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Decay of solutions to the Cauchy problem for the Klein-Gordon equation with a localized nonlinear dissipation

(Dedicated to Professor Rentaro Agemi on his Sixtieth birthday)

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Abstract. We derive a precise decay estimate of the solutions to the Cauchy problem for the Klein-Gordon equation with a nonlinear dissipation:

$$u_{tt} - \Delta u + u + \rho(x, t, u_t) = 0 \quad \text{in} \quad R^N \times [0, \infty),$$

$$u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x),$$

where $\rho(x, t, v)$ is a function like $\rho = a(x)(1+t)^\theta |v|^r v$, $-1 < r$, with $a(x) \geq 0$ supported on $\Omega_R = \{x \in R^N \mid |x| \geq R\}$ for some $R > 0$.

Key words: decay, localized dissipation, wave equation.

1. Introduction

In this paper we are concerned with a decay property of the solutions to the Cauchy problem for the Klein-Gordon equation with a dissipative term:

$$u_{tt} - \Delta u + u + \rho(x, t, u_t) = 0 \quad \text{in} \quad R^N \times [0, \infty), \tag{1.1}$$

$$u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x), \tag{1.2}$$

where $\rho(x, t, v)$ is a function like $\rho = a(x)(1+t)^\theta |v|^r v$, $-1 < r$, with $a(x) \geq 0$ supported on $\Omega_R = \{x \in R^N | \ |x| \geq R\}$ for some $R > 0$.

To explain our problem, let us consider a typical case $\rho = a(x)|v|^r v$.

When $a(x) \geq \varepsilon_0 > 0$ on R^N, we have proved in [7] that the solution $u(t) \in C^1([0, \infty); L^2(R^N)) \cap C([0, \infty); H_1(R^N))$ with $\operatorname{supp} u_0 \cup \operatorname{supp} u_1 \subset B_L \equiv \{x \in R^N \mid |x| \leq L\}$, $L > 0$, satisfies the decay estimate

$$E(t) \equiv \frac{1}{2}\{||u_t(t)||^2 + ||\nabla u(t)||^2 + ||u(t)||^2\}$$

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$$\leq \begin{cases} C_L(E(0))(1+t)^{-(2-Nr)/r} & \text{if} \quad 0 < r < 2/N \\ C_L(E(0))(\log(2+t))^{-2/r} & \text{if} \quad r = 2/N. \end{cases} \tag{1.3}$$

Needless to say, if $r = 0$ we have the usual exponential decay

$$E(t) \leq CE(0)e^{-\lambda t}, \quad \lambda > 0, \tag{1.4}$$

without the support condition on (u_0, u_1).

The estimate (1.3) seems to be sharp, because T. Motai [3] proved that if $r > 2/N$ small amplitude solutions do not decay to 0 as $t \to \infty$. More precisely, it is proved in [3] that if we assume $r > N/2$, $(u_0, u_1) \in H_2 \cap W^{1,s+1} \times H_1 \cap W^{1,s} \cap L^{2(r+1)}$ with $s > N/2$ and $|||u_0|||_{W^{1,s+1}} + |||u_1|||_{W^{1,s}}$ is sufficiently small (but not 0), then $E(t)$ does not converge to 0 as $t \to \infty$. For a generalization or closely related result see also K. Mochizuki and T. Motai [4]. Here, we note that the critical exponent $2/N$ appears only for the case of whole space or exterior domains. Indeed, for the case of bounded domains we know, under the homogeneous Dirichlet boundary condition,

$$E(t) \leq C(E(0))(1+t)^{-2/r} \quad \text{if} \ \ 0 < r \leq 4/(N-2)^+$$

and further,

$$E(t) \leq C(\|u_0\|_{H_2}, \|u_1\|_{H_1})(1+t)^{-2/r} \quad \text{if} \ \ 4/(N-2)^+ < r \leq 8/(N-4)^+,$$

where we use a notation $\alpha \equiv \max\{\alpha, 0\}$. (See [5, 8].) For generalizations in various directions of these results see [10] and the references cited there. The restriction $0 \leq r \leq 2/N$ in the case of whole space comes from the reason that we must estimate $||u_t(t)||$ by $\int_{R^N} \rho(x, u_t)u_t dx$ and for this we must use an inequality like

$$\|u_t(t)\| \leq C_L(1+t)^{Nr/2(r+2)}\|u_t(t)\|_{L^{r+2}}, \tag{1.5}$$

which is shown by the use of the property $\operatorname{supp} u_t(t) \subset B_{L+t}$. We note that the decay property depends in a delicate way on such a time-dependent inequality.

Now, quite recently, Zuazua [16] has treated the linear case $\rho(x, v) = a(x)v$ with $a(x)$ vanishing on a neighbourhood of the origine 0 and proved the exponential decay of $E(t)$. That is, under the assumption $a(x) \geq \varepsilon_0 > 0$ on $\Omega_R, R > 0$, he has proved the estimate (1.4) for the solutions $u(t)$. In fact, a semilinear equation having a term $f(u)$ is treated in [16].

The object of this paper is to combine the ideas in [7] and [16] to derive precise decay estimates for the solutions of the problem (1.1)–(1.2) with $\rho(x,t,u_t)$ like $\rho = a(x)(1+t)^\theta|u_t|^r u_t$, $-1 < r \le 2/N$, where $a(x)$ is supported on $\Omega_R, R > 0$, as in [16].

The decay properties of the solutions to the initial-boundary value problem of the wave equation in a bounded domain with a localized dissipation have been investigated by E. Zuazua [15] and the present author [12, 13, 14] in various situations. Our argument is necessarily related to those papers and also to J. Lions [1], where the boundary control problem for the wave equation is proposed and investigated in detail.

2. Statement of the result

We use only familiar function spaces and omit the definition of them.

Treatment of the nonlinear localized dissipation is delicate and we must consider the so called H_2-solutions rather than usual energy finite solutions. For this we assume:

Hyp. A. $(u_0, u_1) \in H_2 \times H_1$ and

$$\operatorname{supp} u_0 \cup \operatorname{supp} u_1 \subset \{x \in R^N \mid |x| \le L\}$$

for some $L > 0$.

We want to consider a class of $\rho(x,t,v)$ including $a(x)(1+|x|)^{\theta_1}(1+t)^{\theta_2}|v|^r v$, $a(x)(1+|x|+t)^\theta|v|^r v$ etc. with $-1 < r$, and make the following assumptions on ρ.

Hyp. B $\rho(x,t,v)$ is continuous on $R^N \times R^+ \times R$, differentiable in (t,v), $v \ne 0$, and satisfies the conditions:

$$k_0 a(x)(1+t)^\theta |v|^{r+2} \le \rho(x,t,v)v \le k_1 a(x)(1+t)^\theta |v|^{r+2} \tag{1}$$

for $x \in B_{L+t} = \{x \in R^N \mid |x| \le L+t\}$ and $t \in R^+$ if $|v| \le 1$,

and

$$k_0 a(x)(1+t)^\theta |v|^{p+2} \le \rho(x,t,v)v \le k_1 a(x)(1+t)^\theta |v|^{p+2} \tag{2}$$

for $x \in B_{L+t}$ and $t \in R^+$ if $|v| \ge 1$, where k_0, k_1 are positive constants possibly depending on L, the exponents r and p satisfy

$$-1 < r \quad \text{and} \quad -1 \le p \le 2/(N-2)^+,$$

respectively, and $a(x)$ is a nonnegative bounded function such that

$$a(x) \geq \varepsilon_0 > 0 \quad \text{on} \quad \Omega_R = \{x \in R^N \mid |x| \geq R\}$$

for some $R > 0$.

Further, ρ satisfies

$$|\rho_t|^2 \leq k_2 \rho_v \rho v \quad \text{for} \quad x \in B_{L+t},\ t \in R^+ \quad \text{and} \quad v \neq 0 \tag{3}$$

for some positive constant k_2 which may depend on L.

We could treat a little more general class of ρ. But, to make the essential feature clear we restrict ourselves to the class of ρ as in Hyp. B.

