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CL-shellability of ordered structure of reflection systems

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Abstract. We define a "Bruhat" order on a reflection system, and show that each closed interval is CL-shellable. The reflection systems, introduced by M. Dyer, generalize the Coxeter systems. Therefore our result is a generalization of a result of A. Björner and M. Wachs for Coxeter groups.

Key words: Coxeter groups, Bruhat orders, CL-shellability, Cohen-Macaulay posets.

1. Introduction

The lexicographic shellability of posets was first introduced by Björner [Bj]. If a poset P is CL-shellable, then P is Cohen-Macaulay over an arbitrary field [BGS]. Cohen-Macaulay posets have been one of the main interests in combinatorics and have been studied deeply from combinatorial, algebraic, and topological points of view [S2] [Bac] [H] [M1] [M2]. However, directly from its algebraic or geometric definition, it is sometimes difficult to see whether a given poset is Cohen-Macaulay or not. On the other hand, the definition of CL-shellability is completely combinatorial. Consequently the CL-shellabitity makes it easier to see whether or not a given poset is Cohen-Macaulay than the original definitions. One of the most remarkable classes of CL-shellable ordered structures is the Bruhat order of Coxeter systems. CL-shellability of the Bruhat order was first proved for the symmetric groups by Edelman [Ed], and for the classical Weyl groups by Proctor [Pr]. Finally Björner and Wachs showed it for any Coxeter groups [BW1].

The purpose of the present paper is to extend the result of Björner-Wachs for Coxeter systems to reflection systems. The reflection systems were first introduced by Dyer [D] in his study of reflection subgroups of Coxeter groups. Reflection systems are a generalized notion of Coxeter systems. The most noticeable difference lies in the orders of generators: In the case of a Coxeter group, we can choose a system of generators consisting of involutions of order two. On the other hand orders of generators of a reflection system can be an arbitrary nonnegative integer (or even the

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infinity). However, we restrict our attention to the case that generators have finite orders.

This paper is organized as follows: In Section 2, we review the definition of CL-shellability of finite posets and observe the CL-shellability of the Bruhat orders of symmetric groups as an example of the result of Björner-Wachs [BW1]. In Section 3, we define the reflection systems by using Coxeter diagrams, and introduce some terminology which we use later. In Section 4, we define a partial order on a reflection system by formally extending the definition of the Bruhar order for a Coxeter system. We also see some fundamental properties of the partial order. In Section 5, we construct a labeling for each closed interval of a reflection system and show that it is CL-shellable.

Throughout this paper, the following notation is used: For $m \in \mathbf{Z}$, let $\mathbf{Z}_{\geq m}$ (resp. $\mathbf{Z}_{>m}$) denote the set $\{a \in \mathbf{Z} \mid a \geq m\}$ (resp. $\{a \in \mathbf{Z} \mid a > m\}$). For an element x of a group G, $\mathrm{ord}(x)$ denotes the order of x, and for elements $x, y \in G$, $x \cdot y$ denotes xyx^{-1}. The cardinality of a set A is denoted by $\sharp A$.

2. CL-shellability of posets

In this section we review necessary notation on partially ordered sets (posets for short). Consult [S1] (or also [Bir]) for the fundamental terminology. Let P be a finite poset. All posets considered in this paper are finite. A *chain* $\mathcal{C}$ of P is a totally ordered subset $x_0 < x_1 < \cdots < x_r$ of P and the number r is called the *length* of $\mathcal{C}$ and denoted by $\ell(\mathcal{C})$. The set of all the maximal chains of P is denoted by $\mathcal{M}(P)$. If a poset P satisfies the following two conditions, then P is called a *graded* poset of length r:

1. P has the minimum element $\hat{0}$ and the maximum element $\hat{1}$, i.e., $\hat{0} \leq x \leq \hat{1}$ for all $x \in P$,
2. All maximal chains of P have the same length r.

For elements x, $y \in P$ such that $x \leq y$, we say that y *covers* x, and write $y \to x$, if there are no elements $z \in P$ satisfying $x < z < y$. If P is a graded poset, then there exists a function

$$\rho : P \longrightarrow \mathbf{N} = \{0, 1, 2, \ldots\},$$

defined inductively as follows:

1. $\rho(\hat{0}) = 0$,
2. if y covers x, then $\rho(y) = \rho(x) + 1$.

The function ρ is called the *rank function* of P. If P is a graded poset and $\rho(y) - \rho(x) = k$, then a maximal chain from y to x has length k. A *rooted interval* $([x, y], \mathbf{c})$ is a pair of a closed interval $[x, y] = \{z \in P \mid x \leq z \leq y\}$ and a maximal chain $\mathbf{c}$ from $\hat{1}$ to y.

Let P be a graded poset of length r. A *labeling* of P is a map

$$\lambda : \mathcal{M}(P) \longrightarrow (\mathbf{Z}_{>0})^r : \mathcal{C} \mapsto (\lambda_1(\mathcal{C}), \ldots, \lambda_r(\mathcal{C})),$$

where $\mathbf{Z}_{>0}$ denotes the set of positive integers.

Let $\mathbf{m} : \hat{1} = x_0 \to x_1 \to \cdots \to x_r = \hat{0}$ be a maximal chain in P. A *labeling* $\lambda(\mathbf{m})$ of $\mathbf{m}$ is an r-tuple of positive integers $(\lambda_1(\mathbf{m}), \ldots, \lambda_r(\mathbf{m}))$. Here we understand that each integer $\lambda_i(\mathbf{m})$ is assigned to the edge (or the cover relation) $x_{i-1} \to x_i$ of $\mathbf{m}$.

First we impose the following condition on a labeling λ.

(L1) If two maximal chains $\mathbf{m}$ and $\mathbf{m}'$ coincide along their first d edges from the top $\hat{1}$, then $\lambda_i(\mathbf{m}) = \lambda_i(\mathbf{m}')$ for $i = 1, 2, \ldots, d$.

Then the labeling λ satisfying condition (L1) naturally induces a labeling λ' of a rooted interval $([x, y], \mathbf{c})$ by restricting λ to $[x, y]$: Let $\mathbf{m}'$ be a maximal chain in the closed interval $[x, y]$ and $\mathbf{c}'$ be any maximal chain from x to $\hat{0}$. These three chains $\mathbf{c}$, $\mathbf{m}'$, $\mathbf{c}'$ are connected to form a maximal chain $\mathbf{m} := \mathbf{c} * \mathbf{m}' * \mathbf{c}'$ from $\hat{1}$ to $\hat{0}$. If $\rho(\hat{1}) - \rho(y) = \ell$, then the label $\lambda'(\mathbf{m}') = (\lambda'_1(\mathbf{m}'), \ldots, \lambda'_k(\mathbf{m}'))$ is defined by $\lambda'_i(\mathbf{m}) = \lambda_{i+\ell}(\mathbf{m})$. By condition (L1), there is no ambiguity on the choice of an integer for each edge of $\mathbf{m}'$. Note that the induced labeling of a rooted interval also satisfies the condition (L1). We simply write λ for the induced labeling λ' on $([x, y], \mathbf{c})$.

