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Characterizations of Bloch space and Besov spaces by oscillations

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Abstract. We characterize the Bloch space and the Besov spaces on the open unit disc D by using many kinds of oscillations. We give new characterizations with known ones. For example, we use the oscillation and the mean oscillation as the following:

$$\sup_{w\in D(z,r)}(1-|z|^2)^\alpha(1-|w|^2)^\beta\left|\frac{f(z)-f(w)}{z-w}\right|$$

and

$$\frac{1}{|D(z,r)|}\int_{D(z,r)}(1-|z|^2)^\alpha(1-|w|^2)^\beta\left|\frac{f(z)-f(w)}{z-w}\right|dA(w).$$

Key words: Bloch space, Besov space, oscillation, mean oscillation.

1. Introduction

Let $D=\{z\in C;|z|<1\}$ denote the open unit disc in C and let $\partial D=\{z\in C;|z|=1\}$ denote the unit circle. Let $H(D)$ denote the space of analytic functions on D. For $1\leq p<+\infty$, the Lebesgue space $L^p(D,dA)$ is defined to be the Banach space of Lebesgue measurable functions on the open unit disc D with

$$\|f\|_{L^p(dA)}:=\left(\int_D|f(z)|^p dA(z)\right)^{\frac{1}{p}}<+\infty,$$

where $dA(z)$ is the normalized area measure on D. The Bergman space $L_a^p(D)$ is defined to be the subspace of $L^p(D,dA)$ consisting of analytic functions. For $0<p<+\infty$, the Hardy space H^p is defined to be the Banach space of analytic functions f on D with

$$\|f\|_p:=\left(\sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}}<\infty.$$

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For $z, w \in D$, let $\beta(z,w) := \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}$, where $\varphi_z(w) = \frac{z-w}{1-\bar{z}w}$. We will repeatedly use the following properties of φ_z:

$$1 - |\varphi_z(w)|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{z}w|^2},$$

$$\varphi_z(z) = 0, \quad \varphi_z(0) = z, \quad \varphi_z \circ \varphi_z(w) = w.$$

For $0 < r < +\infty$, let $D(z) = D(z,r) = \{w \in D;\, \beta(z,w) < r\}$ denote the Bergman disc. $|D(z,r)|$ denotes the normalized area of $D(z,r)$ and $|D(z,r)|$ is comparable to $(1-|z|^2)^2$. The Bloch space B of D is defined to be the space of analytic functions f on D such that

$$\| f \|_B := \sup\{(1-|z|^2)|f^{(1)}(z)|;\, z \in D\} < +\infty.$$

This defines a semi-norm and it is Möbius invariant in the sense of $\| f \circ \varphi \|_B = \| f \|_B$ for all $f \in B$ and $\varphi \in Aut(D)$, where $Aut(D)$ is the Möbius group of bi-analytic mappings of D. The Bloch functions form a Banach space with the norm $\| f \| = |f(0)| + \| f \|_B$. If f is an analytic function on D, then $f \in B$ if and only if $\sup_{z\in D} \| f \circ \varphi_z - f(z) \|_{L^2(dA)} < +\infty$ ([7, p.101]). The little Bloch space of D, denoted B_0, is the closed subspace of B consisting of functions f with $(1-|z|^2)f^{(1)}(z) \to 0$ $(|z| \to 1^-)$. If f is an analytic function on D, then $f \in B_0$ if and only if $\lim_{|z|\to 1^-} \| f \circ \varphi_z - f(z) \|_{L^2(dA)} = 0$ ([7, p.101]). The space of analytic functions on D of bounded mean oscillation, denoted by $BMOA$, consists of functions f in H^2 for which

$$\| f \|_{BMOA} := \sup\{\| f \circ \varphi_z - f(z) \|_2;\, z \in D\} < +\infty.$$

It is clear that $|g^{(1)}(0)| \leq \| g \|_2$ for every analytic function g on D. Applying $g = f \circ \varphi_z - f(z)$, it follows that $(1-|z|^2)|f^{(1)}(z)| \leq \| f \circ \varphi_z - f(z) \|_2$ for an analytic function f on D and $z \in D$. Thus it follows that the inclusion $BMOA \subset B$.

For $1 < p < +\infty$, the Besov space B_p of D is defined to be the space of analytic functions f on D such that

$$\| f \|_{B_p} := \left\{\int_D (1-|z|^2)^p |f^{(1)}(z)|^p d\lambda(z)\right\}^{\frac{1}{p}} < +\infty,$$

$$\text{where } d\lambda(z) = \frac{dA(z)}{(1-|z|^2)^2}.$$

We will also use the following property:

$$d\lambda(\varphi_z(w)) = d\lambda(w).$$

It is easy to show that $\| * \|_{B_p}$ is a complete semi-norm on B_p. Moreover, it is Möbius invariant in the sense of $\| f \circ \varphi \|_{B_p} = \| f \|_{B_p}$ for all $f \in B_p$ and $\varphi \in Aut(D)$. For convenience of notation, we also write $B_\infty = B$, the Bloch space. Note that B_2 is the classical Dirichlet space. The Besov space B_1 consists of analytic functions f on D such that $f(z) = \sum_{n=1}^{+\infty} a_n \varphi_{\lambda_n}(z)$ with $\sum_{n=1}^{+\infty} |a_n| < +\infty$, where $\lambda_n \in D$. Put

$$\| f \|_{B_1} := \inf \left\{ \sum_{n=1}^{+\infty} |a_n| : f(z) = \sum_{n=1}^{+\infty} a_n \varphi_{\lambda_n}(z) \right\}.$$

It is clear that $B_1 \subset H^\infty$.

Let $\alpha > 0$. Then α-Bloch space B^α of D is defined to be the space of analytic functions f on D such that

$$\| f \|_{B^\alpha} := \sup\{(1 - |z|^2)^\alpha |f^{(1)}(z)|; z \in D\} < +\infty.$$

And the little α-Bloch space of D, denoted B_0^α, is the closed subspace of B^α consisting of functions f with $(1 - |z|^2)^\alpha f^{(1)}(z) \to 0$ $(|z| \to 1^-)$. And for $1 \le p < +\infty$, $p\alpha > 1$, the α-Besov space B_p^α of D is defined to be the space of analytic functions f on D such that

$$\int_D (1 - |z|^2)^{\alpha p} |f^{(1)}(z)|^p d\lambda(z) < +\infty.$$

Note that B^1, B_0^1 and B_p^1 are the Bloch space, the little Bloch space and the Besov space, respectively.

Let $0 < r < +\infty$. Then for a function f on D, let

$$\widehat{f}_r(z) := \frac{1}{|D(z,r)|} \int_{D(z,r)} f(w) dA(w) \quad (z \in D).$$

For a function f on D, the following is called the oscillation of f at z in the Bergman metric ([8, p.327]):

$$\sup_{w \in D(z,r)} |f(z) - f(w)|,$$

and the following is called the mean oscillation of f at z in the Bergman metric:

$$\frac{1}{|D(z,r)|}\int_{D(z,r)}|\widehat{f}_r(z)-f(w)|dA(w).$$

In this paper, we also call the following the oscillation of f at z in the Bergman metric: For $\alpha+\beta=1$, $\alpha,\beta\in\Re$,

$$\sup_{w\in D(z,r)}|\widehat{f}_r(z)-f(w)|,$$
$$\sup_{w\in D(z,r)}(1-|z|^2)^\alpha(1-|w|^2)^\beta\left|\frac{f(z)-f(w)}{z-w}\right|.$$

And we also call the following the mean oscillation of f at z in the Bergman metric: For $\alpha+\beta=1$, $\alpha,\beta\in\Re$,

$$\frac{1}{|D(z,r)|}\int_{D(z,r)}|f(z)-f(w)|\,dA(w),$$
$$\frac{1}{|D(z,r)|}\int_{D(z,r)}(1-|z|^2)^\alpha(1-|w|^2)^\beta\left|\frac{f(z)-f(w)}{z-w}\right|dA(w).$$

Then by elementary calculation, we see that there are some constants $C_1,C_2,K_1,K_2,K_3>0$ such that

$$\begin{aligned}&\sup_{w\in D(z,r)}|\widehat{f}_r(z)-f(w)|\\&\quad\le C_1\sup_{w\in D(z,r)}|f(z)-f(w)|\\&\quad\le C_2\sup_{w\in D(z,r)}(1-|z|^2)^\alpha(1-|w|^2)^\beta\left|\frac{f(z)-f(w)}{z-w}\right|,\end{aligned}$$

$$\begin{aligned}&\frac{1}{|D(z,r)|}\int_{D(z,r)}|\widehat{f}_r(z)-f(w)|dA(w)\\&\quad\le\frac{K_1}{|D(z,r)|}\int_{D(z,r)}|f(z)-f(w)|\,dA(w)\\&\quad\le\frac{K_2}{|D(z,r)|}\int_{D(z,r)}(1-|z|^2)^\alpha(1-|w|^2)^\beta\left|\frac{f(z)-f(w)}{z-w}\right|dA(w)\\&\quad\le K_3\sup_{w\in D(z,r)}|f(z)-f(w)|.\end{aligned}$$

Using these oscillations and mean oscillations, in Section 2 we characterize the Bloch space B as the following:

Theorem 1.1 *Let $p > 0$ and $\alpha + \beta = 1$, $\alpha, \beta \in \Re$. Then for an analytic function f on D and for $r \in (0, +\infty)$, the following statements are equivalent*:

(1) $f \in B$;

(2) $\sup_{z\in D}\Big(\sup_{w\in D(z,r)} |f(z) - f(w)|\Big) < +\infty$;

(3) $\sup_{z\in D}\Big(\frac{1}{|D(z,r)|}\int_{D(z,r)} |f(z) - f(w)|^p dA(w)\Big)^{\frac{1}{p}} < +\infty$;

(4) $\sup_{z\in D}\Big(\frac{1}{|D(z,r)|}\int_{D(z,r)} \log^+(|f(z) - f(w)|)dA(w)\Big) < +\infty$;

(5) $\sup_{z\in D}\Big(\sup_{w\in D(z,r)} |\widehat{f}_r(z) - f(w)|\Big) < +\infty$;

(6) $\sup_{z\in D}\Big(\frac{1}{|D(z,r)|}\int_{D(z,r)} |\widehat{f}_r(z) - f(w)|^p dA(w)\Big)^{\frac{1}{p}} < +\infty$;

(7) $\sup_{z\in D}\Big(\frac{1}{|D(z,r)|}\int_{D(z,r)} \log^+(|\widehat{f}_r(z) - f(w)|)dA(w)\Big) < +\infty$;

(8) $\sup\Big\{\sup_{w\in D(z,r)} (1-|z|^2)^\alpha(1-|w|^2)^\beta\Big|\frac{f(z)-f(w)}{z-w}\Big|;$

$z \in D,\ z \neq w\Big\} < +\infty$;

(9) $\sup_{z\in D}\Big(\frac{1}{|D(z,r)|}\int_{D(z,r)} (1-|z|^2)^{\alpha p}(1-|w|^2)^{\beta p}\Big|\frac{f(z)-f(w)}{z-w}\Big|^p dA(w)\Big)^{\frac{1}{p}}$
$< +\infty$;

(10) $\sup_{z\in D}\Big(\frac{1}{|D(z,r)|}\int_{D(z,r)} \log^+\Big((1-|z|^2)^\alpha(1-|w|^2)^\beta$

$\Big|\frac{f(z)-f(w)}{z-w}\Big|\Big)dA(w)\Big) < +\infty.$

The above equivalences of (1), (2), (3), (6) were proved by K. Zhu ([8, p.328]) in the case of $p > 1$. Similarly, in Section 3 we also characterize the Besov spaces B_p as the following:

Theorem 1.2 *Let $\alpha + \beta = 1$, $\alpha, \beta \in \Re$. Then for an analytic function*

f on D and $p > 1$ and for $r \in (0, +\infty)$, the following statements are equivalent:

(1) $f \in B_p$;

(2) $$\int_D \Big(\sup_{w \in D(z,r)} |f(z) - f(w)| \Big)^p d\lambda(z) < +\infty;$$

(3) $$\int_D \Big(\frac{1}{|D(z,r)|} \int_{D(z,r)} |f(z) - f(w)|^p \, dA(w) \Big) d\lambda(z) < +\infty;$$

(4) $$\int_D \Big(\frac{1}{|D(z,r)|} \int_{D(z,r)} |f(z) - f(w)| \, dA(w) \Big)^p d\lambda(z) < +\infty;$$

(5) $$\int_D \Big(\sup_{w \in D(z,r)} \Big| \widehat{f}_r(z) - f(w) \Big| \Big)^p d\lambda(z) < +\infty;$$

(6) $$\int_D \Big(\frac{1}{|D(z,r)|} \int_{D(z,r)} \Big| \widehat{f}_r(z) - f(w) \Big|^p \, dA(w) \Big) d\lambda(z) < +\infty;$$

(7) $$\int_D \Big(\frac{1}{|D(z,r)|} \int_{D(z,r)} \Big| \widehat{f}_r(z) - f(w) \Big| \, dA(w) \Big)^p d\lambda(z) < +\infty;$$

(8) $$\int_D \Big(\sup_{w \in D(z,r)} (1 - |z|^2)^\alpha (1 - |w|^2)^\beta \Big| \frac{f(z) - f(w)}{z - w} \Big| \Big)^p d\lambda(z) < +\infty;$$

(9) $$\int_D \Big(\frac{1}{|D(z,r)|} \int_{D(z,r)} (1 - |z|^2)^{\alpha p} (1 - |w|^2)^{\beta p} \Big| \frac{f(z) - f(w)}{z - w} \Big|^p dA(w) \Big) d\lambda(z) < +\infty;$$

(10) $$\int_D \Big(\frac{1}{|D(z,r)|} \int_{D(z,r)} (1 - |z|^2)^\alpha (1 - |w|^2)^\beta \Big| \frac{f(z) - f(w)}{z - w} \Big| dA(w) \Big)^p d\lambda(z) < +\infty.$$

The above equivalences of (1), (2), (3), (4), (6), (7) were proved by K. Zhu ([8, p.328]).