For convenience to the readers we give some comments on the assumptions above. Roughly speaking, Hyp. B means that when $x \in B_{L+t}, \rho(x,t,v)$ behaves like $a(x)(1+t)^\theta |v|^r v$ as $|v| \to 0$ and like $a(x)(1+t)^\theta |v|^p v$ as $|v| \to \infty$. Therefore, the restriction on r below, for example, $r \leq 2/N$ (in the case $\theta = 0$) is made on the behaviour of ρ near $v = 0$, while the restriction on p is made on the growth order of ρ as $|v| \to \infty$. Compared with the case of bounded domains, the condition $p \leq 2/(N-2)^+$ might be considered stronger, but this as well as the condition on the initial data $(u_0, u_1) \in H_2 \times H_1$ is required for the estimation of the term

$$\int_t^{t+T} \int_{B_{2R}} |\rho(x,t,u_t)(x \cdot \nabla u)| dx ds$$

(see (4.10), (5.3) and (6.6)), which is not necessary if $a(x)$ is effective in whole space. A further remark on the restriction on p will be given after the proof of Proposition 1 in the Section 4. When $r < 0$ we must require the condition $(u_0, u_1) \in H_2 \times H_1$ even for the case $-1 \leq p \leq 0$ and $a(x) > \epsilon_0 > 0$ in R^N.

Our result reads as follows.

Theorem 1 *Under the hypotheses A and B, the problem* (1.1)–(1.2) *admits a unique solution $u(t)$ in the class*

$$W^{2,\infty}([0,\infty); L^2) \cap W^{1,\infty}([0,\infty); H_1) \cap L^\infty([0,\infty); H_2),$$

with the finite propagation property $\operatorname{supp} u(t) \subset B_{L+t}$, $t \geq 0$, *satisfying the decay estimate below.*

(1) *The case:* $0 \leq r$ *and* $0 \leq p \leq 2/(N-2)^+$.

$(1)_1$ *If* $\max\{-1, \frac{Nr-2}{2}\} < \theta < \min\{1, \frac{2(p+1)}{pN+2}\}$, *then, except for the case*

$p = r = 0$,

$$E(t) \le C_1(1+t)^{-2\eta} \tag{2.1}$$

with

$$\eta = \min\left\{\frac{2+2\theta-Nr}{2r}, \frac{2(p+1)-(pN\theta^+ + 2\theta)}{(N-2)^+p}\right\}.$$

$(1)_2$ *If* $\theta = \frac{Nr-2}{2} < \min\{1, \frac{2p+2}{pN+2}\}$, *then*

$$E(t) \le C_1(\log(2+t))^{-2/r}. \tag{2.2}$$

$(1)_3$ *If* $\frac{Nr-2}{2} < \theta = \frac{2p+2}{pN+2} \le 1$, *then*

$$E(t) \le \begin{cases} C_1(\log(2+t))^{-4(p+1)/(N-2)^+p} & \text{if } p \ne 0 \text{ and } N \ge 3 \\ C_1(1+t)^{-\alpha} & \text{if } p = 0 \text{ or } N = 2 \end{cases} \tag{2.3}$$

for some $\alpha > 0$.
$(1)_4$ *If* $\theta = \frac{Nr-2}{2} = \frac{2p+2}{pN+2}$, *and* $p + r > 0$, *then*

$$E(t) \le C_1(\log(2+t))^{-2\tilde{\eta}} \tag{2.4}$$

with

$$\tilde{\eta} = \min\left\{\frac{1}{r}, \frac{2(p+1)}{(N-2)^+p}\right\}.$$

$(1)_5$ *If* $p = r = 0$ *and* $|\theta| \le 1$, *then*

$$E(t) \le \begin{cases} C_1 \exp\{-\lambda t^{1-|\theta|}\} & \text{if } |\theta| < 1 \\ C_1(1+t)^{-\alpha} & \text{if } |\theta| = 1 \end{cases} \tag{2.5}$$

for some $\lambda > 0$, $\alpha > 0$.
(2) *The case*: $-1 < r < 0$ *and* $0 \le p \le \frac{2}{(N-2)^+}$.
$(2)_1$ *If* $-1 < \theta < Nr/2$, *then*

$$E(t) \le C_1(1+t)^{-2\eta} \tag{2.6}$$

with

$$\eta = \min\left\{\frac{2r+Nr+2-2\theta}{-2r}, \frac{2p+2-2\theta}{(N-2)^+p}\right\}.$$

$(2)_2$ *If* $\max\{-1, Nr/2\} < \theta < \min\{r+1+Nr/2, \frac{4(p+1)+pN^2r}{2(pN+2)}\}$, *then*

$$E(t) \le C_1(1+t)^{-2\eta} \tag{2.7}$$

with

$$\eta = \min\left\{\frac{2r+Nr+2-2\theta}{-2r}, \frac{4(p+1)-2(pN+2)\theta+pN^2r}{2(N-2)^+p}\right\}.$$

$(2)_3$ *If* $\theta = r+1+\frac{Nr}{2} < \frac{4(p+1)+pN^2r}{2(pN+2)}$, *then*

$$E(t) \le C_1(\log(2+t))^{-2(r+1)/(-r)}. \tag{2.8}$$

$(2)_4$ *If* $\theta = \frac{4(p+1)+pN^2r}{2(pN+2)} < r+1+\frac{Nr}{2}$, *then*

$$E(t) \le C_1(\log(2+t))^{-4(p+1)/(N-2)^+p}. \tag{2.9}$$

$(2)_5$ *If* $\theta = r+1+\frac{Nr}{2} = \frac{4(p+1)+pN^2r}{2(pN+2)}$, *then*

$$E(t) \le C_1(\log(2+t))^{-2\tilde{\eta}} \tag{2.10}$$

with

$$\tilde{\eta} = \min\left\{\frac{r+1}{-r}, \frac{2(p+1)}{(N-2)^+p}\right\}.$$

(3) *The case*: $r \ge 0$ *and* $-1 \le p < 0$.
$(3)_1$ *If* $\frac{Nr-2}{2} < \theta < 1$, *then*

$$E(t) \le C_1(1+t)^{-2\eta} \tag{2.11}$$

with

$$\eta = \min\left\{\frac{2-Nr+2\theta}{2r}, \frac{-2(1+\theta)}{(N-2)^+p}\right\}.$$

$(3)_2$ *If* $\theta = \frac{Nr-2}{2} < 1$, *then*

$$E(t) \le C_1(\log(2+t))^{-2/r}. \tag{2.12}$$

(4) *The case*: $-1 < r < 0$ *and* $-1 \le p < 0$.
$(4)_1$ *If* $-1 \le \theta < r+1+\frac{Nr}{2}$, *then*

$$E(t) \le C_1(1+t)^{-2\eta} \tag{2.13}$$

with

$$\eta = \min\left\{\frac{2r + Nr + 2 - 2\theta}{-2r}, \frac{2(1+\theta)}{-(N-2)^+ p}\right\}.$$

(If $\theta = -1$, η should be replaced by some $\alpha > 0$.)
$(4)_2$ *If* $-1 < \theta = r + 1 + \frac{Nr}{2}$, *then*

$$E(t) \leq C_1(\log(2+t))^{-2(r+1)/(-r)}. \tag{2.14}$$

In the above C_1 denotes various constants depending on $||u_0||_{H_2} + ||u_1||_{H_1}$ and L.

Let us restate our result for the most typical case $\rho(x, v) = a(x)|v|^r v$, $-1 < r$, as a corollary.

Corollary 1 *When $\rho(x, u) = a(x)|v|^r v$ we have the estimate:*

$$E(t) \leq \begin{cases} C_1(1+t)^{-2\eta} & \text{if } 0 < r < 2/N \text{ or } -2/(N+2) < r < 0 \\ C_1(\log(2+t))^{-2\tilde{\eta}} & \text{if } r = 2/N \text{ or } r = -2/(N+2) \end{cases}$$

where we take

$$\eta = \min\left\{\frac{2 - Nr}{2r}, \frac{2(r+1)}{(N-2)^+ r}\right\} \quad \text{if } 0 < r < 2/N,$$

$$\eta = \min\left\{\frac{Nr + 2(r+1)}{-2r}, \frac{2}{-(N-2)^+ r}\right\} \quad \text{if } -2/(N+2) < r < 0,$$

$$\tilde{\eta} = \frac{-1}{r} \quad \text{if } r = \frac{2}{N}$$

and

$$\tilde{\eta} = \frac{r+1}{-2r} \quad \text{if } r = -1/(N+2).$$

The proof follows immediately from the cases $(1)_1$, $(1)_2$, $(4)_1$ and $(4)_2$ in the Theorem by taking $\theta = 0$ and $p = r$.