Now we impose another condition on λ:

(L2) For any rooted interval $([x, y], \mathbf{c})$ in P, there exists a unique maximal chain $\mathbf{m_0}$ in $[x, y]$ whose label $\lambda(\mathbf{m_0}) = (\lambda_1(\mathbf{m_0}), \lambda_2(\mathbf{m_0}), \ldots)$ is increasing, i.e., $\lambda_1(\mathbf{m_0}) \leq \lambda_2(\mathbf{m_0}) \leq \cdots$. Moreover the label $\lambda(\mathbf{m_0})$ is smaller than labels $\lambda(\mathbf{m})$ of any other maximal chains $\mathbf{m}$ of $[x, y]$ with respect to the lexicographic order.

Here the lexicographic order $\leq_L$ is defined as follows: For two sequences of integers $a = (a_1, \ldots, a_n)$, $b = (b_1 \ldots, b_n)$, we define $a \leq_L b$ if $a_i < b_i$ in the first coordinate where they differ.

Definition 1 ([Bj] [BW1]) Let P be a finite graded poset. If P has a labeling λ satisfying conditions (L1) and (L2), then P is called a *CL-shellable poset*, and the labeling λ is called a *CL-labeling* of P.

Example 2 Let (W, S) be a Coxeter system and $\leq$ the Bruhat order [Hum]. Then any closed interval $[w', w] = \{v \in W \mid w' \leq v \leq w\}$ is CL-shellable [BW1]. Its CL-labeling λ is constructed as follows:

Let $\mathbf{m} : w = u_0 > u_1 > \cdots > u_t = w'$ be a maximal chain of the closed interval $[w', w]$. Since $[w', w]$ is graded, $\mathbf{m}$ is saturated, i.e., $\ell(u_{k-1}) - \ell(u_k) = 1$ where ℓ is the length function of W. Then a CL-labeling is defined inductively as follows: Fix a reduced expression $s_1 \cdots s_r$ of w. By the subword property [Hum, p. 120], a reduced expression of u_1 is of the form $s_1 \cdots \hat{s}_i \cdots s_r$, where the deleted letter s_i is uniquely determined. Let $\lambda_1(\mathbf{m}) = i$. We can repeat this process to define $\lambda(\mathbf{m}) = (\lambda_1(\mathbf{m}), \ldots, \lambda_r(\mathbf{m}))$. After k steps, we reach u_k and its reduced expression $s_{i_1} \cdots s_{i_{r-k}}$ is uniquely determined as a subexpression of the fixed reduced expression of w. Again by the subword property, a reduced expression of u_{k+1} is of the form $s_{i_1} \cdots \hat{s_{i_p}} \cdots s_{i_{r-k}}$ with uniquely determined s_{i_p}, and $\lambda_k(\mathbf{m})$ is defined by i_p. Thus the label $\lambda(\mathbf{m}) = (\lambda_1(\mathbf{m}), \ldots, \lambda_r(\mathbf{m}))$ is defined, and the labeling λ of the closed interval $[w', w]$ is CL-shellable.

Let $W = S_3$ be the symmetric group of three letters. The group W is generated by transpositions $X = \{s_1, s_2\}$, where $s_1 = (12)$ and $s_2 = (23)$. Then the pair (W, X) is a Coxeter system. Let $g = s_1 s_2 s_1$ be the longest element and $h = e$ be the identity element. A CL-labeling of the closed interval $[h, g]$ is given as follows:

$$
\begin{aligned}
&s_1 s_2 s_1 \xrightarrow{1} s_2 s_1 \xrightarrow{2} s_1 \xrightarrow{3} e \\
&s_1 s_2 s_1 \xrightarrow{1} s_2 s_1 \xrightarrow{3} s_2 \xrightarrow{2} e \\
&s_1 s_2 s_1 \xrightarrow{3} s_1 s_2 \xrightarrow{1} s_2 \xrightarrow{2} e \\
&s_1 s_2 s_1 \xrightarrow{3} s_1 s_2 \xrightarrow{2} s_1 \xrightarrow{1} e.
\end{aligned}
$$

3. Reflection system

Let (X, E) be an undirected finite graph with a vertex set X and an edge set E, and let ϕ, ψ be given functions such that

$$\phi : X \longrightarrow \{n \in \mathbf{N} \mid n \geq 2\}$$
$$\psi : E \longrightarrow \{n \in \mathbf{N} \mid n \geq 3\} \cup \{\infty\}.$$

Assume that $\phi(x) = \phi(y) = 2$ whenever $\{x, y\} \in E$ and $\psi(x, y) \neq \infty$.

Definition 3 The pair (G, X) of a group G and its system of generators X is called a *reflection system* if G satisfies the following fundamental relations:
1. $x^{\phi(x)} = 1$, for $x \in X$,
2. $xy = yx$, if $\{x, y\} \notin E$,
3. $(xy)^{\psi(\{x,y\})} = 1$, if $\{x, y\} \in E$ and $\psi(\{x, y\}) < \infty$.

The function ϕ represents the orders of elements of X, i.e., $\phi(x) = \mathrm{ord}(x)$ for any $x \in X$. Hence Coxeter systems are the reflection systems with $\phi(x) = 2$ for all $x \in X$. Note that a generator $x \in X$ with $\mathrm{ord}(x) \neq 2$ either commutes with y, or has no relations with y, i.e., for any $y \in X$ with $x \neq y$, we have $(xy)^k \neq 1$ for all k. In this paper, such x is said to have *trivial relations* with y, and denoted by $(xy)^\infty = 1$.

A reflection system (G, X) is visualized graphically by its "*Coxeter diagram*". Figure 1 indicates that the group G is generated by $X = \{s, t, u\}$ with fundamental relations $s^2 = t^2 = u^5 = 1$, $(st)^3 = 1$, and $su = us$, i.e., $\phi(s) = \phi(t) = 2$, $\phi(u) = 5$, $\psi(s, t) = 3$, $\psi(s, u) = 2$, and $\psi(t, u) = \infty$.

