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Multiplicity of iterated Jacobian extensions of weighted homogeneous map germs

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Abstract. In this work we study the notion of multiplicity of an analytic map germ $f:(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$ with respect to a Boardman symbol $\mathbf{i}$ and obtain some expressions for this multiplicity when f is a weighted homogeneous map germ. These expressions are consequences of some formulae given in last section for the multiplicity of determinantal rings defined by the maximal or submaximal minors of a matrix with homogeneous entries.

Key words: Thom-Boardman singularities, determinantal rings.

1. Introduction

One of the most known invariants in Singularity Theory is the Milnor number of an analytic map germ $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ with an isolated singularity at the origin. Let $\mathcal{O}_n$ be the ring of analytic map germs from $(\mathbb{C}^n,0)$ to $\mathbb{C}$. Then, the Milnor number is defined as the colength in $\mathcal{O}_n$ of the ideal generated by the partial derivatives of f. Another interesting invariant is the number of cross-caps of a map germ $f:(\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$. It is defined as the colength of the ideal generated by the maximal order minors of its Jacobian matrix. The number of cusps of a map germ $f:(\mathbb{C}^2,0)\to(\mathbb{C}^2,0)$ is also defined as the colength of the ideal generated by the minors of maximal order of the Jacobian matrix of (Jf,f), where Jf is the Jacobian of f. In the work [23], the second author and M.J. Saia defined an invariant that generalizes the above constructions in the general setting of Thom-Boardman singularities. For a Boardman symbol $\mathbf{i}=(i_1,\dots,i_k)$ and an analytic map germ, they define the number $c_{\mathbf{i}}(f)$ as the colength in $\mathcal{O}_n$ of an ideal constructed by an iterative process from the minors of the Jacobian matrix of f. This ideal is called the iterated Jacobian extension of f with respect to $\mathbf{i}$ and is denoted by $J_{\mathbf{i}}(f)$. It is clear that this number is well defined only when the zero set $V(J_{\mathbf{i}}(f))$ is zero dimensional.

If f is a weighted homogeneous map germ, then it is interesting to obtain

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expressions for the invariant $c_{\mathbf{i}}(f)$ that depends only on the weights and degrees of f. There have been some results in this direction, as can be seen in [14], [20] and [22]. Here we give some formulae that are a consequence of a more general result on determinantal rings shown in Section 5. Moreover, we consider an invariant which is finite even in the case that the Jacobian extensions have not finite colength and we also give some expressions for this invariant in the homogeneous case. This invariant is called the *algebraic multiplicity of a map germ with respect to a Boardman symbol* and is denoted by $e_{\mathbf{i}}(f)$. This invariant is equal to the one defined by [23] in the zero dimensional case and has also a geometrical interpretation, as can be seen in [2].

2. Thom-Boardman singularities

The aim of this section is to recall some basic facts about Thom-Boardman singularities we will need. Given an analytic map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, we can make a partition of the source $(\mathbb{C}^n, 0)$ according to the rank of f, which gives the first order singular sets $\Sigma^{i_1}(f)$, with $i_1 = 0, \ldots, \min\{n, p\}$. But we can make a new partition of each one of these sets by looking at the higher order derivatives of f. This lead us to obtain finer invariants. We can do this in a systematic way, so we need some preliminary definitions to make this notion precise.

Let $\mathcal{O}_n$ denote the ring of analytic function germs from $(\mathbb{C}^n, 0)$ to $\mathbb{C}$. For any ideal $I \subseteq \mathcal{O}_n$, the set germ in $(\mathbb{C}^n, 0)$ defined by the zeros of I will be denoted by $V(I)$. Moreover, given a matrix $U = (u_{ij})$ with entries in $\mathcal{O}_n$ and $t \geq 1$, we shall denote by $I_t(U)$ the ideal generated by the t-minors of U. In the case that the matrix U has size $p \times q$ and $t > \min(p, q)$, we will put $I_t(U) = \{0\}$. In particular, if $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is an analytic map germ, $I_t(Df)$ is the ideal generated by the t-minors of its Jacobian matrix $Df = (\partial f_i / \partial x_j)$.