We note that when $-1 < r < 0$, our result is new even for the case $a(x) \geq \varepsilon_0 > 0$ on R^N. For corresponding reslults to the initial-boundary value problem with $\rho = |v|^r v$, $-1 < r < 0$, see [9, 10], where we get the estimate with $\eta = \min\{-(r+1)/r, -2/(N-2)^+ r\}$.

3. Some lemmas

The following is well known and very useful in treating nonlinear terms.

Lemma 1 (Gagliardo-Nirenberg) *Let* $1 \le r < p \le \infty$, $1 \le q \le p$ *and* $k \le m$, *integers. Then, we have the inequality*

$$||u||_{W^{k,p}} \le C||u||_{W^{m,q}}^{\nu}||u||_r^{1-\nu} \quad \textit{for} \quad u \in W^{m,p}$$

with some $C > 0$ *and*

$$\nu = \left(\frac{k}{N} + \frac{1}{r} - \frac{1}{p}\right)\left(\frac{m}{N} + \frac{1}{r} - \frac{1}{q}\right)^{-1}$$

provided that $0 < \nu \le 1$ ($0 < \nu < 1$ *if* $p = \infty$ *and* $mq =$ *integer*).

To derive precise decay rates we use:

Lemma 2 *Let* $\phi(t)$ *be a nonnegative continuous nonincreasing function on* $[0, \infty)$ *satisfying the inequality*

$$\phi(t+T) \le C\sum_{i=1}^{2}(1+t)^{\theta_i}(\phi(t) - \phi(t+T))^{\varepsilon_i} \quad \textit{for} \quad t \ge 0$$

with some $T > 0$, $C > 0$, $0 < \varepsilon_i \le 1$ *and* $\theta_i \le \varepsilon_i$. *Then,* $\phi(t)$ *has the following decay property*:

(1) *If* $0 < \varepsilon_i \le 1$ *with* $\varepsilon_1 + \varepsilon_2 < 1$ *and* $\theta_i < \varepsilon_i$, $i = 1, 2$, *then*

$$\phi(t) \le C_0(1+t)^{-\gamma}$$

with

$$\gamma = \min_{i=1,2}\{(\varepsilon_i - \theta_i)/(1 - \varepsilon_i)\},$$

where we consider as $\frac{\varepsilon_i - \theta_i}{1 - \varepsilon_i} = \infty$ *if* $\varepsilon_i = 1$.

(2) *If* $\theta_1 = \varepsilon_1 < 1$ *and* $\theta_2 < \varepsilon_2 \le 1$, *then*

$$\phi(t) \le C_0(\log(2+t))^{-\varepsilon_1/(1-\varepsilon_1)}.$$

(3) *If* $\varepsilon_1 = \theta_1 < 1$ *and* $\varepsilon_2 = \theta_2 < 1$, *then*

$$\phi(t) \le C_0(\log(2+t))^{-\tilde{\gamma}}$$

with

$$\tilde{\gamma} = \min_{i=1,2}\{\varepsilon_i/(1 - \varepsilon_i)\}.$$

(4) *If* $\varepsilon_1 = \varepsilon_2 = 1$, *then*

$$\phi(t) \le \begin{cases} C_0 \exp\{-\lambda t^{1-\theta}\} & \textit{if } \theta < 1 \\ C_0(1+t)^{-\alpha} & \textit{if } \theta = 1 \end{cases}$$

for some $\lambda > 0$, $\alpha > 0$, *where we set* $\theta = \min\{\theta_1, \theta_2\}$. *In the above* C_0 *denotes constants depending on* $\phi(0)$ *and other known constants.*

For the proof of Lemma 2 see [14]. When $\varepsilon_1 = \varepsilon_2$ and $\theta_1 = \theta_2$, more detailed results are proved in [5] and [6]. Lemma 2 is easily generalized to the difference inequality of the form

$$\phi(t+1) \le C \sum_{i=1}^{m} (1+t)^{\theta_i} (\phi(t) - \phi(t+1))^{\varepsilon_i}.$$

For example, if $0 < \varepsilon_i < 1$ and $\theta_i < \varepsilon_i$ we obtain from this inequality that

$$\phi(t) \le C_0(1+t)^{-\eta}$$

with $\eta = \min_{1 \le i \le m}\{(\varepsilon_i - \theta_i)/(1-\varepsilon_i)\}$. We will also use such a generalization.

For the proof of Theorem we employ multiplyer methods. Here, we list up the identities derived by several multiplyer techniques. We call, for convenience, a solution in Theorem 1 an H_2-solution.

Lemma 3 *Let* $u(t)$ *be an* H_2*-solution of the problem* (1.1)–(1.2). *Then, we have the identities*:

$$\int_t^{t+T} \int_{R^N} \rho(x,s,u_t(s)) u_t(s) dx ds = E(t) - E(t+T) \equiv D(t)^{r+2}. \tag{3.1}$$

$$\begin{aligned} &\int_t^{t+T} \int_{R_N} \varphi\{|\nabla u|^2 + |u|^2\} dx ds \\ &\quad = \int_t^{t+T} \int_{R^N} \left(\frac{1}{2}\Delta\varphi |u|^2 + \varphi |u_t|^2\right) dx ds \\ &\qquad - (u_t(t), \varphi u(t))|_t^{t+T} - \int_t^{t+T} \int_{R^N} \rho(x,s,u_t)\varphi u dx ds. \end{aligned} \tag{3.2}$$

for all $\varphi \in L^\infty([0,\infty); H_{2,loc})$.

$$\frac{1}{2}\int_t^{t+T} \int_{B_r} \operatorname{div}(q)\{|u_t|^2 - |\nabla u|^2 - |u|^2\} dx ds$$

$$+\int_t^{t+T}\int_{B_r}\left\{\frac{\partial q_k}{\partial x_j}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_k}+\rho(x,s,u_t)q\cdot\nabla u\right\}dxds$$
$$=-\int_{B_r}u_tq\cdot\nabla udx\Big|_t^{t+T}+\int_t^{t+T}\int_{S_r}\left\{\frac{1}{2r}(q\cdot x)(|u_t|^2-|\nabla u|^2-|u|^2)\right.$$
$$\left.+\frac{\partial u}{\partial r}(q\cdot\nabla u)\right\}d\Gamma ds \quad (3.3)$$

for all C^1 vector field $q=(q_1,\ldots,q_N)$, where we set

$$B_r=\{x\in R^N \mid |x|<r\} \quad and \quad S_r=\partial B_r, \quad r>0,$$

and we denote by $d\Gamma$ the surface element of S_r and by $\frac{\partial}{\partial r}$ the outward normal derivative at a point on S_r, respectively.

(3.1), (3.2) and (3.3) are derived by multiplying the equation (1.1) by u_t, φu and $q\cdot\nabla u$, respectively, and integrating by parts. For details see Zuazua [16] or Lions [1]. (These papers consider these identities with $t=0$.)

4. Existence

In this section we shall derive some standard a priori estimates for an (assumed) H_2-solution. The existence and uniqueness part in our Theorem follows from these estimates by a rather standard argument (cf. Lions and Strauss [2], Nakao [7].)