Since X generates G, any element of G is written in the form $x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m}$, where $x_i \in X$ and $n_i \in \mathbf{Z}$. However, since each $x \in X$ has the finite order $\phi(x)$ in G, we may assume that all n_i are positive integers. Among such expressions, a *reduced expression* of g is defined to be one with $\sum_{i=1}^m n_i$ minimum. In this case, $\sum_{i=1}^m n_i$ is called the *length* of g,

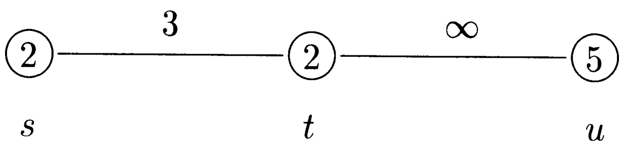

Fig. 1. A Coxeter diagram

and denoted by $\ell(g)$. On the other hand, an expression with the minimum m is called an *ℓ_1-reduced* expression of g, and the integer m is called the *ℓ_1-length*, denoted by $\ell_1(g)$. A reduced expression which is also ℓ_1-reduced is called a *strongly reduced* expression.

For example, let (G, X) be a reflection system which is defined by the Coxeter diagram in Figure 1. An expression $g = stu^2su$ is a reduced expression of length six, but not ℓ_1-reduced. On the other hand, $g = st^3su^8$ is an ℓ_1-reduced expression of ℓ_1-length four of the same element g, but not reduced. An expression $g = stsu^3$ is a strongly reduced.

Let g be an element of G with a strongly reduced expression $x_1^{n_1} \cdots x_m^{n_m}$. A subexpression of $x_1^{n_1} \cdots x_m^{n_m}$ is an expression of the form $x_1^{k_1} \cdots x_m^{k_m}$, where $0 \leq k_i \leq n_i$ for each $i = 1, 2, \ldots, m$. If a subexpression $x_1^{k_1} \cdots x_m^{k_m}$ is consecutive, i.e., $k_i \neq 0$ for each i such that $\min\{j \mid k_j \neq 0\} \leq i \leq \max\{j \mid k_j \neq 0\}$, then it is called a *factor* of $x_1^{n_1} \cdots x_m^{n_m}$. Any factor of a (strongly) reduced expression is again (strongly) reduced.

Remark 4 In the original definition by Dyer [D], the target of ϕ is $\{n \in \mathbf{Z} \mid n \geq 2\} \cup \{\infty\}$, i.e., the orders $\phi(x)$ of generators $x \in X$ in G are not necessarily finite. Hence a reduced expression of $g \in G$ has to be defined as $x_1^{n_1} \cdots x_m^{n_m}$ with $\sum_{i=1}^m |n_i|$ minimum. However, no change is required for the definition of ℓ_1-length if we suppose that each $x \in X$ is of finite order.

The following lemma is a consequence of [D, Lemma (2.9)] and its proof.

Lemma 5 *Any expression of an element of G can be transformed into an ℓ_1-reduced expression only by using the commutativity relations of G.*

In the preceding example, an expression $g = stu^2su \in G$, is transformed into an ℓ_1-reduced expression stu^2s by using the relation $su = us$.

Lemma 6 *Let h and g be elements of G. Suppose that* $\mathrm{ord}(h) = 2$. *Then we have*

$$\ell(g) - \ell(h) \leq \ell(gh) \leq \ell(g) + \ell(h).$$

Proof. It is clear that $\ell(gh) \leq \ell(g) + \ell(h)$. On the other hand, we have $\ell(g) = \ell(ghh^{-1}) \leq \ell(gh) + \ell(h^{-1}) = \ell(gh) + \ell(h)$. Hence $\ell(g) - \ell(h) \leq \ell(gh)$. □

If (G, X) is a reflection system, then elements of the set $T =$

$\bigcup_{g\in G} gXg^{-1}$ are called *reflections* of (G, X). Let

$$M_{(G,X)} = \left\{ \sum_{t\in T} a_t t \mid a_t \in \mathbf{Z}/\mathrm{ord}(t)\mathbf{Z},\ a_t = 0 \text{ for almost all } t \right\}.$$

The group G acts on $M_{(G,X)}$ as follows:

$$g \cdot \left(\sum_{t\in T} a_t t \right) = \sum_{t\in T} a_t \, (g \cdot t).$$

Definition 7 A map $N : G \longrightarrow M_{(G,X)}$ is called a *reflection cocycle* if N satisfies the following two conditions:
1. $N(gh) = h^{-1} \cdot N(g) + N(h)$ for $g, h \in G$,
2. $N(x) = 1x$ for $x \in X$.

We recall the following result which was proved by Dyer [D, Porposition 2.10].

Proposition 8 *For a pair (G, X) of a group G and its system of generators X, (G, X) is a reflection system if and only if (G, X) has the reflection cocycle.*

Let (G, X) be a reflection system. For any element $g \in G$ and a reflection $t \in T$, we define $\mathrm{mult}_t(g) \in \mathbf{Z}/m\mathbf{Z}$ by $N(g) = \sum_{t\in T} \mathrm{mult}_t(g) t$. The proof of the following lemma is similar to [D, Lemma 3.].

Lemma 9 *For $g \in G$, we have*
1. $\ell_1(g) = \sharp\{t \in T \mid \mathrm{mult}_t(g) \neq 0\}$,
2. $\ell(g) = \sum_{t\in T} \mathrm{mult}_t(g)$.

For integers $a \in \mathbf{Z}$ and $m \in \mathbf{Z}_{>0}$, the integer $|a|_m$ is defined by $|a|_m \equiv a \mod m\mathbf{Z}$ and $0 \le |a|_m < m$.

Corollary 10 *Let $g = x_1^{n_1} \cdots x_m^{n_m}$ be an ℓ_1-reduced expression of $g \in G$. Then*

$$g = x_1^{|n_1|_{\mathrm{ord}(x_1)}} \cdots x_m^{|n_m|_{\mathrm{ord}(x_m)}}$$

is a strongly reduced expression of g.

