Hokkaido Mathematical Journal Vol. 27 (1998) p. 51–75

Cauchy problem for nonlinear parabolic equations

Hi Jun CHOE and Jin Ho LEE

(Received February 21, 1996; Revised April 7, 1997)

Abstract. The Cauchy problem for degenerate parabolic equations with bounded measurable coefficients is studied. The existence and uniqueness of initial trace for nonnegative solutions is shown. The Harnack type estimate is fundamental. Moreover the behavior of interface is studied locally and globally. The interface is Hölder continuous graph. Finally the asymptotic behavior of solutions is studied.

Key words: initial trace, Harnack type inequality, degenerate parabolic equation, interface, Hölder continuous.

1. Introduction

In this paper we study the Cauchy problem of a degenerate parabolic equation

$$u_t = (a_{ij}(x)|\nabla u|^{p-2}u_{x_i})_{x_j} \quad \text{in} \quad \mathbb{R}^n \times (0,T], \ (p>2) \tag{1.1}$$

$$u(x,0) = u_0(x) \in L^1_{\text{loc}}(\mathbb{R}^n) \tag{1.2}$$

where $a_{ij}(x)$ are bounded, symmetric and measurable functions which satisfy the ellipticity condition

$$\Lambda^{-1}|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \qquad \text{a.e. in} \quad R^n,$$

for some $\Lambda \ge 1$. The Cauchy problem of the heat equation

$$\begin{cases} u_t = \triangle u \quad \text{in} \quad \mathbb{R}^n \times (0,T], \\ u(x,0) = u_0(x) \end{cases} \tag{1.3}$$

is relatively well understood. Widder [20] proved that to each nonnegative solution u of the heat equation, there corresponds a nonnegative Borel measure μ on $\mathbb{R}^n$ such that

$$\lim_{t\to 0}\int_{R^n} u(x,t)\eta(x)dx = \int_{R^n}\eta\, d\mu$$

1991 Mathematics Subject Classification : 35K65, 35B40, 35B65.

The first author is supported by RIM-GARC, KOSEF and BSRI Program, 1994, N94121. The second author is supported by GARC, KOSEF

for all continuos functions η in $\mathbb{R}^n$ with compact support. Here we call μ as the initial trace of u and the trace μ satisfies the growth condition

$$\int_{R^n} e^{-\frac{|x|^2}{4T}} d\mu(x) < \infty.$$

Moreover if u and v are two nonnegative solutions of (1.3) in $\mathbb{R}^n \times (0, T]$ with equal trace μ, then u is identically equal to v.

For the porous medium equation

$$u_t - \triangle u^m = 0, \quad m > 1 \quad \text{in} \quad \mathbb{R}^n \times (0, \infty) \tag{1.4}$$

Aronson and Caffarelli [1] showed that every nonnegative solution u of (1.4) has a unique initial trace μ and μ satisfies the growth condition

$$\frac{1}{R^n} \int_{|x| \leq R} d\mu = O(R^{\frac{2}{m-1}}) \quad \text{as} \quad R \to \infty. \tag{1.5}$$

Also Benilan, Crandall and Pierre [2] showed that for every measure μ satisfying (1.5), there is a solution u of (1.4) with initial trace μ. On the other hand the uniqueness of nonnegative solution has been studied by Dahlberg and Kenig [8]. They also studied the Cauchy problem of the general porous medium equation

$$u_t = \triangle(\phi(u)) \quad \text{in} \quad \mathbb{R}^n \times (0, T]$$

with suitable growth condition on ϕ.

The evolutionary p-Laplace equation

$$u_t - \text{div}(|\nabla u|^{p-2} \nabla u) = 0 \quad \text{in} \quad \mathbb{R}^n \times (0, T] \quad p > 2 \tag{1.6}$$

has been studied by many authors [7], [11], [12], ... In particular, DiBenedetto and Herrero [12] showed that for every σ-finite Borel measure μ in $\mathbb{R}^n$ satisfying

$$\|\mu\|_r = \sup_{\rho \geq r} \rho^{-n - \frac{p}{p-2}} \int_{|x| \leq \rho} d\mu(x) < \infty \quad \text{for some} \quad r > 0,$$

there exist a weak solution to (1.6) in $\mathbb{R}^n \times (0, T]$, where T is

$$T(\mu) = c_0 \left[\lim_{r \to \infty} \|\mu\|_r \right]^{-(p-2)} \quad \text{if} \quad \|\mu\|_r > 0$$

and $T(\mu) = \infty$ if $\|\mu\|_r = 0$. Furthermore they showed that for each nonnegative weak solution u, there is a unique σ-finite Borel measure μ on $\mathbb{R}^n$

such that

$$\lim_{t\to 0}\int_{R^n} u(x,t)\eta(x)dx = \int_{R^n}\eta d\mu$$

for all continuous and compactly supported function η in $\mathbb{R}^n$. They used the existence of an explicit Barenblatt solution whose initial trace is the Dirac measure at the origin.

We extend the above result to the degenerate parabolic equation (1.1) which has bounded measurable coefficients. A measurable function $(x,t)\mapsto u(x,t)$ defined in $\mathbb{R}^n\times(0,T]$ is a weak solution of (1.1) and (1.2) if for every bounded open set $\Omega\subset\mathbb{R}^n$

$$u\in C(0,T:L^1(\Omega))\cap L^p(0,T:W^{1,p}(\Omega))$$

and

$$\int_\Omega u(x,t)\eta(x,t)dx+\int_0^t\int_\Omega -u\eta_t+a_{ij}(x)|\nabla u|^{p-2}u_{x_i}\eta_{x_j}dx\,ds \\ =\int_\Omega u(x,0)\eta(x,0)dx \tag{1.7}$$

for $0<t<T$ and all test functions $\eta\in W^{1,\infty}(0,T;L^\infty(\Omega))\cap L^\infty(0,T;W_0^{1,\infty}(\Omega))$. If we replace the initial condition (1.2) by a σ-finite Borel measure μ, that is, the right hand side of (1.7) replaced by $\int_\Omega\eta(x,0)d\mu(x)$ for all smooth and compactly supported function η, then we say $u(x,t)$ is a weak solution to (1.1) and (1.2) with the initial trace μ.

Remark. Here we study only the case that a_{ij} is independent of t. When a_{ij} depends on t, all the theorems and lemmas are true except Lemma 3.4.

In section 2 we estimate interior Lipschitz norm in terms of L^p norm of u, by Moser type iterations. After this, the maximum of u can be estimated by L^1 norm of u. We follow the idea of Dahlberg and Kenig [9].

In section 3 we show the growth rate of weak solution u in terms of t. Once we know the growth rate of u, we can show the following estimate

$$\int_0^\tau\int_{R^n}|\nabla u|^{p-1}dx\,dt\le c\tau^{\frac{1}{\kappa}},$$

where $\kappa=n(p-2)+p$. This estimate is useful in showing the uniqueness of the initial trace. A similar estimate has been proved in [12], when $\{a_{ij}\}$ is identity matrix. We shall also prove the Harnack principle, namely the fact that

$$\fint_{|x|<R} u(x,0)dx \le \beta\left[\left(\frac{R^p}{T}\right)^{\frac{1}{p-2}} + \left(\frac{T}{R^p}\right)^{\frac{n}{p}}[u(0,T)]^{\frac{n(p-2)+p}{p}}\right] \tag{1.8}$$

for some constant β. The same estimate for evolutionary p-Laplace equation was proved by DiBenedetto and Herrero [12]. Their proof depends on explicit formula of Barenblatt solution and is not applicable to our problems. We employ a compactness method and scaling argument and the explicit formula of Barenblatt solution is not necessary. This method provides a new proof for Harnack principle for evolutionary p-Laplace equations.

In section 4 we prove the existence and uniqueness of the initial trace of nonnegative weak solution by the use of Harnack inequality (1.8). Also we prove the uniqueness of weak solution with L^1_{loc} data at $t=0$. In section 5 we consider the regularity of interface and the behavior of solutions as t goes to infinity. For the porous medium equation, Caffarelli and Wolanski [4] proved the $C^{1.\alpha}$ regularity of the interface under some nondegeneracy conditions on initial data. On the other hand Choe and Kim [7] have considered the regularity questions of the interface for evolutionary p-Laplace equations. They showed that the interface is Hölder continuous graph if the interface is moving and Lipschitz graph if the initial data satisfy certain nondegeneracy conditions. Here we extend the Hölder regularity result of the inteface to degenerate parabolic equations with bounded measurable coefficients.

The following symbols are used; $\fint_A u dx = \frac{1}{|A|}\int_A u\, dx$, $B_R(x_0) = \{x : |x-x_0| < R\}$, $Q_R(x_0,t_0) = B_R(x_0)\times(t_0-R^p,t_0)$, $S_R(x_0,t_0) = B_R(x_0)\times(t_0-R^p,t_0+R^p)$, $\partial_p Q_R(x_0,t_0) = B_R(x_0)\times\{t_0-R^p\}\cup\partial B_R(x_0)\times[t_0-R^p,t_0]$. If there is no confusion, we drop out (x_0,t_0) in various expressions.

2. Interior estimate

In this section we prove various a priori estimates which are useful in studying pointwise behavior of u. Taking $u^{\alpha+1}\eta^p$ for suitable α and cutoff function η, we find a local maximum principle. A similar estimate for evolutionary p-Laplace equation is known (see [6]).