Proposition 1 *Let $u(t)$ be a (local in time) H_2-solution of the problem (1.1)–(1.2). Then, under the Hypothes A, B, we have*

$$E(t)\equiv\frac{1}{2}\{||u_t(t)||^2+||\nabla u(t)||^2+||u(t)||^2\}$$
$$\leq E(0)\equiv\frac{1}{2}\{||u||^2+||\nabla u_0||^2+||u_0||^2\}<\infty, \quad (4.1)$$

$$||u_{tt}(t)||^2+||\nabla u_t(t)||^2\leq C_1<\infty \quad (4.2)$$

and

$$||\Delta u(t)||^2\leq C_1(1+t)^{2\tilde{\theta}^+}, \quad t\geq 0 \quad (4.3)$$

where C_1 denotes constants depending on L and $||u_0||_{H_2}+||u_1||_{H_1}$ and we *set*

$$\tilde{\theta}^+ = \max\left\{\theta^+, \left(\theta - \frac{Nr}{2}\right)^+\right\}.$$

Proof. (4.1) is an immediate consequence of the identity

$$E(t) + \int_0^t \int_{R^N} \rho(x, s, u_t) u_t dx ds = E(0). \tag{4.4}$$

To prove (4.2) we use the differentiated equation

$$u_{ttt} - \Delta u_t + u_t + \rho_v(x, t, u_t) u_{tt} + \rho_t(x, t, u_t) = 0. \tag{4.5}$$

(More rigorously we must approximate ρ by a smooth ρ^ε appropriately (cf. [8]) and take the limit after establishing the estimates for approximate solutions.)

Multiplying (4.5) by u_{tt} and integrating (note that $\operatorname{supp} u(t) \subset B_{L+t}$, $t \geq 0$) we have

$$E_1(t) + \int_0^t \int_{R^N} \rho_v |u_{tt}|^2 dx ds \leq \int_0^t \int_{R^N} |\rho_t||u_{tt}| dx ds + E_1(0), \tag{4.6}$$

where we set

$$E_1(t) = \frac{1}{2}\{||u_{tt}(t)||^2 + ||\nabla u_t(t)||^2 + ||u_t(t)||^2\}.$$

Here,

$$\begin{aligned}
&\int_0^t \int_{R^N} |\rho_t||u_{tt}| dx ds \\
&\quad = \int_0^t \int_{R^N} \frac{|\rho_t|}{\sqrt{\rho_v}} \sqrt{\rho_v} |u_{tt}| dx ds \\
&\quad \leq \left(\int_0^t \int_{R^N} \frac{|\rho_t|^2}{\rho_v} dx ds\right)^{1/2} \left(\int_0^t \int_{R^N} \rho_v |u_{tt}|^2 dx ds\right)^{1/2} \\
&\quad \leq \frac{1}{2} \int_0^t \int_{R^N} \rho_v |u_{tt}|^2 dx ds + \frac{k_2}{2} \int_0^t \int_{R^N} |\rho u_t| dx ds
\end{aligned} \tag{4.7}$$

where we have used the assumption Hyp. B,(3). It follows from (4.6), (4.7) and (4.4) that

$$E_1(t) \leq E_1(0) + k_2 E(0). \tag{4.8}$$

Further,we note that since $|\rho(x,0,v)| \leq C(1+|v|^{p+1}),\ p \leq \frac{2}{(N-2)^+}$,

$$\begin{aligned} E_1(0) &= \frac{1}{2}\{||u_{tt}(0)||^2 + ||\nabla u_1||^2 + ||u_1||^2\} \\ &\leq C(||\nabla u_1||^2 + ||u_1||^2 + ||u_0||^2 + ||\Delta u_0||^2 + ||\rho(x,0,u_1)||^2) \\ &\leq C(L, ||u_0||_{H_2}, ||u_1||_{H_1}) < \infty. \end{aligned}$$

Finally, we see by the equation itself

$$\begin{aligned} ||\Delta u(t)||^2 &\leq C(||u_{tt}(t)||^2 + ||u(t)||^2 + ||\rho(x,t,u_t)||^2) \\ &\leq C_1\Big\{1 + (1+t)^{2\theta}\int_{R_1^N(t)} |u_t|^{2(r+1)}dx \\ &\qquad + (1+t)^{2\theta}\int_{R_2^N(t)} |u_t|^{2(p+1)}dx\Big\}, \end{aligned} \tag{4.9}$$

where we set

$$\begin{aligned} R_1^N(t) &= \{x \in R^N \mid |u_t(x,t)| \leq 1\} \quad \text{and} \\ R_2^N(t) &= \{x \in R^N \mid |u_t(x,t)| \geq 1\}. \end{aligned}$$

We easily see that if $r \geq 0$,

$$\int_{R_1^N} |u_t|^{2(r+1)}dx \leq \int_{R^N} |u_t|^2 dx \leq E(0)$$

and if $r < 0$,

$$\begin{aligned} \int_{R_1^N(t)} |u_t|^{2(r+1)}dx &\leq \Big\{\int_{B_{L+t}} 1ds\Big\}^{-r}\Big\{\int_{R_1^N(t)} |u_t|^2 dx\Big\}^{r+1} \\ &\leq C_L(1+t)^{-Nr}E(0)^{r+1}. \end{aligned}$$

Similarly, if $0 \leq p < 2/(N-2)^+$ we have, by Sobolev's inequality,

$$\int_{R_2^N(t)} |u_t|^{2(p+1)}dx \leq C_1 < \infty \tag{4.10}$$

and if $p < 0$,

$$\int_{R_2^N(t)} |u_t|^{2(p+1)}dx \leq \int_{R_2^N(t)} |u_t|^2 dx \leq E(0).$$

Hence, we have from (4.9) that

$$||\Delta u(t)||^2 \leq C_1(1+t)^{2\tilde{\theta}^+}.$$

□

Remarks. (1) We note that $\tilde{\theta}^+ = \theta^+$ if $r \geq 0$ and $\tilde{\theta}^+ = (\theta - Nr/2)^+$ if $r < 0$.

(2) The condition $p \leq 2/(N-2)^+$ is made for the proof of (4.3). Once (4.3) is established the arguments below are valid for $-1 \leq p \leq 4/(N-2)^+$.

(3) If we assume, instead of Hyp. B,(3),

$$|\nabla_x \rho|^2 \leq k_2 \rho_v \rho v \tag{3)'$$

we can prove without the condition on p the estimate

$$||\nabla u_t(t)||^2 + ||\Delta u(t)||^2 \leq C_1 < \infty. \tag{4.3)'$$

Indeed, multiplying the equation by $-\Delta u_t(t)$ and integrating by parts we see

$$\frac{1}{2}\frac{d}{dt}\{||\nabla u_t(t)||^2 + ||\Delta u(t)||^2 + ||\nabla u(t)||^2\} + \int_{R^N} \nabla_x \rho \nabla u_t u_t dx + \int_{R^N} \rho_v (\nabla u_t)^2 dx = 0$$

and as in (4.7) we obtain (4.3)$'$. Consequently, it seems at a glance that we could get better result than Theorem 1. However, if we consider $\rho = a(x)|v|^r v$, the condition (3)$'$ means that

$$|\nabla a(x)| \leq k_2 a(x).$$

This is very natural if $a(x) > \epsilon_0 > 0$ on R^N, while it is in fact too restrictive when $a(x)$ may vanish somewhere in R^N, because in this case $a(x)$ must be infinitely differentiable in the neighbourhood of $S \equiv \{x \in R^N \mid a(x) = 0\}$ and all the derivatives of $a(x)$ must be vanish at the points of ∂S. (Consider the Taylor expansions near vanishing points of $a(x)$.) Any way, under the restrictive condition (3)$'$ the estimates in Theorem 1 can be improved into a little simpler form and also applied to $-1 \leq p \leq 4/(N-2)^+$.

5. Inequalities derived by multiplication method

We shall derive some inequalities for the solutions of (1.1)–(1.2) from the identities in Lemma 3. The derivations are essentially included in Zuazua

[16], and we sketch them briefly.

Proposition 2 *Let $T > 0$ and $u(t)$ be an H_2-solution of the problem* (1.1)–(1.2). *Then,*

$$\int_t^{t+T}\int_{\Omega_{2R}}\{|\nabla u|^2+|u|^2\}dxds \le C\Big\{E(t)+E(t+T)+\int_t^{t+T}\int_{\Omega_R}|\rho(x,s,u_t)|\ |u|dxds + \int_t^{t+T}\int_{B_{2R}/B_R}|u|^2dxds+\int_t^{t+T}\int_{\Omega_R}|u_t|^2dxds\Big\}, \tag{5.1}$$

where C is a positive constant independent of u and L.