Let $g = x_1^{n_1} \cdots x_m^{n_m} \in G$ be a strongly reduced expression. Define $t_i = x_m^{-n_m} \cdots x_{i+1}^{-n_{i+1}} \cdot x_i$ $(i = 1, \ldots, m)$ and $T(g) = \{t_1, \ldots, t_m\}$. By a simple observation we see that a reflection t belongs to $T(g)$ if and only if $\mathrm{mult}_t(g) \neq 0$, i.e.,

$$T(g) = \{t \in T \mid \mathrm{mult}_t(g) \neq 0\}.$$

Proposition 11 *Let (G, X) be a reflection system and T be the set of reflections. For a reflection $t \in T$ with $\mathrm{ord}(t) = 2$ and $g \in G$, we have*

$$\ell(gt^{-1}) < \ell(g) \Longleftrightarrow t \in T(g).$$

Proof. If $t \in T(g)$, then it is obvious that $\ell(gt^{-1}) < \ell(g)$. Conversely, suppose that $\mathrm{mult}_t(g) = 0$. If we set $g' = gt^{-1} = gt$, then we have $\mathrm{mult}_t(gt) = \mathrm{mult}_t(g) + 1 \neq 0$ [D, (2.5)]. Thus

$$\ell(g) = \ell(g't^{-1}) < \ell(g') = \ell(gt^{-1}),$$

by the "only if" part of this proposition. □

4. Word problem

In this section, we consider another characterization of reduced expressions of an element of a reflection system. Let (G, X) be a reflection system and F the free group generated by X. Then the canonical surjection $\pi : F \longrightarrow G$ is a surjective map

$$\pi' : F^+ \longrightarrow G$$

from the monoid F^+ on the set X since each generator $x \in X$ has a finite order in G. We consider the following three operations on F^+, called *M-operations*, corresponding to the defining relations of G (i.e., M-operations have no effect on the image of π' of any word in F^+):

(M1) For any $x \in X$, delete the factor $x^{\mathrm{ord}(x)}$ in a word of F^+.

(M2) For $x, y \in X$ with $\psi(x, y) \neq \infty$, replace the factor $\underbrace{xyx\cdots}_{\psi(x,y)\text{ letters}}$ in a word by $\underbrace{yxy\cdots}_{\psi(x,y)\text{ letters}}$.

(M2)$'$ If $x \in X$ satisfies $\phi(x) \neq 2$, then, for any $y \in X$ with $\{x, y\} \notin E$, replace a factor xy in a word by yx.

The following proposition is an analogue of the Tits word theorem for Coxeter groups [T] [Br, p. 49]. (The *length* of an element of F^+ is naturally defined.)

Proposition 12 *Let (G, X) be a reflection system and g an element of G. Then an expression $g = x_1^{n_1} \cdots x_m^{n_m}$ is reduced if and only if the word $x_1^{n_1} \cdots x_m^{n_m} \in F^+$ can not be shortened its length by any finite sequence of M-operations.*

Proof. It suffices to show that any expression can be made into reduced expression by a finite sequence of M-operations. Note first that, by Lemma 5, any expression of g achieves ℓ_1-reducedness only by using commutativity relations which are M-operations of type (M2) and (M2)$'$. By Lemma 10, any ℓ_1-reduced expression of g can be made into a reduced expression only by using M-operations of type (M1). □

Corollary 13 *If the order of $x \in X$ is two, then we have $\ell(gx) = \ell(g) \pm 1$ for any $g \in G$.*

Proof. In view of Lemma 6, it is sufficient to show that the case $\ell(gx) = \ell(g)$ never occurs. Let $g = x_1^{n_1} \cdots x_m^{n_m}$ be a reduced expression of g. If $x \in X$ satisfies the condition $\ell(gx) = \ell(g)$, then the expression $gx = x_1^{n_1} \cdots x_m^{n_m} x$ is not reduced. Hence, by Proposition 12, the length of $x_1^{n_1} \cdots x_m^{n_m} x \in F^+$ should be shortened at least two. This contradicts $\ell(gx) = \ell(g)$. □

5. Reflection orders

Let (G, X) be a reflection system.

Definition 14 For each element $g \in G$, define $S(g) = \{t \in T(g) \mid \ell(gt^{-1}) = \ell(g) - 1\}$. For two elements $g, h \in G$, we write $g \to h$ if there exists $t \in S(g)$ such that $h = gt^{-1}$. We define $g > h$ if there exist $\xi_1, \ldots, \xi_{r-1} \in G$ such that $g = \xi_0 \to \xi_1 \to \cdots \to \xi_{r-1} \to \xi_r = h$.

Then $\leq$ is a partial order on the set G, called the *reflection order* on G. If (G, X) is a Coxeter system, then the reflection order coincides with the Bruhat order [Hil] [Hum]. For elements $g, h \in G$ such that $h < g$, it is obvious from definition that the closed interval $[h, g] = \{\xi \in G \mid h \leq \xi \leq g\}$ is a finite graded poset.

Example 15 Let (G, X) be the reflection system defined by the Coxeter

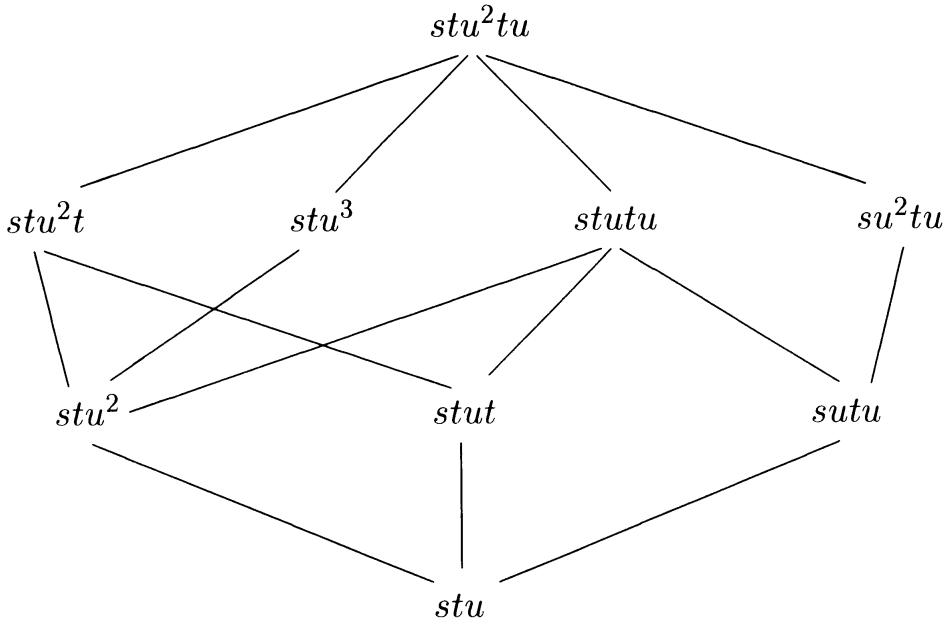

Fig. 2. The Hasse diagram of the closed interval $[h, g]$

diagram in Figure 1. Let $g = stu^2tu$ and $h = stu$. Then $h < g$. In fact, we have the following chain from g to h:

$$g = stu^2tu \to su^2tu \to sutu \to stu = h.$$

The reflections in the first, second, and third steps are given by $(u^2tu)^{-1} \cdot t$, $(tu)^{-1} \cdot u$, and $(tu)^{-1} \cdot u$ respectively. The Hasse diagram of the interval $[h, g]$ is shown in Figure 2.