Definition 2.1 Given $n, p \geq 1$, we define the set $\mathcal{B}(n, p)$ of *Boardman symbols* in dimensions n and p as the set of k-tuples $\mathbf{i} = (i_1, \ldots, i_k)$ of integer numbers such that:

1. $k \geq 1$;
2. $n \geq i_1 \geq \cdots \geq i_k \geq 0$;
3. $i_1 \geq n - p$;
4. if $i_1 = n - p$ then $i_1 = \cdots = i_k$.

If $\mathbf{i} = (i_1, \ldots, i_k)$, we will say that $\mathbf{i}$ has *length* k and we will denote this number by $|\mathbf{i}|$.

For each Boardman symbol $\mathbf{i} = (i_1, \ldots, i_k) \in \mathcal{B}(n,p)$ and an analytic map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ we can associate an ideal of $\mathcal{O}_n$ that we call *iterated Jacobian extension of f with respect to* $\mathbf{i}$. This ideal is constructed by induction on the length of $\mathbf{i}$ as follows. If $k = 1$, then $J_{i_1}(f) = I_{n-i_1+1}(Df)$. For $k > 1$, suppose that $J_{i_1,\ldots,i_{k-1}}(f) = \langle g_1, \ldots, g_r \rangle$, then

$$J_{i_1,\ldots,i_k}(f) = J_{i_1,\ldots,i_{k-1}}(f) + I_{n-i_k+1}\left(D(f,g)\right),$$

where $(f,g) = (f_1, \ldots, f_p, g_1, \ldots, g_r)$.

Suppose that $F : (\mathbb{C}^r \times \mathbb{C}^n, 0) \to (\mathbb{C}^r \times \mathbb{C}^p, 0)$ is a map germ written in the form $F(u,x) = (u, f_u(x))$, where $u \in \mathbb{C}^r$ and $x \in \mathbb{C}^n$. We say that F is an *unfolding* of f if $f_0 = f$. Observe that for such a map we have that $J_{\mathbf{i}}(F) = J_{\mathbf{i}}(f_u, x)$, for any Boardman symbol $\mathbf{i} \in \mathcal{B}(n,p)$. That is, to compute any Jacobian extension of F we only need to consider the partial derivatives of f_u with respect to x.

Now, given an analytic map germ we can define an associated Boardman symbol using the notion of Jacobian extensions.

Definition 2.2 Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be an analytic map germ and let $\mathbf{i} = (i_1, \ldots, i_k)$ be a Boardman symbol of $\mathcal{B}(n,p)$. We say that f is a germ of type $\Sigma^{\mathbf{i}}$ when

1. the rank of f is $n - i_1$;
2. for all $s = 2, \ldots, k$, the rank of (f,g) is $n - i_s$, being $g = (g_1, \ldots, g_r)$ and $g_1, \ldots, g_r$ generators of $J_{i_1,\ldots,i_{s-1}}(f)$.

Let us denote by $J^k(n,p)$ the *k-jet space*, that is, the space of polynomial map germs from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^p, 0)$ of degree $\leq k$. Moreover, the k-jet fiber bundle is defined as $J^k(\mathbb{C}^n, \mathbb{C}^p) = \mathbb{C}^n \times \mathbb{C}^p \times J^k(n,p)$.

Given a map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, the *$k$-jet extension* of f is the map germ $j^k f : (\mathbb{C}^n, 0) \to J^k(\mathbb{C}^n, \mathbb{C}^p)$ given by $j^k f(x) = (x, f(x), \sigma)$ where σ is the Taylor expansion of order k at 0 of the map germ $g(t) = f(t+x) - f(x)$.

