## A criterion for the existence of subobject classifiers

Hiroshi WATANABE\*

(Received January 16, 1998)

**Abstract.** We give a criterion for the existence of subobject classifiers of cocomplete categories with a small, dense subcategory.

Key words: subobject classifier, cocomplete, dense, topos.

#### 1. Introduction

It often occurs that a cocomplete category  $\mathcal{E}$  has a small and dense subcategory  $\mathcal{C}$ . In this case, there is an adjunction between the category  $\mathcal{E}$ and the category  $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$  of presheaves over  $\mathcal{C}$ , which enables us to construct  $\mathcal{E}$  objects from presheaves over  $\mathcal{C}$ .

In this paper, we give a criterion for the existence of a subobject classifier in  $\mathcal{E}$  expressed as a condition in the presheaf category. Moreover we give the subobject classifier concretely by using the presheaves if there exists a subobject classifier.

This criterion is used heavily in the proof of the existence of subobject classifier in the category of functional bisimulations [6]. We expect this applicable to other similar problems.

We proceed as follows. In Section 2 we recall general facts on the adjunction between a cocomplete category  $\mathcal{E}$  and the category of presheaves over the small category  $\mathcal{C}$  when there is a functor from  $\mathcal{C}$  to  $\mathcal{E}$ . Here, we do not assume that  $\mathcal{C}$  is a subcategory of  $\mathcal{E}$ . Then we introduce the notion of dense functors. When a functor A from  $\mathcal{C}$  to  $\mathcal{E}$  is dense, the category  $\mathcal{E}$  is a reflective subcategory of the presheaf category of  $\mathcal{C}$ . Hence  $\mathcal{E}$  is a complete category since presheaf category is complete, and thus a subobject functor Sub :  $\mathcal{E}^{\text{op}} \to \text{Set}$  exists.

In Section 3 we give a criterion for the existence of subobject classifier of  $\mathcal{E}$ , to the effect that the presheaf  $\operatorname{Sub}(A(-))$  is in  $\mathcal{E}$ , i.e., there exists an object  $\Omega \in \mathcal{E}$  such that  $\operatorname{Sub}(A(-)) \cong \mathcal{E}(A(-), \Omega)$  holds.

<sup>1991</sup> Mathematics Subject Classification: 18B25, 18A40, 68Q55.

<sup>\*</sup>This work was done as a doctoral dissertation of Hokkaido University.

When we are dealing with a concrete category, it is usually difficult to find concretely in  $\mathcal{E}$  itself a specific  $\Omega$  which satisfies the above condition. We explain a method of constructing  $\mathcal{E}$  objects via presheaves over  $\mathcal{C}$  by making use of the above adjunction, in Section 4. As an example of this construction, we give a terminal object in  $\mathcal{E}$ . In the case there exists a subobject classifier in  $\mathcal{E}$ , we give it by using presheaf category. This enables us to modify the criterion slightly, which is more effective for checking it in concrete cases.

#### 2. A basic adjunction

Let  $A : \mathcal{C} \to \mathcal{E}$  be a functor from a small category  $\mathcal{C}$  to a cocomplete category  $\mathcal{E}$ . Define a functor  $R : \mathcal{E} \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$  by  $R(E) = \mathcal{E}(A(-), E)$ .

**Proposition 2.1** ([4, pp. 41–42]) The functor R has a left adjoint L :  $\mathbf{Set}^{C^{\mathrm{op}}} \to \mathcal{E}$ . The functor L is given for each presheaf P by

$$LP = \operatorname{Colim}\left(\int P \stackrel{\pi_P}{\to} \mathcal{C} \stackrel{A}{\to} \mathcal{E}\right).$$

Recall that the category  $\int P$  of elements of a presheaf P is defined as follows.

- Its object is a pair (C, p) of an object  $C \in \mathcal{C}$  and  $p \in P(C)$ .
- An arrow  $u : (C, p) \to (C', p')$  is a  $\mathcal{C}$  arrow  $u : C \to C'$  such that  $p' \cdot u = p$ , where  $p' \cdot u := P(u)(p')$ .

The functor  $\pi_P : \int P \to \mathcal{C}$  is the projection  $(C, p) \mapsto C$ , and the composition of functors

$$\int P \stackrel{\pi_P}{\to} \mathcal{C} \stackrel{A}{\to} \mathcal{E} \tag{1}$$

is a diagram in  $\mathcal{E}$  with the indexing category  $\int P$ . We often write the diagram (1) simply by  $A \circ \pi_P$ .