Remark 1.3 For any $z \in D$, let $g(u) = \left| \frac{f(u) - f(z)}{u - z} \right|$, $h(u) = |f(u) - f(z)|$ ($u \in D$) be subharmonic function on D. Then we can prove that the following quantities are equivalent:

(a) $$\sup_{w \in D(z,r)} |f(z) - f(w)|;$$

(b) $$\frac{1}{|D(z,r)|}\int_{D(z,r)} |f(z)-f(w)|\,dA(w);$$

(c) $$\sup_{w\in D(z,r)} (1-|z|^2)^\alpha(1-|w|^2)^\beta \left|\frac{f(z)-f(w)}{z-w}\right|;$$

(d) $$\frac{1}{|D(z,r)|}\int_{D(z,r)} (1-|z|^2)^\alpha(1-|w|^2)^\beta \left|\frac{f(z)-f(w)}{z-w}\right| dA(w).$$

The following Theorem A explains why B_1 as defined above is compatible with the other Besov spaces B_p $(1 < p \leq \infty)$.

Theorem A ([7, p.90]) *If f is an analytic function on D, $1 \leq p \leq +\infty$ and $n \geq 2$ is an integer, then $f \in B_p$ if and only if*

$$(1-|z|^2)^n f^{(n)}(z) \in L^p(D, d\lambda).$$

On account of Theorem A, we would like to give characterizations of the Besov spaces B_p $(1 \leq p \leq \infty)$ using the n times derivative of a function on D by the oscillations which are used in Theorems 1.1 and 1.2. In Section 2 we give Theorems 2.3 and 2.7 generalizing Theorem 1.1. And in Section 3 we give Theorems 3.3 and 3.5 generalizing Theorem 1.2. We should note that Theorem 1.2 does not hold for $p = 1$ while Theorems 3.3 and 3.5 do for $p = 1$.

Finally, we define the spaces $B^{(n)}$, $B_0^{(n)}$, $B_p^{(n)}(p \geq 1)$ as the following: Fix $n \geq 1$; integer, then

$$B^{(n)} := \Big\{ f \in H(D);\ \sup\Big\{(1-|z|^2)^{\frac{n}{2}}(1-|w|^2)^{\frac{n}{2}} \left|\frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w}\right|;\ z, w \in D,\ w \neq z\Big\} < +\infty\Big\},$$

$$B_0^{(n)} := \Big\{ f \in H(D);\ \lim_{|z|\to 1^-} \sup\Big\{(1-|z|^2)^{\frac{n}{2}}(1-|w|^2)^{\frac{n}{2}} \left|\frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w}\right|;\ w \in D,\ w \neq z\Big\} = 0\Big\},$$

$$B_p^{(n)} := \left\{ f \in H(D);\ \int_D \int_D (1-|z|^2)^{\frac{n}{2}p}(1-|w|^2)^{\frac{n}{2}p} \left| \frac{f^{(n-1)}(z) - f^{(n-1)}(w)}{z-w} \right|^p d\lambda(z)d\lambda(w) < +\infty \right\}.$$

For $n = 1$, $B^{(1)} = B$ is just the result which was proved by F. Holland and D. Walsh [3], and $B_0^{(1)} = B_0$ is just the result which was proved by K. Stroethoff [6]. We prove $B^{(2)} = B$ and $B_0^{(2)} = B_0$ in Section 4 (Theorem 4.2 and Corollary 4.3, respectively). However for $n \geq 3$, $B^{(n)} = B$ does not hold because the function $f(z) = \log(1-z)$ is in B but it is not in $B^{(n)}$ (Example 4.1). As a result we see the following: For $n \geq 3$,

$$B_0^{(n)} \subset B_0^{(2)} = B_0^{(1)} = B_0, \quad B^{(n)} \subset B^{(2)} = B^{(1)} = B, \quad B \neq B^{(n)}.$$

Moreover it holds that $BMOA \subset B^{(2)} = B^{(1)} = B$, but it does not hold that $BMOA \subset B^{(n)}$ for $n \geq 3$ because the function $f(z) = \log(1-z)$ is in $BMOA$. On the other hand, for $n = 1$, $B_p^{(1)} = B_p$ $(p > 2)$ is just the result which was proved by K. Stroethoff [6]. For $n \geq 2$, we could not prove $B_p^{(n)} = B_p$, but we see the following: For $n = 2$, $B_p^{(2)} \subset B_p$ $(p > 1)$. For any $n \geq 3$, $B_p^{(n)} \subset B_p$ $(p \geq 1)$.

In Section 5 we will also characterize the α-Bloch space B^α and the little α-Bloch space B_0^α and the α-Besov space B_p^α as well as the Bloch space B and the little Bloch space B_0 and the Besov space B_p, respectively.

Throughout this paper, C_i, K_i for $i = 0, 1, 2$, C, K will denote positive constant whose value is not necessary the same at each occurrence.

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2. Criteria for membership of the Bloch space

In this section we shall use the following Theorems A, B, C, D to prove Theorems 2.1, 2.3 and 2.7 and Corollaries 2.2, 2.5 and 2.9. Theorem 2.1 and Corollary 2.2 were proved by K. Stroethoff [5] in the case of $n = 0$.

Theorem B ([4, p.409]) *Let $n \geq 1$; integer. If f is an analytic function on D, then the following quantities are equivalent*:

(A) $\| f \|_B$;

$$\text{(B)} \quad \sup\{(1-|z|^2)^n|f^{(n)}(z)|; z \in D\} + \sum_{k=1}^{n-1} |f^{(k)}(0)|.$$

Corollary C ([4, p.410]) *If f is an analytic function on D and $n \geq 1$; integer, then $f \in B_0$ if and only if*

$$(1-|z|^2)^n f^{(n)}(z) \to 0 \quad (|z| \to 1^-).$$

Theorem D ([7, p.83]) *If f is an analytic function f on D, then $f \in B_0$ if and only if*

$$\| f - f_t \|_B \to 0 \quad (t \to 1^-),$$

where $f_t(z) = f(tz)$ for all $z \in D$ and $t \in (0,1)$.

Thoerem 2.1 *Let $0 < r < 1$ and fix $n \geq 0$; integer. For an analytic function f on D, the following quantities are equivalent:*

$$\text{(A)} \quad \| f \|_B;$$

$$\text{(B)} \quad \sup_{\lambda \in D} \sup_{z \in \Delta(\lambda, r)} (1-|\lambda|^2)^n \left| f^{(n)}(z) - f^{(n)}(\lambda) \right|,$$

where $\Delta(\lambda, r) = \{z \in D; |\varphi_\lambda(z)| < r\}$.

Proof. When $n = 0$, it was proved by K. Stroethoff [5, p.129]. Since the proof of the case $n = 1$ is simpler than the proof of the case $n = 2$, we shall prove it only for the case $n = 2$. In the case of $n \geq 3$, because it is proved easily by induction, we shall omit to prove it.

When $n = 2$, fix $0 < r < 1$, and let f be an analytic function on D. The following identity is easily proved by a direct calculation:

$$f^{(3)}(0) = \frac{2}{r^4} \int_{\Delta(0,r)} \overline{z} f^{(2)}(z) dA(z).$$

Hence we have

$$|f^{(3)}(0)| \leq \frac{2}{r} \sup_{z \in \Delta(0,r)} |f^{(2)}(z)|.$$

Replacing $f^{(2)}$ by $f^{(2)} \circ \varphi_\lambda - f^{(2)}(\lambda)$, we have the following inequality:

$$(1-|\lambda|^2)|f^{(3)}(\lambda)| \leq \frac{2}{r} \sup_{z \in \Delta(\lambda,r)} |f^{(2)}(z) - f^{(2)}(\lambda)|.$$

Hence we get

$$\sup_{\lambda\in D}(1-|\lambda|^2)^3|f^{(3)}(\lambda)| \le \frac{2}{r}\sup_{\lambda\in D}\sup_{z\in\Delta(\lambda,r)}(1-|\lambda|^2)^2|f^{(2)}(z)-f^{(2)}(\lambda)|.$$

By Theorem A, it follows that $f \in B$.

To prove the converse, let $|w| < r$. Then for all $w \in D$

$$\begin{aligned} \left|f^{(1)}(w)-f^{(1)}(0)\right| &\le \int_0^1 |w|\left|f^{(2)}(tw)\right| dt \\ &\le C\| f \|_B \frac{|w|}{1-|w|^2} \\ &\le C\| f \|_B \frac{r}{1-r^2}. \end{aligned} \qquad \cdots (2.1.1)$$

Replacing f by $f \circ \varphi_\lambda$, we get

$$(1-|\lambda|^2)\left|\frac{f^{(1)}(\varphi_\lambda(w))}{(1-\overline{\lambda}w)^2}-f^{(1)}(\lambda)\right| \le C\| f \|_B \frac{r}{1-r^2}. \qquad \cdots (2.1.2)$$

We have also for all $w \in D$

$$\begin{aligned} \left|f^{(2)}(w)-f^{(2)}(0)\right| &\le \int_0^1 |w|\left|f^{(3)}(tw)\right| dt \\ &\le C\| f \|_B \frac{r}{(1-r^2)^2}. \end{aligned}$$

Replacing f by $f \circ \varphi_\lambda$, we get

$$\begin{aligned} &\Bigg|\frac{f^{(2)}(\varphi_\lambda(w))}{(1-\overline{\lambda}w)^4}(1-|\lambda|^2)^2+2\overline{\lambda}\frac{f^{(1)}(\varphi_\lambda(w))}{(1-\overline{\lambda}w)^3}(|\lambda|^2-1) \\ &\qquad\qquad -f^{(2)}(\lambda)(1-|\lambda|^2)^2-2\overline{\lambda}f^{(1)}(\lambda)(|\lambda|^2-1)\Bigg| \\ &\le C\| f \|_B \frac{r}{(1-r^2)^2}. \end{aligned}$$

Hence we get the following estimation by (2.1.2)

$$\begin{aligned} &(1-|\lambda|^2)^2\left|\frac{f^{(2)}(\varphi_\lambda(w))}{(1-\overline{\lambda}w)^4}-f^{(2)}(\lambda)\right| \\ &\le C\| f \|_B \frac{r}{(1-r^2)^2}+2|\lambda|(1-|\lambda|^2)\left|\frac{f^{(1)}(\varphi_\lambda(w))}{(1-\overline{\lambda}w)^3}-f^{(1)}(\lambda)\right| \end{aligned}$$

$$
\begin{aligned}
&= C\| f \|_B \frac{r}{(1-r^2)^2} + 2\,|\lambda|\,(1-|\lambda|^2) \\
&\qquad\qquad \left| \frac{f^{(1)}(\varphi_\lambda(w))}{(1-\overline{\lambda}w)^3} - \frac{f^{(1)}(\varphi_\lambda(w))}{(1-\overline{\lambda}w)^2} + \frac{f^{(1)}(\varphi_\lambda(w))}{(1-\overline{\lambda}w)^2} - f^{(1)}(\lambda) \right| \\
&\le C\| f \|_B \frac{r}{(1-r^2)^2} + 2\,|\lambda|\,(1-|\lambda|^2) \\
&\qquad\qquad \left| \frac{1}{(1-\overline{\lambda}w)^2} \Big(\frac{f^{(1)}(\varphi_\lambda(w))}{(1-\overline{\lambda}w)} - f^{(1)}(\varphi_\lambda(w)) \Big) \right| \\
&\quad + 2\,|\lambda|\,(1-|\lambda|^2) \left| \frac{f^{(1)}(\varphi_\lambda(w))}{(1-\overline{\lambda}w)^2} - f^{(1)}(\lambda) \right| \\
&\le C\| f \|_B \frac{r}{(1-r^2)^2} + 2\,|\lambda|\, \frac{(1-|\varphi_\lambda(w)|^2)}{(1-|w|^2)|1-\overline{\lambda}w|} \left| f^{(1)}(\varphi_\lambda(w)) \right| \\
&\quad + 2\,|\lambda|\, C\| f \|_B \frac{r}{(1-r^2)^2} \\
&\le CK\| f \|_B.
\end{aligned}
$$

Hence we get for $|w| < r$

$$
\begin{aligned}
&(1-|\lambda|^2)^2 \left| f^{(2)}(\varphi_\lambda(w)) - f^{(2)}(\lambda) \right| \\
&\quad \le (1-|\lambda|^2)^2 \left| f^{(2)}(\varphi_\lambda(w)) \Big(1 - \frac{1}{(1-\overline{\lambda}w)^4} \Big) \right| \\
&\qquad\qquad + (1-|\lambda|^2)^2 \left| \frac{f^{(2)}(\varphi_\lambda(w))}{(1-\overline{\lambda}w)^4} - f^{(2)}(\lambda) \right| \\
&\quad \le K\| f \|_B.
\end{aligned}
$$

Thus we have

$$
\sup_{\lambda\in D}\ \sup_{z\in\Delta(\lambda,r)} (1-|\lambda|^2)^2 \left| f^{(2)}(z) - f^{(2)}(\lambda) \right| < +\infty.
$$

□

Corollary 2.2 *Let $0 < r < 1$ and fix $n \ge 0$; integer. For an analytic function f on D, the following statements are equivalent:*

(A) $f \in B_0$;

(B) $\displaystyle \sup_{z\in\Delta(\lambda,r)} (1-|\lambda|^2)^n \left| f^{(n)}(z) - f^{(n)}(\lambda) \right| \to 0 \quad (|\lambda| \to 1^-)$

where $\Delta(\lambda, r) = \{z \in D; |\varphi_\lambda(z)| < r\}$.

