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# Vector-valued weakly analytic measures

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**Abstract.** A celebrated result of Forelli extends the classical F. & M. Riesz Theorem to representations on spaces of Baire measures on a locally compact Hausdorff topological space. We extend these results to representations on vector valued measures, using methods previously developed by two of the authors. The results contained herein complement a result of Ryan. Our paper is not based upon Forelli's result or methods.

*Key words*: F. & M. Riesz Theorem, Analytic Radon Nikodym Property, vector valued measures, hypothesis $(A)$.

## 1. Introduction

In this paper we study properties of weakly analytic vector-valued measures. To motivate the discussion, let us start with some relevant results for scalar-valued measures. A fundamental result in harmonic analysis, the F. and M. Riesz Theorem, states that if a complex Borel measure $\mu$ on the unit circle $\mathbb{T}$ (in symbols, $\mu \in M(\mathbb{T})$) is analytic that is,

$$\int_{-\pi}^{\pi} e^{-int} d\mu(t) = 0, \quad \text{for all } \ n < 0,$$

then $\mu$ is absolutely continuous with respect to Lebesgue measure $m$, that is $\mu \ll m$. Furthermore, the classical result of Raikov and Plessner [10], shows that $\mu \ll m$ if and only if the measure $\mu$ translates continuously. That is the mapping $t \to \mu(\cdot + t)$ is continuous from $\mathbb{R}$ into $M(\mathbb{T})$. Hence, if $\mu$ is analytic, then $t \to \mu(\cdot + t)$ is continuous.

The F. and M. Riesz Theorem has since been generalized to groups and measure spaces by Helson and Lowdenslager [6], de Leeuw and Glicksberg [3], Forelli [5], Yamaguchi [14], and many other authors.

In recent work in [1], it is shown that under certain conditions on $T$, where $T = (T_t)_{t \in \mathbb{R}}$ is a collection of uniformly bounded invertible isomorphisms of the space of measures on a given measure space, if a measure $\mu$ is analytic, in some weak sense, then the mapping $t \to T_t\mu$ is continuous.

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However, not as much is known for vector-valued measures. First, let us consider measures on the unit circle with values in some dual space $X = Y^*$, that is measures in $M(\mathbb{T}, Y^*)$. R. Ryan [12] obtained that if $\mu$ is weakly analytic i.e. $\int_{-\pi}^{\pi} e^{-int}\, d\langle y, \mu(t)\rangle = 0$, for all $n < 0$, for all $y \in Y$, then $\mu \ll \lambda$. Unlike the scalar case, there exist weakly analytic vector-valued measures that do not translate continuously. As a corollary of our main result, we show that if we further assume that $Y^*$ has the analytic Radon-Nikodým property (ARNP), then every weakly analytic measure translates continuously.

We start with some definitions and notation. Denote the real numbers, the complex numbers and the circle group $\{e^{it} : 0 \leq t < 2\pi\}$ by $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{T}$ respectively. Let $M(\mathbb{R})$ be the Banach space of complex regular Borel measures on $\mathbb{R}$. Denote by $L^1(\mathbb{R})$ the space of Lebesgue integrable functions and by $L^\infty(\mathbb{R})$ the space of essentially bounded Lebesgue measurable functions. Define the spaces $H^1(\mathbb{R})$ and $H^\infty(\mathbb{R})$ as follows:

$$H^1(\mathbb{R}) = \{f \in L^1(\mathbb{R}) : \hat{f}(s) = 0,\ s < 0\},$$

and

$$H^\infty(\mathbb{R}) = \left\{f \in L^\infty(\mathbb{R}) : \int_{\mathbb{R}} f(t)g(t)dt = 0, \text{ for all } \ g \in H^1(\mathbb{R})\right\}.$$

Throughout this paper, let $\Sigma$ denote a $\sigma$-algebra of subsets of a set $\Omega$, and $X$ will be an arbitrary Banach space with norm denoted $\|\cdot\|$. If $\mu$ is a positive measure on $\Omega$, then a vector-valued function $f : \Omega \to X$ is said to be *Bochner* (or *strongly*) *$\mu$-measurable* if there exists a sequence $\{f_n\}$ of $X$-valued simple functions on $\Omega$ such that $\lim_{n\to\infty} f_n(\omega) = f(\omega)$ $\mu$-a.e. on $\Omega$. Let $L^1(\Omega, \mu, X)$ denote the Banach space of all strongly measurable functions which satisfy $\int_\Omega \|f(\omega)\|\, d\mu(\omega) < \infty$, with norm $\|f\| = \int_\Omega \|f(\omega)\|\, d\mu(\omega)$. Such functions $f$ are always limits, in the norm of $L^1(X)$, of simple functions, and this allows us to define the *Bochner integral* of $f$, $\int_\Omega f(\omega)\, d\mu(\omega)$. For the remainder of this paper we will only consider Bochner integrals. A function $f : \Omega \to X$ is called *weakly measurable* if for each $x^* \in X^*$, the map $t \mapsto x^*(f(t))$ is a scalar-valued measurable function.