In the rest of this section, we will show a proposition corresponding to the subword property of Coxeter groups [Hum, p. 120]. The following two lemmas are needed to prove the proposition.

Lemma 16 *Let $g, h \in G$ and $g \geq h$. If the order of $x \in X$ equals two, then we have either $g \geq hx$ or $gx \geq hx$.* (c.f., [Hum, Proposition 5.9])

Proof. It is enough to show the lemma in the case that $g \to h$. Let t be a reflection such that $h = gt^{-1}$ with $t \in S(g)$. If $t = x$, then there is nothing to be proved. Hence we may assume that $t \neq x$.

1. First we consider the case that $\ell(hx) = \ell(h) - 1$. In this case we have $g \to h \to hx$. Hence $hx < g$.
2. If $\ell(hx) = \ell(h) + 1$, then it is enough to show that $\ell(gx) = \ell(hx) + 1$, since we have $gx = htx = hxt'$, where $t' = x^{-1}tx$. Suppose that

$\mathrm{ord}(t) = \mathrm{ord}(t') = 2$ and $x_1^{k_1} \cdots x_m^{k_m}$ is a reduced expression of h. If $\ell(gx) < \ell(hx)$, then we have $t' \in T(hx)$ and

$$gx = \begin{cases} x_1^{k_1} \cdots x_m^{k_m}, & \text{or} \\ x_1^{k_1} \cdots x_i^{k_i - 1} \cdots x_m^{k_m} x, \end{cases}$$

by Proposition 11. The first case contradicts the assumption $t \neq x$. The second case contradicts the condition $\ell(g) > \ell(h)$.

Next we consider the case $\mathrm{ord}(t) \neq 2$. If $g = x_1^{n_1} \cdots x_m^{n_m}$ is a strongly reduced expression of g, then a reduced expression of h is of the form $h = x_1^{n_1} \cdots x_{i-1}^{n_{i-1}} x_i^{n_i - 1} x_{i+1}^{n_{i+1}} \cdots x_m^{n_m}$ for some unique i with $\mathrm{ord}(x_i) \neq 2$. Since $\ell(hx) = \ell(h) + 1$, the expression

$$hx = x_1^{n_1} \cdots x_{i-1}^{n_{i-1}} x_i^{n_i - 1} x_{i+1}^{n_{i+1}} \cdots x_m^{n_m} x$$

is reduced. Since the order of x_i is not two, the expression

$$gx = x_1^{n_1} \cdots x_{i-1}^{n_{i-1}} x_i^{n_i} x_{i+1}^{n_{i+1}} \cdots x_m^{n_m} x$$

is also reduced. Otherwise, by Proposition 12, the above expression of gx can be shortened by any finite sequence of M-operations. This contradicts the reducedness of the expression $hx = x_1^{n_1} \cdots x_{i-1}^{n_{i-1}} x_i^{n_i - 1} x_{i+1}^{n_{i+1}} \cdots x_m^{n_m} x$. Thus we have $hx \leftarrow gx$. □

Lemma 17 *Let $g, h \in G$ and $g = x_1^{n_1} \cdots x_m^{n_m}$ be reduced. Suppose that h can be obtained as a subexpression $x_1^{k_1} \cdots x_m^{k_m}$ of the reduced expression of g with $n_m = k_m$ and $\mathrm{ord}(x_m) \neq 2$. If*

$$h' = x_1^{k_1} \cdots x_m^{k_m - 1} \leq g' = x_1^{n_1} \cdots x_m^{n_m - 1},$$

then we have $h = h' x_m \leq g = g' x_m$.

Proof. It is sufficient to construct a maximal chain in the closed interval $[h', g']$

$$g' = \xi_0 \to \xi_1 \to \cdots \to \xi_r = h',$$

with the condition that $\xi_i x_m \to \xi_i$ for all i. Let us argue by an induction on r. It is enough to construct an element $\xi \in G$ covered by g' such that $\xi x_m \to \xi$. Suppose that any element ξ covered by g' satisfies $\ell(\xi x_m) < \ell(\xi)$, i.e., if $\xi = x_1^{n_1} \cdots x_{i-1}^{n_{i-1}} x_i^{n_i - 1} x_{i+1}^{n_{i+1}} \cdots x_m^{n_m - 1}$ $(i = 1, \ldots m - 1)$, then $\ell(\xi x_m) < \ell(\xi)$. If x_m commutes with all x_i $(i = 1, \ldots m - 1)$, then we have

$\ell(\xi x_m) = \ell(\xi) + 1$. Hence we may assume that there exists a number i $(i = 1, 2, \ldots m-1)$ $(x_i x_m)^\infty = 1$ for some x_i $(i = 1, 2, \ldots m-1)$. Let α be the maximum one among all such i's. Then x_m commutes with x_j for any $j > \alpha$. From the assumption $\ell(\xi x_m) < \ell(\xi)$, we have

$$\sum_{\alpha < j \leq m-1,\ x_j = x_m} n_j = \mathrm{ord}(x_j) - 1.$$

This contradicts the fact that the expression $g = x_1^{n_1} \cdots x_m^{n_m}$ is reduced. □

Proposition 18 *Let (G, X) be a reflection system and g an element of G. Fix a reduced expression $g = x_1^{n_1} \cdots x_m^{n_m}$. Then $h \leq g$ if and only if h can be obtained as a subexpression of the reduced expression of g.*

Proof. Suppose $h \leq g$. It is clear from the definition of the reflection order that h can be obtained as a subexpression of the reduced expression of g. Conversely, suppose that h can be obtained as a subexpression $x_1^{k_1} \cdots x_m^{k_m}$. We prove $h \leq g$ by the induction on $\ell(g)$. First suppose that $k_m < n_m$. Then we have

$$h = x_1^{k_1} \cdots x_m^{k_m} \leq x_1^{n_1} \cdots x_m^{n_m - 1} \leftarrow g,$$

by the induction hypothesis. Next assume that $k_m = n_m$. In this case, we have

$$h' = x_1^{k_1} \cdots x_m^{k_m - 1} \leq g' = x_1^{n_1} \cdots x_m^{n_m - 1}.$$

If $\mathrm{ord}(x_m) = 2$, then it follows from Lemma 16 that either $h = h' x_m \leq x_1^{n_1} \cdots x_m^{n_m - 1}$ or $h = h' x_m \leq x_1^{n_1} \cdots x_m^{n_m}$. If $\mathrm{ord}(x_m) \neq 2$, it is also deduced that $h \leq g$ from Lemma 17. □

6. CL-shellability

In this section we will construct a labeling of the maximal chains of a closed interval $[h, g]$ of a reflection system (G, X) and show the CL-shellability of $[h, g]$.