Definition 2.3 For each Boardman symbol $\mathbf{i} \in \mathcal{B}(n,p)$ of length $\leq k$, we can define the following subset of $J^k(\mathbb{C}^n, \mathbb{C}^p)$:

$$\Sigma^{\mathbf{i}} = \{(x, y, \sigma) \in J^k(\mathbb{C}^n, \mathbb{C}^p) : \ \sigma \text{ has type } \Sigma^{\mathbf{i}}\}.$$

It was proven by Boardman [1] that the sets $\Sigma^{\mathbf{i}}$ are in fact submanifolds of $J^k(\mathbb{C}^n, \mathbb{C}^p)$ of codimension

$$\begin{aligned}\nu(\mathbf{i}) = \;& (p-n+i_1)\mu(i_1,\ldots,i_k) - (i_1 - i_2)\mu(i_2,\ldots,i_k) - \cdots \\ & - (i_{k-1} - i_k)\mu(i_k),\end{aligned}$$

where $\mu(\mathbf{i})$ is the number of symbols $\mathbf{j} = (j_1, \ldots, j_k)$ such that $j_s \le i_s$, $\forall s$, $j_1 \ge \cdots \ge j_k \ge 0$ and $j_1 > 0$.

A different proof of the Boardman result was given by Morin in [21]. For each Boardman symbol $\mathbf{i} \in \mathcal{B}(n,p)$, he constructed an ideal $\Delta^{\mathbf{i}}$ in the polynomial ring associated to $J^k(\mathbb{C}^n, \mathbb{C}^p)$ with the following properties:

1. For each map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, we have that $(j^k f)^*(\Delta^{\mathbf{i}}) = J_{\mathbf{i}}(f)$, where $(j^k f)^*$ is the homomorphism of local rings induced by $j^k f$.
2. $\Sigma^{\mathbf{i}} = V(\Delta^{\mathbf{i}}) \setminus V(\Delta^{\mathbf{i}'})$, where $\mathbf{i}'$ denotes the successor of $\mathbf{i}$ in the lexicographical order.
3. $V(\Delta^{\mathbf{i}})$ is regular of codimension $\nu(\mathbf{i})$ along $\Sigma^{\mathbf{i}}$.

For the sake of simplicity, we do not include here an explicit expression for the Morin ideal. We refer to the cites [2], [11] and [21] to see a construction of this ideal.

Definition 2.4 For each map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ and Boardman symbol $\mathbf{i} \in \mathcal{B}(n,p)$, we define the *algebraic multiplicity of f with respect to* $\mathbf{i}$, denoted by $e_{\mathbf{i}}(f)$, as the the multiplicity, in the Hilbert-Samuel sense, of the local ring $\mathcal{O}_n / J_{\mathbf{i}}(f)$. It can be expressed using the limit formula for the multiplicity (see [3]) as

$$e_{\mathbf{i}}(f) = \lim_{k\to\infty} \frac{d!}{k^d} \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\mathbf{m}_n^k + J_{\mathbf{i}}(f)},$$

where $\mathbf{m}_n$ is the maximal ideal of $\mathcal{O}_n$ and d is the dimension of $\mathcal{O}_n / J_{\mathbf{i}}(f)$.

When the ring $\mathcal{O}_n / J_{\mathbf{i}}(f)$ has dimension zero, this invariant is given by

$$e_{\mathbf{i}}(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{J_{\mathbf{i}}(f)}.$$

In this case, this invariant coincides with the invariant $c_{\mathbf{i}}(f)$ defined by the second author and Saia in [23], which in turn generalizes the invariants mentioned in the Introduction. One of the advantages of this number is that it is finite even in the case in which the singular set $V(J_{\mathbf{i}}(f))$ has positive dimension. In [2] there is a geometric interpretation of this number in terms of the set of singular points in generic deformations of a given map germ, even in the case in which these points are not isolated.

Now, we will expose some preliminary results about multiplicity theory to clarify the number we have just defined. If R is a d dimensional Noetherian local ring and I is an ideal of definition of I, we denote the multiplicity of I, in the Hilbert-Samuel sense, as $e(I, R)$. The multiplicity of the maximal ideal of R is also denoted by $e(R)$.