Also, denote by $M(\Omega, X)$ the space of countably additive $X$-valued measures of bounded variation on $(\Omega, \Sigma)$, with the 1-variation norm $\|\mu\|_1 <$

$\infty$ where

$$\|\mu\|_1 = \sup_{\pi} \sum_{B\in\pi} \|\mu(B)\|_X,$$

and the supremum is over all finite measurable partitions $\pi$ of $\Sigma$. We obtain results for $M(\Omega, X)$, where $X$ is a dual space, that is $X = Y^*$, for some Banach space $Y$. So for the rest of the paper we will consider only $M(\Omega, Y^*)$.

From now on, let $T = (T_t)_{t\in\mathbb{R}}$ denote a collection of uniformly bounded invertible isomorphisms of $M(\Omega, Y^*)$ such that

$$\sup_{t\in\mathbb{R}} \|T_t^{\pm 1}\| \leq c, \tag{1.1}$$

where $c$ is a positive constant.

We can now turn to the notion of analyticity for measures. Two definitions are needed.

**Definition 1.1** Let $(T_t)_{t\in\mathbb{R}}$ be a uniformly bounded collection of isomorphisms of $M(\Omega, Y^*)$. A measure $\mu \in M(\Omega, Y^*)$ is called *weakly measurable* if for every $A \in \Sigma$, the map $t \to T_t\mu(A)$ is Bochner measurable.

**Definition 1.2** Let $(T_t)_{t\in\mathbb{R}}$ be a uniformly bounded collection of isomorphisms of $M(\Omega, Y^*)$. A weakly measurable $\mu \in M(\Omega, Y^*)$ is called *weakly $T$-analytic* (or simply *weakly analytic*) if for all $A \in \Sigma$ and for all $y \in Y$, the map $t \to y(T_t\mu(A))$ is in $H^\infty(\mathbb{R})$.

We will repeatedly use the following two lemmas.

**Lemma 1.3** ([7], Corollary to Theorem 7.5.11) *If $T : X \to Y$ is a bounded linear operator, and $f : \Omega \to X$ is a Bochner integrable function with respect to a positive measure $\lambda$, then $Tf$ is Bochner integrable with respect to $\lambda$, and*

$$\int_\Omega T(f(t))d\lambda(t) = T\Big(\int_\Omega f(t)d\lambda(t)\Big). \tag{1.2}$$

**Lemma 1.4** *Let $T = (T_t)_{t\in\mathbb{R}}$ be a uniformly bounded collection of isomorphisms of $M(\Omega, Y^*)$, and suppose that $\mu \in M(\Omega, Y^*)$ is weakly measurable with respect to $T$. Then for all $f \in L_1(\mathbb{R})$, and all $y \in Y$,*

$$\int_{\mathbb{R}} y\Big(T_t\mu(A)\Big)f(t)dt = y\Big(\int_{\mathbb{R}} T_t\mu(A)f(t)dt\Big). \tag{1.3}$$

*Proof.* Since

$$\int_\Omega \|y\big(T_t\mu(A)\big)f(t)\|dt \le \|y\|\|T\|\,\|\mu\|_1\|f\|$$

then $T_t\mu(A)$ is Bochner integrable and the result follows from Lemma 1.3.

□

The following is a fundamental property that was introduced in [1] for scalar valued measures.

**Definition 1.5** Let $T = (T_t)_{t\in\mathbb{R}}$ be a uniformly bounded collection of isomorphisms of $M(\Omega, Y^*)$. Then we say that $T$ satisfies *hypothesis* $(A)$ if whenever $\mu \in M(\Omega, Y^*)$ is weakly analytic and is such that for every $A \in \Sigma$ we have $T_t\mu(A) = 0$ for almost all $t \in \mathbb{R}$, then $\mu$ is the zero measure.

Next we introduce a property that has been studied extensively by several mathematicians (for instance see [2], [8]). One way of defining it is the following. Let $\lambda$ denote the normalized Haar measure on $\mathbb{T}$.

**Definition 1.6** A complex Banach space $X$ is said to have the *analytic Radon-Nikodým property* ($ARNP$) if every measure $\mu \in M(\mathbb{T}, X)$, such that

$$\int_{\mathbb{T}} e^{-int}d\mu(t) = 0, \quad n < 0$$

has a Radon-Nikodým derivative in $L^1(\mathbb{T}, \lambda, X)$.

The following theorem shows that ARNP passes from $X$ to $M(\Omega, X)$ if $X$ is a dual space.

**Theorem 1.7** [11] *The Banach space $M(\Omega, Y^*)$ has the ARNP whenever $Y^*$ does.*

## 2. Main Lemma

First we will define a subspace of $M(\Omega, Y^*)^*$, which is in effect the simple $Y$-valued functions on $\Omega$. This concept was used by Talagrand [13]. Given $B_1, \ldots, B_n \in \Sigma$ and $y_1, \ldots, y_n \in Y$ define $\varphi^{B_1,B_2,\ldots,B_n}_{y_1,y_2,\ldots,y_n} \in M(\Omega, Y^*)^*$ in the following way:

$$\varphi^{B_1,B_2,\ldots,B_n}_{y_1,y_2,\ldots,y_n}(\mu) = \sum_{i=1}^{n} y_i(\mu(B_i)), \quad \mu \in M(\Omega, Y^*). \tag{2.1}$$

Let $E$ denote the collection of all such linear functionals. The set $E$ is a norming subspace, that is for all $\mu \in M(\Sigma, Y^*)$

$$\|\mu\|_1 = \sup\{\varphi(\mu) : \|\varphi\| = 1,\ \varphi \in E\}.$$

Now we will state the principal result of this section.

