. $\square$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let h be the outer function of X. We set $g(y) = h(iy)$ for $y > 0$. Then by Theorem 3.8 (3) we have

$$g(y) = \alpha(2\pi)^{-1/2}\{y + \check{\gamma}(y)\}^{-1} \qquad (y > 0). \tag{6.1}$$

By Karamata's Tauberian Theorem, for any $q \in (0, 1/2]$ and $l \in \mathcal{R}_0$, $\gamma(t) \sim t^{-q} l(t)$ as $t \to \infty$ if and only if $g(y) \sim y^{-q} l(1/y)^{-1} \alpha(2\pi)^{-1/2} \Gamma(1 - q)^{-1}$ as $y \to 0+$. Therefore (1) (resp. (2)) follows immediately from Theorem 4.1 (resp. Theorem 5.1).

We turn to the assertion (3). Let $l \in \mathcal{R}_0$ such that $\int^\infty l(s) ds/s < \infty$. If we set $l_1(t) = l(t)\bar{l}(t)^{-3}\alpha^2$, then $l_1(t)\tilde{l_1}(t)^{-1/2}2^{-1}\pi^{-1/2} \sim l_2(t)$ as $t \to \infty$, where

$$l_2(t) = l(t)\bar{l}(t)^{-2}\alpha(2\pi)^{-1/2}.$$

Therefore, by Theorem 5.2, $R(t) \sim t^{-1} l(t)\bar{l}(t)^{-3}\alpha^2$ as $t \to \infty$ if and only if $g(1/\cdot) \in \Pi_{l_2}$ with index 1. Suppose $\gamma \in \Pi_l$ with -1. Then Theorem 6.1 yields $\check{\gamma}(1/\cdot) \in \Pi_l$ with index -1. Since $\gamma(t) = \int_t^\infty \{-\dot{\gamma}(s)\} ds$ for $t > 0$, the monotone density theorem of de Haan implies $\gamma(t) \sim \bar{l}(t)$ as $t \to \infty$, hence $\check{\gamma}(1/x) \sim \bar{l}(x)$ as $x \to \infty$. Therefore by (6.1) we have for any $\lambda \geq 1$,

$$\begin{aligned}
&\frac{g(1/\lambda x) - g(1/x)}{l_2(x)} \\
&= -\frac{\bar{l}(x)}{\{(1/\lambda x) + \check{\gamma}(1/\lambda x)\}} \cdot \frac{\bar{l}(x)}{\{(1/x) + \check{\gamma}(1/x)\}} \\
&\quad \times \frac{1}{l(x)} \left[\left(\lambda^{-1} - 1\right) x^{-1} + \{\check{\gamma}(1/\lambda x) - \check{\gamma}(1/x)\} \right] \\
&\to \log \lambda \qquad (x \to \infty),
\end{aligned}$$

hence $g(1/\cdot) \in \Pi_{l_2}$ with index 1. Conversely, suppose $g(1/\cdot) \in \Pi_{l_2}$ with index 1. Then as in the proof of Theorem 5.2, [BGT, Theorem 3.6.10] yields $g(1/x) \sim \widetilde{l_2}(x) \sim \bar{l}(x)^{-1}\alpha(2\pi)^{-1/2}$ as $x \to \infty$. Therefore by (6.1) we have for any $\lambda \geq 1$,

$$\begin{aligned}
&\frac{\check{\gamma}(1/\lambda x) - \check{\gamma}(1/x)}{l(x)} \\
&= -\frac{\alpha^2}{2\pi} \cdot \frac{1}{g(1/\lambda x) g(1/x) \bar{l}(x)^2} \cdot \frac{g(1/\lambda x) - g(1/x)}{l_2(x)} \\
&\quad - (\lambda^{-1} - 1)/\{x l(x)\} \to -\log \lambda \quad (x \to \infty),
\end{aligned}$$

hence, by Theorem 6.1, $\gamma \in \Pi_l$ with index -1. Thus (3) follows. □

Proof of Theorem 1.2. Let E be the canonical representation kernel of X. We set $F(t) = \int_0^t E(s)ds$ for $t \geq 0$. Then $\check{F}(y) = \int_0^\infty e^{-yt} E(t)dt$, and so by [O1, Theorem 2.2],

$$\check{F}(y) = \alpha(2\pi)^{1/2} \{\beta + y + \check{\gamma}(y)\}^{-1} \qquad (y > 0).$$