Lemma 2.1 *Suppose u is a nonnegative smooth solution of* (1.1) *and* (1.2) *in Q_{R_0}, then there exists a constant c_1 and c_2 depending on p, n, Λ*

and R_0 such that

$$\sup_{Q_{\frac{R_0}{2}}} u \leq c_1 \left[\int_{Q_{R_0}} u^p dx\, dt \right]^{\frac{1}{2}} + c_2.$$

Now we estimate $\iint_{Q_R} u^p dx\, dt$ in terms of $\iint_{Q_{2R}} u dx\, dt$.

Lemma 2.2 *Suppose u is a smooth nonnegative solution of (1.1) in Q_{2R_0}. Then there are constants σ and c depending only on Λ, n and p such that*

$$\sup_{Q_{\frac{R_0}{2}}} u \leq c \left[\sup_t \int_{|x|<2R_0} u(x,t) dx + 1 \right]^{\sigma}.$$

Proof. From Hölder inequality we have

$$\iint u^{p/n+(\alpha+p)} \eta^{p(1+p/n)}\, dx\, dt \tag{2.1}$$
$$\leq \left[\sup_t \int u \eta^p\, dx \right]^{p/n} \int \left[\int u^{(\alpha+p)n/(n-p)} \eta^{np/(n-p)}\, dx \right]^{(n-p)/n} dt.$$

We let η be a smooth cut off function such that $0 \leq \eta \leq 1$, $\eta = 0$ on ∂S_R and $\eta(z) = 1$ on for all $z \in S_\rho$. Now we note that $u_1 = u + 1$ is again a solution to the same equation. The advantage of taking u_1 is that we can take negative exponent α in (2.1). By Sobolev embedding theorem and (2.1),

$$\begin{aligned}
\int \left[\int {u_1}^{\frac{\alpha+p}{p} \frac{np}{n-p}} \eta^{\frac{np}{n-p}} dx \right]^{\frac{n-p}{np} \cdot p} dt &\leq \iint |\nabla({u_1}^{\frac{\alpha+p}{p}} \eta)|^p dx\, dt \\
&\leq c \iint_{S_R} ({u_1}^{\alpha+2} + {u_1}^{\alpha+p}) dx\, dt \\
&\leq c \iint_{S_R} u^{\alpha+p} + 1 dx\, dt.
\end{aligned}$$

Then we can write (2.1) as

$$\iint_{S_\rho} {u_1}^{\frac{p}{n}+(\alpha+p)} dx\, dt \leq c \left[\sup_t \int_{|x|<R} u_1 dx \right]^{\frac{p}{n}} \iint_{S_R} {u_1}^{\alpha+p} dx\, dt. \tag{2.2}$$

We let $\frac{p}{n} + \alpha_i + p = \alpha_{i+1} + p$ with $\alpha_0 + p = 1$ then $\alpha_i = \alpha_0 + \frac{p}{n} i$. Define

$R_i = R_0(1+2^{-i})$ and $\rho = R_{i+1}$, $R = R_i$. Hence iterating (2.2) we obtain

$$\iint_{S_{R_{i+1}}} u_1^{\frac{p}{n}+\alpha_i+p} dx\, dt \le c \left[\sup_t \int_{|x|<R} u_1 dx\right]^{\frac{p}{n}} \iint_{S_{R_{i+1}}} u_1^{(\alpha_i+p)} dx\, dt$$

and

$$\iint_{S_{R_i}} u_1^{\alpha_i+p} dx\, dt \le c \left[\sup_t \int_{|x|<R_0} u_1 dx\right]^{\sigma} \left[\iint_{S_{R_0}} u_1 dx\, dt\right] \tag{2.3}$$

for some σ depending only on n, p and Λ. Therefore combining Lemma 2.1 and (2.3) we prove the Lemma. □

Now we improve Lemma 2.2.

Theorem 2.3 *Let u be a smooth nonnegative solution of (1.1) in S_{2R}. Then there are constants c, γ and σ depending on p, n and Λ such that*

$$\sup_{S_{\frac{R_0}{2}}} u \le c\,[I^\sigma + I^\gamma]\,, \tag{2.4}$$

where $I = \sup_t \int_{|x|<2R_0} u(x,t)dx$.

Proof. If $I \ge \varepsilon_0$ for some fixed $\varepsilon_0 > 0$, then Lemma 2.2 implies (2.4) with a constant c depending on ε_0. When $0 \le I < \varepsilon_0$, there is $c_0(R_0)$ such that $0 \le u \le c_0$ in Q_{R_0}. Since $\sup|u| \le c_0$ from (2.1) we can deduce that

$$\sup_t \int u^{\alpha+2}\eta^p dx + c \iint |\nabla(u^{\frac{\alpha+p}{p}})\eta|^p dx\, dt$$
$$\le c \iint u^\alpha g(|\eta_t| + |\nabla\eta|^p) dx\, dt \tag{2.5}$$

for suitable function g and some constant c depending on c_0. Hence iterating (2.5) with similar methods as Lemma 2.1 and 2.2 we prove the Theorem.

□

3. Harnack estimate

Here we employ the idea of Dahlberg and Kenig [9] which uses the scaling properties of solutions. We denote by S the class of all nonnegative weak solutions of (1.1) and (1.2) in $\mathbb{R}^n \times (0,T]$ and $P(N) = \{u \in S : \sup_t \int_{R^n} u(x,t)dx \le N\}$

Lemma 3.1 *Let $u \in P(N)$, then there is a constant c depending only on*

p, n and Λ such that for all $x \in \mathbb{R}^n$

$$u(x,t) \le cN^{\frac{p}{n(p-2)+p}} t^{\frac{-n}{n(p-2)+p}} \quad \textit{for} \quad 0 < t < T-1.$$

Proof. We define $v(\xi,\tau)$ as

$$v(\xi,\tau) = \frac{1}{\gamma} u(x+\rho\xi, t\tau), \tag{3.1}$$

where γ is defined by $\gamma^{(p-2)} = \frac{\rho^p}{t}$. Then $v(\xi,\tau)$ is solution to

$$\frac{\gamma}{t} v_\tau(\xi,\tau) = \frac{\gamma^{p-1}}{\rho^p} (a_{ij}(x+\rho\xi, t\tau)|\nabla v|^{p-2} v_{\xi_i})_{\xi_j}$$

which is $v_\tau(\xi,\tau) = (\bar{a}_{ij}|\nabla v|^{p-2} v_{\xi_i})_{\xi_j}$. Note $\bar{a}_{ij}$ is bounded and has the same ellipticity modulus. If we choose ρ so that $\gamma\rho^n = N$, then $\rho = N^{\frac{p-2}{\kappa}} t^{\frac{1}{\kappa}}$ and

$$\int_{R^n} v(\xi,\tau) d\xi = \frac{1}{\gamma} \int_{R^n} u(x+\rho\xi, t\tau) d\xi = \frac{1}{\gamma\rho^n} \int_{R^n} u(y, t\tau) dy = 1$$

and $v \in P(1)$. Therefore from Theorem 2.3, we get

$$\begin{aligned} u(x,t) &= \gamma v(0,1) = \left(\frac{\rho^p}{t}\right)^{\frac{1}{p-2}} v(0,1) \le c \left(\frac{\rho^p}{t}\right)^{\frac{1}{p-2}} \\ &= cN^{\frac{p}{n(p-2)+p}} t^{\frac{-n}{n(p-2)+p}}. \end{aligned}$$

□

Lemma 3.2 *Let $\sigma \in (0,T)$ be fixed and $u \in P(N)$, then there exists some positive constant c depending on σ, p and n such that*

$$\int_0^\tau \int_{B_R} |\nabla u|^{p-1} dx\, dt \le cN^{1+\frac{p-2}{\kappa}} \tau^{\frac{1}{\kappa}} \tag{3.2}$$

for all $\tau \in (0, T-\sigma)$, where $\kappa = n(p-2)+p$

Proof. As in the case of Lemma 3.1 we define $v(x,t)$ as $v(x,t) = \frac{1}{\gamma} u(\rho x, t)$, with $\gamma = \rho^{\frac{p}{p-2}}$. Then $v(x,t)$ is solution to $v_t(x,t) = (\bar{a}_{ij}|\nabla v|^{p-2} v_{x_i})_{x_j}$. Note $\bar{a}_{ij}$ is bounded and has the same ellipticity modulus. If we choose $\rho = N^{\frac{p-2}{\kappa}}$, then $v \in P(1)$ and $||v||_{L^1(R^n)} = 1$. By Hölder's inequality we get

$$\int_0^\tau \int_{B_R} |\nabla v|^{p-1} dx\, dt |\nabla v|^{p-1} t^{-\delta} v^{\varepsilon} dx\, dt \tag{3.3}$$

$$\leq \left[\int_0^\tau \int_{B_R} (t^\delta v^{-\varepsilon})^{\frac{p}{p-1}} |\nabla v|^p dz\right]^{\frac{p-1}{p}} \left[\int_0^\tau \int_{B_R} \left(t^{-\delta} v^{\varepsilon}\right)^p dz\right]^{\frac{1}{p}}$$

Let $A = \int_0^\tau \int_{B_R} (t^\delta v^{-\varepsilon})^{\frac{p}{p-1}} |\nabla v|^p dz$ and $B = \int_0^\tau \int_{B_R} (t^{-\delta} v^{\varepsilon})^p dz$.