**Lemma 2.1** *Suppose $Y^*$ has ARNP. Let $E$ be as above, and let $f : \mathbb{R} \to M(\Omega, Y^*)$ be such that $\sup_{t\in\mathbb{R}} \|f(t)\|_1 < \infty$. Suppose further that for all $A \in \Sigma$, $t \to f(t)(A)$ is Bochner measurable and that for all $y \in Y$, for all $A \in \Sigma$ the map $t \to y(f(t)(A))$ is in $H^\infty(\mathbb{R})$. Then there exists a Bochner measurable essentially bounded $g : \mathbb{R} \to M(\Omega, Y^*)$ such that for all $\varphi \in E$ we have*

$$\varphi(g(t)) = \varphi(f(t)),$$

*for almost all $t \in \mathbb{R}$ .*

*Proof.* Let $\Phi(z) = i\frac{1-z}{1+z}$ be the conformal mapping of the unit disk onto the upper half plane, mapping $\mathbb{T}$ onto $\mathbb{R}$. Let $F = f \circ \Phi$. Consider the expression

$$y\Big(\int_A F(\theta)(B)\frac{d\theta}{2\pi}\Big),$$

where $A$ is a Borel subset of $\mathbb{T}$, $B \in \Sigma$. Since $F(\theta)(B)$ is Bochner integrable, one can apply Lemma 1.4 to obtain that $y(F(\theta)(B))$ is Lebesgue integrable and

$$y\Big(\int_A F(\theta)(B)\frac{d\theta}{2\pi}\Big) = \int_A y(F(\theta)(B))\frac{d\theta}{2\pi}. \tag{2.2}$$

Using (2.2), in the notation of (2.1), we see that for all $\varphi = \varphi^{B_1,\dots,B_n}_{y_1,\dots,y_n} \in E$:

$$\begin{aligned} y_1\Big(\int_A F(\theta)(B_1)\frac{d\theta}{2\pi}\Big) + \cdots + y_n\Big(\int_A F(\theta)(B_n)\frac{d\theta}{2\pi}\Big) \\ = \int_A \varphi^{B_1,\dots,B_n}_{y_1,\dots,y_n}(F(\theta))\frac{d\theta}{2\pi}, \end{aligned} \tag{2.3}$$

and that $\varphi(F(\theta))$ is Lebesgue integrable for every $\varphi \in E$. Hence, $\theta \to \varphi(F(\theta)) \in H^\infty(\mathbb{T})$ by (2.3) and by the assumption that $y(f(t)(A)) \in H^\infty(\mathbb{R})$.

Define a scalar-valued measure $\mu_\varphi$ on the Borel sets of $\mathbb{T}$ by

$$\mu_\varphi(A) = \int_A \varphi(F(\theta))\frac{d\theta}{2\pi}, \quad (\varphi \in E) \tag{2.4}$$

Then for all continuous functions $h$ on $\mathbb{T}$, one has that

$$\int_{\mathbb{T}} h(\theta)d\mu_\varphi(\theta) = \int_{\mathbb{T}} h(\theta)\varphi(F(\theta))\frac{d\theta}{2\pi}.$$

Now we will restrict to $\varphi_y^B \in E$, for the clarity of the proof, and later we will generalize the formulas for any $\varphi \in E$. First we want to prove that the $Y^*$-valued set function defined by:

$$m(B) = \int_A F(\theta)(B)\frac{d\theta}{2\pi}, \quad (B \in \Sigma)$$

is a countably additive measure of bounded variation, that is $m \in M(\Omega, Y^*)$. To show that $m$ is countably additive, it is enough ([4], Corollary 1.4.7) to prove that the scalar measure, defined by:

$$\eta_y(B) = y\Big(\int_A F(\theta)(B)\frac{d\theta}{2\pi}\Big),$$

is countably additive , for every $y \in Y$. Hence we need to show that for any sequence of disjoint sets $\{B_k\}$ in $\Sigma$, $\bigcup_{k=1}^\infty B_k \in \Sigma$:

$$\lim_{n\to\infty} y\Big(\int_A F(\theta)\Big(\bigcup_{k=1}^n B_k\Big)\frac{d\theta}{2\pi}\Big) = y\Big(\int_A F(\theta)\Big(\bigcup_{k=1}^\infty B_k\Big)\frac{d\theta}{2\pi}\Big). \tag{2.5}$$