Proof. Take a function $\varphi\in L^\infty([0,\infty);H_2)$ such that $0\le\varphi\le 1$ on R^N, $\varphi=0$ on B_R and $\varphi=1$ on Ω_{2R}. Then, the identity (3.2) implies (5.1) immediately. □

Proposition 3 *For an H_2-solution $u(t)$ we have*

$$\int_t^{t+T}\int_{B_{2R}}\{|u_t|^2+|\nabla u|^2+|u|^2\}dxds \le C\Big\{\int_t^{t+T}\int_{B_{4R}}|\rho|(|u|+|\nabla u|)dxds+\int_t^{t+T}\int_{B_{4R}/B_R}(|u_t|^2)dxds + \int_t^{t+T}\int_{B_{4R}}|u|^2dxds+E(t)+E(t+T)\Big\}, \tag{5.2}$$

where C is a constant independent of L and u.

Proof. First, take $q=x$ and $r=2R$ in (3.3). Then,

$$\frac{N}{2}\int_t^{t+T}\int_{B_{2R}}\{|u_t|^2-|\nabla u|^2-|u|^2\}dxds+\int_t^{t+T}\int_{B_{2R}}|\nabla u|^2dxds = -\int_t^{t+T}\int_{B_{2R}}\rho x\cdot\nabla u dxds-\int_{B_{2R}}u_t x\cdot\nabla u dx\Big|_t^{t+T} + R\int_t^{t+T}\int_{S_{2R}}\{|u_t|^2-|\nabla u|^2-|u|^2\}d\Gamma ds + 2R\int_t^{t+T}\int_{S_{2R}}\left|\frac{\partial u}{\partial r}\right|^2 d\Gamma ds. \tag{5.3}$$

Next, take $\varphi\equiv 1$ in a similar identity to (3.2) with R^N replaced by

B_{2R}. Then,

$$\begin{aligned}\int_t^{t+T}&\int_{B_{2R}}\{|\nabla u|^2+|u|^2\}dxds\\&=\int_t^{t+T}\int_{B_{2R}}|u_t|^2dxds-\int_t^{t+T}\int_{B2R}\rho udxds\\&\quad+\int_t^{t+T}\int_{S_{2R}}\frac{\partial u}{\partial r}ud\Gamma ds-\int_{B_{2R}}u_tudx\Big|_t^{t+T}.\end{aligned}\tag{5.4}$$

It follows from (5.3) and (5.4) that

$$\begin{aligned}\int_t^{t+T}&\int_{B_{2R}}\{|u_t|^2+|\nabla u|^2+|u|^2\}dxds\\&\le C\Big\{E(t)+E(t+T)+\int_t^{t+T}\int_{B_{2R}}|\rho|(|\nabla u|+|u|)dxds\\&\qquad+\int_t^{t+T}\int_{B_{2R}}|u|^2dxds\Big\}\\&\quad+CR\int_t^{t+T}\int_{S_{2R}}\{|u_t|^2-|\nabla u|^2-|u|^2\}d\Gamma ds\\&\quad+2R\int_t^{t+T}\int_{S_{2R}}\left|\frac{\partial u}{\partial r}\right|d\Gamma ds+C\left|\int_t^{t+T}\int_{S_{2R}}\frac{\partial u}{\partial r}ud\Gamma ds\right|\end{aligned}\tag{5.5}$$

for some $C>0$.

To estimate the boundary integrals in (5.5) we take a function $\varphi(x)$ such that

$$0\le\varphi\le1\ \text{ in }\ B_{2R},\varphi=0\ \text{ in }\ B_{3R/2}\ \text{ and }\ \varphi=1\ \text{ on }\ S_{2R},$$

and set $q(x)=\varphi(x)x$ in (3.3) to get

$$\begin{aligned}\frac{N}{2}&\int_t^{t+T}\int_{B_{2R}}\varphi\{|u_t|^2-|\nabla u|^2-|u|^2\}dxds\\&+\int_t^{t+T}\int_{B_{2R}}\varphi|\nabla u|^2dxds+\int_t^{t+T}\int_{B_{2R}}(\nabla\varphi\cdot\nabla u)(x\cdot\nabla u)dxds\\&+\int_t^{t+T}\int_{B_{2R}}(x\cdot\nabla\varphi)\{|u_t|^2-|\nabla u|^2-|u|^2\}dxds\\=&-\int_t^{t+T}\int_{B_{2R}}\rho\varphi(x\cdot\nabla u)dxds-\int_{B_{2R}}u_t(s)\varphi(x\cdot\nabla u)dx\Big|_t^{t+T}\end{aligned}$$

$$+R\int_t^{t+T}\int_{S_{2R}}\{|u_t|^2-|\nabla u|^2-|u|^2\}+2R\int_t^{t+T}\int_{S_{2R}}\left|\frac{\partial u}{\partial r}\right|^2 d\Gamma ds \tag{5.6}$$

Also, by a similar identity to (3.2) (see (5.4)),

$$\begin{aligned}\int_t^{t+T}\int_{B_{2R}}\varphi\{|\nabla u|^2+|u|^2\}dxds &= \int_t^{t+T}\int_{B_{2R}}\{\varphi|u_t|^2-u\nabla\varphi\cdot\nabla u\}dxds+\int_t^{t+T}\int_{S_{2R}}\frac{\partial u}{\partial r}ud\Gamma ds\\ &-\int_{B_{2R}}\varphi u_t udx\Big|_t^{t+T}-\int_t^{t+T}\int_{B_{2R}}\rho\varphi udxds.\end{aligned}\tag{5.7}$$

We have from (5.6) and (5.7) that

$$\begin{aligned}&\left|R\int_t^{t+T}\int_{S_{2R}}\{|u_t|^2-|\nabla u|^2-|u|^2\}+2R\int_t^{t+T}\int_{S_{2R}}\left|\frac{\partial u}{\partial r}\right|^2 d\Gamma ds\right|\\ &\quad+\left|\int_t^{t+T}\int_{S_{2R}}\frac{\partial u}{\partial r}ud\Gamma ds\right|\\ &\le C\Big\{\int_t^{t+T}\int_{B_{2R}/B_{\tilde R}}(|\nabla u|^2+|u|^2)dxds+\int_t^{t+T}\int_{B_{2R}/B_{\tilde R}}|u_t|^2dxds\\ &\quad+\int_t^{t+T}\int_{B_{2R}/B_{\tilde R}}|\rho|(|u|+|\nabla u|)dxds+E(t)+E(t+T)\Big\},\\ &\qquad \tilde R\equiv 3R/2.\end{aligned}\tag{5.8}$$

Finally, to treat the first integral of the righthand side of (5.8) we take a function φ such that $0\le\varphi\le 1$ on B_{4R}, $\varphi=1$ on $B_{2R}/B_{\tilde R}$, $\operatorname{supp}\varphi\subset B_{4R}/B_R$ and $|\nabla\varphi|^2/\varphi\in L^\infty$.

Then, by the identity (5.7) with $2R$ replaced by $4R$ and with φ defined just above, we see

$$\begin{aligned}&\int_t^{t+T}\int_{B_{4R}}\varphi(|\nabla u|^2+|u|^2)dxds\\ &\le\int_t^{t+T}\int_{\operatorname{supp}\varphi}(\varphi|u_t|^2+|\nabla\varphi|/\sqrt{\varphi}\cdot|u|\cdot\sqrt{\varphi}|\nabla u|)dxds\\ &\quad+\int_t^{t+T}\int_{B_{4R}/B_R}|\rho||u|dxds+E(t)+E(t+T)\end{aligned}$$

and hence,

$$\begin{aligned}
&\int_t^{t+T}\int_{B_{2R}/B_{\tilde R}}(|\nabla u|^2+|u|^2)dxds\\
&\le \int_t^{t+T}\int_{B_{4R}}\varphi(|\nabla u|^2+|u|^2)dxds\\
&\le C\Big\{\int_t^{t+T}\int_{B_{4R}/B_R}(|u_t|^2+|u|^2+|\rho||u|)dxds+E(t)+E(t+T)\Big\}.
\end{aligned} \tag{5.9}$$

From (5.5), (5.8) and (5.9) we obtain (5.2). □

Proposition 4 *There exists $T_0>0$ such that if $T>T_0$, we have*

$$\begin{aligned}
E(t)\ \le\ C(T)\Big\{D(t)^{r+2}&+\int_t^{t+T}\int_{\Omega_R}|u_t|^2dxds+\int_t^{t+T}\int_{B_{4R}}|u|^2dxds\\
&+\int_t^{t+T}\int_{R^N}|\rho(x,s,u_t)|(|u|+|u_t|)dxds\Big\},\quad t\ge 0
\end{aligned} \tag{5.10}$$

for a H_2-solution $u(t)$, where $C(T)$ is a constant independent of L and u.