From Lemma 3.1, B can be estimated as

$$\begin{aligned} B &\leq \int_0^\tau t^{-\delta p} \|v^{\varepsilon p-1}\|_\infty \int_{B_R} v dx\, dt \\ &\leq c\int_0^\tau t^{-p\delta} t^{(\varepsilon p-1)\frac{-n}{\kappa}} dt = c\int_0^\tau t^{-p\delta - \frac{n}{\kappa}(\varepsilon p-1)} dt. \end{aligned}$$

Hence if we choose δ and ε satisfying $p\delta + \frac{n}{\kappa}(\varepsilon p - 1) < 1$ and $\varepsilon > \frac{1}{p}$, then

$$B \leq c\tau^{1-p\delta-\frac{n}{\kappa}(\varepsilon p-1)}.$$

for some c depending only on ρ, σ, n and p. To estimate A, take $t^{\frac{\delta p}{p-1}} v^{1-\frac{\varepsilon p}{p-1}} \phi^2$ as a test function to (1.1), where ϕ is a piecewise smooth cutoff function in B_{R+1} with $|\nabla \phi| \leq c$. Here we assume $1 - \frac{\varepsilon p}{p-1} \geq 0$. Hence we obtain

$$\iint v_t (t^{\frac{\delta p}{p-1}} v^{1-\frac{\varepsilon p}{p-1}} \phi^2) + \bar{a}_{ij}(x,t) |\nabla v|^{p-2} v_{x_i} (t^{\frac{\delta p}{p-1}} v^{1-\frac{\varepsilon p}{p-1}} \phi^2)_{x_j} dz = 0$$

and

$$\begin{aligned} &\frac{p-1}{2(p-1)-\varepsilon p} \iint (v^{2-\frac{\varepsilon p}{p-1}} t^{\frac{\delta p}{p-1}} \phi^2)_t dz \\ &- \frac{\delta p}{2(p-1)-\varepsilon p} \iint v^{2-\frac{\varepsilon p}{p-1}} t^{\frac{\delta p}{p-1}-1} \phi^2 dz + \lambda \iint |\nabla v|^p v^{\frac{-\varepsilon p}{p-1}} t^{\frac{\delta p}{p-1}} \phi^2 dz \\ &\quad \leq c \iint |\nabla v|^{p-1} v^{1-\frac{-\varepsilon p}{p-1}} t^{\frac{\delta p}{p-1}} \phi |\nabla \phi| dz. \end{aligned}$$

From the definition of A, we get

$$\begin{aligned} A &\leq c\iint v^{2-\frac{\varepsilon p}{p-1}} t^{\frac{\delta p}{p-1}-1} \phi^2 dz + c\iint |\nabla v|^{p-1} v^{1-\frac{\varepsilon p}{p-1}} t^{\frac{\delta p}{p-1}} \phi |\nabla\phi| dz \\ &\leq c\int_0^\tau \|v^{1-\frac{\varepsilon p}{p-1}}\|_\infty t^{\frac{\delta p}{p-1}-1} \int_{B_{R+1}} v\phi^2 dx\, dt \\ &\qquad + c\iint |\nabla v|^{p-1} v^{1-\frac{\varepsilon p}{p-1}} t^{\frac{\delta p}{p-1}} \phi |\nabla\phi| dz \\ &\leq c\int_0^\tau t^{\frac{-n}{\kappa}(1-\frac{\varepsilon p}{p-1})} t^{\frac{\delta p}{p-1}-1} dt + c\iint |\nabla v|^{p-1} v^{1-\frac{\varepsilon p}{p-1}} t^{\frac{\delta p}{p-1}} \phi|\nabla\phi| dz \\ &\leq c_1 \tau^{-\frac{n}{\kappa}(1-\frac{\varepsilon p}{p-1}) + \frac{\delta p}{p-1}} + c_2 \iint |\nabla v|^{p-1} v^{1-\frac{\varepsilon p}{p-1}} t^{\frac{\delta p}{p-1}} \phi|\nabla\phi| dz. \end{aligned}$$

Here δ satisfies

$$\frac{\delta p}{p-1}-\frac{n}{\kappa}\left(1-\frac{\varepsilon p}{p-1}\right)>0.$$

On the other hand by Youngs inequality

$$\iint |\nabla v|^{p-1}v^{1-\frac{\varepsilon p}{p-1}}t^{\frac{\delta p}{p-1}}\phi|\nabla\phi|dz$$
$$\leq \lambda\iint |\nabla v|^{p}v^{\frac{-\varepsilon p}{p-1}}t^{\frac{\delta p}{p-1}}\phi^{\frac{p}{p-1}}dz+c(\lambda)\iint v^{p-\frac{\varepsilon p}{p-1}}t^{\frac{\delta p}{p-1}}|\nabla\phi|^{p}dz.$$

Hence taking sufficiently small λ, we see that

$$A\leq c_1\tau^{-\frac{n}{\kappa}(1-\frac{\varepsilon p}{p-1})+\frac{\delta p}{p-1}}+c\iint v^{p-2}v^{1-\frac{\varepsilon p}{p-1}}t^{\frac{\delta p}{p-1}}v\,dz. \tag{3.4}$$

From Lemma 3.1, we see that

$$\iint v^{p-2}v^{1-\frac{\varepsilon p}{p-1}}t^{\frac{\delta p}{p-1}}v\,dx\,dt$$
$$\leq c\int_0^{\tau}t^{\frac{\delta p}{p-1}-\frac{n}{\kappa}(1-\frac{\varepsilon p}{p-1})-1}dt\leq c\tau^{\frac{\delta p}{p-1}-\frac{n}{\kappa}(1-\frac{\varepsilon p}{p-1})},$$

provided that ε and δ satisfy $\frac{\delta p}{p-1}-\frac{n}{\kappa}(1-\frac{\varepsilon p}{p-1})>0$. Thus from (3.4) we deduce that

$$A\leq c\tau^{-\frac{n}{\kappa}(1-\frac{\varepsilon p}{p-1})+\frac{\delta p}{p-1}}.$$

Now we need to find δ and ε which satisfies following conditions; $\frac{1}{p}\leq\varepsilon\leq\frac{p-1}{p}$, $p\delta+\frac{n}{\kappa}(\varepsilon p-1)\leq 1$, $\frac{\delta p}{p-1}-\frac{n}{\kappa}(1-\frac{\varepsilon p}{p-1})>0$, $\frac{\delta p}{p-1}-\frac{n}{\kappa}(p-1-\frac{\varepsilon p}{p-1})>-1$. In particular $\delta=\varepsilon=\frac{1}{p}$ satisfy the above inequalities. So with a suitable choice of δ and ε we conclude that

$$\int_0^{\tau}\int_{B_R}|\nabla v|^{p-1}dx\,dt$$
$$\leq\left[c\tau^{\frac{\delta p}{p-1}-\frac{n}{\kappa}(1-\frac{\varepsilon p}{p-1})}\right]^{\frac{p-1}{p}}\left[c\tau^{1-(\delta p+\frac{n}{\kappa}(\varepsilon p-1))}\right]^{\frac{1}{p}}=c\tau^{\frac{1}{\kappa}},$$

where c depends on Λ, n and p. Thus scaling back we prove the Theorem. □

Theorem 3.3 *Suppose $u_k\in P(N)$ and μ_k is the initial trace of u_k at $t=0$. Suppose also that μ_k converges to μ weakly as $t\to 0$. Then there is $u\in P(N)$ such that u_k converges to u uniformly on each compact subset of $\mathbb{R}^n\times(0,T)$ and u has initial trace μ.*

Proof. Suppose $K \subset R^n \times (0,T)$ is compact. From Lemma 3.1 we know that $\{u_k\}$ are uniformly bounded in K and hence u_k are uniformly Hölder continuous (see [11]) and hence $\{u_k\}$ converge to a Hölder continuous function u in K. The fact u is a weak solution follows from weak convergence in $L^p(0,t;W^{1,p}(K))$ and equicontinuity of u_k. Hence it is enough to show that whenever u is locally the uniform limit of $u_k, u \in P(N)$ and u has initial trace μ. Let $0 < \tau < t < T$ and fix $\eta \in C_0^\infty(\mathbb{R}^n)$. From Lemma 3.2

$$\left|\int_{R^n} u_k(x,t)\ \eta(x)\,dx - \int_{R^n} u_k(x,\tau)\eta(x)\,dx\right|$$
$$\leq c\int_\tau^t \int_{R^n} |\nabla u_k|^{p-1}|\nabla\eta|\,dx\,dt \leq c\|\nabla\eta\|_\infty (t-\tau)^{1/\kappa}. \tag{3.5}$$

Therefore sending $\tau \to 0$ and $k \to \infty$ we prove the Theorem. □

Lemma 3.4 *There exists a solution $u(x,t) \in P(1)$ of (1.1) such that the initial trace of u is $\delta(0)$. Moreover there is $T_0 > 0$ such that*

$$u(0,T_0) \geq \frac{1}{4}.$$

Proof. First we consider the case that $\{a_{ij}(x)\}$ is a constant matrix. Let initial data $u_0^k(x)$ of u as $u_0^k(x) = \frac{1}{\omega_n}k^n\chi_{B(\frac{1}{k})}$ and $\int_{R^n} u_0^k(x)dx = 1$, where ω is the measure of $B_1(0)$ Then a nonnegative solution u of (1.1) with initial data $u(x,0) = u_0^k(x)$ will exist. So the existence of solution with Dirac measure as initial trace follows from Theorem 3.3. Let $\eta \in C_0^\infty(\mathbb{R}^n)$ be nonnegative with $\eta(0) = \max_{x\in R^n} = 1$, $||\nabla\eta||_{L^\infty} \leq 1$ and $\int_{R^n}\eta(x)dx = 1$. From (3.5) we have for each t

$$\left|\int_{R^n} u(x,t)\eta(x)dx - \eta(0)\right| < ct^{\frac{1}{\kappa}}$$

for some c depending on Λ, p and n. Furthermore since $\{a_{ij}\}$ is a constant symmetric matrix, it is easy to see that $u(0,t) = \max_{R^n} u(x,t)$ for each t (see [19]). Thus if $T \leq (2c)^{-k}$, then

$$u(0,T) = \int_{R^n} u(0,T)\eta(x)dx \geq \int_{R^n} u(x,T)\eta(x)dx \geq \frac{1}{2}.$$