Proof. When $n = 0$, it was proved by K. Stroethoff [5, p.131]. When $n \geq 1$, this is an immediate consequence of Theorem 2.1. $\square$

Theorem 2.3 *Let $p > 0$ and fix $n \geq 0$; integer such that $n = \alpha + \beta$, $\alpha, \beta \in \Re$. Then for an analytic function f on D and for $r \in (0, +\infty)$, the following statements are equivalent:*

(1) $f \in B$;

(2) $$\sup_{z\in D}\Big(\sup_{w\in D(z,r)} (1-|z|^2)^{\alpha}(1-|w|^2)^{\beta}|f^{(n)}(z) - f^{(n)}(w)|\Big) < +\infty;$$

(3) $$\sup_{z\in D}\Big(\frac{1}{|D(z,r)|}\int_{D(z,r)} (1-|z|^2)^{\alpha p}(1-|w|^2)^{\beta p} |f^{(n)}(z) - f^{(n)}(w)|^p dA(w)\Big)^{\frac{1}{p}} < +\infty;$$

(4) $$\sup_{z\in D}\Big(\frac{1}{|D(z,r)|}\int_{D(z,r)} \log^+\Big((1-|z|^2)^{\alpha}(1-|w|^2)^{\beta} |f^{(n)}(z) - f^{(n)}(w)|\Big)dA(w)\Big) < +\infty;$$

(5) $$\sup_{z\in D}\Big(\sup_{w\in D(z,r)} (1-|z|^2)^{\alpha}(1-|w|^2)^{\beta}|\widehat{f^{(n)}}_r(z) - f^{(n)}(w)|\Big) < +\infty;$$

(6) $$\sup_{z\in D}\Big(\frac{1}{|D(z,r)|}\int_{D(z,r)} (1-|z|^2)^{\alpha p}(1-|w|^2)^{\beta p} |\widehat{f^{(n)}}_r(z) - f^{(n)}(w)|^p dA(w)\Big)^{\frac{1}{p}} < +\infty;$$

(7) $$\sup_{z\in D}\Big(\frac{1}{|D(z,r)|}\int_{D(z,r)} \log^+\Big((1-|z|^2)^{\alpha}(1-|w|^2)^{\beta} |\widehat{f^{(n)}}_r(z) - f^{(n)}(w)|\Big)dA(w)\Big) < +\infty.$$

Proof. Firstly, we prove that (2) implies (1). Fix $0 < r < +\infty$, and let f be an analytic function on D. The following identity is easily proved by a direct calculation:

$$f^{(n+1)}(0) = \frac{2}{s^4}\int_{D(0,r)} \overline{u} f^{(n)}(u)dA(u), \qquad \cdots (2.3.1)$$

where $D(0,r) = \{u \in D; |u| < s\}$, $s = \tanh r \in (0,1)$. Hence we have

$$\left|f^{(n+1)}(0)\right| \leq \frac{2}{s}\sup_{u\in D(0,r)}\left|f^{(n)}(u)\right|.$$

Replacing $f^{(n)}$ by $f^{(n)} \circ \varphi_z - f^{(n)}(z)$, we have

$$(1-|z|^2)\left|f^{(n+1)}(z)\right| \le \frac{2}{s} \sup_{w\in D(z,r)} \left|f^{(n)}(w) - f^{(n)}(z)\right|.$$

Since $(1-|z|^2)$ is comparable to $(1-|w|^2)$ for $w \in D(z,r)$,

$$\begin{aligned}
&\sup_{z\in D}(1-|z|^2)^{n+1}\left|f^{(n+1)}(z)\right| \\
&\quad \le \frac{2}{s} \sup_{z\in D} \sup_{w\in D(z,r)} (1-|z|^2)^n \left|f^{(n)}(w) - f^{(n)}(z)\right| \\
&\quad \le C \sup_{z\in D} \sup_{w\in D(z,r)} (1-|z|^2)^\alpha (1-|w|^2)^\beta \left|f^{(n)}(w) - f^{(n)}(z)\right|.
\end{aligned}$$

By Theorem A, it follows that $f \in B$. Thus we have proved that (2) implies (1).

To prove that (1) implies (3), assume that $f \in B$. Then by Theorem 5.6 in [2] and by making a change of variable formura and by using $\int_{D(w,r)} d\lambda(z) \le C < +\infty$ and the fact that $(1-|z|^2)$ is comparable to $|1-\overline{w}z|$, $(1-|w|^2)$ and $|D(z,r)|^{\frac{1}{2}}$ when $w \in D(z,r)$, there exists a constant $K > 0$ (independent of f) such that

$$\begin{aligned}
&\sup_{z\in D}\Big(\frac{1}{|D(z,r)|}\int_{D(z,r)} (1-|z|^2)^{\alpha p}(1-|w|^2)^{\beta p}|f^{(n)}(w) - f^{(n)}(z)|^p dA(w)\Big)^{\frac{1}{p}} \\
&\le C\sup_{z\in D}\Big(\int_{D(z,r)} (1-|z|^2)^{np}|f^{(n)}(w) - f^{(n)}(z)|^p \frac{(1-|z|^2)^2}{|1-z\overline{w}|^4} dA(w)\Big)^{\frac{1}{p}} \\
&= C\sup_{z\in D}\Big(\int_{D(0,r)} (1-|z|^2)^{np}|f^{(n)}\circ\varphi_z(w) - f^{(n)}\circ\varphi_z(0)|^p dA(w)\Big)^{\frac{1}{p}} \\
&\le K_1\sup_{z\in D}\Big(\int_{D(0,r)} (1-|z|^2)^{np}(1-|w|^2)^p|(f^{(n)}\circ\varphi_z)^{(1)}(w)|^p dA(w)\Big)^{\frac{1}{p}} \\
&= K_1\sup_{z\in D}\Big(\int_{D(0,r)} (1-|z|^2)^{np}(1-|\varphi_z(w)|^2)^p|f^{(n+1)}\circ\varphi_z(w)|^p dA(w)\Big)^{\frac{1}{p}} \\
&= K_1\sup_{z\in D}\Big(\int_{D(z,r)} (1-|z|^2)^{np}(1-|w|^2)^p|f^{(n+1)}(w)|^p \frac{(1-|z|^2)^2}{|1-z\overline{w}|^4} dA(w)\Big)^{\frac{1}{p}} \\
&\le K_2\sup_{z\in D}\Big(\sup_{w\in D(z,r)} (1-|w|^2)^{n+1}|f^{(n+1)}(w)|\Big(\int_{D(z,r)} \frac{(1-|z|^2)^2}{|1-z\overline{w}|^4} dA(w)\Big)^{\frac{1}{p}}\Big)
\end{aligned}$$

$$\leq C_1 K_2 \sup_{z\in D}(1-|w|^2)^{(n+1)}|f^{(n+1)}(w)|.$$

Hence we have by Theorem A

$$\sup_{z\in D}\left(\frac{1}{|D(z,r)|}\int_{D(z,r)}(1-|z|^2)^{\alpha p}(1-|w|^2)^{\beta p}\right.$$
$$\left.|f^{(n)}(w)-f^{(n)}(z)|^p dA(w)\right)^{\frac{1}{p}} < +\infty.$$

Thus we have proved that (1) implies (3).

We prove that (3) implies (2). In fact, since $|g(u)|^p$ is subharmonic for all analytic function g on D for $0<p<+\infty$,

$$|g(w)|^p \leq \frac{C}{|D(w,r)|}\int_{D(w,r)}|g(u)|^p dA(u), \qquad \cdots (2.3.2)$$

for all analytic function g on D. Replacing g by $f^{(n)}-f^{(n)}(z)$,

$$|f^{(n)}(w)-f^{(n)}(z)|^p \leq \frac{C}{|D(w,r)|}\int_{D(w,r)}|f^{(n)}(u)-f^{(n)}(z)|^p dA(u).$$

Since $D(w,r)\subset D(z,2r)$ for $w\in D(z,r)$ and there is a constant $K>0$ such that

$$\frac{C}{|D(w,r)|}\leq\frac{K}{|D(z,2r)|},$$

and that $(1-|z|^2)$ is comparable to $(1-|w|^2)$ for $w\in D(z,r)$, we have

$$\begin{aligned}
&\sup_{w\in D(z,r)}(1-|z|^2)^{\alpha}(1-|w|^2)^{\beta}|f^{(n)}(w)-f^{(n)}(z)| \\
&\leq C_1\sup_{w\in D(z,r)}(1-|z|^2)^{n}|f^{(n)}(w)-f^{(n)}(z)| \\
&\leq\left(\frac{K_1}{|D(z,2r)|}\int_{D(z,2r)}(1-|z|^2)^{np}|f^{(n)}(u)-f^{(n)}(z)|^p dA(u)\right)^{\frac{1}{p}} \\
&\leq\left(\frac{K}{|D(z,2r)|}\int_{D(z,2r)}(1-|z|^2)^{\alpha p}(1-|u|^2)^{\beta p}|f^{(n)}(u)-f^{(n)}(z)|^p dA(u)\right)^{\frac{1}{p}}.
\end{aligned}$$
$$\cdots (2.3.3)$$

Thus we have proved that (3) implies (2).

Next we prove that (2) implies (5). Since $(1-|z|^2)$ is comparable to

$(1-|w|^2)$ and $|D(z,r)|^{\frac{1}{2}}$ for $w \in D(z,r)$, we see

$$\begin{aligned}
&\sup_{w\in D(z,r)}\Big((1-|z|^2)^\alpha(1-|w|^2)^\beta|\widehat{f^{(n)}}_r(z)-f^{(n)}(w)|\Big)\\
&\le C(1-|z|^2)^n \sup_{w\in D(z,r)}\Big(\frac{1}{|D(z,r)|}\Big|\int_{D(z,r)}\Big(f^{(n)}(w)-f^{(n)}(u)\Big)dA(u)\Big|\Big)\\
&\le C(1-|z|^2)^n \sup_{w\in D(z,r)}\Big(\sup_{u\in D(z,r)}|f^{(n)}(w)-f^{(n)}(u)|\Big)\\
&\le C(1-|z|^2)^n \sup_{w\in D(z,r)}\Big\{\sup_{u\in D(z,r)}(|f^{(n)}(w)-f^{(n)}(z)|+|f^{(n)}(z)-f^{(n)}(u)|)\Big\}\\
&\le C(1-|z|^2)^n\Big(\sup_{w\in D(z,r)}|f^{(n)}(w)-f^{(n)}(z)|+\sup_{u\in D(z,r)}|f^{(n)}(u)-f^{(n)}(z)|\Big).
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\sup_{w\in D(z,r)}\Big((1-|z|^2)^\alpha(1-|w|^2)^\beta|\widehat{f^{(n)}}_r(z)-f^{(n)}(w)|\Big)\\
&\qquad\le 2C\sup_{w\in D(z,r)}(1-|z|^2)^n|f^{(n)}(w)-f^{(n)}(z)|\\
&\qquad\le 2CK\sup_{w\in D(z,r)}\Big((1-|z|^2)^\alpha(1-|w|^2)^\beta|f^{(n)}(w)-f^{(n)}(z)|\Big).
\end{aligned} \qquad \cdots (2.3.4)$$

Thus we have proved that (2) implies (5). That (5) implies (6) is trivial.

Next we prove that (6) implies (5). Applying $g = f^{(n)} - \widehat{f^{(n)}}_r(z)$ to (2.3.2),

$$|f^{(n)}(w)-\widehat{f^{(n)}}_r(z)|^p \le \frac{C}{|D(w,r)|}\int_{D(w,r)}|f^{(n)}(u)-\widehat{f^{(n)}}_r(z)|^p dA(u).$$

Hence we have the following as well as (2.3.3):

$$\begin{aligned}
&\sup_{w\in D(z,r)}(1-|z|^2)^\alpha(1-|w|^2)^\beta|f^{(n)}(w)-\widehat{f^{(n)}}_r(z)|\\
&\quad\le\Big(\frac{K_1}{|D(z,2r)|}\int_{D(z,2r)}(1-|z|^2)^{\alpha p}(1-|u|^2)^{\beta p}\\
&\qquad\qquad |f^{(n)}(u)-\widehat{f^{(n)}}_r(z)|^p dA(u)\Big)^{\frac{1}{p}}.
\end{aligned}$$

Thus we have proved that (6) implies (5).

Next we prove that (5) implies (1). In fact, we also get the following equation as well as (2.3.1): For all analytic function g on D,

$$g^{(1)}(0) = \frac{2}{s^4} \int_{D(0,r)} \overline{u} g(u) dA(u),$$

where $D(0,r) = \{u \in D; |u| < s\}$, $s = \tanh r \in (0,1)$. By using the above equation, we get

$$|g^{(1)}(0)| \leq \frac{2}{s^4} \int_{D(0,r)} |g(u)|\, dA(u). \qquad \cdots (2.3.5)$$

Applying $g = f^{(n)} \circ \varphi_z - \widehat{f^{(n)}}_r(z)$ and using Hölder's inequality and using the fact that $(1-|z|^2)$ is comparable to $|1-\overline{z}u|$ and $|D(z,r)|^{\frac{1}{2}}$ for $u \in D(z,r)$, for $p \geq 1$

$$\begin{aligned}
&(1-|z|^2)|f^{(n+1)}(z)| \\
&\leq K\left(\int_{D(0,r)} \left|f^{(n)} \circ \varphi_z(u) - \widehat{f^{(n)}}_r(z)\right|^p dA(u)\right)^{\frac{1}{p}} \\
&= K\left(\int_{D(z,r)} \left|f^{(n)}(u) - \widehat{f^{(n)}}_r(z)\right|^p \frac{(1-|z|^2)^2}{|1-\overline{z}u|^4} dA(u)\right)^{\frac{1}{p}} \\
&\leq K\left(\frac{C}{|D(z,r)|} \int_{D(z,r)} \left|f^{(n)}(u) - \widehat{f^{(n)}}_r(z)\right|^p dA(u)\right)^{\frac{1}{p}} \\
&\leq KC \sup_{u \in D(z,r)} \left|f^{(n)}(u) - \widehat{f^{(n)}}_r(z)\right|. \qquad \cdots (2.3.6)
\end{aligned}$$

Multiplying the both sides by $(1-|z|^2)^n$, we see that

$$\begin{aligned}
&(1-|z|^2)^{(n+1)} \left|f^{(n+1)}(z)\right| \\
&\qquad \leq K_1 \sup_{u \in D(z,r)} (1-|z|^2)^\alpha (1-|u|^2)^\beta \left|f^{(n)}(u) - \widehat{f^{(n)}}_r(z)\right|.
\end{aligned}$$

By Theorem A, that (5) implies (1) is proved.