Let g, h be elements of G such that $g > h$ and $\ell(g) - \ell(h) = r$. Since the closed interval $[h, g]$ is a finite graded poset of length r, each maximal chain $\mathbf{m}$ has length r, say

$$\mathbf{m} : g = \xi_0 \to \xi_1 \cdots \to \xi_{r-1} \to \xi_r = h.$$

Fix a strongly reduced expression

$$g = x_1^{n_1} \cdots x_m^{n_m}$$

of g. First we will construct a label $\lambda(\mathbf{m}) = (\lambda_1(\mathbf{m}), \ldots, \lambda_r(\mathbf{m}))$ of $\mathbf{m}$. Since ξ_1 is covered by g, it can be expressed in the form

$$x_1^{n_1} \cdots x_{i-1}^{n_{i-1}} x_i^{n_i - 1} x_{i+1}^{n_{i+1}} \cdots x_m^{n_m},$$

and the deleted letter is uniquely determined since $g = x_1^{n_1} \cdots x_m^{n_m}$ is ℓ_1-reduced, and we define

$$\lambda_1(\mathbf{m}) = n_1 + \cdots + n_{i-1} + 1,$$

i.e., we understand that each n_j letters x_j in the strongly reduced expression of g carries positive integers

$$n_1 + \cdots + n_{j-1} + 1, \ldots, n_1 + \cdots + n_{j-1} + n_j$$

respectively, and the deleted x_i is the one with the minimum number among them.

Example 19 In Example 15, consider a strongly reduced expression $g = stu^2$. We understand that the letter s has a label 1, t has 2, the first u has 3 and the last u has 4. There exist three elements in G that are covered by g, namely $h_1 = tu^2$, $h_2 = su^2$, and $h_3 = stu$. Then the cover relations $g \to h_i$ $(i = 1, 2, 3)$ have the labels 1, 2, and 3, respectively.

The second component $\lambda_2(\mathbf{m})$ of $\lambda(\mathbf{m})$ is defined in a similar way, but we have to note that the reduced expression

$$\xi_1 = x_1^{n_1} \cdots x_{i-1}^{n_{i-1}} x_i^{n_i - 1} x_{i+1}^{n_{i+1}} \cdots x_m^{n_m}$$

is not necessarily ℓ_1-reduced. Hence, for some $h \leftarrow \xi_1$, uniqueness of j such that $h = x_1^{n_1} \cdots x_i^{n_i - 1} \cdots x_j^{n_j - 1} \cdots x_m^{n_m}$ is violated. However, by Lemma 4, if $x_1^{n_1} \cdots x_i^{n_i - 1} \cdots x_m^{n_m}$ is not ℓ_1-reduced, then it is possible to make it strongly reduced only by using the commutativity relations.

Now we can determine the second component $\lambda_2(\mathbf{m})$. There is no problem if

$$x_1^{n_1} \cdots x_{i-1}^{n_{i-1}} x_i^{n_i - 1} x_{i+1}^{n_{i+1}} \cdots x_m^{n_m}$$

is strongly reduced. If not, then make it strongly reduced by Lemma 4, say $x_{j_1}^{p_1} \cdots x_{j_n}^{p_n}$ $(\sum p_i = (\sum n_i) - 1,\ n \leq m)$. A reduced expression of ξ_2 is of the form $x_{j_1}^{p_1} \cdots x_{j_k}^{p_k-1} \cdots x_{j_n}^{p_n}$ for unique k, and $\lambda_2(\mathbf{m})$ is defined to be the minimum number which is possessed by x_{j_k} in $x_{j_1}^{p_1} \cdots x_{j_n}^{p_n}$.

Example 20 Let (G, X) be a reflection system in Example 15, and g, h elements of G given by stu^2tu, stu, respectively. Then the label of the maximal chain $stu^2tu \to stutu \to stu^2 \to stu$ is

$$g = stu^2tu \xrightarrow{3} stutu \xrightarrow{5} stu^2 \xrightarrow{4} stu = h.$$

Now we are in a position to prove that the labeling λ defined above is CL-shellable. First we prove the uniqueness of a maximal chain with an increasing label.

Proposition 21 *For each closed interval $[h, g]$, if there exists a maximal chain $\mathbf{m}$ whose label $\lambda(\mathbf{m}) = (\lambda_1(\mathbf{m}), \ldots, \lambda_r(\mathbf{m}))$ is increasing, i.e., $\lambda_1(\mathbf{m}) \leq \cdots \leq \lambda_r(\mathbf{m})$, then it is uniquely determined.*

Proof. Fix a strongly reduced expression $g = x_1^{n_1} \cdots x_m^{n_m}$, and let λ be the labeling of the closed interval $[h, g]$ defined by this strongly reduced expression. Suppose that two maximal chains

$$\mathbf{m} : g = \xi_0 \to \xi_1 \to \cdots \to \xi_r = h$$
$$\mathbf{m}' : g = \eta_0 \to \eta_1 \to \cdots \to \eta_r = h,$$

both have increasing labels

$$\lambda(\mathbf{m}) = (i_1, \ldots, i_r) \ ; \ i_1 < \cdots < i_r,$$
$$\lambda(\mathbf{m}') = (j_1, \ldots, j_r) \ ; \ j_1 < \cdots < j_r.$$

Let $h = x_1^{n_1-p_1} \cdots x_\alpha^{n_\alpha - p_\alpha} x_{\alpha+1}^{n_{\alpha+1}} \cdots x_m^{n_m}$ be the reduced expression obtained by going down from g to h along with the maximal chain $\mathbf{m}$. Also $h = x_1^{n_1-q_1} \cdots x_\beta^{n_\beta - q_\beta} x_{\beta+1}^{n_{\beta+1}} \cdots x_m^{n_m}$ be the one with respect to the maximal chain $\mathbf{m}'$.