Since $F(\theta) \in M(\Omega, Y^*)$, then

$$\lim_{n\to\infty} F(\theta)\Big(\bigcup_{k=1}^n B_k\Big) = F(\theta)\Big(\bigcup_{k=1}^\infty B_k\Big), \quad (\theta \in \mathbb{T})$$

and also

$$\|F(\theta)\Big(\bigcup_{k=1}^n B_k\Big)\|_{Y^*} \leq \|F(\theta)\|_1 < \infty, \quad (n > 0).$$

Therefore one can use the Dominated Convergence Theorem ([7], Theorem 3.7.9) to obtain

$$\lim_{n\to\infty} \int_A F(\theta)\Big(\bigcup_{k=1}^n B_k\Big)\frac{d\theta}{2\pi} = \int_A F(\theta)\Big(\bigcup_{k=1}^\infty B_k\Big)\frac{d\theta}{2\pi}.$$

Next we can apply $y \in Y$ to both sides to get (2.5), which shows that $m$ is countably additive. All that is left to show is that $\|m\|_1 < \infty$:

$$\begin{aligned}
\|m\|_1 &= \sup_{\pi} \sum_{B \in \Sigma} \| \, m(B) \, \|_{Y^*} = \sup_{\pi} \sum_{B \in \Sigma} \| \int_A F(\theta)(B) \frac{d\theta}{2\pi} \, \|_{Y^*} \\
&\leq \sup_{\pi} \sum_{B \in \Sigma} \int_A \|F(\theta)(B)\|_{Y^*} \frac{d\theta}{2\pi} \\
&\leq \sup_{\pi} \int_A \sum_{B \in \Sigma} \|F(\theta)(B)\|_{Y^*} \frac{d\theta}{2\pi} \\
&\leq \sup_{\pi} \int_A \|F(\theta)(B)\|_1 \frac{d\theta}{2\pi} \\
&\leq \sup_{\theta \in \mathbb{T}} \|F(\theta)\|_1 < \infty.
\end{aligned}$$

Thus we have shown that $m \in M(\Omega, Y^*)$. Denoting $\mu(A) = m \in M(\Omega, Y^*)$, it follows from (2.3) and (2.4) that

$$\mu_{\varphi_y^B}(A) = y(m(B)) = \varphi_y^B(m) = \varphi_y^B(\mu(A)).$$

Similarly, for any $\varphi \in E$ using the definition of $\mu_\varphi(A)$, and the formula (2.3) we can show that

$$\mu_\varphi(A) = \varphi(\mu(A)). \tag{2.6}$$

We show now that the above defined $\mu$ is a $M(\Omega, Y^*)$-valued countably additive measure of bounded variation. First, the fact that $\mu$ is countably additive follows easily from the definition of $\mu$. Next, we will obtain the inequality:

$$\|\mu(A)\|_1 \leq \int_A \|F(\theta)\|_1 \frac{d\theta}{2\pi}. \tag{2.7}$$

This follows because $E$ is a norming subspace and hence, given $A \in \Sigma$ and $\epsilon > 0$, there exists $\varphi \in E$, with $\|\varphi\| \leq 1$ and $\|\mu(A)\|_1 \leq |\varphi(\mu(A))| + \epsilon$, and hence (2.7) follows since

$$|\varphi(\mu(A))| = \Big| \int_A \varphi(F(\theta)) \frac{d\theta}{2\pi} \Big| \leq \int_A \|F(\theta)\|_1 \frac{d\theta}{2\pi}.$$

Therefore we have established that $\mu$ is a measure of bounded variation, that is $\mu \in M(\mathbb{T}, M(\Omega, Y^*))$.

Next we show that

$$\int_{\mathbb{T}} e^{-in\theta} d\mu(\theta) = 0, \quad (n < 0).$$

This follows because $E$ is norming, and if $\varphi \in E$ and $n < 0$, then

$$\varphi\Big(\int_{\mathbb{T}} e^{-in\theta} d\mu(\theta)\Big) = \int_{\mathbb{T}} e^{-in\theta} d\mu_\varphi(\theta) = \int_{\mathbb{T}} e^{-in\theta} \varphi(F(\theta)) \frac{d\theta}{2\pi} = 0.$$

Since we assume that $M(\Omega, Y^*)$ has ARNP, one can find a Bochner integrable function $G \in L^1(\mathbb{T}, M(\Omega, Y^*))$ such that

$$\mu(A) = \int_A G(\theta) \frac{d\theta}{2\pi}, \tag{2.8}$$

for all Borel subsets $A$ of $\mathbb{T}$. Using (2.4), (2.6), and (2.8) one gets for all $A \in \mathcal{B}$, $\varphi \in E$,

$$\int_A \varphi(G(\theta)) \frac{d\theta}{2\pi} = \varphi(\mu(A)) = \mu_\varphi(A) = \int_A \varphi(F(\theta)) \frac{d\theta}{2\pi}.$$