Outline of proof. First we define a bijective correspondence

$$\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}(P, R(E)) \cong \mathcal{E}(LP, E)$$

for each presheaf P and an object  $E \in \mathcal{E}$ .

Let  $\tau \in \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}(P, R(E))$ . Consider the diagram  $A \circ \pi_P$  and take a cocone  $\tilde{\tau} : A \circ \pi_P \to E$  with  $\tilde{\tau}_{(C,p)} = \tau_C(p)$  for each object  $(C,p) \in \int P$ , where  $\tau_C$  is the component of the natural transformation  $\tau$  at  $C \in \mathcal{C}$ . Since

the category  $\mathcal{E}$  is cocomplete, there exists a universal cocone  $\mu : A \circ \pi_P \to \operatorname{Colim}(A \circ \pi_P) = LP$ . By the universality of the cocone  $\mu$ , we have a unique  $\mathcal{E}$  arrow  $g: \operatorname{Colim}(A \circ \pi_P) \to E$  satisfying  $g \circ \mu_{(C,p)} = \tilde{\tau}_{(C,p)}$  for each object  $(C,p) \in \int P$ . It is easy to show that the correspondence  $\tau \mapsto g$  is bijective and natural for P and E.

**Definition 2.2** For each object  $E \in \mathcal{E}$ , we call the functor

$$(A(-)/E) \xrightarrow{\partial} \mathcal{C} \xrightarrow{A} \mathcal{E}$$

the **canonical diagram** of E. Here A(-)/E is the comma category. Its object is a pair of  $C \in \mathcal{C}$  and an  $\mathcal{E}$  arrow  $f : A(C) \to E$ , and its arrow from  $(C, f : A(C) \to E)$  to  $(C', g : A(C') \to E)$  is a  $\mathcal{C}$  arrow  $u : C \to C'$  such that  $f = g \circ A(u)$ . The functor  $\partial : (A(-)/E) \to \mathcal{C}$  is the projection:  $(C, f) \mapsto C$ .

For each object  $E \in \mathcal{E}$ , there always exists a cocone  $\nu : A \circ \partial \to E$ defined by  $\nu_{(C,f)} = f$  for each object  $(C,f) \in A(-)/E$ , which we call the **canonical cocone** of E.

**Definition 2.3** A functor  $A : \mathcal{C} \to \mathcal{E}$  is called **dense** if the canonical cocone of E is universal for each object  $E \in \mathcal{E}$ .

We call a subcategory is dense when the inclusion functor is dense.

Note that the canonical diagram of E is nothing but the diagram  $A \circ \pi_{R(E)} : \int R(E) \to \mathcal{E}$ . Hence if the functor  $A : \mathcal{C} \to \mathcal{E}$  is dense, then

$$LR(E) = \operatorname{Colim}\left(\int R(E) \stackrel{\pi_{R(E)}}{\to} \mathcal{C} \stackrel{A}{\to} \mathcal{E}\right)$$
$$= \operatorname{Colim}((A(-)/E) \stackrel{\partial}{\to} \mathcal{C} \stackrel{A}{\to} \mathcal{E}) \cong E$$

The following fact on the above adjunction  $L \dashv R$  is known.

**Proposition 2.4** (See [2]) Let  $L \dashv R$  be the adjunction of Proposition 2.1. Then the following three conditions are equivalent to each other.

- 1. The functor  $A : C \to \mathcal{E}$  is dense.
- 2. The right adjoint functor R is full and faithful.
- 3. The counit  $\varepsilon : LR \to Id_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$  of the adjunction is a natural isomorphism, i.e., the E-component  $\varepsilon_E : LR(E) \to E$  is isomorphic for each object  $E \in \mathcal{E}$ .

*Proof.* Because the equivalence between 2 and 3 is well-known, we have

only to show the equivalence between 1 and 3. By the adjunction  $L \dashv R$  in Proposition 2.1, we have the bijective correspondence

$$\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}(R(E), R(E)) \cong \mathcal{E}(LR(E), E)$$

for object  $E \in \mathcal{E}$  and  $R(E) \in \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ , and the counit  $\varepsilon_E : LR(E) \to E$  corresponds to the identity transformation  $\mathrm{id}_{R(E)}$  under this bijective correspondence.