Proof. By Propositions 2 through to 4 and the identity (3.1) we have

$$\begin{aligned}
&TE(t+T)\\
&\le \frac{1}{2}\int_t^{t+T}\int_{R^N}\{|u_t|^2+|\nabla u|^2+|u|^2\}dxds\\
&\le C\Big\{D(t)^{r+2}+E(t+T)+\int_t^{t+T}\int_{R^N}|\rho|(|u|+|\nabla u|)dxds\\
&\qquad+\int_t^{t+T}\int_{\Omega_R}|u_t|^2dxds+\int_t^{t+T}\int_{B_{4R}}|u|^2dxds\Big\}
\end{aligned} \tag{5.11}$$

Therefore, for $T>T_0\equiv C$, we see by (5.11) and (3.1) that

$$\begin{aligned}
E(t)\ &=\ E(t+T)+D(t)^{r+2}\\
&\le\ C(T)\Big\{D(t)^{r+2}+\int_t^{t+T}\int_{\Omega_R}|u_t|^2dxds\\
&\qquad+\int_t^{t+T}\int_{B_{4R}}|u|^2dxds+\int_t^{t+T}\int_{R^N}|\rho|(|u|+|\nabla u|)dxds\Big\}.
\end{aligned}$$

□

6. Estimation of the nonlinear term

In what follows we fix $T > T_0$ and we denote by C the positive constants depending on T and L as well as other known constants and denote by C_1 positive constants depending continuously on L and $||u_0||_{H_2} + ||u_1||_{H_1}$. Estimating the integral including $\rho(x, t, u_t)$ in (5.10) we have:

Proposition 5 *For an H_2-solution $u(t)$ the inequality*

$$E(t) \leq C_1\Big\{A_i(t)^2 + \int_t^{t+T}\int_{\Omega_R}|u_t|^2dxds + \int_t^{t+T}\int_{B_{4R}}|u|^2dxds\Big\} \quad t \geq 0, \tag{6.1}$$

holds, where $A_i(t)$, $i = 1,2,3,4$, correspond to the cases (i) in Theorem 1 *and are defined as follows: For the case* (1) : $0 \leq r$ *and* $0 \leq p \leq 2/(N-2)^+$,

$$A_i(t)^2 = D(t)^{r+2} + (1+t)^\theta D(t)^{r+2} + (1+t)^{2(pN\theta^+ + 2\theta)/(2p+Np+4)}D(t)^{4(r+2)(p+1)/(2p+Np+4)} \tag{6.2}$$

For the case (2) : $-1 < r < 0$ *and* $0 \leq p \leq 2/(N-2)^+$,

$$A_2(t)^2 = D(t)^{r+2} + (1+t)^{(2\theta - Nr)/(r+2)}D(t)^{2(r+1)} + (1+t)^{2(pN\tilde{\theta}^+ + 2\theta)/(2p+Np+4)}D(t)^{4(r+2)(p+1)/(2p+Np+4)}. \tag{6.3}$$

For the case (3) : $0 \leq r$ *and* $-1 \leq p < 0$,

$$A_3(t)^2 = D(t)^{r+2} + (1+t)^\theta D(t)^{r+2}. \tag{6.4}$$

For the case (4) : $-1 < r < 0$ *and* $-1 \leq p < 0$,

$$A_4(t)^2 = D(t)^{r+2} + (1+t)^{(2\theta - Nr)/(r+2)}D(t)^{2(r+1)} + (1+t)^\theta D(t)^{r+2}. \tag{6.5}$$

Proof. We set

$$\Omega^1(t) \equiv \{x \in R^N \mid |u_t(x,t)| \leq 1\} \cap B_{L+t} \text{ and } \Omega^2(t) = R^N/\Omega^1(t).$$

Then, by Hyp. B,

$$\begin{aligned}
&\int_t^{t+T}\int_{R^N}|\rho|(|u|+|\nabla u|)dxds \\
&\leq C\int_t^{t+T}\int_{\Omega^1(s)}(1+s)^\theta a(x)|u_t|^{r+1}(|u|+|\nabla u|)dxds \\
&\quad + C\int_t^{t+T}\int_{\Omega^2(s)}(1+s)^\theta a(x)|u_t|^{p+1}(|u|+|\nabla u|)dxds \\
&\equiv I_1+I_2. \qquad (6.6)
\end{aligned}$$

For the case (1) we see

$$\begin{aligned}
I_1 &\leq C\Big(\int_t^{t+T}\int_{\Omega^1(s)}(1+s)^{2\theta}a(x)|u_t|^{2(r+1)}dxds\Big)^{1/2}\sqrt{E(t)} \\
&\leq C(1+t)^{\theta/2}\Big(\int_t^{t+T}\int_{\Omega^1(s)}(1+s)^\theta a(x)|u_t|^{r+2}dxds\Big)^{1/2}\sqrt{E(t)} \\
&\leq C(1+t)^{\theta/2}D(t)^{(r+2)/2}\sqrt{E(t)}. \qquad (6.7)
\end{aligned}$$

And,

$$\begin{aligned}
I_2 &\leq C(1+t)^{\theta/(p+2)} \\
&\quad\times\Big(\int_t^{t+T}\int_{\Omega^2(s)}(1+s)^\theta a(x)|u_t|^{p+2}dxds\Big)^{(p+1)/(p+2)} \\
&\quad\times\Big(\int_t^{t+T}\int_{\Omega^2(s)}(|u|^{p+2}+|\nabla u|^{p+2})dxds\Big)^{1/(p+2)}. \qquad (6.8)
\end{aligned}$$

Here, by Gagliardo-Nirenberg inequality and Proposition 1,

$$||u||_{p+2}\leq C(||u||+||\nabla u||)\leq C\sqrt{E(t)},$$

and

$$\begin{aligned}
||\nabla u||_{p+2} &\leq C||\nabla u||^{1-\nu}||u||_{H_2}^{\nu} \qquad (\nu=pN/2(p+2)) \\
&\leq C_1(1+t)^{pN\theta^+/2(p+2)}E(t)^{(2p+4-Np)/4(p+2)}. \qquad (6.9)
\end{aligned}$$

Thus, we see

$$I_2 \leq C(1+t)^{\theta/(p+2)}D(t)^{(p+1)(r+2)/(p+2)}\sqrt{E(t)}$$

$$+ C_1(1+t)^{(pN\theta^+ + 2\theta)/2(p+2)} D(t)^{(p+1)(r+2)/(p+2)} \times E(t)^{(2p+4-Np)/4(p+2)}. \tag{6.10}$$

Applying Young's inequality to (6.7) and (6.10) we obtain from (5.10) the desired estimate, where we have used the inequality

$$(1+t)^{2\theta/(p+2)} D(t)^{2(p+1)(r+2)/(p+2)} \le C_1(D(t)^{r+2} + (1+t)^{\theta} D(t)^{r+2}). \tag{6.11}$$

For the case (2) we have, instead of (6.7),

$$\begin{aligned} I_1 &\le C(1+t)^{\theta/(r+2)} \\ &\quad \times \left(\int_t^{t+T} \int_{\Omega^1(s)} (1+s)^{\theta} a(x) |u_t|^{r+2} dx ds \right)^{(r+1)/(r+2)} \\ &\quad \times \left(\int_t^{t+T} \int_{\Omega^1(s)} (|u|^{r+2} + |\nabla u|^{r+2}) dx ds \right)^{1/(r+2)} \\ &\le C(1+t)^{\theta/(r+2)} D(t)^{r+1} (1+t)^{-Nr/2(r+2)} \\ &\quad \times \left(\int_t^{t+T} \int_{\Omega^1(s)} (|u|^2 + |\nabla u|^2) dx ds \right)^{1/2} \\ &\le C(1+t)^{(2\theta - Nr)/2(r+2)} D(t)^{r+1} \sqrt{E(t)}. \end{aligned} \tag{6.12}$$

I_2 is estimated as in (6.10) with θ^+ replaced by $\tilde{\theta}^+ = (\theta - \frac{Nr}{2})^+$. Hence, we obtain (6.1) with $A_2(t)^2$ defined by (6.3).