Since this is true for all $\varphi \in E$ and $A \in \mathcal{B}$, one can conclude that for given $\varphi \in E$,

$$\varphi(G(\theta)) = \varphi(F(\theta)) \quad \text{for almost all } \theta.$$

Since we have (2.7), and (2.8), we can apply Lemma 2.3 in [1], from which it follows that $G$ is essentially bounded. Let $g(t) = G(\Phi^{-1}(t))$. Then $g$ is a Bochner measurable, essentially bounded function and for every $\varphi \in E$,

$$\varphi(g(t)) = \varphi(G(\Phi^{-1}(t))) = \varphi(F(\Phi^{-1}(t))) = \varphi(f(t)), \quad \text{for almost all } t \in \mathbb{R}$$

completing the proof of the lemma. □

We will use the lemma in the following setting. Let $T = \{T_t\}_{t\in\mathbb{R}}$ be a family of uniformly bounded isomorphisms of $M(\Omega, Y^*)$ such that $\|T_t\| \le c$. Suppose that $\mu \in M(\Omega, Y^*)$ is weakly analytic and let $f(t) = T_t\mu$ for $t \in \mathbb{R}$. Then we have that $\|f(t)\|_1 \le c\|\mu\|_1$, and that the map $t \to y(f(t)(A)) \in H^\infty(\mathbb{R})$ for all $y \in Y$ and for all $A \in \Sigma$.

**Corollary 2.2** *Let $Y^*$ have ARNP. Let $T = \{T_t\}_{t\in\mathbb{R}}$ be a family of uniformly bounded isomorphisms of $M(\Omega, Y^*)$. Let $\mu \in M(\Omega, Y^*)$ be a weakly analytic measure. Then there exists a Bochner measurable essen-*

*tially bounded function* $g : \mathbb{R} \to M(\Omega, Y^*)$ *such that for all* $A \in \Sigma$,

$$g(t)(A) = T_t\mu(A) \quad \textit{for almost all} \quad t \in \mathbb{R}.$$

*Remark* 2.3. Since the results in this section hold for Banach spaces which are dual spaces and have ARNP, we give some examples of these spaces. Examples of Banach spaces which are dual spaces and have ARNP are:

(1) Orlicz spaces $L^\Phi(\mu)$, where $(\Omega, \Sigma, \mu)$ is a $\sigma$-finite measure space, and $\Phi$ is an Orlicz function for which it and its Young's complementary function satisfy the $\Delta_2$-condition (see [4] and [9]).

(2) All dual spaces that are Banach lattices not containing $c_0$, or preduals of a von Neuman algebra (see [8]).

## 3. Bochner measurability of weakly analytic measures

For all the results of this section, which include the main results of this paper, we will suppose that $T = (T_t)_{t\in\mathbb{R}}$ is a one-parameter group of uniformly bounded invertible isomorphisms of $M(\Sigma)$ for which (1.1) holds. We will also suppose that

$$T_t^* : E \to \bar{E}, \tag{3.1}$$

where $\bar{E}$ denotes the closure of $E$ in $M(\Omega, Y^*)^*$.

We begin with some properties of the convolution, which are technical, but needed for proving the main theorem.

Consider a weakly measurable $\mu \in M(\Omega, Y^*)$ and $\nu \in M(\mathbb{R})$, where $\nu$ is absolutely continuous with respect to Lebesgue measure. Define:

$$\nu *_T \mu(A) = \int_{\mathbb{R}} T_{-t}\mu(A) d\nu(t). \tag{3.2}$$

When there is no risk of confusion, we will simply write $\nu * \mu$ for $\nu *_T \mu$. The integral is well defined since $T_{-t}\mu(A)$ is $\nu$-measurable and bounded, hence $\nu$-integrable. One can check that the formula defines a countably additive vector-valued measure of bounded variation, using properties of Bochner integral and the fact that $\mu \in M(\Omega, Y^*)$. Hence, $\nu *_T \mu \in M(\Omega, Y^*)$.

From now on assume that $\nu, \sigma \in M(\mathbb{R})$ and let $\mu \in M(\Omega, Y^*)$ be weakly measurable. Let $\nu$, $\sigma$ be absolutely continuous with respect to Lebesgue measure. Let $E$ be defined as in the previous section.

**Lemma 3.1** *Suppose that* $\varphi \in \bar{E}$. *Then the mapping* $t \to \varphi(T_t\mu)$ *is*

*Lebesgue measurable on* $\mathbb{R}$. *Futhermore,*

$$\int_{\mathbb{R}} \varphi(T_{-s}\mu)d\nu(s) = \varphi(\nu *_T \mu).$$

*Proof.* It is sufficient to prove it in the case that $\varphi \in E$. The mapping $t \to \varphi(T_t\mu)$ is Lebesgue measurable, because $\mu$ is weakly measurable. Consider $\varphi_y^A \in E$. Then

$$\begin{aligned}\int_{\mathbb{R}} \varphi_y^A(T_{-s}\mu)d\nu(s) &= y\Big(\int_{\mathbb{R}} (T_{-s}\mu)(A)d\nu(s)\Big) \\ &= y\Big(\nu *_T \mu(A)\Big) = \varphi_y^A(\nu *_T \mu).\end{aligned}$$

Similarly, one can show the result holds for any $\varphi \in E$, so the lemma follows. □