This proves our claim when $\{a_{ij}\}$ is constant matrix. We employ a blow up method along time, to consider general (a_{ij}). We let $\{\tau_s\}$ be a decreasing sequence of time which converges to zero and set $u_s(x,t) = \tau_s^{\frac{n}{\kappa}} u(\tau_s^{\frac{1}{\kappa}}x, \tau_s t)$, then $u_s(x,t)$ is a nonnegative solution to

$$v_t - (b^s_{ij}(x)|\nabla v|^{p-2}v_{x_i})_{x_j} = 0,$$

where $b^s_{ij}(x) = a_{ij}(\tau_s^{\frac{1}{\kappa}}x)$. Moreover $||u_s||_{L^1} = 1$ for each s and the initial trace of u_s is Dirac measure for each s. Observe that since $b^s_{ij}(x)$ is bounded, it converges weak* L^∞ to a constant symmetric matrix b^∞_{ij} with the same modulus of ellipticity as a_{ij}. Hence it follows from Theorem 3.3 that u_s converges uniformly on each compact subset to u_∞ which is a solution to

$$v_t - (b^\infty_{ij}|\nabla v|^{p-2}v_{x_i})_{x_j} = 0$$

with Dirac delta measure as initial trace. Since b^∞_{ij} is a constant matrix, we have $u_\infty(0,T) \geq \frac{1}{2}$ and hence for sufficiently large number s,

$$u_s(0,T) = \tau_s^{\frac{n}{\kappa}}u(0,\tau_s T) \geq \frac{1}{4}$$

and this completes the proof. □

With the obvious modification of the previous proof, we find the following Corollary.

Corollary 3.5 *For every $N > 0$, there is a unique solution $u_N \in P(N)$ with initial trace $N\delta$. Furthermore, there is N such that* $\inf\{u_N(0,1)\} > 0$.

Lemma 3.6 *Suppose that u is a continuous nonnegative solution to* (1.1) *in $\mathbb{R}^n \times (0,\infty)$. Define $J(s) = s^{\frac{p}{p-2}}$ and*

$$H(s) = \begin{cases} 1 & \text{if } \ 0 < s \leq 1 \\ s\left[J^{-1}(s)\right]^n & \text{if } \ s > 1, \end{cases}$$

then there is a constant c such that

$$\int_{|x|\leq 1} u(x,0)dx \leq cH\left(u(0,1)\right) \tag{3.6}$$

Proof. We shall prove the result under the following additional assumptions that

$$\operatorname{supp}\{u(x,0)\} \subset \{x : |x| < 1\} \quad \text{and} \quad \sup_{0<t<\infty}\int u(x,t)\,dx < \infty. \tag{3.7}$$

The general cases can be handled by cutoff method (see [9]). Suppose

(3.6) does not hold, then there exist a sequence of continuous nonnegative solutions u_k satisfying

$$I_k \equiv \int_{R^n} u_k(x,0)dx \geq kH\left(u_k(0,1)\right)$$

for $k = 1,2,3,\ldots$. Let $v_k(x,t) = \frac{1}{\gamma_k}u_k(\alpha_k x,t) \in P(1)$, then $v_k(x,t)$ is solution to $(v_k)_t = \frac{\gamma_k^{p-2}}{\alpha_k^p}$ div $\left(a_{ij}(x,t)|\nabla v_k|^{p-2}\nabla v_k\right)$. We define α_k and γ_k such that $\alpha_k^{n+1}\alpha_k^{\frac{2}{p-2}} = I_k$ and $\gamma_k = \alpha_k\alpha_k^{\frac{2}{p-2}} = \alpha_k^{\frac{p}{p-2}}$. By the definition of v_k we get $\int v_k(x,0)dx = \frac{1}{\gamma_k}\int u_k(\alpha_k x,0)dx = 1$ and $\operatorname{supp}\{v_k(x,0)\} \subset \{x;\alpha_k|x| < 1\}$, and $v_k(x,0)$ converges weakly to Dirac delta measure centered at origin. By compactness argument, there is a $v(x,t)$ such that a subsequence $v_k(x,t)$ converges uniformly to $v(x,t)$ and $v(x,t)$ solves $v(x,t) - (b_{ij}(x)|\nabla v|^{p-2}v_{x_i})_{x_j} = 0$, where $b_{ij}(x)$ is weak*- L^∞ limit of $a_{ij}(\alpha_k x)$. We claim that there is T_0 such that $v(0,T_0) > \varepsilon$. If the claim is not true, then there is a sequence T_k such that $v(0,T_k) \to 0$ as $T_k \to 0$. Let $w_k(x,t) = \frac{1}{\gamma(T_k)}v((T_k)^{\frac{1}{\kappa}}x, T_k t)$. We define $\gamma(T_k)$ as $\gamma(T_k) = (T_k)^{\frac{p}{p-2}}$, then $w_k(x,t) = \frac{1}{\gamma(T_k)}v((T_k)^{\frac{1}{\kappa}}x,T_k t), w_k(x,t)_t = \frac{T_k}{\gamma(T_k)}v_t((T_k)^{\frac{1}{\kappa}}x,T_k t)$, and $\nabla w_k(x,t) = \frac{(T_k)^{\frac{1}{\kappa}}}{\gamma(T_k)}v((T_k)^{\frac{1}{\kappa}}x,T_k t)$ and hence $w_k(x,t)$ is solution to

$$w_k(x,t)_t = \left(b_{ij}((T_k)^{\frac{1}{\kappa}}x)|\nabla w_k|^{p-2}w_{kx_i}\right)_{x_j}.$$

Since $(T_k)^{\frac{1}{\kappa}}x \to 0$ as k goes to ∞, it is easy to see that $b_{ij}((T_k)^{\frac{1}{\kappa}}x) \to b_{ij}(0)$ in weak*- L^∞ sense. Hence there is $w(x,t)$ such that $w_k(x,t)$ converges uniformly to $w(x,t)$ as $T_k \to 0$ and $w(x,t)$ solves the equation $w(x,t) - (b_{ij}(0)|\nabla w|^{p-2}w_{x_i})_{x_j} = 0$ with Dirac measure as initial trace. Since $a_{ij}(x)$ is symmetric, $b_{ij}(0)$ is symmetric and $w(0,t)$ is the maximum of $w(x,t)$. Then there is T such that $w(0,T) > \mu$ for any μ and $v(0,T_k) > \mu\gamma(T_k) > \varepsilon$ with suitable choice of μ. But this contradicts to the assumption that $v(0,T_k) \to 0$ as $T_k \to 0$ and the claim follows. Without loss of generality we may assume $T_0 = 1$ and hence $v(0,1) > \varepsilon$. Recall $v_k(0,1) = \frac{u_k(0,1)}{\gamma_k}$. Hence from the uniform convergence of v_k to v, $u_k(0,1) \geq \varepsilon\gamma_k \to \infty$. On the other hand

$$\begin{aligned} u_k(0,1)[J^{-1}(u_k(0,1))]^n &= H[u_k(0,1)] \leq \frac{1}{k}\alpha_k^n\alpha_k^{\frac{p}{p-2}} \\ &= \frac{1}{k}\gamma_k[J^{-1}(\gamma_k)]^n \end{aligned}$$

and $\frac{u_k(0,1)}{\gamma_k} \le \frac{1}{k} \to 0$ as $k \to \infty$. □

Theorem 3.7 *Suppose $u \ge 0$ is a weak solution to $u_t = (a_{ij}(x)|\nabla u|^{p-2}u_{x_i})_{x_j}$ in $\mathbb{R} \times (0, T_0)$, $T_0 > 0$. We fix $0 < T < T_0 - \delta$. Then there is a constant β depending only on n, p, Λ and δ such that*

$$\int_{|x|<R} u(x,0)dx \le \beta \left[\left(\frac{R^p}{T} \right)^{\frac{1}{p-2}} + \left(\frac{T}{R^p} \right)^{\frac{n}{p}} [u(0,T)]^{\frac{n(p-2)+p}{p}} \right] \quad (3.8)$$

and the Theorem follows from scaling back.

Proof. Let $\gamma = (\frac{R^p}{T})^{\frac{1}{p-2}}$ and $v(x,t) = \frac{1}{\gamma}u(Rx, Tt) = \frac{1}{\gamma}u(y,s)$. Then, v is solution to $v_t(x,t) = (a_{ij}(Rx, Tt)|\nabla v|^{p-2}v_{x_i})_{x_j}$. We find that $\|v\|_{L^1}(t) = \|u\|_{L^1}(t)$. Hence from Lemma 3.6 we obtain for $\kappa = n(p-2) + p$

$$\int_{|y|\le 1} v(y,0)dy \le \beta \left[1 + v(0,1)^{\frac{\kappa}{p}} \right] = \beta \left[1 + (\frac{u(0,T)}{\gamma})^{\frac{\kappa}{p}} \right].$$

□

4. Existence and Uniqueness

In this section we will show the existence of an initial trace for any nonnegative weak solution u in $\mathbb{R}^n \times (0, T)$, and solutions are uniquely determined by their initial trace. The Harnack inequality and compactness argument are main ingredients.