If we assume A is dense, then we have  $\varepsilon_E$  is isomorphic. Conversely if we assume  $\varepsilon_E$  is isomorphic, then the canonical cocone has the universality. Hence 1 and 3 are equivalent.

When the functor A is dense, the functor  $R : \mathcal{E} \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$  is full and faithful by Proposition 2.4. Consequently the category  $\mathcal{E}$  is a reflective subcategory of  $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ . Since the category  $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$  is complete, we obtain by using [1, Proposition 3.5.3]:

**Proposition 2.5** If the functor  $A : C \to \mathcal{E}$  is dense from a small category C to a cocomplete category  $\mathcal{E}$ , then the category  $\mathcal{E}$  is complete. Hence a cocomplete category with a small, dense subcategory is complete.

The monic arrows in  $\mathcal{E}$  are preserved by the functor R.

**Lemma 2.6** The  $\mathcal{E}$  arrow  $m: E_1 \to E_2$  is monic if and only if the arrow  $R(m): R(E_1) \to R(E_2)$  is monic in **Set**<sup>Cop</sup>.

*Proof.* First we assume  $m : E_1 \to E_2$  be a monic arrow in  $\mathcal{E}$ . Let  $\alpha, \beta : P \to R(E_1)$  be  $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$  arrows. Suppose that  $R(m) \circ \alpha = R(m) \circ \beta$ . We must show  $\alpha_C = \beta_C$  at each object  $C \in \mathcal{C}$ . Let x be any element of P(C). Then we have  $R(m)_C(\alpha_C(x)) = R(m)_C(\beta_C(x))$ . Since  $R(m)_C(\alpha_C(x)) = m \circ \alpha_C(x)$  and  $R(m)_C(\beta_C(x)) = m \circ \beta_C(x)$ , we have  $m \circ \alpha_C(x) = m \circ \beta_C(x)$ , whence  $\alpha_C(x) = \beta_C(x)$  because m is monic. It follows then  $\alpha_C = \beta_C$  at each  $C \in \mathcal{C}$  since x was arbitrary. Thereby we have  $\alpha = \beta$  and conclude that R(m) is monic in  $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ .

The reverse direction follows from the fact that a faithful functor reflects monics.  $\hfill \square$ 

#### 3. The subobject classifier

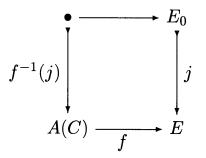
Let  $\mathcal{E}$  be a cocomplete category and assume, throughout this section, the existence of a dense functor  $A : \mathcal{C} \to \mathcal{E}$  from a small category  $\mathcal{C}$  to  $\mathcal{E}$ . We consider the condition under which such  $\mathcal{E}$  has a subobject classifier. As we have seen in the previous section, the category  $\mathcal{E}$  is complete by Proposition 2.5. Hence we have the following two properties for the category  $\mathcal{E}$ :

- There exists a terminal object 1 in  $\mathcal{E}$ .
- The pullbacks exist in  $\mathcal{E}$ . In particular, the pullbacks of monic arrows exist in  $\mathcal{E}$ . Hence there exists a subobject functor Sub :  $\mathcal{E}^{\text{op}} \to \text{Set}$ . We first define the  $\text{Set}^{\mathcal{C}^{\text{op}}}$  arrows.

Let  $j: E_0 \to E$  be any subobject of  $E \in \mathcal{E}$ . We define a  $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$  arrow  $\chi_j: R(E) \to \mathrm{Sub}(A(-))$  whose C component  $\chi_{j_C}: R(E)(C) \to \mathrm{Sub}(A(C))$  is given by

$$\chi_{j_C}(f) = f^{-1}(j)$$

for  $f \in R(E)(C) = \mathcal{E}(A(C), E)$ . In the above definition,  $f^{-1}(j)$  is a subobject of A(C) defined by the following pullback:



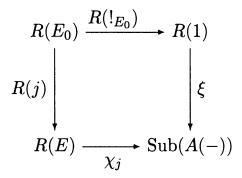
Next we define a  $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$  arrow  $\xi : R(1) \to \mathrm{Sub}(A(-))$  whose C component  $\xi_C$  is given by

 $\xi_C(!_{A(C)}) = \mathrm{id}_{A(C)},$ 

where  $!_E$  denote the unique  $\mathcal{E}$  arrow from E to 1.

Then we have the following result.