**Corollary 3.2** *For all* $t \in \mathbb{R}$, *we have*

$$T_t(\nu * \mu) = \nu * (T_t\mu).$$

*Moreover, the measure* $\nu * \mu$ *is weakly measurable.*

*Proof.* Using lemma 1.3 we have that for any $A \in \Sigma$, $y \in Y$:

$$\begin{aligned}y\Big(\nu * (T_t\mu)(A)\Big) &= y\Big(\int_{\mathbb{R}} T_{-s+t}\mu(A)d\nu(s)\Big) \\ &= \int_{\mathbb{R}} y\Big(T_{-s+t}\mu(A)\Big)d\nu(s) \\ &= \int_{\mathbb{R}} \varphi_y^A\Big(T_t(T_{-s}\mu)\Big)d\nu(s) \\ &= \int_{\mathbb{R}} T_t^*(\varphi_y^A)(T_{-s}\mu)d\nu(s).\end{aligned}$$

Applying the previous lemma:

$$\begin{aligned}\int_{\mathbb{R}} T_t^*(\varphi_y^A)(T_{-s}\mu)d\nu(s) &= T_t^*(\varphi_y^A)(\nu *_T \mu) \\ &= \varphi_y^A\Big(T_t(\nu *_T \mu)\Big) \\ &= y\Big(T_t(\nu *_T \mu)(A)\Big),\end{aligned}$$

hence $\nu * (T_t\mu) = T_t(\nu * \mu)$.

Next we want to show that $\nu *_T \mu$ is weakly measurable, that is we need to prove that $f(t) = T_t(\nu * \mu)(A)$ is Bochner measurable for all $A \in \Sigma$. By the previous part we have that

$$f(t) = T_t(\nu * \mu)(A) = \nu * (T_t\mu)(A) = \int_{\mathbb{R}} T_{t-s}\mu(A)d\nu(s).$$

By the Pettis measurability theorem [4, Theorem II.2], we need to show that $f(t)$ is weakly measurable and essentially separably valued. Since $T_t\mu(A)$ is Bochner measurable, there exists a sequence of simple functions $\{f_n\}$ such that $f_n(t) \to T_t\mu(A)$ in $Y^*$-norm. Let

$$f_n(t) = \sum_{i=1}^{k} y^*_{i,n}\chi_{A_{i,n}}(t).$$

Note that for all $n \in \mathbb{N}$

$$\begin{aligned} \nu * f_n(t) &= \int_{\mathbb{R}} \sum_{i=1}^{k} y^*_{i,n}\chi_{A_{i,n}}(t-s)d\nu(s) \\ &= \sum_{i=1}^{k} y^*_{i,n} \int_{\mathbb{R}} \chi_{A_{i,n}}(t-s)d\nu(s) \\ &= \sum_{i=1}^{k} y^*_{i,n}\Big(\chi_{A_{i,n}} * \nu(t)\Big). \end{aligned}$$

This calculation shows that $\nu * f_n$ has finite dimensional range, hence $\nu * f_n(\mathbb{R})$ is separable for all $n \in \mathbb{N}$. It can be easily checked that $\nu * (T_t\mu)(A) = \lim_{n\to\infty}(\nu * f_n)(t)$ in $Y^*$-norm, therefore $\nu * (T_t\mu)(A)$ has separable range. To show that $f(t)$ is weakly measurable, we need to verify that $y(f(t))$ is measurable for all $y \in Y$ . It follows from Lemma 1.3 that

$$y(f(t)) = \int_{\mathbb{R}} y\Big(T_{t-s}\mu(A)\Big)d\nu(s),$$

and hence $y(f(t))$ is measurable being the convolution of a measure in $M(\mathbb{R})$ and a bounded measurable function on $\mathbb{R}$. This finishes the proof of the corollary . □

**Corollary 3.3** *With the above notation, one has*

$$(\sigma * \nu) * \mu = \sigma * (\nu * \mu).$$

*Proof.* For $A \in \Sigma$, $y \in Y$, one can reduce it to the scalar case using

Lemma 1.3:

$$\begin{aligned} y\Big[(\sigma * \nu) * \mu(A)\Big] &= y\Big[\int_{\mathbb{R}} (T_{-s}\mu(A) d(\sigma * \nu)(s)\Big] \\ &= \int_{\mathbb{R}} y(T_{-s}\mu(A)) d(\sigma * \nu)(s). \end{aligned}$$

Next we can use the results known in scalar case [5, Theorem 19.10] to obtain that

$$\int_{\mathbb{R}} y(T_{-s}\mu(A)) d(\sigma * \nu)(s) = \int_{\mathbb{R}} y\Big(\nu * T_{-s}\mu(A)\Big) d\sigma(s).$$

Finally, using Lemma 1.3 again gives us that:

$$\int_{\mathbb{R}} y\Big(\nu * T_{-s}\mu(A)\Big) d\sigma(s) = y\Big(\sigma * (\nu * \mu)(A)\Big).$$

□

Let $g$ be the Bochner measurable function defined on $\mathbb{R}$ with values in $M(\Omega, Y^*)$ given by Corollary 2.2. Let $\mu$ be a weakly analytic measure in $M(\Omega, Y^*)$. For $y > 0$, let $P_y$ be the Poisson kernel on $\mathbb{R}$:

$$P_y(x) = \frac{1}{\pi} \frac{y}{x^2 + y^2} \quad (x \in \mathbb{R}).$$

We can form the Poisson integral of $g$ as follows:

$$P_y * g(t) = \int_{\mathbb{R}} g(t - x) P_y(x) dx,$$

where the integral exists as a Bochner integral.