It is well known that if I is a parameter ideal and R is Cohen-Macaulay, this multiplicity can be expressed as the length of R/I (see [3] for basic notions about multiplicity theory and its relation with Cohen-Macaulay rings). Next proposition shows that the multiplicity of an arbitrary definition ideal can be always computed as the multiplicity of a parameter ideal (see also [3] for details).

Proposition 2.5 *Suppose that (R, m) is a local ring of dimension d such that the ground field $k = R/m$ is infinite. Let I be a definition ideal of R. Then, there exists elements $a_1, \ldots, a_d$ such that*

1. $e(\langle a_1, \ldots, a_d\rangle, R) = e(I, R)$;
2. *a_i can be found among k-linear combinations of any given set of generators of I.*

Now, consider a map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ and a Boardman symbol $\mathbf{i} \in \mathcal{B}(n,p)$ such that the ring $\mathcal{O}_n/J_{\mathbf{i}}(f)$ has dimension d. If we apply the previous proposition to the maximal ideal of $\mathcal{O}_n/J_{\mathbf{i}}(f)$, we have that there exists a family of d linear forms $g_1, \ldots, g_d$ such that $e_{\mathbf{i}}(f)$ is expressed as

$$e_{\mathbf{i}}(f) = e\left(\frac{\mathcal{O}_n}{J_{\mathbf{i}}(f)}\right) = e\left(\langle \overline{g}_1, \ldots, \overline{g}_d\rangle, \frac{\mathcal{O}_n}{J_{\mathbf{i}}(f)}\right),$$

where $\overline{g}_i$ denotes the image of g_i in $\mathcal{O}_n/J_{\mathbf{i}}(f)$. Moreover, it is well known that this expression holds for almost all choice of d linear forms. We will say that $L = V(g_1, \ldots, g_d)$ is a *generic plane* for $\mathcal{O}_n/J_{\mathbf{i}}(f)$, when $g_1, \ldots, g_d$ is such a family of linear forms. The ideal in $\mathcal{O}_n/J_{\mathbf{i}}(f)$ generated by these forms is also called a *reduction* of the maximal ideal.

If in addition, we suppose that the ring $\mathcal{O}_n/J_{\mathbf{i}}(f)$ is Cohen-Macaulay, we have that

$$e_{\mathbf{i}}(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{J_{\mathbf{i}}(f) + \langle g_1, \dots, g_d \rangle}.$$

We can interpret geometrically $e_{\mathbf{i}}(f)$ as $i(L \cdot V(J_{\mathbf{i}}(f))_0$, the local intersection number of $V(J_{\mathbf{i}}(f))$ with a generic plane L at 0 (see [13]). But there are some other intersection numbers that can give interesting information. For instance, if $\sigma = j^k f(0)$, we can also consider $i(j^k f(L) \cdot V(\Delta^{\mathbf{i}}))_\sigma$, the intersection number of $j^k f(L)$ and $V(\Delta^{\mathbf{i}})$ in $J^k(\mathbb{C}^n, \mathbb{C}^p)$ at σ. It has been shown in [2] that this number does not depend on the generic plane L and is called the *deformation multiplicity* of f with respect to $\mathbf{i}$. Finally, if $F(u, x) = (u, f_u(x))$ is an r-parameter unfolding of f, we can look at $i(\{0\} \times L \cdot V(J_{\mathbf{i}}(F)))_0$. The following lemma shows the relation between these three intersection numbers.

Lemma 2.6 *Let $\mathbf{i} \in \mathcal{B}(n,p)$ be a Boardman symbol and let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be an analytic map germ such that* $\operatorname{codim} V(J_{\mathbf{i}}(f)) = \operatorname{codim}_\sigma V(\Delta^{\mathbf{i}}) = n-d$*, where $\sigma = j^k f(0)$. For any r-parameter unfolding $F(u,x) = (u, f_u(x))$ and for any generic plane L we have*

$$i(j^k f(L) \cdot V(\Delta^{\mathbf{i}}))_\sigma \le i(\{0\} \times L \cdot V(J_{\mathbf{i}}(F)))_0 \le i(L \cdot V(J_{\mathbf{i}}(f)))_0.$$

Moreover, $V(\Delta^{\mathbf{i}})$ is Cohen-Macaulay at σ if and only if both inequalities are actually equalities and $V(J_{\mathbf{i}}(f))$ is Cohen-Macaulay.