Theorem 4.1 *Suppose u is a nonnegative weak solution of (1.1) in $\mathbb{R}^n \times (0, T)$, then there is a unique nonnegative σ-finite Borel measure μ such that*

$$\lim_{t\to 0} \int_{R^n} u(x,t)\eta(x)\, dx = \int \eta d\mu,$$

for all $\eta(x) \in C_0^\infty(\mathbb{R}^n)$. Furthermore μ satisfies $\|\mu\|_1 \equiv \sup_{R>1} R^{-\frac{\kappa}{p-2}} \int_{B_R} d\mu < c(u(0,T))$ for some constant c.

Proof. As a consequence of our Harnack inequality (Theorem 3.7) we get

$$\sup_t \int_{|x|<R} u(x,t)dx \le c(T, p, R, u(0,T)) < \infty.$$

Thus there exists a sequence $t_j \to 0$ and a σ-finite Borel measure μ on $\mathbb{R}^n$ such that $u(x, t_j)$ converges weakly to μ on $\mathbb{R}^n$. If $\eta \in C_0^\infty(\mathbb{R}^n)$, then for

$0 < \tau < t$

$$\int_{R^n} u(x,t)\eta(x)\,dx - \int_{R^n} u(x,\tau)\eta(x)\,dx = \int_\tau^t \int_{R^n} a_{ij}(x)|\nabla u|^{p-2}u_{x_i}\eta_{x_j}\,dx\,ds. \tag{4.1}$$

From Lemma 3.2 and (4.1) it follows that

$$\left|\int_{R^n} u(x,t)\eta(x)dx - \int_{R^n} u(x,\tau)\eta(x)dx\right| \le c(t-\tau)^{\frac{1}{\kappa}}. \tag{4.2}$$

Hence taking τ along t_j, we get $\lim_{t_j\to 0}\int_{R^n} u(x,t_j)\eta(x)dx = \int \eta d\mu$. Now we assume that there are σ-finite measure ν and a sequence $s_j \to 0$ such that $\lim_{j\to\infty}\int_{R^n} u(x,s_j)\eta(x)dx = \int \eta d\nu$ for all $\eta \in c_0^\infty(\mathbb{R}^n)$. Let ζ be a nonnegative cutoff function in $B_{(1+\varepsilon)R}$ such that $\zeta \equiv 1$ in B_R and $|\nabla\zeta| \le \frac{c}{\varepsilon R}$. Taking ζ as a test function to (1.1) we get

$$\int_{B_{(1+\varepsilon)R}} u(x,t)dx - \int_{B_R} u(x,\tau)dx \ge -\frac{c}{\varepsilon R}\int_\tau^t \int_{B_{(1+\varepsilon)R}} |\nabla u|^{p-2}|\nabla u|dxds$$

and

$$\int_{B_{(1+\varepsilon)R}} u(x,t)dx \ge \int_{B_R} u(x,\tau)dx - \frac{c}{\varepsilon R}(t-\tau)^{\frac{1}{\kappa}}.$$

Taking $\tau = t_i$ and $t = s_j$ and sending i and j to infinity, we get

$$\int_{B_{(1+\varepsilon)R}} d\nu \ge \int_{B_R} d\mu \quad \text{for all} \quad R > 0$$

Since this inequality holds for all R and ε, letting $\varepsilon \to 0$ and interchanging the role of μ and ν we conclude $\nu = \mu$. This completes the proof. □

From a usual energy estimate, we obtain an improved convergence.

Lemma 4.2 *Suppose that u and v are two weak solutions of (1.1) in $\mathbb{R}^n \times (0,T)$ for some $0 < T < \infty$. If $\sup_{t\in(0,T)}\|u(t)\|_1 + \|v(t)\|_1 < \infty$ and $\lim_{t\to 0}[u(\cdot,t) - v(\cdot,t)] = 0$ in $L^1_{\text{loc}}(R^n)$, then*

$$\lim_{t\to 0}[u(\cdot,t) - v(\cdot,t)] = 0 \quad in \quad L^{1+\varepsilon}_{\text{loc}}(\mathbb{R}^n)$$

for all $0 < \varepsilon < \frac{1}{n}$.

Once we know the higher integrability lemma, we can prove uniqueness of nonnegative weak solutions. Indeed DiBenedetto and Herrero proved the uniqueness considering some weighted norm (see [12]). Here Gronwall type inequality is established and hence uniqueness follows easily.

Theorem 4.3 *Suppose that u and v are two nonnegative weak solutions of (1.1) in $\mathbb{R}^n \times (0,T)$ for some $0 < T < \infty$. If $\sup_{t\in(0,T)} (\|u(t)\|_1 + \|v(t)\|_1) < \infty$, and $\lim_{t\to 0} [u(\cdot,t) - v(\cdot,t)] = 0$ in $L^1_{\rm loc}(R^n)$, then $u \equiv v$ in $\mathbb{R} \times (0,t)$.*

Proof. Let $w = u - v$ and we may assume $w \geq 0$. Let η be a standard cutoff function which is compactly supported in B_{R+1} and $\eta \equiv 1$ in B_R and $|\nabla\eta| < c$ for some constant c. We take $w^\varepsilon \eta^2$ as a test function to (1.1). We know that $\lim_{\tau\to 0} \int w(x,\tau)^{1+\varepsilon} dx = 0$. Thus from this we obtain

$$\begin{aligned}\int_{R_R} w(x,t)^{1+\varepsilon}\, dx &\leq c\int_0^t \int_{B_{R+1}} (|\nabla u| + |\nabla v|)^{p-2} w^{\varepsilon+1}\, dx\, ds \\ &\leq c\left[\int_0^t \int_{B_{R+1}} (|\nabla u| + |\nabla v|)^{p-1}\, dx\, ds\right]^{(p-2)/(p-1)} \\ &\qquad \left[\int_0^t \int_{B_{R+1}} w^{(\varepsilon+1)(p-1)}\, dx\, ds\right]^{1/(p-1)}. \qquad (4.3)\end{aligned}$$

From Lemma 3.2, we know that $\int_0^t \int (|\nabla u| + |\nabla v|)^{p-1}\, dxds \leq ct^{\frac{1}{\kappa}}$. On the other hand, from Lemma 3.1, $\sup w \leq ct^{-\frac{n}{\kappa}}$. Hence we obtain that for $\sigma = (p-2)/\kappa(p-1)$

$$\begin{aligned}&t^{-\sigma} \int_{B_R} w^{1+\varepsilon}\, dx \\ &\leq c\left[\int_0^t s^{-n/\kappa(1+\varepsilon)(p-2)+\sigma} s^{-\sigma} \int_{B_{R+1}} w^{1+\varepsilon}\, dx\, ds\right]^{1/(p-1)}. \qquad (4.4)\end{aligned}$$

Now we establish Gronwall type inequality. This implies uniqueness. Define

$$H(R,t) = t^{-\frac{p-2}{\kappa(p-1)}} \int_{B_R} w^{1+\varepsilon}(x,t)dx.$$

Then (4.4) becomes

$$H(R,t) \leq c\left[\int_0^t s^{-\frac{n}{\kappa}(1+\varepsilon)(p-2)+\frac{p-2}{\kappa(p-1)}} H(R+1,s)ds\right]^{\frac{1}{p-1}} \qquad (4.5)$$

Let $\delta \equiv -\frac{n}{\kappa}(1+\varepsilon)(p-2) + \frac{p-2}{\kappa(p-1)}$, then $\delta > -1$ for small ε. Moreover we note that from the proof of Lemma 4.2 and scaling there is a smooth function $G : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\frac{G(R+1)}{G(R)} \le c$ for some c and $\frac{H(R,t)}{G(R)} \to 0$ as $R \to 0$ (see [9]). Since $\lim_{R\to 0} \frac{H(R,t)}{G(R)} = 0$ for all t, we can find R_1 such that $\frac{H(R_1,t)}{G(R_1)} = \sup_{R\ge 1} \frac{H(R,t)}{G(R)}$. Hence we obtain

$$\frac{H(R,t)}{G(R)} \le c\frac{G(R+1)}{G(R)} \left[\int_0^t s^\delta \frac{H(R+1,s)}{G(R+1)} ds\right]^{\frac{1}{p-1}}. \tag{4.6}$$

Let $\int_0^t s^\delta \sup_{R\ge 1} \frac{H(R,s)}{G(R)} ds = A(t)$. Since $\frac{G(R+1)}{G(R)} < c$ and $\frac{H(R,t)}{G(R)} \to 0$ as $R \to \infty$, we get from (4.6)

$$A'(t)t^{-\delta} \le cA(t)^{\frac{1}{p-1}}$$

and this implies $A(t) = 0$ since $\delta > -1$. Therefore we conclude that $u \equiv v$. □

5. Regularity of the interface and asymptotic behavior

We define a cylindrical domain $Q_R^h(x_0, t_0) = B_R(x_0) \times (t_0, t_0+h)$, $\Omega(t) = \{(x,t) : u(x,t) > 0\}$, $\Gamma(t) =$ the boundary of $\Omega(t)$, and $\Gamma = \bigcup_{t\ge 0} \Gamma(t)$. Then $\Gamma(0)$ is the boundary of $\{x \in \mathbb{R}^n : u_0 > 0\}$. We show that interface Γ consists of two parts that moving part Γ_1 and nonmoving part Γ_2. Moreover Γ_1 is Hölder continuous graph. The Harnack principle is a main tool in studying the behavior of interface.