**Proposition 3.4** *We have that*

$$\lim_{y \to 0} P_y * g(t) = g(t) \tag{3.3}$$

*in $M(\Omega, Y^*)$-norm, for almost all $t \in \mathbb{R}$.*

*Proof.* Since $g$ is measurable and essentially bounded, the proof is similar to the classical proof for scalar-valued functions. □

**Lemma 3.5** *For all $t \in \mathbb{R}$, one has that*

$$P_y * g(t) = P_y * T_t\mu.$$

*Proof.* For any $A \in \Sigma$, $y_1 \in Y$, one has by Lemma 1.3:

$$\begin{aligned} y_1\Big(P_y * g(t)(A)\Big) &= y_1\Big(\int_{\mathbb{R}} g(s)(A)P_y(t-s)ds\Big) \\ &= \int_{\mathbb{R}} y_1\Big(g(s)(A)\Big)P_y(t-s)ds. \end{aligned}$$

By Lemma 1.3 again,

$$\begin{aligned} \int_{\mathbb{R}} y_1\Big(g(s)(A)\}\Big)P_y(t-s)ds &= \int_{\mathbb{R}} y_1\Big(T_s\mu(A)\}\Big)P_y(t-s)ds \\ &= \int_{\mathbb{R}} y_1\Big(T_{t-s}\mu(A)\Big)P_y(s)ds \\ &= y_1\Big(\int_{\mathbb{R}} T_{t-s}\mu(A)P_y(t-s)ds\Big) \\ &= y_1\Big(P_y * (T_t\mu)(A)\Big), \end{aligned}$$

which completes the proof of the lemma. □

**Lemma 3.6** *Let $t_0$ be any real number such that* (3.3) *holds. Then for all $t \in \mathbb{R}$, we have*:

$$\lim_{y\to 0} P_y * T_t\mu = T_{t-t_0}(g(t_0))$$

*in $M(\Omega, Y^*)$-norm.*

*Proof.* Since $P_y * g(t_0) \to g(t_0)$, it follows that

$$T_{t-t_0}\Big((P_y * g(t_0)\Big) \to T_{t-t_0}(g(t_0)).$$

By Lemma 3.5 and Corollary 3.2 we get

$$T_{t-t_0}\Big(P_y * g(t_0)\Big) = T_{t-t_0}\Big(P_y * T_{t_0}\mu\Big) = P_y * T_t\mu,$$

establishing the lemma. □

**Theorem 3.7** (Main theorem) *Suppose $T = (T_t)_{t\in\mathbb{R}}$ satisfies hypothesis $(A)$. Let $g$ be as above. Then we have*

$$T_t\mu = g(t), \quad \textit{for almost all } \ t \in \mathbb{R}.$$

*Consequently, the mapping $t \to T_t\mu$ is Bochner measurable.*

*Proof.* It is enough to show that the equality in the theorem holds for all $t = t_0$ where (3.3) holds. Fix such a $t_0$ and let $A \in \Sigma$. Since $t \to T_t\mu(A)$ is a bounded, measurable function on $\mathbb{R}$, then, as in Proposition 3.4, one can show that

$$P_y * (T_t\mu)(A) \to T_t\mu(A), \quad \text{for almost all } t \in \mathbb{R}.$$

By Lemma 3.6:

$$P_y * (T_t\mu)(A) \to T_{t-t_0}\Big(g(t_0)(A)\Big), \quad (t \in \mathbb{R}).$$

Hence

$$T_t\mu(A) = T_{t-t_0}g(t_0)(A), \quad \text{for almost all } t \in \mathbb{R}. \tag{3.4}$$

From Lemma 3.5, one has that $T_{t_0}g(t_0) = \lim_{t\to 0} P_y * \mu$ in $M(\Omega, Y^*)$ norm. Since $\mu$ is weakly analytic, it follows that $P_y * \mu$ is weakly analytic, because for all $y_1 \in Y$ and all $A \in \Sigma$ and all $h(t) \in H^1(\mathbb{R})$ we have that

$$\begin{aligned}
&\int_{\mathbb{R}} y_1\Big(T_t(P_y * \mu)(A)\Big)h(t)dt \\
&\quad = \int_{\mathbb{R}} y_1\left(T_t \int_{\mathbb{R}} T_{-s}\mu(A)P_y(s)ds\right)h(t)dt \\
&\quad = \int_{\mathbb{R}}\int_{\mathbb{R}} y_1\Big(T_{t-s}\mu(A)\Big)P_y(s)h(t)dsdt \\
&\quad = \int_{\mathbb{R}}\int_{\mathbb{R}} y_1\Big(T_{t-s}\mu(A)\Big)P_y(s)h(t)dtds \\
&\quad = \int_{\mathbb{R}} y_1\left(T_{-s}\int_{\mathbb{R}} T_t\mu(A)h(t)dt\right)P_y(s)ds = 0.
\end{aligned}$$