Suppose that $i_r < j_r$. It follows from the definition of the labeling that $\alpha \leq \beta$. First we consider the case $\alpha < \beta$. In this case, if we let $t = x_m^{-n_m} \cdots x_{\beta+1}^{-n_{\beta+1}} \cdot x_\beta$, then we have

$$\begin{aligned} \eta_{r-1} &= ht \\ &= x_1^{n_1-q_1} \cdots x_\beta^{n_\beta - q_\beta} x_{\beta+1}^{n_{\beta+1}} \cdots x_m^{n_m} t \\ &= x_1^{n_1-p_1} \cdots x_\alpha^{n_\alpha - p_\alpha} x_{\alpha+1}^{n_{\alpha+1}} \cdots x_m^{n_m} t \end{aligned}$$

$$= x_1^{n_1-p_1} \cdots x_\alpha^{n_\alpha-p_\alpha} x_{\alpha+1}^{n_{\alpha+1}} \cdots x_{\beta-1}^{n_{\beta-1}} x_\beta^{n_\beta+1} x_{\beta+1}^{n_{\beta+1}} \cdots x_m^{n_m}.$$

If the order $\phi(x_\beta)$ of x_β equals two, then we have $\ell(\eta_{r-1}) < \ell(h)$, which is a contradiction. If $\mathrm{ord}(x_\beta) > 2$, then the above expression of η_{r-1} cannot be rewritten as a subexpression of $g = x_1^{n_1} \cdots x_\beta^{n_\beta} \cdots x_m^{n_m}$, since x_β has the trivial relations with $y \in X$, $y \neq x_\beta$. This contradicts Proposition 18.

Next we consider the case where $\alpha = \beta$, which we call γ. Note that the order of x_γ can not be two in this case. Then we have

$$\begin{aligned} h &= x_1^{n_1-p_1} \cdots x_\gamma^{n_\gamma-p_\gamma} x_{\gamma+1}^{n_{\gamma+1}} \cdots x_m^{n_m} \\ &= x_1^{n_1-q_1} \cdots x_\gamma^{n_\gamma-q_\gamma} x_{\gamma+1}^{n_{\gamma+1}} \cdots x_m^{n_m}. \end{aligned}$$

It follows from the assumption $i_r < j_r$ that $p_\gamma < q_\gamma$. If we set $d = q_\gamma - p_\gamma$, then we have

$$x_1^{n_1-p_1} \cdots x_{\gamma-1}^{n_{\gamma-1}} x_\gamma^d = x_1^{n_1-q_1} \cdots x_{\gamma-1}^{n_{\gamma-1}},$$

both of which are reduced. By the assumption $i_r < j_r$, there exists a number k, $1 \leq k \leq \gamma - 1$, such that $(x_k x_\gamma)^\infty = 1$, and $x_l \neq x_\gamma$ for all $k+1 \leq l \leq \gamma - 1$. If not, by the definition of the reflection order, we have a descent in the label $(i_1, \ldots, i_r)$ of $\mathbf{m}$. Thus we cannot remove the letter x_γ from the above equation. Hence we have a non-trivial relation involving the letter x_γ. This is a contradiction to the fact that x_γ has trivial relations with $x \in X$, $x \neq x_\gamma$. Now we have $i_r \geq j_r$. By symmetry we have $i_r \leq j_r$, and hence we obtain $i_r = j_r$. □

Proposition 22 *Let g, h be elements of G such that $h < g$ and $\ell(g) - \ell(h) = 2$. Fix a reduced expression $g = x_1^{n_1} \cdots x_m^{n_m}$. Suppose that h has a reduced expression*

$$h = x_1^{n_1} \cdots x_{i-1}^{n_{i-1}} x_i^{n_i-1} x_{i+1}^{n_{i+1}} \cdots x_{j-1}^{n_{j-1}} x_j^{n_j-1} x_{j+1}^{n_{j+1}} \cdots x_m^{n_m}$$

for some $i \neq j$. Then there exists in the closed interval $[h, g]$ a unique maximal chain $g \to \xi \to h$ with increasing label $a < b$. Also there exists a maximal chain $g \to \eta \to h$ with decreasing label $p > q$. Moreover we have $a < p$.

Proof. Suppose that $h = x_1^{n_1} \cdots x_i^{n_i-1} \cdots x_j^{n_j-1} \cdots x_m^{n_m}$ is a reduced expression with j minimal among all such expressions. The uniqueness in the first statement is proved from Proposition 21. To prove the existence of such a maximal chain, it is enough to show that

$$h' = x_1^{n_1} \cdots x_{i-1}^{n_{i-1}} x_i^{n_i - 1} x_{i+1}^{n_{i+1}} \cdots x_j^{n_j} \cdots x_m^{n_m}$$

is reduced.

1. If $\mathrm{ord}(x_j) \neq 2$, then it is clear from the reducedness of the expression

$$x_1^{n_1} \cdots x_{i-1}^{n_{i-1}} x_i^{n_i - 1} x_{i+1}^{n_{i+1}} \cdots x_j^{n_j} \cdots x_m^{n_m},$$

and the assumption.

2. Suppose that $\mathrm{ord}(x_j) = 2$.
   (a) If $n_i - 1 \neq 0$, then it follows that the order of x_i is not equal to two. Hence $x_1^{n_1} \cdots x_i^{n_i - 1} \cdots x_j^{n_j} \cdots x_m^{n_m}$ is reduced, since $x_1^{n_1} \cdots x_i^{n_i} \cdots x_j^{n_j} \cdots x_m^{n_m}$ is reduced.
   (b) Suppose that $n_i - 1 = 0$. Let $t = x_m^{-n_m} \cdots x_{j+1}^{-n_{j+1}} \cdot x_j$. Note that $\mathrm{ord}(t) = \mathrm{ord}(x_j) = 2$. If $\ell(ht) < \ell(h)$, then

$$ht = ht^{-1} = \begin{cases} x_1^{n_1} \cdots \hat{x}_i \cdots \hat{x}_j \cdots x_p^{n_p - 1} \cdots x_m^{n_m} & \text{(A)} \\ x_1^{n_1} \cdots \hat{x}_i \cdots x_p^{n_p - 1} \cdots \hat{x}_j \cdots x_m^{n_m} & \text{(B)} \\ x_1^{n_1} \cdots x_p^{n_p - 1} \cdots \hat{x}_i \cdots \hat{x}_j \cdots x_m^{n_m} & \text{(C)}, \end{cases}$$

   by Proposition 11.
   In the case (A), let t' be a reflection

$$t' = x_m^{-n_m} \cdots x_{p+1}^{-n_{p+1}} \cdot x_p.$$

   Then we have $ht = ht^{-1} = ht'^{-1}$ and $\mathrm{ord}(t') = 2$. Hence we have $g = g1 = gtt' = x_1^{n_1} \cdots x_i^{n_i - 1} \cdots x_p^{n_p - 1} \cdots x_m^{n_m}$. This contradicts the assumption.
   In the case (B), let t' denote the reflection $x_m^{n_m} \cdots \hat{x}_j \cdots x_{p+1}^{n_{p+1}} \cdot x_p$. Then we have $ht = ht'$, and $\mathrm{ord}(t') = 2$. Hence

$$\begin{aligned} h &= ht't \\ &= x_1^{n_1} \cdots x_i^{n_i - 1} \cdots \hat{x}_p \cdots x_j \cdots x_m^{n_m}. \end{aligned}$$

   This contradicts the minimality of j.
   The case (C) is similar.