Lemma 5.1 *Suppose that $u(x, t_0) = 0$ for all $x \in B_{R_0}(x_0)$ Then we have*

$$\sup_{Q^h_{\frac{R_0}{2}}} u \le c\left(\frac{h}{R_0{}^p}\right)^{\frac{1}{2}} \left[\fint\!\!\fint_{Q^h_{R_0}(x_0,t_0)} u^p dx\, dt\right]^{\frac{1}{2}}.$$

for some constant c depending on p, n and Λ.

Proof. Let $0 < \rho < R < R_0$ and η be a standard cutoff function such that $0 \le \eta \le 1$, $\eta(x) = 1$ on B_ρ, $\eta = 0$ on ∂B_R and $\eta \in C_0^\infty(B_R)$ with $|\nabla\eta| \le \frac{c}{(R-\rho)}$ for some constant c. We take $u^{\alpha+1}\eta^p$ as a test function to (1.1). Since $u \equiv 0$ on $B_R \times \{t_0\}$ and η is independent of t, we have

$$\frac{1}{(\alpha+2)} \iint_{Q_R^h} (u^{\alpha+2}\eta^p)_t dx\, dt + (\alpha+1) \iint_{Q_R^h} |\nabla u|^p u^\alpha \eta^p dx\, dt$$

$$\leq \frac{c}{\alpha+1} \iint_{Q_R^h} u^{\alpha+p} |\nabla \eta|^p dx\, dt.$$

From the energy estimate, we have

$$\sup_t \int_{B_R} u^{\alpha+2} \eta^p dx + c \iint_{Q_R^h} |\nabla (u^{\frac{\alpha+p}{p}}) \eta|^p dx\, dt$$
$$\leq c \iint_{Q_R^h} u^{\alpha+p} |\nabla \eta|^p dx\, dt.$$

From Hölder inequality and Sobolev inequality we have

$$\iint_{Q_\rho^h} u^{(\alpha+2)\frac{p}{n}+(\alpha+p)} dx\, dt \leq \left[\frac{c}{(R-\rho)^p} \iint_{Q_R^h} u^{\alpha+p} dx\, dt \right]^{1+\frac{p}{n}} \tag{5.1}$$

for some constant c depending only n, Λ, α and p. Define $\alpha_{i+1} = (\alpha_i + 2)\frac{p}{n} + \alpha_i$ and $\alpha_0 = 0$. Setting $\gamma = 1 + \frac{p}{n}$, we can write $\alpha_i = 2(\gamma^i - 1)$. Let $R_i = \frac{1}{2} R_0 (1 + 2^{-i})$ and take $\alpha = \alpha_i, \rho = R_{i+1}$ and $R = R_i$ in (5.1).

Hence if we define $\Psi_i = ⨍⨍_{Q_{R_i}^h} u^{\alpha_i + p} dx\, dt$, then from (5.1) we get

$$\Psi_{i+1} \leq c^i \left(\frac{h}{R_0^p} \right)^{\frac{p}{n}} \Psi_i^\gamma. \tag{5.2}$$

Iterating (5.2) we prove the Lemma. □

Lemma 5.2 *Suppose that $u(x, t_0) = 0$ in $B_R(x_0)$. Then there exists a constant c such that if*

$$⨍⨍_{Q_R^h(x_0,t_0)} u^p dx\, dt \leq c \left(\frac{h}{R^p} \right)^{-\frac{p}{p-2}},$$

then $u \equiv 0$ in $Q_{R/4}^h$.

Proof. We will show that if $y \in B_{R/4}(x_0)$, then $\sup_{t_0 \leq t \leq t_0+h} u(y,t) = 0$. Let $y \in B_{R/4}(x_0)$ be given. We define for $M_\rho = \sup_{Q_\rho^h(x_0)} u$ for $\rho \in (0, R/4)$. From Lemma 5.1 we have $M_{\rho/2} \leq c(\frac{h}{\rho^p})^{\frac{1}{2}} M_\rho^{\frac{p}{2}}$ for $\rho \in (0, R/2)$. It is a rather standard argument to prove that $\sup_{t_0 \leq t \leq t_0+h} u(y,t) = \lim_{\rho \to 0} M_\rho = 0$ if $M_{R/4} \leq c^{-\frac{4}{(p-2)^2}} [c(\frac{h}{R^p})^{\frac{1}{2}}]^{-\frac{2}{p-2}}$. By Lemma 5.1 and the assumption, we have

$$M_{R/4} \leq \sup_{Q_{R/2}^h(x_0,t_0)} u \leq \left(\frac{h}{R^p} \right)^{\frac{1}{2}} \left[⨍⨍_{Q_R^h(x_0,t_0)} u^p dx\, dt \right]^{\frac{1}{2}}$$

$$\leq c\left(\frac{c}{R^p}\right)^{-\frac{1}{p-2}}. \tag{5.3}$$

So we conclude that $\sup_{t_0\leq t\leq t_0+h} u(y,t)=0$. Since y could be any point in $B_{R/4}(x_0)$, the proof is completed. □

Lemma 5.3 *Suppose $u(x,t_0)=0$ in $B_R(x_0)$, then for any $\varepsilon_0>0$, there exists a constant c depending only on Λ, n, p and ε_0 such that*

$$\fint\!\!\fint_{Q^h_{R/2}} u^p dx\,dt \leq c\left[\sup_{t_0\leq t\leq t_0+h}\fint_{B_R} u(x,t)dx\right]^p + \varepsilon_0\left(\frac{R^p}{h}\right)^{\frac{p}{p-2}}. \tag{5.4}$$

Proof. Define $v(x,t)=(\frac{h}{R^p})^{\frac{1}{p-2}}u(Rx+x_0,ht+t_0)$, then v is solution to

$$v_t=(a_{ij}(Rx+x_0)|\nabla v|^{p-2}v_{x_i})_{x_j}. \tag{5.5}$$

Let $\frac{1}{2}<r_2<r_1\leq 1$ and denote $v_\varepsilon=v+\varepsilon$. We take $v_\varepsilon^{\alpha+1}\eta^p$ as a test function to (5.5), where η is a standard cutoff function as $\eta\equiv 1$ on B_{r_2} and $\eta=0$ on ∂B_{r_1} with $|\nabla\eta|\leq c$. Since $v(x,0)=0$, we have

$$\begin{aligned}(\alpha+1)\iint_{Q^1_{r_1}}&|\nabla v|^p v_\varepsilon^\alpha\eta^p dx\,dt+\frac{1}{\alpha+2}\int_{B_{r_1}} v_\varepsilon^{\alpha+2}(x,1)\eta^p dx\\ &\leq c\iint_{Q^1_{r_1}}|\nabla v|^{p-1}v_\varepsilon^{\alpha+1}\eta^{p-1}|\nabla\eta|dx\,dt+\int_{B_{r_1}}\frac{\varepsilon^{\alpha+2}}{\alpha+2}\eta^p(x)dx.\end{aligned}$$

Hence if $\alpha<-2$ or $\alpha>-1$, then

$$\begin{aligned}&\iint_{Q^1_{r_1}}|\nabla v|^p v_\varepsilon^\alpha\eta^p dx\,dt\\ &\quad\leq c(\alpha)\left[\iint_{Q^1_{r_1}}|\nabla v|^{p-1}v_\varepsilon^{\alpha+1}\eta^{p-1}|\nabla\eta|dx\,dt+\varepsilon^{\alpha+2}|B_{r_1}|\right]\end{aligned} \tag{5.6}$$

and if $-2<\alpha<-1$, then

$$\begin{aligned}&\iint_{Q^1_{r_1}}|\nabla v|^p v_\varepsilon^\alpha\eta^p dx\,dt\\ &\leq c(\alpha)\left[\iint_{Q^1_{r_1}}|\nabla v|^{p-1}v_\varepsilon^{\alpha+1}\eta^{p-1}|\nabla\eta|dx\,dt+\int_{B_{r_1}}v_\varepsilon^{\alpha+2}(x,1)\eta^p dx\right].\end{aligned} \tag{5.7}$$

Combining (5.6) and (5.7), from the ellipticity condition and Young's inequality we get for some small δ

$$\begin{aligned}\iint_{Q^1_{r_1}} &|\nabla v|^p v_\varepsilon^\alpha \eta^p dx\,dt \\ &\le c(\alpha,\Lambda)\left[\iint_{Q^1_{r_1}} \delta^{p/(p-1)}|\nabla v|^p v_\varepsilon^\alpha \eta^p + \delta^{-p} v_\varepsilon^{\alpha+p}|\nabla\eta|^p dx\,dt \right.\\ &\qquad\left. + \int_{B_{r_1}} (c_1 v_\varepsilon^{\alpha+2} + c_2 v_\varepsilon)\eta^p dx + \varepsilon^{\alpha+2}\right]. \qquad (5.8)\end{aligned}$$