To prove that (4) implies (2), suppose that

$$\sup_{z \in D}\left(\frac{1}{|D(z,r)|} \int_{D(z,r)} \log^+\Big((1-|z|^2)^\alpha (1-|w|^2)^\beta \right.$$

$$\left. |f^{(n)}(z) - f^{(n)}(w)|\Big) dA(w)\right) < +\infty.$$

Since $\log^+ |g(w)|$ is subharmonic for all analytic function g on D,

$$\log^+ |g(w)| \le \frac{C}{|D(w,r)|} \int_{D(w,r)} \log^+ |g(u)| dA(u).$$

Applying $g = (1-|z|^2)^n \left(f^{(n)} - f^{(n)}(z)\right)$, there is a constant $C > 0$ such that

$$\begin{aligned}&\log^+ \left((1-|z|^2)^n |f^{(n)}(w) - f^{(n)}(z)|\right) \\ &\le \frac{C}{|D(w,r)|} \int_{D(w,r)} \log^+ \left((1-|z|^2)^n |f^{(n)}(u) - f^{(n)}(z)|\right) dA(u).\end{aligned}$$

Since $D(w,r) \subset D(z,2r)$ for $w \in D(z,r)$ and there is a constant $K > 0$ such that

$$\frac{C}{|D(w,r)|} \le \frac{K}{|D(z,2r)|},$$

$$\begin{aligned}&\log^+\left((1-|z|^2)^\alpha (1-|w|^2)^\beta |f^{(n)}(w) - f^{(n)}(z)|\right) \\ &\le C_1 \log^+\left((1-|z|^2)^n |f^{(n)}(w) - f^{(n)}(z)|\right) \\ &\le \frac{C_1 C}{|D(w,r)|} \int_{D(w,r)} \log^+\left((1-|z|^2)^n |f^{(n)}(u) - f^{(n)}(z)|\right) dA(u) \\ &\le \left(\frac{K_1}{|D(z,2r)|} \int_{D(z,2r)} \log^+\Big(K(1-|z|^2)^\alpha (1-|u|^2)^\beta \right. \\ &\qquad\qquad\qquad \left. |f^{(n)}(u) - f^{(n)}(z)|\Big) dA(u)\right) \\ pt &= K_1 \left(\frac{1}{|D(z,2r)|} \int_{D(z,2r)} \log^+\Big((1-|z|^2)^\alpha (1-|u|^2)^\beta \right. \\ &\qquad\qquad\qquad \left. |f^{(n)}(u) - f^{(n)}(z)|\Big) dA(u) + \log^+ K\right) \\ &=: M < +\infty. \qquad \cdots (2.3.7)\end{aligned}$$

Since $x \le \exp\left(\log^+ x\right)$ $(x \ge 0)$, for any $z \in D$

$$\sup_{w \in D(z,r)} \left((1-|z|^2)^\alpha (1-|w|^2)^\beta |f^{(n)}(w) - f^{(n)}(z)|\right) \le \exp M < +\infty.$$

Thus we have proved that (4) implies (2). That (2) implies (4) follows from $\log^+ x \le x$ $(x \ge 0)$ easily. We can also prove the equivalence of (5) and

(7) as well as the equivalence of (2) and (4). This completes the proof of Theorem 2.3. □

Remark 2.4 Carefully examining the proof of the above theorem, we see that they also hold for (4*) and (7*) instead of (4) and (7), respectively: For a monotonically increasing convex function φ on R^1 such that $\varphi(0) = 0$ and for some constants $C, K > 0$,

$$(4^*) \quad \sup_{z\in D} \varphi^{-1}\left(\frac{K}{|D(z,r)|}\int_{D(z,r)} \varphi\Big(C(1-|z|^2)^{\alpha}(1-|w|^2)^{\beta} |f^{(n)}(z) - f^{(n)}(w)|\Big) dA(w)\right) < +\infty,$$

$$(7^*) \quad \sup_{z\in D} \varphi^{-1}\left(\frac{K}{|D(z,r)|}\int_{D(z,r)} \varphi\Big(C(1-|z|^2)^{\alpha}(1-|w|^2)^{\beta} |\widehat{f^{(n)}}_r(z) - f^{(n)}(w)|\Big) dA(w)\right) < +\infty.$$

Corollary 2.5 *Let $p > 0$ and fix $n \geq 0$; integer such that $n = \alpha + \beta$, $\alpha, \beta \in \Re$. Then for an analytic function f on D and for $r \in (0, +\infty)$, the following statements are equivalent:*

(1) $f \in B_0$;

(2) $$\lim_{|z|\to 1^-}\left(\sup_{w\in D(z,r)} (1-|z|^2)^{\alpha}(1-|w|^2)^{\beta}|f^{(n)}(z) - f^{(n)}(w)|\right) = 0;$$

(3) $$\lim_{|z|\to 1^-}\left(\frac{1}{|D(z,r)|}\int_{D(z,r)} (1-|z|^2)^{\alpha p}(1-|w|^2)^{\beta p} |f^{(n)}(z) - f^{(n)}(w)|^p dA(w)\right) = 0;$$

(4) $$\lim_{|z|\to 1^-}\left(\sup_{w\in D(z,r)} (1-|z|^2)^{\alpha}(1-|w|^2)^{\beta}|\widehat{f^{(n)}}_r(z) - f^{(n)}(w)|\right) = 0;$$

(5) $$\lim_{|z|\to 1^-}\left(\frac{1}{|D(z,r)|}\int_{D(z,r)} (1-|z|^2)^{\alpha p}(1-|w|^2)^{\beta p} |\widehat{f^{(n)}}_r(z) - f^{(n)}(w)|^p dA(w)\right) = 0.$$

Proof. This is an immediate consequence of Theorem 2.3. In fact, let $0 < r < +\infty$ and fix $n \geq 1$; integer such that $n = \alpha + \beta$, $\alpha, \beta \in \Re$. For an analytic function f on D and for $2r > 0$, suppose that

$$\lim_{|z|\to 1^-}\left(\frac{1}{|D(z,2r)|}\int_{D(z,2r)}(1-|z|^2)^{\alpha p}(1-|w|^2)^{\beta p}\right.$$

$$\left.|f^{(n)}(z)-f^{(n)}(w)|^p dA(w)\right)=0.$$

Then by (2.3.3)

$$\sup_{w\in D(z,r)}(1-|z|^2)^{\alpha}(1-|w|^2)^{\beta}|f^{(n)}(z)-f^{(n)}(w)|$$

$$\leq C\left(\frac{1}{|D(z,2r)|}\int_{D(z,2r)}(1-|z|^2)^{\alpha p}(1-|w|^2)^{\beta p}\right.$$

$$\left.|f^{(n)}(z)-f^{(n)}(w)|^p dA(w)\right)^{\frac{1}{p}}.$$

Thus we proved that (3) implies (2).

To prove that (1) implies (2), suppose that $f\in B_0$. Let $f_t(z):=f(tz)$ $(0<t<1)$. Then for $\beta(z,w)<r$, by Theorem 2.3

$$\sup_{w\in D(z,r)}(1-|z|^2)^{\alpha}(1-|w|^2)^{\beta}\left|(f-f_t)^{(n)}(z)-(f-f_t)^{(n)}(w)\right|$$

$$\leq C\|f-f_t\|_B.$$

Hence

$$\sup_{w\in D(z,r)}(1-|z|^2)^{\alpha}(1-|w|^2)^{\beta}|f^{(n)}(z)-f^{(n)}(w)|$$

$$\leq C\|f-f_t\|_B+\sup_{w\in D(z,r)}(1-|z|^2)^{\alpha}(1-|w|^2)^{\beta}|f_t^{(n)}(z)-f_t^{(n)}(w)|$$

$$\leq C\|f-f_t\|_B+Kt^n\frac{(1-|z|^2)^n}{(1-t^2)^n}$$

$$\sup_{w\in D(z,r)}(1-|tz|^2)^{\alpha}(1-|tw|^2)^{\beta}|f^{(n)}(tz)-f^{(n)}(tw)|$$

$$\leq C\|f-f_t\|_B+Kt^n\frac{(1-|z|^2)^n}{(1-t^2)^n}\|f\|_B.$$

In the above inequality, letting $|z|\to 1^-$, the second term on the right side converges to 0, and moreover letting $t\to 1^-$, by Theorem D the first term on the right side also converges to 0. Thus we have proved that (1) implies (2). That (2) implies (3) is trivial. That (2) implies (4) follows from

(2.3.4). That (4) implies (5) is trivial. That (5) implies (1) follows from (2.3.6). That (4) implies (2) follows from (2.3.7). That (2) implies (4) is trivial. This completes the proof of Corollary 2.5. □

Remark 2.6 Carefully examining the proof of the above corollary, we see that the following are also the equivalent conditions: For a monotonically increasing convex function φ on R^1 such that $\varphi(0) = 0$ and for some constants $C, K > 0$,

$$(6)\quad \lim_{|z|\to 1^-} \varphi^{-1}\left(\frac{K}{|D(z,r)|}\int_{D(z,r)} \varphi\Big(C(1-|z|^2)^{\alpha}(1-|w|^2)^{\beta} |f^{(n)}(z) - f^{(n)}(w)|\Big) dA(w)\right) = 0,$$

$$(7)\quad \lim_{|z|\to 1^-} \varphi^{-1}\left(\frac{K}{|D(z,r)|}\int_{D(z,r)} \varphi\Big(C(1-|z|^2)^{\alpha}(1-|w|^2)^{\beta} |\widehat{f^{(n)}}_r(z) - f^{(n)}(w)|\Big) dA(w)\right) = 0.$$

Theorem 2.7 *Let $p > 0$ and fix $n \geq 1$; integer such that $n = \alpha + \beta$, $\alpha, \beta \in \Re$. Then for an analytic function f on D and for $r \in (0, +\infty)$, the following statements are equivalent*:

(1) $f \in B$;

(2) $$\sup\left\{ \sup_{w\in D(z,r)} (1-|z|^2)^{\alpha}(1-|w|^2)^{\beta} \left|\frac{f^{(n-1)}(z) - f^{(n-1)}(w)}{z-w}\right|;\ z \in D,\ z \neq w\right\} < +\infty;$$

(3) $$\sup_{z\in D}\left(\frac{1}{|D(z,r)|}\int_{D(z,r)} (1-|z|^2)^{\alpha p}(1-|w|^2)^{\beta p} \left|\frac{f^{(n-1)}(z) - f^{(n-1)}(w)}{z-w}\right|^p dA(w)\right)^{\frac{1}{p}} < +\infty;$$

(4) $$\sup_{z\in D}\left(\frac{1}{|D(z,r)|}\int_{D(z,r)} \log^+\left((1-|z|^2)^{\alpha}(1-|w|^2)^{\beta} \left|\frac{f^{(n-1)}(z) - f^{(n-1)}(w)}{z-w}\right|\right) dA(w)\right) < +\infty.$$

Proof. This theorem can be proved in the same order with that of Theorem 2.3, but we prove it in different order. Firstly, we prove that (1) implies (2) by using the induction. Suppose $f \in B$. By Theorem B, we have

$$\begin{aligned}\left|f^{(n-1)}(u) - f^{(n-1)}(0)\right| &\leq \int_0^1 |u| \left|f^{(n)}(tu)\right| dt \\ &\leq C\| f \|_B \int_0^1 \frac{|u|}{(1-t^2|u|^2)^n} dt \\ &= C\| f \|_B \int_0^{|u|} \frac{ds}{(1-s^2)^n} \\ &\leq C\| f \|_B \frac{|u|}{(1-|u|^2)^{n-1}}\end{aligned}$$

for all $u \in D$. For any $z, w \in D$, replacing f by $f \circ \varphi_w$ and applying $u = \varphi_w(z)$, we have

$$\begin{aligned}&\left|(f\circ\varphi_w)^{(n-1)}(\varphi_w(z)) - (f\circ\varphi_w)^{(n-1)}(0)\right| \\ &\qquad \leq C\| f\circ\varphi_w \|_B \frac{|\varphi_w(z)|}{(1-|\varphi_w(z)|^2)^{n-1}}.\end{aligned}$$

Moreover by an elementary calculation, we get

$$\begin{aligned}&(1-|z|^2)^{\frac{n}{2}}(1-|w|^2)^{\frac{n}{2}} \left|\frac{(f\circ\varphi_w)^{(n-1)}(\varphi_w(z)) - (f\circ\varphi_w)^{(n-1)}(0)}{z-w}\right| \\ &\leq C\| f \|_B (1-|z|^2)^{\frac{n}{2}}(1-|w|^2)^{\frac{n}{2}} \\ &\qquad\qquad \frac{1}{|w-z|}\frac{|w-z|}{|1-\overline{w}z|}\frac{|1-\overline{w}z|^{2(n-1)}}{(1-|z|^2)^{(n-1)}(1-|w|^2)^{(n-1)}} \\ &= C\| f \|_B \frac{|1-\overline{w}z|^{2n-3}}{(1-|z|^2)^{(\frac{n}{2}-1)}(1-|w|^2)^{(\frac{n}{2}-1)}} \\ &\leq C\| f \|_B |1-\overline{w}z|^{n-1}.\end{aligned}$$