**Lemma 3.1** For each subobject  $j : E_0 \rightarrow E$  of  $E \in \mathcal{E}$ , the following diagram is a pullback in  $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ :



*Proof.* We examine only the commutativity of the above diagram, since the other assertion is shown by routine arguments.

We fix a subobject  $j : E_0 \to E$ . In order to show the commutativity of the above diagram, we check the commutativity at each C component. Take any  $f \in R(E_0)(C) = \mathcal{E}(A(C), E_0)$ . Then we have

$$\chi_{j_C}(R(j)_C(f)) = \chi_{j_C}(j \circ f) = \mathrm{id}_{A(C)}$$

since  $(j \circ f)^{-1}(j) = \mathrm{id}_{A(C)}$ , and

$$\xi_C(R(!_{E_0})_C(f)) = \xi_C(!_{E_0} \circ f) = \xi_C(!_{A(C)}) = \mathrm{id}_{A(C)}$$

Then  $\chi_{j_C} \circ R(j)_C = \xi_C \circ R(!_{E_0})_C$  since f was arbitrary. Hence the commutativity  $\chi_j \circ R(j) = \xi \circ R(!_{E_0})$  because C was also arbitrary.  $\Box$ 

Recall the condition that the category  $\mathcal{E}$  has a subobject classifier is given by the statement that the subobject functor Sub :  $\mathcal{E}^{\text{op}} \to \mathbf{Set}$  is representable. Here, the assumption on  $\mathcal{E}$ , that it has a dense functor A :  $\mathcal{C} \to \mathcal{E}$  from a small category  $\mathcal{C}$ , cut down the condition as follows.

**Theorem 3.2** Let  $A : C \to \mathcal{E}$  be a dense functor from a small category C to a cocomplete category  $\mathcal{E}$ . Then the following two conditions are equivalent:

1. The presheaf  $\operatorname{Sub}(A(-)) \in \operatorname{Set}^{\mathcal{C}^{\operatorname{op}}}$  is in  $\mathcal{E}$ , i.e., there exists an object  $\Omega \in \mathcal{E}$  such that the following isomorphism holds.

$$\operatorname{Sub}(A(-)) \cong \mathcal{E}(A(-), \Omega) = R(\Omega)$$
 (2)

2. There exists a subobject classifier in  $\mathcal{E}$ .

*Proof.*  $1. \Rightarrow 2.$ 

Assume that the presheaf  $\operatorname{Sub}(A(-))$  is in  $\mathcal{E}$ . Consider a  $\operatorname{Set}^{\mathcal{C}^{\operatorname{op}}}$  arrow from R(1) to  $R(\Omega)$  given by the composition of the arrow  $\xi$  and the isomorphism of the assumption (2). Since the functor R is full and faithful, this

 $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$  arrow determines a unique  $\mathcal{E}$  arrow:  $\mathbf{t} : 1 \to \Omega$ . This  $\mathbf{t}$  is obviously monic by Lemma 2.6.

Now we show the following to show the existence of the subobject classifier in  $\mathcal{E}$ :

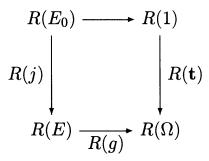
(i) For each object  $E \in \mathcal{E}$ , there is a bijection

$$\operatorname{Sub}(E) \cong \mathcal{E}(E,\Omega),$$

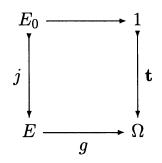
(ii) This bijection is natural for  $E \in \mathcal{E}$ .

First we show the existence of bijections. Define a map  $\Phi_E : \operatorname{Sub}(E) \to \mathcal{E}(E,\Omega)$  by sending a subobject  $j \in \operatorname{Sub}(E)$  to an  $\mathcal{E}$  arrow  $g : E \to \Omega$  such that R(g) is equal to the composite of  $\operatorname{Set}^{\mathcal{C}^{\operatorname{op}}}$  arrow  $\chi_j$  and the isomorphism of the assumption (2), where the well-definedness of this map  $\Phi_E$  is assured by the fullness and the faithfulness of the functor R.

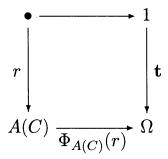
Then, for the above subobject j of E and the arrow  $g = \Phi_E(j)$ , the following diagram is a pullback in **Set**<sup> $\mathcal{C}^{op}$ </sup> by Lemma 3.1.