We set $c = \alpha\beta^{-2}(2\pi)^{1/2}$. Then in the same way as the proof of Theorem 1.1, $\gamma \in \Pi_l$ with index -1 if and only if $\check{F}(1/\cdot) \in \Pi_l$ with index c. By de Haan's Abel-Tauber Theorem, $\check{F}(1/\cdot) \in \Pi_l$ with index c if and only if $F \in \Pi_l$ with index c, which, by the monotone density theorem of de Haan, is equivalent to

$$E(t) \sim ct^{-1}l(t) \qquad (t \to \infty). \tag{6.2}$$

Since $\int_0^\infty E(t)dt = \alpha\beta^{-1}(2\pi)^{1/2}$, Lemma 3.8 of [I1] implies that (6.2) is equivalent to $R(t) \sim t^{-1}l(t)\alpha^2\beta^{-3}$ as $t \to \infty$. Thus the theorem follows. □

Proof of Theorem 3.6. If $\gamma(t) \sim t^{-q}l(t)$ as $t \to \infty$ for $q \in (1,\infty)$, then $m_{-1}(\rho) = \int_0^\infty \gamma(t)dt < \infty$, and so $\rho \notin M^1$, which yields (4) with $q \in (1,\infty)$. In the same way, if $\gamma(t) \sim t^{-1}l(t)$ as $t \to \infty$ for $l \in \mathcal{R}_0$ such that $\int^\infty l(s)ds/s < \infty$, then $\rho \notin M^1$. For any $\rho \in M^1$, we define $\nu \in \Sigma^1$ by $\nu = L^{-1}(1,\rho)$. For any $\mu \in \Sigma^1$, K_μ is bounded, hence we have the equality

$$N^{10} = \{\mu \in \Sigma^1 : K_\mu \in L^2[1,\infty)\}.$$

Therefore $\rho \in C$ if and only if $K_\nu \in L^2[1,\infty)$. By the definition of L,

$$\int_0^\infty e^{-yt}K_\nu(t)dt = (2\pi)^{1/2}\{y + \check{\gamma}(y)\}^{-1} \qquad (y > 0). \tag{6.3}$$

If $0 < q < 1$ and $\gamma(t) \sim t^{-q}l(t)$ as $t \to \infty$, then by Karamata's Tauberian Theorem $K_\nu(t) \sim ct^{-(1-q)}l(t)^{-1}$ as $t \to \infty$ for some $c > 0$, hence (4) with $q \in (1/2, 1)$, (2) and (3). We set $g(y) = (2\pi)^{-1}\int_0^\infty e^{-yt}K_\nu(t)dt$ for $y > 0$. If $\int^\infty l(s)ds/s < \infty$ and $\gamma \in \Pi_l$ with index -1, then, as in the proof of theorem 1.1, $g(1/\cdot) \in \Pi_{l_2}$ for some $l_2 \in \mathcal{R}_0$, so that $K_\nu(t) \sim ct^{-1}l_2(t)$ as $t \to \infty$ for some $c > 0$. Thus (1) follows. Finally suppose $\gamma(t) \sim t^{-1}l(t)$ as $t \to \infty$ for $l \in \mathcal{R}_0$ such that $\int^\infty l(s)ds/s = \infty$. If we set $f(t) = \int_0^t \gamma(s)ds$ for $t \geq 0$, then by (6.3)

$$\check{K}_\nu(y) = (2\pi)^{1/2}\{1 + \check{f}(y)\}^{-1} \qquad (y > 0).$$

In the same way as the proof of Theorem 1.1, we have $\check{K}_\nu(1/\cdot) \in \Pi_{l_1}$ with index -1, where $l_1(t) = l(t)\tilde{l}(t)^{-2}(2\pi)^{1/2}$, which by Theorem 6.1 is equivalent to $K_\nu \in \Pi_{l_1}$ with index -1. Therefore $K_\nu(t) \sim \tilde{l}(t)^{-1}(2\pi)^{1/2}$, hence (4). □

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