For the case (3) we have, instead of (6.10),

$$\begin{aligned} I_2 &\le C\left(\int_t^{t+T} \int_{\Omega^2(s)} (1+s)^{2\theta} a(x) |u_t|^{2(p+1)} dx ds \right)^{1/2} \\ &\quad \times \left(\int_t^{t+T} \int_{\Omega^2(s)} (|u|^2 + |\nabla u|^2) dx ds \right)^{1/2} \\ &\le C(1+t)^{\theta/2} \left(\int_t^{t+T} \int_{\Omega^2(s)} (1+t)^{\theta} a(x) |u_t|^{p+2} dx ds \right)^{1/2} \sqrt{E(t)} \\ &\le C(1+t)^{\theta/2} D(t)^{(r+2)/2} \sqrt{E(t)}. \end{aligned} \tag{6.13}$$

Hence, we obtain (6.1) with $A_3(t)^2$ defined by (6.4).

The estimate for the case (4) follows immediately from the argument above. □

We proceed to the estimation of the last term in the inequality (6.1). For this we utilize the 'unique continuation property for the wave equation'.

Proposition 6 *The inequality*

$$\int_t^{t+T}\int_{B_{4R}} |u|^2 dxds \le C_1\Big\{A_i(t)^2 + \int_t^{t+T}\int_{\Omega_R} |u_t|^2 dxds\Big\} \tag{6.14}$$

holds for some constant $C_1 > 0$, $i = 1, 2, 3, 4$.

Proof. We prove (6.14) by contradiction as in [13, 16]. If (6.14) were false, there would exist a sequence $\{t_n\}$ and a sequence of solutions $\{u_n(t)\}$ such that

$$\int_{t_n}^{t_n+T}\int_{B_{4R}} |u_n|^2 dxds \ge n\Big\{A_i(t_n)^2 + \int_{t_n}^{t_n+T}\int_{\Omega_R} |u_{nt}|^2 dxds\Big\}. \tag{6.15}$$

Setting

$$\lambda_n^2 = \int_{t_n}^{t_n+T}\int_{B_{4R}} |u_n|^2 dxds \quad \text{and} \quad v_n(t) = u(t + t_n)/\lambda_n,$$

we see

$$\int_0^T\int_{B_{4R}} |v_n(t)|^2 dxdt = 1 \tag{6.16}$$

and

$$Q_n^2 \equiv \int_0^T\int_{\Omega_R} |v_{n_t}(t)|^2 dxdt + \left(\frac{A_i(t_n)}{\lambda_n}\right)^2 \to 0 \text{ as } n \to \infty. \tag{6.17}$$

Further, dividing the both sides of (6.1) in Proposition 5 by λ_n^2 we know

$$||v_{n_t}(0)||^2 + ||\nabla v_n(0)||^2 + ||v_n(0)||^2 \le C_1(Q_n^2 + 1) \le C_1 < \infty$$

and hence (note that $E(t)$ is decreasing for $v_n(t)$)

$$||v_{nt}(t)||^2 + ||\nabla v_n(t)||^2 + ||v_n(t)||^2 \le C_1 < \infty \quad \text{for } t \in [0, T]. \tag{6.18}$$

Thus, we have by a standard compactness argument that

$$v_n(t) \to v(t) \text{ strongly in } L^2(B_{4R} \times [0, T]), \tag{6.19}$$

$$v_n(t) \to v(t) \text{ weakly}^* \text{ in } L^\infty([0, T]; H_1(R^N)), \tag{6.20}$$

and

$$v_{n_t}(t) \to v_t(t) \text{ weakly}^* \text{ in } L^\infty([0,T];L^2(R^N)) \tag{6.21}$$

along a subsequence of $\{v_n(t)\}$.

The limit function $v(t)$ belongs to

$$W^{1,\infty}([0,T];L^2(R^N)) \cap L^\infty([0,T];H_1(R^N))$$

and satisfies, in particular,

$$\int_0^T \int_{B_{4R}} |v(t)|^2 dxdt = 1, \quad \text{and} \quad \int_0^T \int_{\Omega_R} |v_t(t)|^2 dxdt = 0. \tag{6.22}$$

Moreover, we can show

$$\lim_{n\to\infty} \frac{\rho(x,t+t_n,u_{nt}(t+t_n))}{\lambda_n} = 0 \quad \text{in } L^1_{loc}([0,T]\times R^N). \tag{6.23}$$

Indeed, we see, for any compact set $K \subset R^N$,

$$\begin{aligned}
&\int_{t_n}^{t_n+T} \int_K |\rho(x,t,u_{nt}(t)|dxdt \\
&\quad \le C\Big\{ \int_{t_n}^{t_n+T} \int_{B^1} (1+t)^\theta a(x)|u_{nt}(t)|^{r+1} dxdt \\
&\qquad + \int_{t_n}^{t_n+T} \int_{B^2} (1+t)^\theta a(x)|u_{nt}(t)|^{p+1} dxdt \Big\} \\
&\quad \equiv \tilde{I}_1 + \tilde{I}_2,
\end{aligned} \tag{6.24}$$

where we set

$$B^1 = \Omega^1(t) \cap K \quad \text{and} \quad B^2 = \Omega^2(t) \cap K.$$

For the case (1),

$$\begin{aligned}
\tilde{I}_1 &\le C(K)(1+t_n)^{\theta/2}\Big(\int_{t_n}^{t_n+T} \int_{B^1} (1+t)^\theta a(x)|u_{nt}|^{2(r+1)} dxdt \Big)^{1/2} \\
&\le C(K)(1+t_n)^{\theta/2} D(t_n)^{(r+2)/2} \le CA_1(t_n) \quad \text{(see (6.2))},
\end{aligned} \tag{6.25}$$

and

$$\tilde{I}_2 \le C \int_{t_n}^{t_n+T} \int_{B^2} (1+t)^\theta a(x)|u_{nt}|^{p+2} dxdt$$

$$\leq CD(t)^{r+2} \leq C(K)A_1(t_n). \tag{6.26}$$

The last inequality (6.26) is valid for other cases $i = 2, 3, 4$ if we replace $A_1(t_n)$, by $A_i(t_n)$, $i = 2, 3, 4$, respectively. We also note that (6.25) is valid for $i = 3$ if $A_1(t_n)$ replaced by $A_3(t_n)$. Further, for the cases (2) and (4) we easily see

$$\begin{aligned} \tilde{I}_1 &\leq C(K)(1+t_n)^{\theta/(r+2)} \\ &\quad \times \left(\int_{t_n}^{t_n+T} \int_{B^1} (1+t)^\theta a(x) |u_{nt}(t)|^{r+2} dx dt \right)^{(r+1)/(r+2)} \\ &\leq C(K)(1+t_n)^{\theta/(r+2)} D(t_n)^{r+1} \\ &\leq C(K) A_i(t_n), \quad i = 2, 4, \end{aligned} \tag{6.27}$$

(see (6.3), (6.5)).

Thus, in any case we see

$$\int_{t_n}^{t_n+T} \int_K \frac{|\rho(x,t,u_{nt}(t))|}{\lambda_n} dx dt \leq C(K) \frac{A_i(t_n)}{\lambda_n} \to 0$$

for each compact set $K \subset R^N$.