The last inequality follows from the fact that $|1-\overline{w}z|$ is comparable to $(1-|z|^2)$, $(1-|w|^2)$ because $\beta(z,w) < r$. By an elementary calculation, there are some constants $\{C_k\}_{k=1}^{n-1}$ such that

$$\frac{(1-|z|^2)^{\frac{n}{2}}(1-|w|^2)^{\frac{n}{2}}}{|z-w|}$$
$$\left|\sum_{k=1}^{n-1}\left\{f^{(k)}(z)\frac{(\overline{w}z-1)^{n-1+k}}{(|w|^2-1)^{n-1}}-f^{(k)}(w)(|w|^2-1)^k\right\}(\overline{w})^{n-k-1}C_k\right|$$
$$\leq C\| f\|_B|1-\overline{w}z|^{n-1},$$

where $C_{n-1}=1$. Dividing the both sides by $|1-\overline{w}z|^{n-1}$, we have

$$\frac{(1-|z|^2)^{\frac{n}{2}}(1-|w|^2)^{\frac{n}{2}}}{|z-w|}$$
$$\left|\sum_{k=1}^{n-1}\left\{f^{(k)}(z)\frac{(\overline{w}z-1)^{k}}{(|w|^2-1)^{n-1}}-f^{(k)}(w)\frac{(|w|^2-1)^k}{(\overline{w}z-1)^{n-1}}\right\}(\overline{w})^{n-k-1}C_k\right|$$
$$\leq C\| f\|_B.$$

Moreover

$$\frac{(1-|z|^2)^{\frac{n}{2}}(1-|w|^2)^{\frac{n}{2}}}{|z-w|}\frac{|\overline{w}z-1|^{n-1}}{(1-|w|^2)^{n-1}}$$
$$\left|\sum_{k=1}^{n-1}\left\{f^{(k)}(z)(\overline{w}z-1)^{k-n+1}-f^{(k)}(w)\frac{(|w|^2-1)^{k+n-1}}{(\overline{w}z-1)^{2(n-1)}}\right\}(\overline{w})^{n-k-1}C_k\right|$$
$$\leq C\| f\|_B.$$

Since $\frac{|\overline{w}z-1|^{n-1}}{(1-|w|^2)^{n-1}}\geq C$ for $\beta(z,w)<r$, we get

$$\frac{(1-|z|^2)^{\frac{n}{2}}(1-|w|^2)^{\frac{n}{2}}}{|z-w|}$$
$$\left|\sum_{k=1}^{n-1}\{f^{(k)}(z)(\overline{w}z-1)^{k-n+1}-f^{(k)}(w)\frac{(|w|^2-1)^{k+n-1}}{(\overline{w}z-1)^{2(n-1)}}\}(\overline{w})^{n-k-1}C_k\right|$$
$$\leq C\| f\|_B.$$

By the way, we put

$$N:=\sum_{k=1}^{n-1}\left\{f^{(k)}(z)(\overline{w}z-1)^{k-n+1}-f^{(k)}(w)\frac{(|w|^2-1)^{k+n-1}}{(\overline{w}z-1)^{2(n-1)}}\right\}(\overline{w})^{n-k-1}C_k$$

$$= \sum_{k=1}^{n-2} \Big\{ f^{(k)}(z)(\overline{w}z-1)^{k-n+1} - f^{(k)}(w)\frac{(|w|^2-1)^{k+n-1}}{(\overline{w}z-1)^{2(n-1)}} \Big\} (\overline{w})^{n-k-1} C_k$$

$$+ f^{(n-1)}(z) - f^{(n-1)}(w)\frac{(|w|^2-1)^{2(n-1)}}{(\overline{w}z-1)^{2(n-1)}}$$

$$=: M + f^{(n-1)}(z) - f^{(n-1)}(w)\frac{(|w|^2-1)^{2(n-1)}}{(\overline{w}z-1)^{2(n-1)}}.$$

Then we have

$$(1-|z|^2)^{\frac{n}{2}}(1-|w|^2)^{\frac{n}{2}}\left|\frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w}\right|$$

$$= (1-|z|^2)^{\frac{n}{2}}(1-|w|^2)^{\frac{n}{2}}\left|\frac{N + f^{(n-1)}(w)\frac{(|w|^2-1)^{2(n-1)}}{(\overline{w}z-1)^{2(n-1)}} - M - f^{(n-1)}(w)}{z-w}\right|$$

$$\leq C\| f \|_B + (1-|z|^2)^{\frac{n}{2}}(1-|w|^2)^{\frac{n}{2}}$$

$$\left|\frac{f^{(n-1)}(w)\frac{(|w|^2-1)^{2(n-1)}}{(\overline{w}z-1)^{2(n-1)}} - M - f^{(n-1)}(w)}{z-w}\right|$$

$$=: C\| f \|_B \ + \ L.$$

The triangular inequality shows that

$$L \leq \frac{(1-|z|^2)^{\frac{n}{2}}(1-|w|^2)^{\frac{n}{2}}}{|z-w|}\left\{\left|\frac{(|w|^2-1)^{2(n-1)}}{(\overline{w}z-1)^{2(n-1)}} - 1\right| |f^{(n-1)}(w)| + |M|\right\}.$$

The following inequalities are proved by using an elementary induction:

$$\frac{(1-|z|^2)^{\frac{n}{2}}(1-|w|^2)^{\frac{n}{2}}}{|z-w|}\left|\frac{(|w|^2-1)^{2(n-1)}}{(\overline{w}z-1)^{2(n-1)}} - 1\right| |f^{(n-1)}(w)| \leq C\| f \|_B.$$

$$\frac{(1-|z|^2)^{\frac{n}{2}}(1-|w|^2)^{\frac{n}{2}}}{|z-w|}|M| \leq C\| f \|_B.$$

Hence we have that

$$\sup\left\{ \sup_{w\in D(z,r)} (1-|z|^2)^{\frac{n}{2}}(1-|w|^2)^{\frac{n}{2}}\left|\frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w}\right|; \right.$$

$$\left. z\in D, \ z\neq w \right\} < +\infty.$$

Since $1-|z|^2$ is comparable to $1-|w|^2$ for $w \in D(z,r)$, we have

$$\sup\left\{\sup_{w\in D(z,r)}(1-|z|^2)^\alpha(1-|w|^2)^\beta\left|\frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w}\right|;\right.$$
$$\left. z\in D,\ z\neq w\right\} < +\infty.$$

Thus we have proved that (1) implies (2). That (2) implies (3) is trivial.

We prove that (3) implies (1). In fact, since $|g(z)|^p$ is subharmonic for all analytic function g on D and $p>0$, we have

$$|g(z)| \le \left(\frac{C}{|D(z,r)|}\int_{D(z,r)}|g(w)|^p\,dA(w)\right)^{\frac{1}{p}}.$$

for all analytic function g on D. Applying $g(w)=\frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w}$, then

$$\left|f^{(n)}(z)\right| \le \left(\frac{C}{|D(z,r)|}\int_{D(z,r)}\left|\frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w}\right|^p dA(w)\right)^{\frac{1}{p}}. \qquad \cdots (2.7.1)$$

Multiplying the both sides by $(1-|z|^2)^n$, we have

$$\begin{aligned}
&\sup_{z\in D}(1-|z|^2)^n\left|f^{(n)}(z)\right| \\
&\le \sup_{z\in D}\left(\frac{C}{|D(z,r)|}\int_{D(z,r)}(1-|z|^2)^{\alpha p}(1-|w|^2)^{\beta p}\right. \\
&\qquad\qquad \left.\left|\frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w}\right|^p dA(w)\right)^{\frac{1}{p}} < +\infty.
\end{aligned}$$

By Theorem A, we have $f\in B$. Thus we have proved that (3) implies (1).

To prove that (4) implies (1), suppose that

$$\begin{aligned}
&\sup_{z\in D}\left(\frac{1}{|D(z,r)|}\int_{D(z,r)}\log^+\left((1-|z|^2)^\alpha(1-|w|^2)^\beta\right.\right. \\
&\qquad\qquad \left.\left.\left|\frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w}\right|\right)dA(w)\right) < +\infty.
\end{aligned}$$

Since $\log^+|g(w)|$ is subharmonic for all analytic function g on D,

$$\log^+|g(z)| \le \frac{C}{|D(z,r)|}\int_{D(z,r)}\log^+|g(w)|dA(w).$$

Applying $g(w) = (1-|z|^2)^n \frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w}$, there is a constant $C > 0$ such that

$$
\begin{aligned}
&\log^+\left((1-|z|^2)^n |f^{(n)}(z)|\right) \\
&\le \frac{C}{|D(z,r)|}\int_{D(z,r)} \log^+\left((1-|z|^2)^n \left|\frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w}\right|\right) dA(w) \\
&\le C \sup_{z\in D}\left(\frac{1}{|D(z,r)|}\int_{D(z,r)} \log^+\left(K(1-|z|^2)^\alpha (1-|w|^2)^\beta \right.\right. \\
&\qquad\qquad \left.\left.\left|\frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w}\right|\right) dA(w)\right) \\
&= C \sup_{z\in D}\left(\frac{1}{|D(z,r)|}\int_{D(z,r)} \log^+\left((1-|z|^2)^\alpha (1-|w|^2)^\beta \right.\right. \\
&\qquad\qquad \left.\left.\left|\frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w}\right|\right) dA(w)\right) + C\log^+ K \\
&=: M < +\infty. \qquad\qquad \cdots (2.7.2)
\end{aligned}
$$

Since $x \le \exp\left(\log^+ x\right)$ $(x \ge 0)$, we have

$$(1-|z|^2)^n \left|f^{(n)}(z)\right| \le \exp M < +\infty \quad (z \in D).$$

By Theorem A, we have $f \in B$. Thus we have proved that (4) implies (1).

That (2) implies (4) follows from the fact that $\log^+ x \le x$ $(x \ge 0)$ easily. This completes the proof of Theorem 2.7. □

Remark 2.8 Carefully examining the proof of the above theorem, we see that the following are also the equivalent conditions: For a monotonically increasing convex function φ on R^1 such that $\varphi(0) = 0$ and for some constants $C, K > 0$,

$$
\begin{aligned}
(5) \quad &\sup_{z\in D}\varphi^{-1}\left(\frac{K}{|D(z,r)|}\int_{D(z,r)} \varphi\left(C(1-|z|^2)^\alpha (1-|w|^2)^\beta \right.\right. \\
&\qquad\qquad \left.\left.\left|\frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w}\right|\right) dA(w)\right) < +\infty.
\end{aligned}
$$

Corollary 2.9 *Let $p > 0$ and fix $n \ge 1$; integer such that $n = \alpha + \beta$, $\alpha, \beta \in \Re$. Then for an analytic function f on D and for $r \in (0, +\infty)$, the* *following statements are equivalent*:

(1) $f \in B_0$;

(2) $$\lim_{|z|\to 1^-} \left\{ \sup_{w\in D(z,r)} (1-|z|^2)^{\alpha}(1-|w|^2)^{\beta} \left| \frac{f^{(n-1)}(z) - f^{(n-1)}(w)}{z-w} \right|;\ z \in D,\ z \neq w \right\} = 0;$$

(3) $$\lim_{|z|\to 1^-} \left(\frac{1}{|D(z,r)|} \int_{D(z,r)} (1-|z|^2)^{\alpha p}(1-|w|^2)^{\beta p} \left| \frac{f^{(n-1)}(z) - f^{(n-1)}(w)}{z-w} \right|^p dA(w) \right) = 0.$$

Proof. This is an immediate consequence of Theorem 2.7. In fact, let $0 < r < +\infty$ and fix $n \geq 1$; integer such that $n = \alpha + \beta$, $\alpha, \beta \in \Re$. For an analytic function f on D, suppose that

$$\lim_{|z|\to 1^-} \sup \left\{ (1-|z|^2)^{\alpha}(1-|w|^2)^{\beta} \left| \frac{f^{(n-1)}(z) - f^{(n-1)}(w)}{z-w} \right|; \ w \in D,\ \beta(z,w) < r,\ z \neq w \right\} = 0.$$

Then we have

$$\begin{aligned}
&(1-|z|^2)^n |f^{(n)}(z)| \\
&\quad = \lim_{w\to z} (1-|z|^2)^{\alpha}(1-|w|^2)^{\beta} \left| \frac{f^{(n-1)}(z) - f^{(n-1)}(w)}{z-w} \right| \\
&\quad \leq \sup \left\{ (1-|z|^2)^{\alpha}(1-|w|^2)^{\beta} \left| \frac{f^{(n-1)}(z) - f^{(n-1)}(w)}{z-w} \right|; \ w \in D, \beta(z,w) < r, z \neq w \right\} \to 0 \quad (|z| \to 1^-).
\end{aligned}$$

Hence Thorem C implies that $f \in B_0$. Thus we proved that (2) implies (1).