Hence we obtain the following pullback in  $\mathcal{E}$  since R is full and faithful.



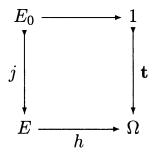
In particular, for each  $C \in \mathcal{C}$  the map  $\Phi_{A(C)}$  coincides with the *C*-component of the natural isomorphism (2). Moreover the following pullback holds for each  $r \in \text{Sub}(A(C))$ :



This means that each correspondence under each C component  $\operatorname{Sub}(A(C)) \cong \mathcal{E}(A(C), \Omega)$  of (2) gives a pullback as above.

Now we show that  $\Phi_E$  is injective. Suppose  $i, j \in \text{Sub}(E)$  and satisfies  $\Phi_E(i) = \Phi_E(j) = g$ . Then there exist two pullback diagrams: One is a diagram which pulls back **t** along g producing i, and the other is pulling back **t** along g producing j. Hence  $i \cong j$  and  $\Phi_E$  is injective.

Next we show that  $\Phi_E$  is surjective. Fix any  $g \in \mathcal{E}(E, \Omega)$  and take a subobject j of E given by pulling back  $\mathbf{t}$  along g. Put  $h = \Phi_E(j)$  and we show g = h for this h. By the previous observation for  $\Phi_E$ , we have the following pullback diagram:



Remember that each object  $E \in \mathcal{E}$  is given by the colimit of the canonical diagram

$$A(-)/E \xrightarrow{\partial} \mathcal{C} \xrightarrow{A} \mathcal{E}$$

since the functor  $A : \mathcal{C} \to \mathcal{E}$  is dense. Let  $\mu : A \circ \partial \to E$  be the canonical cocone of above diagram. Consider the subobjects given by pulling back j along the coprojections of  $\mu$ . Here each subobject determines an  $\mathcal{E}$  arrow with codomain  $\Omega$  by the isomorphism of assumption (2), which amounts to determine a cocone  $\nu : A \circ \partial \to \Omega$  of a canonical diagram of E with the vertex  $\Omega$ .

As we have seen in the previous observation, each pair of a subobject and an arrow under the bijective correspondence given by the C-component of (2) constitute a pullback diagram. Hence we have

$$h \circ \mu_{(C,f)} = \nu_{(C,f)} = g \circ \mu_{(C,f)}$$

for each object  $(C, f) \in A(-)/E$  because  $\nu_{(C,f)}$  is the unique arrow which corresponds both to the subobject pulling back **t** along  $g \circ \mu_{(C,f)}$  and the one pulling back **t** along  $h \circ \mu_{(C,f)}$ .

Since the colimit of the canonical diagram  $A \circ \partial$  is E, the cocone  $\nu$  determines a unique arrow from E to  $\Omega$ . Hence g = h and we have shown that  $\Phi_E$  is surjective.

The naturality of the bijection follows by routine diagram chasing. 2.  $\Rightarrow 1$ .

The reverse direction is obvious.

Note that when the presheaf  $\operatorname{Sub}(A(-))$  is in  $\mathcal{E}$ , namely the isomorphism  $\operatorname{Sub}(A(-)) \cong R(\Omega)$  holds, then the subobject classifier is given by the  $\mathcal{E}$  arrow  $\mathbf{t} : 1 \to \Omega$  such that  $R(\mathbf{t})$  is equal to the composite of  $\xi : R(1) \to \operatorname{Sub}(A(-))$  and the isomorphism.

By Theorem 3.2, we obtain

**Corollary 3.3** Under the same assumption as in Theorem 3.2, the unit  $\eta_{\operatorname{Sub}(A(-))} : \operatorname{Sub}(A(-)) \to RL \operatorname{Sub}(A(-))$  of the adjunction has a retraction if and only if there exists a subobject classifier in  $\mathcal{E}$ .

**Proof.** First we assume  $\eta_{\operatorname{Sub}(A(-))}$  has a retraction. Then the unit  $\eta_{\operatorname{Sub}(A(-))}$  becomes an isomorphism since R is full and faithful. By using the Theorem 3.2, the category  $\mathcal{E}$  has a subobject classifier.