For the second statement, the existence of such chain is similar. In this case, we choose a reduced expression $h = x_1^{n_1} \cdots x_p^{n_p - 1} \cdots x_q^{n_q - 1} \cdots x_m^{n_m}$ with p maximum. Hence we have $a < p$. □

Remark 23 Note that the uniqueness does not holds for a maximal chain whose label is decreasing. See the interval $[stu^2, stu^2tu]$ in Example 15.

Now we are in a position to prove our main theorem.

Theorem 24 *Let (G, X) be a reflection system. Let the symbol $\leq$ denote the reflection order of (G, X). Then each closed interval $[h, g]$ $(h \leq g)$ is CL-shellable. More precisely, if λ is the labeling of $[h, g]$ defined by a fixed strongly reduced expression of g, then λ is a CL-labeling.*

Proof. It is clear from the definition that the labeling λ satisfies condition (L1). It suffices to verify the condition (L2). Let $([u, v], \mathbf{c})$ be a rooted interval in the closed interval $[h, g]$, where $\mathbf{c} : g = \xi_0 \to \xi_1 \to \cdots \to \xi_r = v$ is a saturated chain from g to v. Since the number attached to the letter deleted is uniquely determined in each step of the saturated chain $\mathbf{c}$, the set $\mathcal{S}$ of all the numbers possessed by the letters appearing in the reduced expression of v obtained by going down from g along with the chain $\mathbf{c}$ is uniquely determined for $\mathbf{c}$. Hence the induced labeling on maximal chains of $[u, v]$ as a rooted interval of $[h, g]$ is equivalent to the labeling constructed by starting directly from the reduced expression of g, i.e., an order-preserving bijection

$$\varphi : \mathcal{S} \longrightarrow \{1, 2, \ldots, \ell(g) - d\},$$

$(d = \ell(g) - \ell(v))$ induces a bijection

$$\{\lambda'(\mathbf{m}) \mid \mathbf{m} \in \mathcal{M}(([h, g], \mathbf{c}))\} \longrightarrow \{\lambda(\mathbf{m}) \mid \mathbf{m} \in \mathcal{M}([h, g])\},$$

where $(\lambda'_1(\mathbf{m}), \ldots, \lambda'_t(\mathbf{m})) \mapsto (\varphi(\lambda'_1(\mathbf{m})), \ldots, \varphi(\lambda'_t(\mathbf{m})))$. Therefore it is sufficient to confirm the condition (L2) for the 'full' interval $[h, g]$.

The uniqueness of the maximal chain with increasing label is proved in Proposition 21. It remains to show that the maximal chain with the minimum label in the lexicographic order has an increasing label. Let $\mathbf{m_0} = (\xi_i)_{i=0}^{r}$ be the maximal chain of $[h, g]$ such that the label $\lambda(\mathbf{m_0})$ is minimum in the set $\{\lambda(\mathbf{m}) \mid \mathbf{m} \in \mathcal{M}([h, g])\}$. Suppose that there is a descent $\lambda_i(\mathbf{m}) > \lambda_{i+1}(\mathbf{m})$ for some i. Remark that if we express ξ_{i+1} as a subexpression $x_1^{k_1} \cdots x_i^{n_i - 1} \cdots x_j^{n_j - 1} \cdots x_m^{k_m}$ for a reduced expression $x_1^{k_1} \cdots x_m^{k_m}$ of ξ_{i-1}, then we always have $i \neq j$, since there is no maximal chain with decreasing label in the interval $[\xi_{i+1}, \xi_{i-1}]$.

Let $\xi_{i-1} \to \xi_i \to \xi_{i+1}$ be a maximal chain with decreasing label. By

Proposition 22, there exists a unique maximal chain $\xi_{i-1} \to \zeta \to \xi_{i+1}$ with increasing label $a < b$ satisfying the condition $a < \lambda_i(\mathbf{m})$. Thus following maximal chain of $[h, g]$

$$g = \xi_0 \to \cdots \to \xi_{i-1} \to \zeta \to \xi_{i+1} \to \cdots \to \xi_r = h$$

has a labeling strictly smaller than $\lambda(\mathbf{m_0})$. This contradicts the choice of $\mathbf{m_0}$. $\square$

Example 25 The CL-labeling of maximal chains of the poset in Figure 2 is indicated below:

$$stu^2tu \begin{cases} \xrightarrow{2} su^2tu \xrightarrow{3} sutu \xrightarrow{4} stu \\ \xrightarrow{3} stutu \begin{cases} \xrightarrow{2} sutu \xrightarrow{4} stu \\ \xrightarrow{6} stut \xrightarrow{5} stu \\ \xrightarrow{5} stu^2 \xrightarrow{4} stu \end{cases} \\ \xrightarrow{5} stu^3 \xrightarrow{3} stu^2 \xrightarrow{4} stu \\ \xrightarrow{6} stu^2t \begin{cases} \xrightarrow{3} stut \xrightarrow{5} stu \\ \xrightarrow{5} stu^2 \xrightarrow{3} stu. \end{cases} \end{cases}$$

Our main theorem generalizes a restricted version of a result by Björner and Wachs [BW1]. Their result says that the "ordinary quotient" defined below is also CL-shellable. Let (W, S) be a Coxeter system and J a subset of S. Then a subset W^J of W is defined by

$$W^J := \{w \in W \mid \ell(ws) > \ell(w) \text{ for any } s \in J\},$$

and is called the ordinary quotient of W by J. Björner and Wachs showed that each closed interval of W^J is CL-shellable. The subset W^J is also called the minimal coset representatives which plays a fundamental role in the theory of Coxeter systems. Later they extended their result to 'generalized quotients' [BW2]. It would be interesting to consider an analogous setting for reflection systems.

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