To prove that (1) implies (2), suppose that $f \in B_0$. Let $f_t(z) := f(tz)$ $(0 < t < 1)$. Then for $\beta(z,w) < r$, by the proof of Theorem 2.7

$$\begin{aligned}
&(1-|z|^2)^{\alpha}(1-|w|^2)^{\beta} \left| \frac{(f-f_t)^{(n-1)}(z) - (f-f_t)^{(n-1)}(w)}{z-w} \right| \\
&\quad \leq C \| f - f_t \|_B.
\end{aligned}$$

Hence we have

$$
\begin{aligned}
&(1-|z|^2)^\alpha(1-|w|^2)^\beta\left|\frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w}\right| \\
&\leq C\| f-f_t \|_B+(1-|z|^2)^\alpha(1-|w|^2)^\beta \\
&\qquad\left|\frac{t^{n-1}f^{(n-1)}(tz)-t^{n-1}f^{(n-1)}(tw)}{z-w}\right|.
\end{aligned}
$$

Next estimating the second term on the right side of the above inequality, by the proof of Theorem 2.7

$$
\begin{aligned}
&(1-|z|^2)^\alpha(1-|w|^2)^\beta\left|\frac{t^{n-1}f^{(n-1)}(tz)-t^{n-1}f^{(n-1)}(tw)}{z-w}\right| \\
&= t^n\frac{(1-|z|^2)^\alpha(1-|w|^2)^\beta}{(1-|tz|^2)^\alpha(1-|tw|^2)^\beta}(1-|tz|^2)^\alpha(1-|tw|^2)^\beta \\
&\qquad\left|\frac{f^{(n-1)}(tz)-f^{(n-1)}(tw)}{tz-tw}\right| \\
&\leq K\frac{t^n}{(1-t^2)^n}(1-|z|^2)^\alpha(1-|w|^2)^\beta\| f \|_B.
\end{aligned}
$$

Since $(1-|z|^2)$ is comparable to $(1-|w|^2)$ for $\beta(z,w)<r$, we see that

$$
\begin{aligned}
&\sup\left\{(1-|z|^2)^\alpha(1-|w|^2)^\beta\left|\frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w}\right|;\right. \\
&\qquad\qquad\left. w\in D,\ \beta(z,w)<r,\ w\neq z\right\} \\
&\leq C\| f-f_t \|_B+K\frac{t^n}{(1-t^2)^n}(1-|z|^2)^n\| f \|_B.
\end{aligned}
$$

In the above inequality, letting $|z|\to 1^-$, the second term on the right side converges to 0, and moreover letting $t\to 1^-$, by Theorem D the first term on the right side also converges to 0. Thus we have proved that (1) implies (2). That (2) implies (3) is trivial. That (3) implies (1) is clear by (2.7.1). This completes the proof of Corollary 2.9. □

Remark 2.10 Carefully examining the proof of the above corollary, we see that the following is also the equivalent condition: For a monotonically increasing convex function φ on R^1 such that $\varphi(0) = 0$ and for some constants $C, K > 0$,

$$(4) \quad \lim_{|z|\to 1^-} \varphi^{-1}\left(\frac{K}{|D(z,r)|}\int_{D(z,r)} \varphi\left(C(1-|z|^2)^\alpha(1-|w|^2)^\beta \left|\frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w}\right|\right)dA(w)\right) = 0.$$

3. Criterior for membership of the Besov space

The following Lemma 3.1, Lemma 3.2 are used to prove Theorems 3.3 and 3.5.

Lemma 3.1 ([7, p.75]) *Let $0 < r < +\infty$. For any $p \geq 1$, there is a constant $C_p > 0$ such that*

$$\int_{sD} |f(z)|^p \, dA(z) \leq C_p \int_{sD} |zf(z)|^p \, dA(z)$$

for any analytic function f on D, where $sD = \{w \in D; |w| < s\} = D(0,r)$, $s = \tanh r$.

Lemma 3.2 *Let $0 < r < +\infty$. For any $p \geq 1$, there is a finite constant $C > 0$ such that*

$$\int_{sD} \left|\frac{g(z)-g(0)}{z}\right|^p dA(z) \leq C \int_{sD} \left|g^{(1)}(z)\right|^p (1-|z|^2)^p dA(z)$$

for any analytic function g on D, where $sD = \{w \in D; |w| < s\} = D(0,r)$, $s = \tanh r$.

Proof. Let g be an analytic function on D and let

$$sD = \{w \in D; |w| < s\} = D(0,r), \; s = \tanh r.$$

Then for any $p \geq 1$, there is a constant $C_p > 0$ such that

$$\begin{aligned}\int_{sD} \left|\frac{g(z)-g(0)}{z}\right|^p dA(z) &\leq C_p \int_{sD} |g(z)-g(0)|^p \, dA(z) \\ &\leq K \int_{sD} \left|g^{(1)}(z)\right|^p (1-|z|^2)^p dA(z).\end{aligned}$$

The first inequality above follows from Lemma 3.1, and the second inequality follows from Theorem 5.6 in [2]. □

Theorem 3.3 *Fix $n \geq 0$; integer such that $n = \alpha + \beta$, $\alpha, \beta \in \Re$. Then for an analytic function f on D and for $r \in (0, +\infty)$, for $p > 1$ in the case of $n = 0$, and for $p \geq 1$ in the case of $n \geq 1$, the following statements are equivalent*:

(1) $f \in B_p$;

(2) $$\int_D \Big(\sup_{w \in D(z,r)} (1-|z|^2)^\alpha (1-|w|^2)^\beta |f^{(n)}(z) - f^{(n)}(w)| \Big)^p d\lambda(z) < +\infty;$$

(3) $$\int_D \Big(\frac{1}{|D(z,r)|} \int_{D(z,r)} (1-|z|^2)^{\alpha p} (1-|w|^2)^{\beta p} |f^{(n)}(z) - f^{(n)}(w)|^p dA(w) \Big) d\lambda(z) < +\infty;$$

(4) $$\int_D \Big(\frac{1}{|D(z,r)|} \int_{D(z,r)} (1-|z|^2)^\alpha (1-|w|^2)^\beta |f^{(n)}(z) - f^{(n)}(w)| dA(w) \Big)^p d\lambda(z) < +\infty;$$

(5) $$\int_D \Big(\sup_{w \in D(z,r)} (1-|z|^2)^\alpha (1-|w|^2)^\beta |\widehat{f^{(n)}}_r(z) - f^{(n)}(w)| \Big)^p d\lambda(z) < +\infty;$$

(6) $$\int_D \Big(\frac{1}{|D(z,r)|} \int_{D(z,r)} (1-|z|^2)^{\alpha p} (1-|w|^2)^{\beta p} |\widehat{f^{(n)}}_r(z) - f^{(n)}(w)|^p dA(w) \Big) d\lambda(z) < +\infty;$$

(7) $$\int_D \Big(\frac{1}{|D(z,r)|} \int_{D(z,r)} (1-|z|^2)^\alpha (1-|w|^2)^\beta |\widehat{f^{(n)}}_r(z) - f^{(n)}(w)| dA(w) \Big)^p d\lambda(z) < +\infty.$$

Proof. Let $0 < r < +\infty$ and fix $n \geq 0$; integer such that $n = \alpha + \beta$, $\alpha, \beta \in \Re$. Let $0 < s < 1$ such that $s = \frac{e^r - e^{-r}}{e^r + e^{-r}} = \tanh r \in (0,1)$. In the case of $n = 0$, this theorem does not hold for $p = 1$ because of the definition of B_p, but in the case $n \geq 1$ it does for all $p \geq 1$.

Firstly, we show that (1) implies (3). Assume that $f \in B_p$. Then by Theorem 5.6 in [2] and by making a change of variable formura and by using $\int_{D(w,r)} d\lambda(z) \leq C < +\infty$ and the fact that $(1-|z|^2)$ is comparable to $|1 - \overline{w}z|$, $(1-|w|^2)$ and $|D(z,r)|^{\frac{1}{2}}$ when $w \in D(z,r)$, there exists a constant

$K > 0$ (independent of f) such that

$$\int_D \left(\frac{1}{|D(z,r)|} \int_{D(z,r)} (1-|z|^2)^{\alpha p}(1-|w|^2)^{\beta p} |f^{(n)}(w) - f^{(n)}(z)|^p dA(w) \right) d\lambda(z)$$
$$\leq KC \int_D \int_{D(z,r)} (1-|z|^2)^{np}(1-|w|^2)^p |f^{(n+1)}(w)|^p d\lambda(w) d\lambda(z)$$
$$= KC \int_D \int_D \chi_{D(w,r)}(z)(1-|z|^2)^{np}(1-|w|^2)^p |f^{(n+1)}(w)|^p d\lambda(z) d\lambda(w)$$
$$\leq C_1 KC \int_D (1-|w|^2)^{(n+1)p} |f^{(n+1)}(w)|^p d\lambda(w).$$

The first inequality follows from the proof that (1) implies (3) in Theorem 2.3. And the second equation follows from $\chi_{D(w,r)}(z) = \chi_{D(z,r)}(w)$ and Fubini's theorem. Thus we have proved that (1) implies (3). That (3) implies (4) follows from Hölder's inequality easily.

To prove that (4) implies (1), we assume that

$$\int_D \left(\frac{1}{|D(z,r)|} \int_{D(z,r)} (1-|z|^2)^{\alpha}(1-|w|^2)^{\beta} |f^{(n)}(z) - f^{(n)}(w)| dA(w) \right)^p d\lambda(z) < +\infty.$$

By an elementary calculation, we have the following

$$s^4 \left| g^{(1)}(0) \right| \leq 2 \int_{sD} \left| g(w) - g(0) \right| dA(w)$$

for all analytic function g on D, where $sD = \{z \in D; |z| < s\}$. Replacing g by $f^{(n)} \circ \varphi_z$,

$$s^4(1-|z|^2)|f^{(n+1)}(z)| \leq 2 \int_{D(0,r)} |f^{(n)} \circ \varphi_z(w) - f^{(n)}(z)| dA(w).$$

By making a change of variable formura, we have

$$\int_D (1-|z|^2)^{(n+1)p} |f^{(n+1)}(z)|^p d\lambda(z)$$
$$\leq C \int_D \left(\int_{D(0,r)} (1-|z|^2)^n |f^{(n)} \circ \varphi_z(w) - f^{(n)}(z)| dA(w) \right)^p d\lambda(z)$$

$$
\begin{aligned}
&= C\int_D\Big(\int_{D(z,r)}(1-|z|^2)^n|f^{(n)}(w)-f^{(n)}(z)|\frac{(1-|z|^2)^2}{|1-z\overline{w}|^4}dA(w)\Big)^p d\lambda(z)\\
&\le CK\int_D\Big(\frac{1}{|D(z,r)|}\int_{D(z,r)}(1-|z|^2)^\alpha(1-|w|^2)^\beta\\
&\qquad\qquad |f^{(n)}(w)-f^{(n)}(z)|dA(w)\Big)^p d\lambda(z)\\
&< +\infty.
\end{aligned}
$$

By Theorem A, we have $f \in B_p$. Thus we have proved that (4) implies (1). That (2) implies (4) is trivial. That (4) implies (2) follows from (2.3.3). That (2) implies (5) is clear by using (2.3.4). That (5) implies (6) is trivial. And by using Hölder's inequality, that (6) implies (7) is proved easily.

It remains to prove that (7) implies (1). In fact, applying $g = f^{(n)} \circ \varphi_z - \widehat{f^{(n)}}_r(z)$ to (2.3.5) and multiplying the both sides by $(1-|z|^2)^n$ and making a change of variable formula, we get

$$
\begin{aligned}
&(1-|z|^2)^{(n+1)}|f^{(n+1)}(z)|\\
&\le K_1\Big(\frac{C}{|D(z,r)|}\int_{D(z,r)}(1-|z|^2)^\alpha(1-|u|^2)^\beta|f^{(n)}(u)-\widehat{f^{(n)}}_r(z)|dA(u)\Big).
\end{aligned}
$$

Thus we have proved that (7) implies (1). This completes the proof of Theorem 3.3. □

Remark 3.4 Carefully examining the proof of the above theorem, we see that the following are also the equivalent conditions: For a monotonically increasing convex function φ on R^1 such that $\varphi(0) = 0$ and for some constants $C, K > 0$,

$$
\begin{aligned}
(8)\quad &\int_D\Big(\varphi^{-1}\Big(\frac{K}{|D(z,r)|}\int_{D(z,r)}\varphi\Big(C(1-|z|^2)^\alpha(1-|w|^2)^\beta\\
&\qquad |f^{(n)}(z)-f^{(n)}(w)|\Big)dA(w)\Big)\Big)^p d\lambda(z) < +\infty,\\
(9)\quad &\int_D\Big(\varphi^{-1}\Big(\frac{K}{|D(z,r)|}\int_{D(z,r)}\varphi\Big(C(1-|z|^2)^\alpha(1-|w|^2)^\beta\\
&\qquad |\widehat{f^{(n)}}_r(z)-f^{(n)}(w)|\Big)dA(w)\Big)\Big)^p d\lambda(z) < +\infty.
\end{aligned}
$$

Theorem 3.5 *Fix $n \ge 1$; integer such that $n = \alpha + \beta$, $\alpha, \beta \in \Re$. Then* *for an analytic function f on D and for $r \in (0,+\infty)$, for $p>1$ in the case of $n=1$, and for $p \geq 1$ in the case of $n \geq 2$, the following statements are equivalent*:

(1) $f \in B_p$;

(2) $$\int_D \left(\sup_{w \in D(z,r)} (1-|z|^2)^{\alpha p}(1-|w|^2)^{\beta p} \left| \frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w} \right| \right)^p d\lambda(z) < +\infty;$$

(3) $$\int_D \left(\frac{1}{|D(z,r)|} \int_{D(z,r)} (1-|z|^2)^{\alpha p}(1-|w|^2)^{\beta p} \left| \frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w} \right|^p dA(w) \right) d\lambda(z) < +\infty;$$

(4) $$\int_D \left(\frac{1}{|D(z,r)|} \int_{D(z,r)} (1-|z|^2)^{\alpha}(1-|w|^2)^{\beta} \left| \frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w} \right| dA(w) \right)^p d\lambda(z) < +\infty.$$