Then choosing δ small in (5.8) we obtain

$$\begin{aligned}\iint_{Q^1_{r_1}} &|\nabla v|^p v_\varepsilon^\alpha \eta^p dx\,dt \\ &\le c\left[\iint_{Q^1_{r_1}} v_\varepsilon^{\alpha+p}(\eta^p + |\nabla\eta|^p)dx\,dt + I + \varepsilon^{\alpha+2}\right], \qquad (5.9)\end{aligned}$$

where $I = \sup_{t\in[0,1]} \int_{B_{r_1}} v(x,t)dx$. From Sobolev inequality, Hölder inequality and (5.9), we have

$$\iint_{Q^1_{r_2}} v_\varepsilon^{\frac{p}{n}+p+\alpha} dx\,dt \le cI^{\frac{p}{n}}\left[\iint_{Q^1_{r_1}} v_\varepsilon^{\alpha+p} dx\,dt + I + \varepsilon^{\alpha+2}\right] + \varepsilon^{\frac{p}{n}+p}. \qquad (5.10)$$

Iterating (5.10) in a similar manner as Lemma 2.2 we get

$$⨍⨍_{Q^1_{1/2}} v^p dx\,dt \le c\left[\sup_{t\in[0,1]} ⨍_{B_1} v(x,t)dx\right]^p + \varepsilon_0$$

for some constant c depending on Λ, p and n. Therefore scaling back we prove the lemma. □

Theorem 5.4 *Suppose that* $\operatorname{dist}(x_0, \operatorname{supp}\{u_0\}) > R$, *then there is a constant* c *depending on* Λ, p *and* n *such that* $u(x_0,t) = 0$ *for all* $t \le \frac{c}{\|u_0\|^{p-2}} R^{n(p-2)+p}$.

Proof. From Lemma 5.2 we know that if $⨍⨍_{Q^h_R} u^p dx\,dt \le c(\frac{h}{R^p})^{\frac{p}{p-2}}$, then $u \equiv 0$in $Q^h_{R/4}$. Hence if $u(x_0,h) \ne 0$ for some $h > 0$, then

$$⨍⨍_{Q^h_R} u^p dx\,dt > c\left(\frac{h}{R^p}\right)^{\frac{p}{p-2}}. \qquad (5.11)$$

On the other hand we know that from Lemma 5.3, for any $\varepsilon_0 > 0$,

$$\fint\!\!\fint_{Q_R^h} u^p dx\, dt \le c \left[\sup_{t\in[0,h]} \fint_{B_{2R}} u(x,t)dx \right]^p + \varepsilon_0 \left(\frac{R^p}{h} \right)^{\frac{p}{p-2}}. \qquad (5.12)$$

Hence if we choose $\varepsilon_0 \le \frac{c}{2}$, then from (5.11), (5.12) we have

$$\frac{c}{2}\left(\frac{R^p}{h}\right)^{\frac{p}{p-2}} < c \left[\frac{1}{R^n} \sup_{t\in[0,h]} \int_{B_{2R}} u(x,t)dx \right]^p.$$

Since $\|u\|_{L^1}(t) = \|u_0\|_{L^1}$ for all t, we conclude that $h > \frac{c}{\|u_0\|_{L^1}^{p-2}} R^{n(p-2)+p}$.

□

Now we prove the converse of Theorem 5.4. These are direct consequences of Harnack principle.

Corollary 5.5 *Suppose that* $\operatorname{supp}\{u_0\} \subset B_R(0)$, *Then there is a constant* c *such that if*

$$h \ge \frac{c}{\|u_0\|_{L^1}^{p-2}} (|x_0| + R)^{n(p-2)+p},$$

then $u(x_0, h) > 0$.

Proof. The Harnack principle (see Theorem 3.7) implies that if $\fint_{B_R(x_0)} u_0(x)dx \ge 2\beta(\frac{R^p}{h})^{\frac{1}{p-2}}$, then $u(x_0,h) \ge c(\frac{R^p}{h})^{\frac{1}{p-2}}$. We note that $B_{|x_0|+R}(x_0) \supset B_{R(0)} \supset \operatorname{supp} u_0$ and hence we have $\fint_{B_{|x_0|+R}(x_0)} u_0(x)dx = \frac{c}{(|x_0|+R)^n}\|u_0\|_{L^1}$. Thus if $\frac{c}{(|x_0|+R)^n}\|u_0\|_{L^1} \ge 2\beta(\frac{(|x_0|+R)^p}{h})^{\frac{1}{p-2}}$, that is, $h \ge c\frac{(\|x_0\|+R)^{n(p-2)+p}}{\|u_0\|_{L^1}^{p-2}}$, then $u(x_0,h) > 0$. □

From the Harnack principle we can observe the support expands as time goes. Now we show that the interface is Hölder continuous. First we find that the interface Γ consists of moving part Γ_1 and nonmoving part Γ_2, that is, $\Gamma = \Gamma_1 \cup \Gamma_2$ and Γ_1 is relatively open in Γ. We refer Caffarelli and Friedmann [3] for the porous medium equations and Choe and Kim [7] for the evolutionary p-Laplace equations.

Theorem 5.6 *Suppose that* $(x_0, t_0) \in \Gamma_1$, *that is, the vertical segment does not contain any point of* Γ, *then there exist constants* c, h *and* α *such* *that*

$$u(x,t)=0 \quad on \quad \{(x,t): t\in(t_0-h,t_0), |x-x_0|\le c|t_0-t|^\alpha\}$$

and

$$u(x,t)>0 \quad on \quad \{(x,t): t\in(t_0,t_0+h), |x-x_0|\le c|t_0-t|^\alpha\}.$$

Consequently Γ_1 *is locally Hölder continuous.*

Proof. Let $t_1<t_0$ be fixed and set $h=t_0-t_1$. From the monotonicity of the support there exists $R>0$ such that $B_{2R}(x_0)\cap\Omega(t_1)$ is empty, that is, $u(x_1,t_1)=0$ for all $x\in B_{2R}(x_0)$. For $t\in[t_1,t_0]$, we write $t=t_1+\delta h$. Our aim is to find a relation between δ and $\mathrm{dist}(x_0,\Gamma(t))=d(\delta)R$.

First we claim that there exists a $\delta_0<1/2$ such that $d(\delta_0)>1/3$. It follows from Lemma 5.2 that if $\frac{|x_1-x_0|}{R}=d(\delta)\le 1/3$, then

$$\left[\fint_{t_1}^{t_1+\delta h}\fint_{B_{(1-d)R}(x_1)} u^p(x,s)dxds\right]^{\frac{1}{p}} \ge c\left[\frac{(1-d)^pR^p}{\delta h}\right]^{\frac{1}{p-2}}.$$

Thus we obtain

$$\left[\fint_{t_1}^{t_1+\delta h}\fint_{B_R(x_0)} u^p(x,s)dxds\right]^{\frac{1}{p}} \ge c\frac{(1-d)^{\frac{n}{p}+\frac{p}{p-2}}}{\delta^{\frac{1}{p-2}}}\left(\frac{R^p}{h}\right)^{\frac{1}{p-2}}. \tag{5.13}$$

By Harnack principle and Lemma 5.3 we get

$$\begin{aligned}
&c\frac{(1-d)^{\frac{n}{p}+\frac{p}{p-2}}}{\delta^{\frac{1}{p-2}}}\left(\frac{R^p}{h}\right)^{\frac{1}{p-2}}\\
&\le\left[\fint_{t_1}^{t_1+\delta h}\fint_{B_R(x_0)} u^p(x,s)dxds\right]^{\frac{1}{p}}\\
&\le c(\varepsilon)\left[\sup_{t_1\le s\le t_1+\delta h}\fint_{B_R(x_0)} u(x,s)dx\right]+\varepsilon\left(\frac{R^p}{\delta h}\right)^{\frac{1}{p-2}}\\
&\le c(\varepsilon)\left(\frac{R^p}{(1-\delta)h}\right)^{\frac{1}{p-2}}+\varepsilon\left(\frac{R^p}{\delta h}\right)^{\frac{1}{p-2}}
\end{aligned}$$

for any small ε since $u(x_0, t_0) = 0$. Hence we get

$$(1-d)^{\frac{n}{p}+\frac{p}{p-2}} \leq c\left(\frac{\delta}{1-\delta}\right)^{\frac{1}{p-2}} + \varepsilon.$$

Since ε is arbitrary, d is close to 1 if δ is small. Now we let $\frac{1}{3} = (1-\delta_0)^{\alpha}$ for some $a > 0$. Hence we have

$$\operatorname{dist}(x_0, \Gamma(t_0 - (1-\delta_0)h)) \geq (1-\delta_0)^{\alpha} R.$$

Repeating this process we get

$$\operatorname{dist}(x_0, \Gamma(t_0 - (1-\delta_0)^k h)) \geq (1-\delta_0)^{k\alpha} R$$

for each $k = 1, 2, \ldots$. Finally varying h we conclude that

$$\operatorname{dist}(x_0, \Gamma(t)) \geq \left(\frac{t_0 - t}{h}\right)^{\alpha} R.$$

This completes the proof of the first statement. The second statement can be proved in the same way. □

Now we study the behavior of a solution u to (1.1) as t goes to infinity. From Lemma 3.1 we know that

$$u(x,t) \leq cN^{\frac{p}{n(p-2)+p}} t^{\frac{-n}{n(p-2)+p}} \quad \text{for} \quad 0 < t < T-1.$$

Considering the Harnack estimate we can state a Lemma which is converse to Lemma 3.1.