One can also show that $T_{-t_0}g(t_0)$ is weakly analytic, because $\mu$ is weakly analytic, that is for all $y_1 \in Y$ and all $A \in \Sigma$ and all $h(t) \in H^1(\mathbb{R})$ we have that

$$\begin{aligned}
\int_{\mathbb{R}} y_1\Big(T_{t-t_0}g(t_0)(A)\Big)h(t)dt &= \int_{\mathbb{R}} y_1\left(T_t\Big\{\lim_{y\to 0} P_y * \mu(A)\Big\}\right)h(t)dt \\
&= \int_{\mathbb{R}} \lim_{y\to 0} y_1\Big(T_t(P_y * \mu(A))\Big)h(t)dt \\
&= \lim_{y\to 0}\int_{\mathbb{R}} y_1\Big(T_t(P_y * \mu(A))\Big)h(t)dt.
\end{aligned}$$

Taking $t = 0$ in (3.5) one has that

$$\mu(A) - T_{-t_0}g(t_0)(A) = 0, \quad (A \in \Sigma).$$

Since the measure $\mu - T_{-t_0}g(t_0)$ is weakly analytic, then using hypothesis

$(A)$, we have that

$$\mu = T_{-t_0} g(t_0).$$

Applying $T_{t_0}$ to both sides of this equality completes the proof. □

**Theorem 3.8** *Let $T$ and $\mu$ be as in the main theorem. Then*

$$\lim_{y \to 0} P_y * \mu = \mu \quad \textit{in the } M\textit{-norm.}$$

*Proof.* Let $t_0$ be such that (3.3) holds. Then

$$T_{-t_0}(P_y * g(t_0)) \to T_{-t_0}(g(t_0))$$

in the $M(\Omega, Y^*)$-norm. By Lemma 3.2, we have that $T_{-t_0}(P_y * g(t_0)) = P_y * \mu$. By Theorem 3.7 we have that $T_{-t_0} g(t_0) = \mu$. The result follows. □

**Theorem 3.9** *Let $\mu$ and $T$ be as in the main theorem. Then the mapping $t \to T_t \mu$ is uniformly continuous from $\mathbb{R}$ into $M(\Omega, Y^*)$.*

*Proof.* For each $y > 0$, consider the map $t \to P_y * g(t)$. Since $g$ is essentially bounded, one can easily see that this map is continuous. Hence by Lemma 3.2 we have that the map $t \to P_y * T_t \mu = T_t(P_y * \mu)$ is continuous. By Theorem 3.5. one has that

$$T_t(P_y * \mu) \to T_t \mu$$

uniformly in $t$. It follows that $t \to T_t \mu$ is uniformly continuous. □

**Definition 3.10** Let $\mu \in M(\Omega, Y^*)$, where $\Omega$ is a $\sigma$-field and $\lambda \in M(\Omega)$. The measure $\mu$ is called $\lambda$-continuous (in symbols $\mu \ll \lambda$), if $\mu$ vanishes on sets of $|\lambda|$-measure zero.

Let $\Omega = \mathbb{T}$, take $T_t$ to be translation, that is $T_t \mu(\cdot) = \mu(\cdot + t)$. It is possible to show (see [1] and Proposition 1.7, [5]), similarly as in scalar case that, the definition of weak analyticity of the measure $\mu$ can be written as a condition:

$$\int_{-\pi}^{\pi} e^{-int} d \langle y, \mu(t) \rangle = 0, \ \forall n < 0, \ \forall y \in Y.$$

This definition coincides with the definition of the analyticity in [12] by R. Ryan. Although in [12] it is referred to as a Pettis integral, it is an example of what [4] calls a Gelfand or weak* integral. R. Ryan [12] obtained that if $\mu$ is weakly analytic, then $\mu \ll \lambda$. Unlike the scalar case, there exist weakly analytic vector-valued measures that do not translate continuously. For example, let $g(t) = \exp\left(-i \cot \frac{t}{2}\right)$. Then the measure defined by $d\mu(t) = g(t - \cdot)dt$, is a weakly analytic measure in $M(\mathbb{T}, L^\infty(\mathbb{T}))$, but it does not translate continuously, although $\mu \ll \lambda$. We have the following as a corollary of Theorem 3.9.

**Corollary 3.11** *Let $\mu \in M(\mathbb{T}, Y^*)$. Assume that $Y^*$ has ARNP. If $\mu$ is a measure such that*

$$\int_{-\pi}^{\pi} e^{-int} d\left\langle y, \mu(t)\right\rangle = 0, \ \forall n < 0, \ \forall y \in Y.$$

*then the mapping $t \to \mu(e^{i(\cdot+t)})$ is continuous.*

Hence we have obtained that if we further assume that $Y^*$ has the analytic Radon-Nikodým property (ARNP), then every weakly analytic measure translates continuously.

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