Proof. We will denote the coordinates in $J^k(\mathbb{C}^n, \mathbb{C}^p)$ by x_i, z^j_α with $i = 1, \dots, n$, $j = 1, \dots, p$ and α a multiindex such that $0 \le |\alpha| \le k$ and so that $z^j_\alpha \circ j^k f = \partial^{|\alpha|} f_j / \partial x^\alpha$. Then, it follows that $i(j^k f(L) \cdot V(\Delta^{\mathbf{i}}))_\sigma = e(I_1 A + I_2 A, A)$, where

$$A = \frac{\mathcal{O}_{J^k(\mathbb{C}^n,\mathbb{C}^p),\sigma}}{\Delta^{\mathbf{i}} \mathcal{O}_{J^k(\mathbb{C}^n,\mathbb{C}^p),\sigma}}, \quad I_1 = \langle g_1, \dots, g_d \rangle, \quad I_2 = \left\langle z^j_\alpha - \frac{\partial^{|\alpha|} f_j}{\partial x^\alpha}(x) \right\rangle,$$

and $g_1, \dots, g_d$ are the defining linear forms of L. Since $I_1 A + I_2 A$ is a parameter ideal of A we have that $e(I_1 A + I_2 A, A) \le e(I_1(A/I_2 A), A/I_2 A)$. Moreover, A is Cohen-Macaulay if and only if the equality holds and $A/I_2 A$ is Cohen-Macaulay (see Corollary 4.7.11 of [3]). But note that $A/I_2 A = \mathcal{O}_n / J_{\mathbf{i}}(f)$ and the intersection number $i(L \cdot V(J_{\mathbf{i}}(f)))_0$ is equal to $e(I_1(A/I_2 A), A/I_2 A)$.

Let us consider now $\mathcal{O}_{J^k(\mathbb{C}^n,\mathbb{C}^p),\sigma}$ as a subring of $\mathcal{O}_{\mathbb{C}^r \times J^k(\mathbb{C}^n,\mathbb{C}^p),(0,\sigma)}$.

Then, $u_1, \ldots, u_r$ is a regular sequence in

$$B = \frac{\mathcal{O}_{\mathbb{C}\times J^k(\mathbb{C}^n,\mathbb{C}^p),(0,\sigma)}}{\Delta^{\mathbf{i}}\mathcal{O}_{\mathbb{C}\times J^k(\mathbb{C}^n,\mathbb{C}^p),(0,\sigma)}},$$

and thus $e(I_1A + I_2A, A) = e(\mathbf{u}B + I_1B + I_2B, B)$, where $\mathbf{u}B$ denotes $\langle u_1, \ldots, u_r\rangle B$. Moreover, $\mathbf{u}B + I_1B + I_2B = \mathbf{u}B + I_1B + I_2'B$, being

$$I_2' = \left\langle z_\alpha^j - \frac{\partial^{|\alpha|} f_{u,j}}{\partial x^\alpha}(x)\right\rangle.$$

This implies that $e(I_1A + I_2A, A) \leq e(\mathbf{u}C + I_1C, C)$, where $C = B/I_2'B = \mathcal{O}_{n+1}/J_{\mathbf{i}}(F)$. Observe that this last multiplicity is equal to $i(\{0\} \times L \cdot V(J_{\mathbf{i}}(F)))_0$. Finally, taking quotient by $\mathbf{u}C$, we get

$$\begin{aligned} e(\mathbf{u}C + I_1C, C) &\leq e(I_1(C/\mathbf{u}C), C/\mathbf{u}C) \\ &= e\Big(\langle \overline{g}_1, \ldots, \overline{g}_d\rangle; \frac{\mathcal{O}_n}{J_{\mathbf{i}}(f)}\Big) = i(L \cdot V(J_{\mathbf{i}}(f)))_0. \end{aligned}$$