Next we assume that  $\mathcal{E}$  has a subobject classifier. Then there exists an object  $\Omega \in \mathcal{E}$  with an isomorphism  $\operatorname{Sub}(A(-)) \cong R(\Omega)$  by Theorem 3.2. Because the unit  $\eta$  is a natural transformation, we have the following commutative diagram for the isomorphism:

$$\begin{array}{c|c} \operatorname{Sub}(A(-)) & \longrightarrow & R(\Omega) \\ & & & & & \\ \eta_{\operatorname{Sub}(A(-))} & & & & & \\ & & & & & \\ & & & \\ & & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & &$$

In the above diagram, both the top and the bottom horizontal arrows are isomorphisms. Moreover the right vertical  $\eta_{R(\Omega)}$  is an isomorphism because

the counit  $\varepsilon_{\Omega} : LR(\Omega) \to \Omega$  is an isomorphism by Proposition 2.4. This implies  $\eta_{Sub(A(-))}$  is an isomorphism, and hence has a retraction.

## 4. The construction of $\mathcal{E}$ objects via presheaf category

#### 4.1. Preliminary

Now we study the construction of  $\mathcal{E}$  objects from the category  $\mathbf{Set}^{\mathcal{C}^{\mathsf{op}}}$ of presheaves by using the left adjoint functor  $L: \mathbf{Set}^{\mathcal{C}^{\mathsf{op}}} \to \mathcal{E}$ .

To begin with, we introduce a notation for the "components" of the unit  $\eta$  of the adjunction  $L \dashv R$ . Because the P component of the unit  $\eta$  is a natural transformation  $\eta_P : P \to RLP$ , the component of  $\eta_P$  at object  $C \in \mathcal{C}$  is a map  $\eta_{PC} : P(C) \to RLP(C) (= \mathcal{E}(A(C), LP))$ . Define an  $\mathcal{E}$  arrow  $\kappa_p^{LP} : A(C) \to LP$  by

$$\kappa_p^{LP} := \eta_{PC}(p)$$

for each object  $C \in \mathcal{C}$  and  $p \in P(C)$ . Then we have the following Corollary to Proposition 2.1.

**Corollary 4.1** Let P be a presheaf over C. The collection of  $\mathcal{E}$  arrows

$$\left\{\kappa_p^{LP}: A(C) \to LP \mid (C, p) \in \int P\right\}$$
(3)

constitutes a universal cocone of the diagram  $A \circ \pi_P$ .

*Proof.* By the naturality condition on the unit  $\eta_P : P \to RLP$ , the collection (3) is a cocone of  $A \circ \pi_P$ . From Proposition 2.1, we have a bijective correspondence

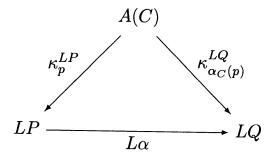
$$\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}(P, RLP) \cong \mathcal{E}(LP, LP)$$
 (4)

for object  $P \in \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$  and  $LP \in \mathcal{E}$ . The identity arrow  $\mathrm{id}_{LP}$  on  $LP \in \mathcal{E}$  corresponds to  $\eta_P$  under the correspondence (4), which means that the cocone (3) is universal.

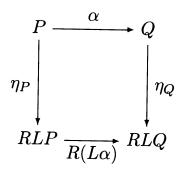
For each  $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$  arrow  $\alpha: P \to Q$ , the  $\mathcal{E}$  arrow  $L\alpha: LP \to LQ$  has a following property.

**Lemma 4.2** Let  $\alpha: P \to Q$  be a  $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$  arrow. For each  $p \in P(C)$  with

# $C \in \mathcal{C}$ , the following diagram commutes:



*Proof.* From the naturality condition on the unit  $\eta : \mathrm{id}_{\mathbf{Set}^{C^{\mathrm{op}}}} \to RL$  for  $\mathbf{Set}^{C^{\mathrm{op}}}$  arrow  $\alpha$ , we have the following commutative diagram:



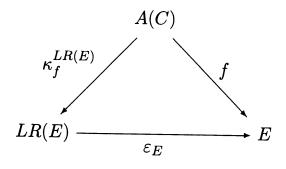
By considering the C-component of the above diagram, we get

$$\kappa^{LQ}_{\alpha_C(p)} = Llpha \circ \kappa^{LP}_p$$

for each  $p \in P(C)$ .

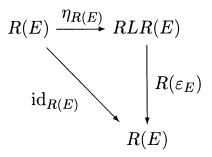
The counit  $\varepsilon_E$  has a following property.