Lemma 5.7 *There exist constants c_1 and c_2 depending only on Λ, n and p such that*

$$u(0,t) \geq c_1 \|u_0\|_{L^1}^{\frac{p}{n(p-2)+p}} t^{\frac{-n}{n(p-2)+p}}$$

if t is so large that $\fint_{B_R(0)} u_0(x)dx \geq \frac{1}{2}\|u_0\|_{L^1}$, where $R = c_2\|u_0\|_{L^1}^{\frac{p}{n(p-2)+p}}$.

Proof. From the Harnack estimate we get

$$\fint_{B_R(x_0)} u_0(x)dx \leq \beta\left[\left(\frac{R^p}{t}\right)^{\frac{1}{p-2}} + \left(\frac{t}{R^p}\right)^{\frac{n}{p}} [u(x_0,t)]^{\frac{n(p-2)+p}{p}}\right].$$

We fix R so that $\fint_{B_R(x_0)} u_0(x)dx \geq 12R^n\|u_0\|_{L^1}$. If t is so large that

$\beta(\frac{R^p}{t})^{\frac{1}{p-2}} \leq \frac{1}{4R^n}\|u_0\|_{L^1}$, that is $t \geq \frac{c}{\|u_0\|_{L^1}^{p-2}} R^{n(p-2)+p}$, then

$$\begin{aligned}\frac{1}{2R^n}\|u_0\|_{L^1} &\leq \fint_{B_R(x_0)} u_0(x)dx \\ &\leq \frac{1}{4R^n}\|u_0\|_{L^1} + \beta\left(\frac{t}{R^p}\right)^{\frac{n}{p}} [u(x_0,t)]^{\frac{n(p-2)+p}{p}}.\end{aligned}$$

Hence

$$u(x_0,t) \geq \left[\frac{c}{4R^n}\|u_0\|_{L^1}\left(\frac{R^p}{t}\right)^{\frac{n}{p}}\right]^{\frac{p}{n(p-2)+p}} = \|u_0\|_{L^1}^{\frac{p}{n(p-2)+p}} t^{\frac{-n}{n(p-2)+p}}.$$

The proof ended. □

Finally we study the asymptotic behavior of u in term of fundamental solutions. There are some results for porous medium equations and p-Laplace equations using explicit form of Barenblatt solutions (see [13] and [15]). Let $F(\Lambda, M)$ be the class of all fundamental solutions Q to $Q_t - (a_{ij}(x)|\nabla Q|^{p-2}Q_{x_i})_{x_j} = 0$, with $Q(x,0) = M\delta(0)$, where a_{ij} satisfies the ellipticity condition $\Lambda^{-1}|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$ for all $x, \xi \in \mathbb{R}^n$.

Theorem 5.8 *Let $R(t) = t^{\frac{1}{n(p-2)+p}}$, then*

$$\lim_{t\to\infty} \inf_{Q\in F(\Lambda,\|u\|_{L^1})} \left(\sup_{B_{R(t)}(0)} t^{\frac{n}{n(p-2)+p}}|u(x,t) - Q(x,t)|\right) = 0.$$

Proof. We prove by contradiction. Suppose the assertion is not true, then there is a sequence of time $\tau_i \to \infty$ such that for some ε and for all $Q \in F(\Lambda, ||u||_{L^1})$ $\sup_{B_\tau(0)} \tau_i^{\frac{n}{\kappa}}|u(x,\tau_i) - Q(x,\tau_i)| > \varepsilon$. We define $v^i(x,t) = \tau_i^{\frac{n}{\kappa}} u(\tau_i^{\frac{1}{\kappa}}x, \tau_i t)$, then v^i is solution to $v_t^i - (a_{ij}(\tau_i^{1/\kappa}x, \tau_i t)|\nabla v^i|^{p-2}v_{x_i}^i)_{x_j} = 0$, with $v^i(x,0) = \tau_i^{n/\kappa}u_0(\tau_i^{\frac{1}{\kappa}}x)$. We may assume that $[a_{ij}(\tau_i^{1/\kappa}x)]$ converges to $[b_{ij}(x)]$ weakly, where $[b_{ij}(x)]$ satisfies the same ellipticity condition. Since $v^i(x,0) \to \|u_0\|_{L^1}\delta(0)$ as $i \to \infty$, $v^i \to Q$ uniformly on each compact subset $K \subset R^n \times R^+$ for some $Q \in F(\Lambda, \|u_0\|_{L^1})$. In particular

$$\lim_{i\to\infty} \sup_{B_1(0)} |v^i(x,1) - Q(x,1)| = 0.$$

We define $\bar{Q}^i(x,t)$ by

$$Q(x,t) = \tau_i^{\frac{n}{k}} \bar{Q}^i(\tau_i^{\frac{1}{k}} x, \tau_i t).$$

Now we see that $\bar{Q}^i \in F(\Lambda, \|u_0\|_{L^1})$ and

$$\begin{aligned}
&\lim_{i\to\infty} \left(\sup_{B_{\frac{1}{\tau_i^{k}}}(0)} \tau_i^{\frac{n}{k}} |u(x,\tau_i) - \bar{Q}^i(x,\tau_i)| \right) \\
&\quad = \lim_{i\to\infty} \sup_{B_1(0)} |\tau_i^{\frac{n}{k}} u(\tau^{\frac{1}{k}} x, \tau_i) - \tau^{\frac{n}{k}} \bar{Q}^i(\tau_i^{\frac{1}{k}} x, \tau_i)| \\
&\quad = \lim_{i\to\infty} \sup_{B_1(0)} |v^i(x,1) - Q(x,1)| = 0
\end{aligned}$$

and this completes the proof □

We note that if a_{ij} is identity matrix, then $F(\Lambda, \|u_0\|_{L^1})$ has only a single element, that is, the explicit Barenblatt solution to evolutionary p-Laplace equation. This will corresponds to the known results (see [15]).

References

[1] Aronson D.G. and Cafarelli L.A., *The initial trace of a solution of porous medium equation.* Trans. Amer. Math. Soc. **280** (1983), 351–366.

[2] Bénilan P., Crandall M.G. and Pierre M., *Solutions of the porous medium equation under optimal conditions on the initial values.* Indiana Univ. Math. J. **33** (1984), 51–84.

[3] Caffarelli L.A. and Friedman A., *Regularity of the free boundary of a gas flow in an n-dimensional porous medium.* Indiana Univ. Math. J. **29** (1980), 361–389.

[4] Caffarelli L.A. and Wolanski N.I., *$C^{1,\alpha}$ regularity of the free boundary for the N-dimensional porous medium equation.* Comm. Pure Appl. Math **XLIII** (1990), 885–902.

[5] Cho C.K. and Choe H.J., *The initial trace of a solution of a degenerate parabolic equation with bounded measurable coefficients.* preprint

[6] Choe H.J., *Hölder continuity for solutions of certain degenerate parabolic systems.* Nonlinear Analysis, Theory, Methods and Applications **18** (1992), 235–243.

[7] Choe H.J. and Kim J.S., *Regularity for the interface of Evolutionary P-Laplacian Functions.* SIAM J. Math. Anal. **26** No.4 (1995), 791–819.

[8] Dahlberg B.E.J. and Kenig C.E., *Non negaative solutions of the porous medium equation.* Comm. in Partial Differential Equations **9** (1984), 409–437.

[9] Dahlberg B.E.J. and Kenig C.E., *Non negaative solutions of generalized porous medium equation.* Revista Math. Iber **2** (1986), 267–305.

[10] DiBenedetto E., *Intrinsic Harnack type inequalities for solutions of certain degen-*

erate parabolic equations. Arch. Rational Mech. Anal **100** (1988), 129–147.

[11] DiBenedetto E., *On the local behavior of solutions of degenerate parabolic equations with measurable coefficients.* Ann. Scoula Norm. Sup. Pisa Cl. Sci(4) **13** (1986), 487–535.

[12] DiBenedetto E. and Herrero M.A., *On the Cauchy problem and initial traces for a degenerate parabolic equation.* Trans. Amer. Math. Soc **314** (1989), 187–224.

[13] Friedman A. and Kamin S., *The asymptotic behavior of gas in an n-dimensional porous medium.* Trans. Amer. Math. Soc **262** (1980), 551–563.

[14] Juning Z., *Source-type solutions of degenerate quasilinear parabolic equations.* J. Diff. Eq **92** (1991), 179–198.

[15] Juning Z., *The asymptotic behavior of solutions of a quasilinear degenerate parabolic equation.* J. Diff. Eq **102** (1993), 33–52.

[16] Ladyzenskaja O.A., Solonnikov V.A. and Uraltzeva N.N. *Linear and quasilinear equations of parabolic type Trans. Math. Monographs 23.* Amer. Math. Soc. Providence RI 1968.

[17] Moser J., *A Harnack inequality for parabolic equations.* Comm. Pure. Appl. Math **17** (1964), 101–134.

[18] Sacks P.E., *Continuity of solutions of a singular parabolic equation.* Nonlinear Analysis **7** (1983), 387–409.

[19] Vazquez J.L., *Symétrization pour* $u_t = \triangle\phi(u)$ *et applications.* C. R. Acad. Paris **295** (1982), 71–74.

[20] Widder D., *Positive temperatures on the infinite rod.* Trans. Amer. Math. Soc. **55** (1944), 84–95.

Hi Jun Choe
Department of Mathematics
KAIST, Taejon
Republic of Korea

Jin Ho Lee
Department of Mathematics
Sookmyung Women's University
Seoul, 140-742 Republic of Korea
E-mail: jhlee@sookmyung.ac.kr