□

Remark 2.7 *From the above lemma we can deduce that, under the same hypotheses, $V(\Delta^{\mathbf{i}})$ is Cohen-Macaulay at σ if and only if $\mathcal{O}_{n+r}/J_{\mathbf{i}}(F)$ is Cohen-Macaulay for any r-parametric unfolding F of f.*

3. Invariance of the multiplicity

Given an r-parametric unfolding $F(u,x) = (u, f_u(x))$ of a map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ and a Boardman symbol $\mathbf{i} \in \mathcal{B}(n,p)$, we can consider the problem of determining the invariance of the number $e_{\mathbf{i}}(f_u)$ for all u in a neighbourhood of $0 \in \mathbb{C}^r$. These problem makes sense only if $\mathbb{C}^r \times \{0\} \subseteq V(J_{\mathbf{i}}(F))$. As we shall see, the concept of equimultiplicity along subvarieties will be the key to characterize this invariance. Now, we give some preliminary definitions and results that can be found in the work [19] of J. Lipman.

Definition 3.1 Let $V \subseteq \mathbb{C}^n$ an analytic subset, v a point of V and $R = \mathcal{O}_{V,v}$ the ring of germs of analytic functions near v. If $W \subseteq V$ is an analytic subspace such that the germ of W at v is irreductible and $P \subseteq R$ denotes the prime ideal of germs of analytic functions vanishing on W, we say that V *is equimultiple along* W *at* v if R and R_P have the same multiplicity, where R_P denotes the localization of R at P. This is equivalent to say that $e(\mathcal{O}_{V,v}) =$

$e(\mathcal{O}_{V,y})$, for all y in a neighbourhood of v in W (see [19], Proposition 4.1). Moreover, we also have some other interesting equivalences:

1. Let $C(V, y)$ be the Zariski tangent cone of V at y. Then, V is equimultiple along W at v if and only if there exists a linear projection $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^d, 0)$, where $d = \dim_v V$, such that for each point y in a neighbourhood of v in W,
$$\phi^{-1}(\phi(y)) \cap V = \{y\} \quad \text{and} \quad \phi^{-1}(\phi(y)) \cap C(V, y) = \{y\}.$$
2. Let $\pi : X \to \mathbb{C}^n$ denote the blowup of $\mathbb{C}^n$ at W and let V' be the strict transform of V by π. Then, V is equimultiple along W at v if and only if all the fibers over W near v of the restriction $\pi : V' \to V$ are pure dimensional, of dimension equal to $\operatorname{codim}_V W - 1$.

If we suppose that $\mathbb{C}^r \times \{0\} \subseteq V(J_{\mathbf{i}}(F))$, it makes sense to ask about the equimultipicity of $V(J_{\mathbf{i}}(F))$ along $\mathbb{C}^r \times \{0\}$ at 0. Let $\pi : X \to \mathbb{C}^r \times \mathbb{C}^n$ denote the blowup of $\mathbb{C}^r \times \mathbb{C}^n$ at $\mathbb{C}^r \times \{0\}$ and let V' be the strict transform of $V(J_{\mathbf{i}}(F))$ by π. Then, in view of the second characterization of equimultiplicity given above, $V(J_{\mathbf{i}}(F))$ is equimultiple along $\mathbb{C}^r \times \{0\}$ at 0 if and only if the fibers over $\mathbb{C}^r \times \{0\}$ in a neighbourhood of 0 of the restriction $\pi : V' \to V(J_{\mathbf{i}}(F))$ are pure dimensional of dimension $d - 1$, where $d = \dim \mathcal{O}_n / J_{\mathbf{i}}(f)$. Now, we will see that this is also equivalent to the constancy of $e_{\mathbf{i}}(f_u)$ with respect to u.