**Lemma 4.3** The following diagram commutes for each  $\mathcal{E}$  arrow f:  $A(C) \rightarrow E$  with  $C \in \mathcal{C}$ :



*Proof.* From one of the triangular identities of the adjunction  $L \dashv R$ , we

have the following commutative diagram.



The C component of above diagram gives

$$f = \varepsilon_E \circ \kappa_f^{LR(E)}$$

for every  $f \in \mathcal{E}(A(C), E)$ .

## 4.2. The terminal object

We can give the terminal object in  $\mathcal{E}$  as follows.

**Lemma 4.4** Under the same assumption as in Theorem 3.2, let **1** be a terminal object in **Set**<sup> $\mathcal{C}^{\text{op}}$ </sup>. Then the object  $L\mathbf{1} \in \mathcal{E}$  is a terminal object in  $\mathcal{E}$ .

*Proof.* Let 1 be a terminal object in  $\mathcal{E}$ . Then  $\mathbf{1} \cong R(1)$ . By applying the functor L and by using denseness of A, we have  $L\mathbf{1} \cong LR(1) \cong 1$ . Hence  $L\mathbf{1}$  is a terminal object in  $\mathcal{E}$ .

## 4.3. The subobject classifier

In case there exists a subobject classifier in  $\mathcal{E}$ , we can give it by using the presheaves. This enables us to restate the Theorem 3.2 in a bit different form, which is more useful for us to check its existence in a concrete category.

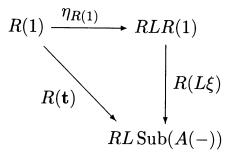
**Corollary 4.5** Under the same assumption as in Theorem 3.2, the  $\mathcal{E}$  arrow  $L\xi : LR(1) \to L \operatorname{Sub}(A(-))$  is a subobject classifier if the category  $\mathcal{E}$  has a subobject classifier.

**Proof.** Suppose that we have a subobject classifier in  $\mathcal{E}$ . By using Corollary 3.3, the unit  $\eta_{\operatorname{Sub}(A(-))} : \operatorname{Sub}(A(-)) \to RL \operatorname{Sub}(A(-))$  is an isomorphism. Now recall the construction of the subobject classifier in the proof of Theorem 3.2, according to which, the subobject classifier is given by the  $\mathcal{E}$  arrow  $\mathbf{t} : 1 \to L \operatorname{Sub}(A(-))$  which satisfies

$$R(\mathbf{t}) = \eta_{\operatorname{Sub}(A(-))} \circ \xi.$$
(5)

Since R is full and faithful, both the existence and the uniqueness of such an arrow are guaranteed.

By the naturality diagram of the unit  $\eta$  for the **Set**<sup> $\mathcal{C}^{\circ p}$ </sup> arrow  $\xi$  and by the equation (5), we have the following commutative diagram:

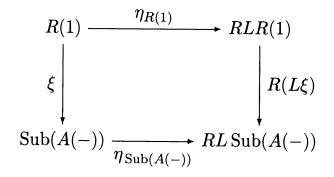


Because the unit  $\eta_{R(1)}$  is an isomorphism and the functor R is full and faithful, the underlying subobjects  $\mathbf{t}$  and  $L\xi$  of  $L\operatorname{Sub}(A(-))$  are isomorphic. Hence we conclude that the  $\mathcal{E}$  arrow  $L\xi : LR(1) \to L\operatorname{Sub}(A(-))$  is a subobject classifier.

By Corollary 3.3, if there exists a subobject classifier in  $\mathcal{E}$ , then the unit  $\eta_{\operatorname{Sub}(A(-))}$  has a retraction. Now we characterize this retraction.

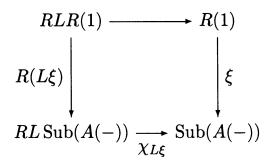
**Corollary 4.6** Under the same assumptions as in Theorem 3.2, there exists a subobject classifier in  $\mathcal{E}$  if and only if  $\chi_{L\xi}$  is a retraction of the unit  $\eta_{\operatorname{Sub}(A(-))} : \operatorname{Sub}(A(-)) \to RL \operatorname{Sub}(A(-)).$ 

**Proof.** Assume the existence of a subobject classifier in  $\mathcal{E}$ . Then the unit  $\eta_{\operatorname{Sub}(A(-))} : \operatorname{Sub}(A(-)) \to RL \operatorname{Sub}(A(-))$  is an isomorphism by Corollary 3.3. Since the unit  $\eta$  is a natural transformation, the following diagram commutes, and since both  $\eta_{R(1)}$  and  $\eta_{\operatorname{Sub}(A(-))}$  are isomorphic, the diagram is a pullback in  $\operatorname{Set}^{\mathcal{C}^{\operatorname{op}}}$ .