From the above argument $v(x,t)$ becomes a solution of the linear Klein-Gordon equation

$$v_{tt} - \Delta v + v = 0 \quad \text{in} \quad R^N \times [0,T]. \tag{6.28}$$

It is well known that $v(t)$, a solution of (6.28), belongs in fact to $C^1([0,T]; L^2(R^N)) \cap C([0,T]; H_1(R^N))$, and the latter condition of (6.22) implies $v_t(x,t) \equiv 0$ in $\Omega_R \times [0,T]$. Thus, by a standard unique continuation property due to Holmgren's theorem, we conclude that $v(x,t) \equiv 0$ in $R^N \times [0,T]$, which is a contradiction to the former condition of (6.22). □

7. Completion of the proof of Theorem 1

By Propositions 5 and 6 we have the estimate

$$E(t) \leq C_1 \left\{ A_i(t)^2 + \int_t^{t+T} \int_{\Omega_R} |u_t|^2 dx ds \right\}. \tag{7.1}$$

To complete the proof of Theorem 1 we must estimate the last term of

(7.1). Here, we see

$$\int_t^{t+T}\int_{\Omega_R}|u_t|^2dxds = \int_t^{t+T}\int_{\Omega_R^1}|u_t|^2dxds + \int_t^{t+T}\int_{\Omega_R^2}|u_t|^2dxds$$
$$= \hat{I}_1 + \hat{I}_2, \tag{7.2}$$

where we set

$$\Omega_R^1 = \{x \in \Omega_R | |u_t(x,t)| \le 1\} \cap B_{L+t} \quad \text{and} \quad \Omega_R^2 = \Omega_R/\Omega_R^1.$$

For the case (1) we have

$$\hat{I}_1 \le C(1+t)^{-2\theta/(r+2)}\left(\int_t^{t+T}\int_{\Omega_R^1}(1+t)^\theta a(x)|u_t|^{r+2}dxds\right)^{2/(r+2)}$$
$$\times\left(\int_t^{t+T}\int_{\Omega_R^1}1dxds\right)^{r/(r+2)}$$
$$\le C(1+t)^{(Nr-2\theta)/(r+2)}D(t)^2, \tag{7.3}$$

and

$$\hat{I}_2 \le C\int_t^{t+T}\int_{\Omega_R}a(x)|u_t|^{p+2}dxds \le C(1+t)^{-\theta}D(t)^{r+2}. \tag{7.4}$$

For the case (2) we see, instead of (7.3),

$$\hat{I}_1 \le C\int_t^{t+T}\int_{\Omega_R}a(x)|u_t|^{r+2}dxds \le C(1+t)^{-\theta}D(t)^{r+2}, \tag{7.5}$$

and for the case (3) we have, instead of (7.4),

$$\hat{I}_2 \le C\int_t^{t+T}\left(\int_{\Omega_R}|u_t|^{p+2}dx\right)^{2(1-\nu)/(p+2)}$$
$$\times\left(\int_{\Omega_R}(|u_t|^2+|\nabla u|^2)dx\right)^{(2\nu+p)/(p+2)}ds$$
$$\le C_1(1+t)^{-2\theta(1-\nu)/(p+2)}$$
$$\times\left(\int_t^{t+T}\int_{\Omega_R}(1+s)^\theta a(x)|u_t|^{p+2}dxds\right)^{2(1-\nu)/(p+2)}$$
$$\le C_1(1+t)^{-4\theta/(2p-Np+4)}D(t)^{4(r+2)/(2p-Np+4)}.$$
$$\left(\nu \equiv \frac{-Np}{2p+4-Np}\right) \tag{7.6}$$

Thus, we obtain for the case (1),

$$\begin{aligned} &E(t+T)\\ &\quad\le E(t)\\ &\quad\le C_1\{A_1(t)^2+(1+t)^{(Nr-2\theta)/(r+2)}D(t)^2+(1+t)^{-\theta}D(t)^{r+2}\}\\ &\quad\le C_1\{(1+t)^{|\theta|}D(t)^{r+2}+(1+t)^{(Nr-2\theta)/(r+2)}D(t)^2\\ &\qquad+(1+t)^{2(pN\theta^+ +2\theta)/(2p+Np+4)}D(t)^{4(r+2)(p+1)/(2p+Np+4)}\}. \end{aligned} \tag{7.7}$$

Recalling (3.1) and applying Lemma 2 to (7.7) carefully, we can derive the estimates from $(1)_1$ through to $(1)_5$ in Theorem 1. Indeed,for the case $(1)_1$, we apply a generalization of the case (1) in Lemma 2 with

$$\begin{aligned} &\theta_1=|\theta|,\quad \theta_2=\frac{Nr-2\theta}{r+2},\quad \theta_3=\frac{2(pN\theta^+ +2\theta)}{2p+Np+4},\\ &\varepsilon_1=1,\quad \varepsilon_2=\frac{2}{r+2}\quad\text{and}\quad \varepsilon_3=\frac{4(p+1)}{2p+Np+4} \end{aligned}$$

to get the desired estimate (2.1) with

$$\begin{aligned} \eta &= \frac{1}{2}\min\left\{\frac{1-|\theta|}{+0},\ \frac{\frac{2}{r+2}-\frac{Nr-2\theta}{r+2}}{1-\frac{2}{r+2}},\ \frac{\frac{4(p+1)}{2p+Np+4}-\frac{2(pN\theta^+ +2\theta)}{2p+Np+4}}{1-\frac{4(p+1)}{2p+Np+4}}\right\}\\ &= \min\left\{\frac{2+2\theta-Nr}{2r},\ \frac{2(p+1)-(pN\theta^+ +2\theta)}{(N-2)^+p}\right\}. \end{aligned}$$

For the case $(1)_2$, we apply a generalization of the case (2) in Lemma 2 to get the estimate (2.2). Other cases can be treated similarly.

For the case (2) we obtain from (7.1), (7.4) and (7.5)

$$\begin{aligned} E(t)\ \le\ &C_1\{(1+t)^{|\theta|}D(t)^{r+2}+(1+t)^{(2\theta-Nr)/(r+2)}D(t)^{2(r+1)}\\ &+(1+t)^{2(pN\tilde{\theta}^+ +2\theta)/(2p+Np+4)}D(t)^{4(r+2)(p+1)/(2p+Np+4)}\}. \end{aligned} \tag{7.8}$$

Applying Lemma 2 to (7.8) we have the estimates for the cases from $(2)_1$ to $(2)_4$ in Theorem 1.

For the case (3) we obtain from (7.1), (7.3) and (7.6),

$$E(t)\ \le\ C_1\{D(t)^{r+2}+(1+t)^{\theta}D(t)^{r+2}+(1+t)^{(Nr-2\theta)/(r+2)}D(t)^2$$

$$+ (1+t)^{-4\theta/((2p-Np+4)}D(t)^{4(r+2)/(2p-Np+4)}\}. \qquad (7.9)$$

Applying Lemma 2 to (7.9) we have the estimates $(3)_1$ and $(3)_2$ in Theorem 1.

Finally, for the case (4) we obtain

$$E(t) \le C_1\{(1+t)^{|\theta|}D(t)^{r+2} + (1+t)^{(2\theta-Nr)/(r+2)}D(t)^{2(r+1)} + (1+t)^{-4\theta/(2p-Np+4)}D(t)^{4(r+2)/(2p-Np+4)}\}. \qquad (7.10)$$

Applying Lemma 2 to (7.10) we have the estimates $(4)_1$ and $(4)_2$ in Theorem 1. The proof of Theorem 1 is now complete.

8. A remark on the equation in an exterior domain

Let Ω be an exterior domain in R^N with a compact boundary $\partial\Omega$. Zuazua [16] also states a result on the exponential decay of the solutions for the equation with a linear dissipation $a(x)u_t$ and a semilinear term $f(u)$ in an exterior domain under the additional condition

$$a(x) \ge \varepsilon_0 > 0 \quad \text{on a neighbourhood of } \partial\Omega. \qquad (8.1)$$

Combining the results in [13, 14] with the argument in previous sections, we can easily extend Theorem 1 to the equation in an exterior domain.

We consider the problem

$$(P^*) \quad \begin{cases} u_{tt} - \Delta u + u + \rho(x,t,u_t) = 0 & \text{in } [0,\infty)\times\Omega \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{and } \quad u|_{\partial\Omega} = 0, \end{cases}$$

where Ω is an exterior domain in R^N with a compact C^2 class boundary $\partial\Omega$.

We make the same hypothesis B on $\rho(x,t,v)$ and (8.1). Then, we obtain essentially the same result as in Theorem 1 with H_2 and H_1 replaced by $H_2(\Omega)\cap H_1^o(\Omega)$ and $H_1^o(\Omega)$, respectively.

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