Proof. Firstly, we show that (1) implies (3). This can be proved by using the induction as well as the proof of Theorem 2.7, but we prove it in another way. Suppose that $f \in B_p$. For any $p \geq 1$, the following inequality follows from lemma 3.2:

$$\int_{sD} \left| \frac{g(u)-g(0)}{u} \right|^p dA(u) \leq C \int_{sD} \left| g^{(1)}(u) \right|^p (1-|u|^2)^p dA(u)$$

for all analytic function g on D, where $sD = \{w \in D; |w| < s\} = D(0,r)$, $s = \tanh r$. Applying g to $f^{(n-1)} \circ \varphi_z$, we have

$$\int_{D(0,r)} \left| \frac{(f^{(n-1)} \circ \varphi_z)(w) - (f^{(n-1)} \circ \varphi_z)(0)}{w} \right|^p dA(w) \leq C \int_{D(0,r)} |(f^{(n-1)} \circ \varphi_z)^{(1)}(w)|^p (1-|w|^2)^p dA(w). \quad \cdots (3.5.1)$$

Then by using (3.5.1) and by making a change of variable formura and by using $\int_{D(w,r)} d\lambda(z) \leq C < +\infty$ and the fact that $(1-|z|^2)$ is comparable to $|1-\overline{w}z|$, $(1-|w|^2)$ and $|D(z,r)|^{\frac{1}{2}}$ when $w \in D(z,r)$, there exists a constant

$C, C_1, C_2, C_3, K, K_1, K_2 > 0$ (independent of f) such that

$$\int_D \left(\frac{1}{|D(z,r)|} \int_{D(z,r)} (1-|z|^2)^{\alpha p} (1-|w|^2)^{\beta p} \left| \frac{f^{(n-1)}(z) - f^{(n-1)}(w)}{z-w} \right|^p dA(w) \right) d\lambda(z)$$

$$\leq C \int_D \left(\frac{1}{|D(z,r)|} \int_{D(z,r)} (1-|z|^2)^{np} \left| \frac{f^{(n-1)}(z) - f^{(n-1)}(w)}{z-w} \right|^p dA(w) \right) d\lambda(z)$$

$$\leq CC_1 \int_D \left(\int_{D(z,r)} (1-|z|^2)^{np} \left| \frac{f^{(n-1)}(w) - f^{(n-1)}(z)}{\varphi_z(w)} \right|^p |1-\overline{z}w|^p \frac{(1-|z|^2)^2}{|1-\overline{z}w|^4} dA(w) \right) d\lambda(z)$$

$$= CC_1 \int_D \left(\int_{D(0,r)} (1-|z|^2)^{np} \left| \frac{f^{(n-1)} \circ \varphi_z(w) - f^{(n-1)} \circ \varphi_z(0)}{w} \right|^p |1-\overline{z}\varphi_z(w)|^p dA(w) \right) d\lambda(z)$$

$$\leq K \int_D \left(\int_{D(0,r)} (1-|z|^2)^{np} \frac{(1-|z|^2)^p}{|1-\overline{z}w|^p} |(f^{(n-1)} \circ \varphi_z)^{(1)}(w)|^p (1-|w|^2)^p dA(w) \right) d\lambda(z)$$

$$= K \int_D \left(\int_{D(0,r)} (1-|z|^2)^{np} \frac{(1-|z|^2)^{2p}}{|1-\overline{z}w|^{3p}} |f^{(n)}(\varphi_z(w))|^p (1-|w|^2)^p dA(w) \right) d\lambda(z)$$

$$\leq K_1 \int_D \left(\int_{D(0,r)} (1-|z|^2)^{np} |f^{(n)}(\varphi_z(w))|^p dA(w) \right) d\lambda(z)$$

$$\leq K_2 \int_D \left(\int_{D(z,r)} (1-|z|^2)^{np} |f^{(n)}(w)|^p \frac{(1-|z|^2)^2}{|1-\overline{z}w|^4} dA(w) \right) d\lambda(z)$$

$$\leq C_2 \int_D \left(\int_{D(z,r)} (1-|w|^2)^{np} |f^{(n)}(w)|^p d\lambda(w) \right) d\lambda(z)$$

$$= C_2 \int_D \left(\int_{D(w,r)} d\lambda(z) \right) (1-|w|^2)^{np} |f^{(n)}(w)|^p d\lambda(w)$$
$$\leq C_3 \int_D (1-|w|^2)^{np} |f^{(n)}(w)|^p d\lambda(w).$$

The last equation follows from $\chi_{D(w,r)}(z) = \chi_{D(z,r)}(w)$ and Fubini's theorem. That (1) implies (3) follows from Theorem A. That (3) implies (4) follows from Hölder's inequality easily.

To prove that (4) implies (1), we assume that

$$\int_D \left(\frac{1}{|D(z,r)|} \int_{D(z,r)} (1-|z|^2)^\alpha (1-|w|^2)^\beta \left| \frac{f^{(n-1)}(z) - f^{(n-1)}(w)}{z-w} \right| dA(w) \right)^p d\lambda(z) < +\infty.$$

By (2.7.1), we have

$$\left| f^{(n)}(z) \right| \leq \frac{1}{|D(z,r)|} \int_{D(z,r)} \left| \frac{f^{(n-1)}(z) - f^{(n-1)}(w)}{z-w} \right| dA(w).$$

Hence we have

$$\int_D (1-|z|^2)^{np} \left| f^{(n)}(z) \right|^p d\lambda(z)$$
$$\leq C \int_D \left((1-|z|^2)^n \frac{1}{|D(z,r)|} \int_{D(z,r)} \left| \frac{f^{(n-1)}(z) - f^{(n-1)}(w)}{z-w} \right| dA(w) \right)^p d\lambda(z)$$
$$\leq C \int_D \left(\frac{1}{|D(z,r)|} \int_{D(z,r)} (1-|z|^2)^\alpha (1-|w|^2)^\beta \left| \frac{f^{(n-1)}(z) - f^{(n-1)}(w)}{z-w} \right| dA(w) \right)^p d\lambda(z) < +\infty.$$

By Theorem A, we have $f \in B_p$. Thus we have proved that (4) implies (1). That (2) implies (4) is trivial.

It remains to prove that (4) implies (2). In fact, we have for all analytic function g on D

$$|g(w)| \leq \frac{C}{|D(w,r)|} \int_{D(w,r)} |g(u)| dA(u).$$

Applying $g(u) = \frac{f^{(n-1)}(u)-f^{(n-1)}(z)}{u-z}$, then

$$\begin{aligned}
&\left|\frac{f^{(n-1)}(w) - f^{(n-1)}(z)}{w - z}\right| \\
&\qquad \le \frac{C}{|D(w,r)|}\int_{D(w,r)}\left|\frac{f^{(n-1)}(u) - f^{(n-1)}(z)}{u - z}\right| dA(u)
\end{aligned}$$

for all analytic function f on D. Since $D(w,r) \subset D(z,2r)$ for $w \in D(z,r)$ and there is a constant $K > 0$ such that

$$\frac{1}{|D(w,r)|} \le \frac{K}{|D(z,2r)|},$$

$$\begin{aligned}
&\sup_{w\in D(z,r)} (1 - |z|^2)^n\left|\frac{f^{(n-1)}(w) - f^{(n-1)}(z)}{w - z}\right| \\
&\qquad \le \frac{CK}{|D(z,2r)|}\int_{D(z,2r)} (1 - |z|^2)^n\left|\frac{f^{(n-1)}(u) - f^{(n-1)}(z)}{u - z}\right| dA(u).
\end{aligned}$$

Since $(1-|z|^2)$ is comparable to $(1-|w|^2)$ for $w \in D(z,r)$ and that $(1-|z|^2)$ is comparable to $(1 - |u|^2)$ for $u \in D(z,2r)$,

$$\begin{aligned}
&\int_D\left(\sup_{w\in D(z,r)} (1 - |z|^2)^\alpha(1 - |w|^2)^\beta\left|\frac{f^{(n-1)}(w) - f^{(n-1)}(z)}{w - z}\right|\right)^p d\lambda(z) \\
&\quad \le C\int_D\left(\frac{1}{|D(z,2r)|}\int_{D(z,2r)} (1 - |z|^2)^\alpha(1 - |u|^2)^\beta\right. \\
&\qquad\qquad \left.\left|\frac{f^{(n-1)}(u) - f^{(n-1)}(z)}{u - z}\right| dA(u)\right)^p d\lambda(z) \\
&\quad < +\infty.
\end{aligned}$$

Thus we have proved that (4) implies (2). This completes the proof of Theorem 3.5. □

Remark 3.6 Carefully examining the proof of the above theorem, we see that the following is also the equivalent condition: For a monotonically increasing convex function φ on R^1 such that $\varphi(0) = 0$ and for some constants $C, K > 0$,

$$(5)\quad \int_D \left(\varphi^{-1}\left(\frac{K}{|D(z,r)|}\int_{D(z,r)} \varphi\left(C(1-|z|^2)^{\alpha}(1-|w|^2)^{\beta} \left|\frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w}\right|\right) dA(w)\right)\right)^p d\lambda(z) < +\infty.$$

4. The spaces $B^{(n)}$, $B_0^{(n)}$ and $B_p^{(n)}$

In this section we study the spaces $B^{(n)}$, $B_0^{(n)}$ and $B_p^{(n)}$.

Example 4.1 The function $f(z) = \log(1-z)$ is in B but it is not in $B^{(n)}$ for $n \geq 3$. In fact for $n \geq 3$,

$$\begin{aligned}(1-|z|^2)^{\frac{n}{2}}&(1-|w|^2)^{\frac{n}{2}}\left|\frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w}\right| \\ &= (1-|z|^2)^{\frac{n}{2}}(1-|w|^2)^{\frac{n}{2}}(n-1)!\frac{\left|\sum_{k=0}^{n-2}\binom{n-2}{k}(1-w)^{n-2-k}(1-z)^k\right|}{|1-z|^{n-1}|1-w|^{n-1}}.\end{aligned}$$

Fix w such that $|w| = s \in [0,1)$ and let $|z| = r \in [0,1)$. Then we have

$$\begin{aligned}\sup&\left\{(1-|z|^2)^{\frac{n}{2}}(1-|w|^2)^{\frac{n}{2}}\left|\frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w}\right|; z,\ w \in D,\ z \neq w\right\} \\ &\geq (1-r^2)^{\frac{n}{2}}(1-s^2)^{\frac{n}{2}}(n-1)!\frac{\sum_{k=0}^{n-2}\binom{n-2}{k}(1-s)^{n-2-k}(1-r)^k}{(1-r)^{n-1}(1-s)^{n-1}} \\ &\to +\infty \quad (r \to 1^-).\end{aligned}$$

□

Theorem 4.2 *For an analytic function f on D, $f \in B$ if and only if*

$$\sup\left\{(1-|z|^2)(1-|w|^2)\left|\frac{f^{(1)}(z)-f^{(1)}(w)}{z-w}\right|; z, w \in D,\ z \neq w\right\} < +\infty.$$

Proof. Suppose that

$$\sup\left\{(1-|z|^2)(1-|w|^2)\left|\frac{f^{(1)}(z)-f^{(1)}(w)}{z-w}\right|; z, w \in D,\ z \neq w\right\} < +\infty.$$

Then for any $z \in D$, we have

$$\begin{aligned}
&(1-|z|^2)^2 \left| f^{(2)}(z) \right| \\
&= \lim_{w \to z} (1-|z|^2)(1-|w|^2) \left| \frac{f^{(1)}(z) - f^{(1)}(w)}{z-w} \right| \\
&\le \sup \left\{ (1-|z|^2)(1-|w|^2) \left| \frac{f^{(1)}(z) - f^{(1)}(w)}{z-w} \right| ; z, w \in D,\ z \neq w \right\} \\
&< +\infty.
\end{aligned}$$

Thus Theorem A implies that $f \in B$.

To prove the converse, assume that $f \in B$. By (2.1.1),

$$\left| f^{(1)}(u) - f^{(1)}(0) \right| \le C \| f \|_B \frac{|u|}{1-|u|^2}$$

for all $u \in D$. For any $z, w \in D$, replacing f by $f \circ \varphi_w$ and applying $u = \varphi_w(z)$, we see that

$$\begin{aligned}
\left| (f \circ \varphi_w)^{(1)}(\varphi_w(z)) - (f \circ \varphi_w)^{(1)}(0) \right| &\le C \| f \circ \varphi_w \|_B \frac{|\varphi_w(z)|}{1-|\varphi_w(z)|^2} \\
&= C \| f \|_B \frac{|\varphi_w(z)|}{1-|\varphi_w(z)|^2}.
\end{aligned}$$

Hence by an elementary calculation, we get

$$\frac{(1-|z|^2)(1-|w|)}{|w-z|} \left| f^{(1)}(z) - \frac{(1-|w|^2)^2}{(1-\overline{w}z)^2} f^{(1)}(w) \right| \le C \| f \|_B.$$

Thus we have

$$\begin{aligned}
&\frac{(1-|z|^2)(1-|w|^2)}{|w-z|} \left| f^{(1)}(z) - f^{(1)}(w) \right| \\
&= \frac{(1-|z|^2)(1-|w|^2)}{|w-z|} \Bigg| f^{(1)}(z) - \frac{(1-|w|^2)^2}{(1-\overline{w}z)^2} f^{(1)}(w) \\
&\qquad\qquad + \frac{(1-|w|^2)^2}{(1-\overline{w}z)^2} f^{(1)}(w) - f^{(1)}(w) \Bigg| \\
&\le \frac{(1-|z|^2)(1-|w|^2)}{|w-z|} \Bigg\{ \left| f^{(1)}(z) - \frac{(1-|w|^2)^2}{(1-\overline{w}z)^2} f^{(1)}(w) \right| \\
&\qquad\qquad + \left| \frac{(1-|w|^2)^2}{(1-\overline{w}z)^2} f^{(1)}(w) - f^{(1)}(w) \right| \Bigg\}
\end{aligned}$$

$$\leq (1+|w|)C\| f \|_B + (1-|z|^2)\| f \|_B \frac{|1-|w|^2+1-\overline{w}z|}{|1-\overline{w}z|^2}$$
$$\leq 2C\| f \|_B + (1-|z|^2)\| f \|_B \left(\frac{|1-|w|^2|}{|1-\overline{w}z|^2} + \frac{|1-\overline{w}z|}{|1-\overline{w}z|^2} \right)$$
$$\leq (2C+6)\| f \|_B.$$

Hence

$$\sup\left\{ (1-|z|^2)(1-|w|^2) \left| \frac{f^{(1)}(z)-f^{(1)}(w)}{z-w} \right| ; z, w \in D, \ z \neq w \right\} < +\infty.$$

□

Theorem 4.2 implies that $B^{(2)} = B$. The equivalence of (1) and (2) in Thorem 2.7, Example 4.1 and Theorem 3 of [3] imply that

$$B^{(n)} \subset B^{(2)} = B^{(1)} = B, \quad B^{(n)} \neq B \quad (n \geq 3).$$

Corollary 4.3 *For an analytic function f on D, $f \in B_0$ if and only if*

$$\lim_{|z|\to 1^-} \sup\left\{ (1-|z|^2)(1-|w|^2) \left| \frac{f^{(1)}(z)-f^{(1)}(w)}{z-w} \right| ; w \in D, \ z \neq w \right\} = 0.$$

Proof. This is an immediate consequence of Theorem 4.2. □

Corollary 4.3 implies that $B_0^{(2)} = B_0$. The equivalence of (1) and (2) in Corollary 2.9 and Theorem 2 of [6] imply that

$$B_0^{(n)} \subset B_0^{(2)} = B_0^{(1)} = B_0 \quad (n \geq 3).$$

For the space $B_p^{(n)}$, we don't know whether $B_p = B_p^{(2)}$ holds or not. The equivalence of (1) and (3) in Theorem 3.5 implies that $B_p^{(n)} \subset B_p$ $(n \geq 1)$. In the case of $n = 1$, $B_p^{(1)} = B_p$ for any $p > 2$. In the case of $n = 2$, $B_p^{(2)} \subset B_p$ for any $p > 1$. In the case of $n \geq 3$, $B_p^{(n)} \subset B_p$ for any $p \geq 1$.