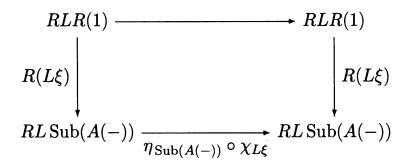


By using Lemma 3.1 for monic arrow  $L\xi$  we have the following pullback

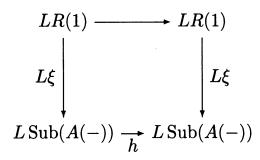
diagram:



By the above two diagrams and the pullback lemma, we get the following pullback diagram:



Because the functor R is full and faithful, we have the following pullback diagram, where  $h: L \operatorname{Sub}(A(-)) \to L \operatorname{Sub}(A(-))$  is a unique  $\mathcal{E}$  arrow satisfying  $R(h) = \eta_{\operatorname{Sub}(A(-))} \circ \chi_{L\xi}$ :



Since  $L\xi$  is a subobject classifier by Lemma 4.5, and since the characteristic arrow of the subobject classifier  $L\xi$  is an identity on  $L\operatorname{Sub}(A(-))$ , we have  $h = \operatorname{id}_{L\operatorname{Sub}(A(-))}$ . Hence we obtain  $\eta_{\operatorname{Sub}(A(-))} \circ \chi_{L\xi} = \operatorname{id}_{RL\operatorname{Sub}(A(-))}$ , i.e.,  $\chi_{L\xi}$  is an inverse of  $\eta_{\operatorname{Sub}(A(-))}$ . Thereby  $\chi_{L\xi}$  is a retraction of  $\eta_{\operatorname{Sub}(A(-))}$ . The reverse direction follows by Theorem 3.3.

The above Corollary 4.6 enable us to check effectively the existence of subobject classifier in concrete cases. In order to show the existence of subobject classifier in  $\mathcal{E}$ , we have only to check the following equation holds

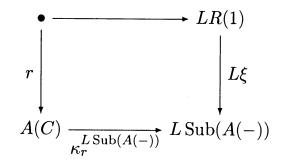
in  $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ .

$$\chi_{L\xi} \circ \eta_{\operatorname{Sub}(A(-))} = \operatorname{id}_{\operatorname{Sub}(A(-))} \tag{6}$$

Here the equation (6) is equivalent to the following condition: When the universal cocone of the diagram  $\int \operatorname{Sub}(A(-)) \to \mathcal{C} \to \mathcal{E}$  is given, in accordance with the notation in Section 4.1, by the collection

$$\left\{\kappa_r^{L\operatorname{Sub}(A(-))}: A(C) \to L\operatorname{Sub}(A(-)) \mid (C,r) \in \int \operatorname{Sub}(A(-))\right\},\$$

the following diagram is a pullback for each coprojection  $\kappa_r^{L\operatorname{Sub}(A(-))}$  of the cocone.



**Acknowledgment** I wish to express my gratitude to Professor Toru Tsujishita for his warm and constant encouragement. I also wish to thank Professor John Power for his useful advice.

#### References

- [1] Borceux F., Handbook of Categorical Algebra 1: Basic Category Theory. Cambridge U. P., Cambridge, 1994.
- [2] Kelly G.M., Basic concepts of enriched category theory. London Math. Soc. Lecture Notes Series 64, Cambridge U. P., Cambridge, 1982.
- [3] Mac Lane S., Categories for the Working Mathematician. Springer-Verlag, New York, 1971.
- [4] Mac Lane S. and Moerdijk I., Sheaves in Geometry and Logic: a first introduction to topos theory. Springer-Verlag, New York, 1992.
- [5] Tsujishita T. and Watanabe H., Monoidal closedness of the category of simulations. Hokkaido University Preprint Series in Mathematics, Series #392, 1997.
- [6] Watanabe H., The subobject classifier of the category of functional bisimulations. Hokkaido Mathematical Journal, to appear (1999).

Mail Box 1503, Semantics Group Electrotechnical Laboratory Tsukuba 305-8568, Japan E-mail: hirowata@etl.go.jp