For $p \geq 1$, it is clear that $B_p \subset B_0 \subset B$ $(p \geq 1)$. And we also see that $B_p^{(n)} \subset B_0^{(n)} \subset B^{(n)}$ $(p \geq 1)$ for $n \geq 1$. As we stated in Example 4.1, for $n \geq 3$, the function $f(z) = \log(1-z)$ $(z \in D)$ gives an example that the implication $B^{(n)} \subset B$ is strict. Moreover it holds that $BMOA \subset B^{(2)} = B^{(1)} = B$, but in the case of $n \geq 3$, $BMOA$ is not contained in $B^{(n)}$ because $f(z) = \log(1-z) \in BMOA$ is not in $B^{(n)}$. Moreover for $n \geq 3$, we also see that the little Bloch space B_0 is not contained in $B^{(n)}$ too. In fact, letting $f(z) = (1-z)^{\frac{1}{4}}$, then

$$\begin{aligned}(1-|z|^2)|f^{(1)}(z)| &= \frac{1}{4}(1-|z|^2)|1-z|^{-\frac{3}{4}} \\ &\leq \frac{1}{4}\frac{1-|z|^2}{(1-|z|)^{\frac{3}{4}}} \to 0 \quad (|z| \to 1^-).\end{aligned}$$

Hence $f \in B_0$, but

$$\begin{aligned}&(1-|z|^2)^{\frac{3}{2}}(1-|w|^2)^{\frac{3}{2}}\left|\frac{f^{(2)}(z)-f^{(2)}(w)}{z-w}\right| \\ &= \frac{1}{16}(1-|z|^2)^{\frac{3}{2}}(1-|w|^2)^{\frac{3}{2}}\frac{1}{|1-z|^{\frac{7}{4}}|1-w|^{\frac{7}{4}}}\frac{|(1-w)^{\frac{7}{4}}-(1-z)^{\frac{7}{4}}|}{|z-w|}.\end{aligned}$$

Fix w such that $|w| = s \in [0,1)$ and let $|z| = r \in [0,1)$. Then we have

$$\begin{aligned}&\sup\left\{(1-|z|^2)^{\frac{3}{2}}(1-|w|^2)^{\frac{3}{2}}\left|\frac{f^{(2)}(z)-f^{(2)}(w)}{z-w}\right|; z,w \in D,\ z \neq w\right\} \\ &\geq \frac{1}{16}(1-r^2)^{\frac{3}{2}}(1-s^2)^{\frac{3}{2}}\frac{1}{(1-s)^{\frac{7}{4}}(1-r)^{\frac{7}{4}}}\frac{|(1-s)^{\frac{7}{4}}-(1-r)^{\frac{7}{4}}|}{|s-r|} \\ &\to +\infty \quad (r \to 1^-).\end{aligned}$$

Hence $f \notin B^{(3)}$. And for $n \geq 3$, this also gives an example that the implication $B_0^{(n)} \subset B_0$ is strict.

5. The α-Bloch space B^α, the little α-Bloch space B_0^α and the α-Besov space B_p^α

In Section 2 and Section 3 we characterized the Bloch space B and the little Bloch space B_0 and the Besov space B_p, but similar characterizations also hold for the α-Bloch space B^α and the little α-Bloch space B_0^α and the α-Besov space B_p^α by using the following Theorem E and Corollary F and Theorem G as well. Since most proofs are also similar to ones of the case of $\alpha = 1$, we'll only present results. For example, we give Theorems 5.1, 5.2, 5.3 and 5.4 which are correspond to Theorems 2.3, 2.7, 3.3 and 3.5, respectively. On the other hand, we could not characterize ones which are correspond to the results of Section 4.

Theorem E ([9, p.1148]) *Let $\alpha > 0$ and $n \geq 2$; integer. For an analytic* *function f on D, $f \in B^{\alpha}$ if and only if*

$$(1-|z|^2)^{\alpha+n-1} f^{(n)}(z) \in L^{\infty}(D, dA).$$

Corollary F ([9, p.1148]) *Let $\alpha > 0$ and $n \geq 2$; integer. For an analytic function f on D, $f \in B_0^{\alpha}$ if and only if*

$$(1-|z|^2)^{\alpha+n-1} f^{(n)}(z) \to 0 \quad (|z| \to 1^-).$$

Theorem G ([9, p.1174]) *Let $\alpha > 0$ and $n \geq 2$; integer. For an analytic function f on D and for $1 \leq p < +\infty$, $p(\alpha+n-1) > 1$, $f \in B_p^{\alpha}$ if and only if t*

$$(1-|z|^2)^{\alpha+n-1} f^{(n)}(z) \in L^{p}(D, d\lambda).$$

Theorem 5.1 *Let $\alpha > 0$ and $p > 0$ and fix $n \geq 0$; integer such that $n = \alpha_1 + \beta_1 - \alpha + 1$, $\alpha_1, \beta_1 \in \Re$. Then for an analytic function f on D and for $r \in (0, +\infty)$, the following statements are equivalent:*

(1) $f \in B^{\alpha}$;

(2) $$\sup_{z\in D}\left(\sup_{w\in D(z,r)} (1-|z|^2)^{\alpha_1}(1-|w|^2)^{\beta_1}|f^{(n)}(z)-f^{(n)}(w)|\right) < +\infty;$$

(3) $$\sup_{z\in D}\left(\frac{1}{|D(z,r)|}\int_{D(z,r)} (1-|z|^2)^{\alpha_1 p}(1-|w|^2)^{\beta_1 p} |f^{(n)}(z)-f^{(n)}(w)|^p dA(w)\right)^{\frac{1}{p}} < +\infty;$$

(4) $$\sup_{z\in D}\left(\frac{1}{|D(z,r)|}\int_{D(z,r)} \log^+\left((1-|z|^2)^{\alpha_1}(1-|w|^2)^{\beta_1} |f^{(n)}(z)-f^{(n)}(w)|\right)dA(w)\right) < +\infty;$$

(5) $$\sup_{z\in D}\left(\sup_{w\in D(z,r)} (1-|z|^2)^{\alpha_1}(1-|w|^2)^{\beta_1}|\widehat{f^{(n)}}_r(z)-f^{(n)}(w)|\right) < +\infty;$$

(6) $$\sup_{z\in D}\left(\frac{1}{|D(z,r)|}\int_{D(z,r)} (1-|z|^2)^{\alpha_1 p}(1-|w|^2)^{\beta_1 p} |\widehat{f^{(n)}}_r(z)-f^{(n)}(w)|^p dA(w)\right)^{\frac{1}{p}} < +\infty;$$

$$(7)\quad \sup_{z\in D}\left(\frac{1}{|D(z,r)|}\int_{D(z,r)}\log^+\Big((1-|z|^2)^{\alpha_1}(1-|w|^2)^{\beta_1}\right.$$
$$\left.|\widehat{f^{(n)}}_r(z)-f^{(n)}(w)|\Big)dA(w)\right)<+\infty.$$

Theorem 5.2 *Let $\alpha>0$ and $p>0$ and fix $n\geq 1$; integer such that $n=\alpha_1+\beta_1-\alpha+1$, $\alpha_1,\beta_1\in\Re$. Then for an analytic function f on D and for $r\in(0,+\infty)$, the following statements are equivalent:*

(1) $f\in B^\alpha$;

$$(2)\quad \sup\left\{\sup_{w\in D(z,r)}(1-|z|^2)^{\alpha_1}(1-|w|^2)^{\beta_1}\left|\frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w}\right|;\ z\in D,\ z\neq w\right\}<+\infty;$$

$$(3)\quad \sup_{z\in D}\left(\frac{1}{|D(z,r)|}\int_{D(z,r)}(1-|z|^2)^{\alpha_1 p}(1-|w|^2)^{\beta_1 p}\left|\frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w}\right|^p dA(w)\right)^{\frac{1}{p}}<+\infty;$$

$$(4)\quad \sup_{z\in D}\left(\frac{1}{|D(z,r)|}\int_{D(z,r)}\log^+\left((1-|z|^2)^{\alpha_1}(1-|w|^2)^{\beta_1}\left|\frac{f^{(n-1)}(z)-f^{(n-1)}(w)}{z-w}\right|\right)dA(w)\right)<+\infty.$$

Theorem 5.3 *Let $\alpha>0$ and fix $n\geq 0$; integer such that $n=\alpha_1+\beta_1-\alpha+1$, $\alpha_1,\beta_1\in\Re$. Then for an analytic function f on D and for $r\in(0,+\infty)$, for $1\leq p<+\infty$, $p(\alpha+n)>1$, the following statements are equivalent:*

(1) $f\in B_p^\alpha$;

$$(2)\quad \int_D\left(\sup_{w\in D(z,r)}(1-|z|^2)^{\alpha_1}(1-|w|^2)^{\beta_1}|f^{(n)}(z)-f^{(n)}(w)|\right)^p d\lambda(z)<+\infty;$$

$$(3)\quad \int_D\left(\frac{1}{|D(z,r)|}\int_{D(z,r)}(1-|z|^2)^{\alpha_1 p}(1-|w|^2)^{\beta_1 p}|f^{(n)}(z)-f^{(n)}(w)|^p dA(w)\right)d\lambda(z)<+\infty;$$

(4) $$\int_D \left(\frac{1}{|D(z,r)|} \int_{D(z,r)} (1-|z|^2)^{\alpha_1} (1-|w|^2)^{\beta_1} |f^{(n)}(z) - f^{(n)}(w)| dA(w) \right)^p d\lambda(z) < +\infty;$$

(5) $$\int_D \left(\sup_{w \in D(z,r)} (1-|z|^2)^{\alpha_1} (1-|w|^2)^{\beta_1} |\widehat{f^{(n)}}_r(z) - f^{(n)}(w)| \right)^p d\lambda(z) < +\infty;$$

(6) $$\int_D \left(\frac{1}{|D(z,r)|} \int_{D(z,r)} (1-|z|^2)^{\alpha_1 p} (1-|w|^2)^{\beta_1 p} |\widehat{f^{(n)}}_r(z) - f^{(n)}(w)|^p dA(w) \right) d\lambda(z) < +\infty;$$

(7) $$\int_D \left(\frac{1}{|D(z,r)|} \int_{D(z,r)} (1-|z|^2)^{\alpha_1} (1-|w|^2)^{\beta_1} |\widehat{f^{(n)}}_r(z) - f^{(n)}(w)| dA(w) \right)^p d\lambda(z) < +\infty.$$

Theorem 5.4 *Let $\alpha > 0$ and fix $n \geq 1$; integer such that $n = \alpha_1 + \beta_1 - \alpha + 1$, $\alpha_1, \beta_1 \in \Re$. Then for an analytic function f on D and for $r \in (0, +\infty)$, for $1 \leq p < +\infty$, $p(\alpha + n - 1) > 1$, the following statements are equivalen:*

(1) $f \in B_p^\alpha$;

(2) $$\int_D \left(\sup_{w \in D(z,r)} (1-|z|^2)^{\alpha_1 p} (1-|w|^2)^{\beta_1 p} \left| \frac{f^{(n-1)}(z) - f^{(n-1)}(w)}{z-w} \right| \right)^p d\lambda(z) < +\infty;$$

(3) $$\int_D \left(\frac{1}{|D(z,r)|} \int_{D(z,r)} (1-|z|^2)^{\alpha_1 p} (1-|w|^2)^{\beta_1 p} \left| \frac{f^{(n-1)}(z) - f^{(n-1)}(w)}{z-w} \right|^p dA(w) \right) d\lambda(z) < +\infty;$$

(4) $$\int_D \left(\frac{1}{|D(z,r)|} \int_{D(z,r)} (1-|z|^2)^{\alpha_1} (1-|w|^2)^{\beta_1} \left| \frac{f^{(n-1)}(z) - f^{(n-1)}(w)}{z-w} \right| dA(w) \right)^p d\lambda(z) < +\infty.$$

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