Proposition 3.2 *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be an analytic map germ and let $\mathbf{i} \in \mathcal{B}(n,p)$ be a Boardman symbol such that $V(\Delta^{\mathbf{i}})$ is Cohen-Macaulay at $\sigma = j^k f(0)$ and $\operatorname{codim} V(J_{\mathbf{i}}(f)) = \operatorname{codim}_\sigma V(\Delta^{\mathbf{i}})$. Let $F(u,x) = (u, f_u(x))$ be any r-parameter unfolding such that $\mathbb{C}^r \times \{0\} \subseteq V(J_{\mathbf{i}}(F))$. Then, $e_{\mathbf{i}}(f) = e_{\mathbf{i}}(f_u)$ for all u, if and only if $V(J_{\mathbf{i}}(F))$ is equimultiple along $\mathbb{C}^r \times \{0\}$ at 0.*

Proof. Suppose that $L = V(g_1, \dots, g_d)$ is a generic plane for $V(J_{\mathbf{i}}(f))$ and consider the map $\phi : V(J_{\mathbf{i}}(F)) \to \mathbb{C}^r \times \mathbb{C}^n$ defined by $\phi(u,x) = (u, g_1(x), \dots, g_d(x))$. Let R be the ring $\mathcal{O}_{n+r}/J_{\mathbf{i}}(F)$ and let $\mathcal{I}$ the sheaf of ideals on $V(J_{\mathbf{i}}(F))$ that extends the stalk $\mathcal{I}_0 = \mathbf{x}R$ in a natural form, where $\mathbf{x} = (x_1, \dots, x_n)$ denote the coordinates in $\mathbb{C}^n$. If we apply [19] Example 3.4 to this setup, we observe that the multiplicity that Lipman denotes by $e_{(u,0)}(\mathcal{I})$, for a given $(u, 0)$ close enough to 0 in $\mathbb{C}^r \times \mathbb{C}^d$, is equal to the multiplicity $e_{\mathbf{i}}(f_u)$. Now, by [19] Theorem 4b we have that if the multiplicity $e_{\mathbf{i}}(f_u)$ is independent of u (in a neighbourhood of 0 in $\mathbb{C}^r$) then $V(J_{\mathbf{i}}(F))$

is equimultiple along $\mathbb{C}^r \times \{0\}$ at 0. Reciprocally, if this equimultiplicity condition holds and we add that R is Cohen-Macaulay, as a consequence of [19] Theorem 4a (see also scholium on page 135) we have that $e_{\mathbf{i}}(f) = e_{\mathbf{i}}(f_u)$ for all u in a neighbourhood of 0 in $\mathbb{C}^r$. $\square$

Now, we apply the notion of equimultiplicity to study the multiplicity of weighted homogeneous map germs with respect to a Boradman symbol. A map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is said to be *weighted homogeneous* with degrees $d_1, \ldots, d_p$ and weights $w_1, \ldots, w_n$ if for each $i = 1, \ldots, p$ we have:

$$f_i(\lambda^{w_1} x_1, \ldots, \lambda^{w_n} x_n) = \lambda^{d_i} f_i(x), \qquad \forall x \in \mathbb{C}^n, \quad \lambda \in \mathbb{C}.$$

When f is weighted homogeneous, the generators of the ideal $J_{\mathbf{i}}(f)$ are also weighted homogeneous with the same weights, for any Boardman symbol $\mathbf{i}$. Thus, one expects that the number $e_{\mathbf{i}}(f)$ is determined by the weights and degrees of f.

Given vectors $\mathbf{w} = (w_1, \ldots, w_n)$ and $\mathbf{d} = (d_1, \ldots, d_p)$ such that their coordinates are nonnegative integer numbers, we will denote by $\mathcal{H}(\mathbf{w}; \mathbf{d})$ the finite dimensional complex vector space of weighted homogeneous maps of weights $w_1, \ldots, w_n$ and degrees $d_1, \ldots, d_p$. When $w_1 = \cdots = w_n = 1$, in the homogeneous case, we will denote $\mathcal{H}(\mathbf{w}; \mathbf{d})$ by $\mathcal{H}_n(\mathbf{d})$.

As we shall see, it is important to remark that the Zariski tangent cone of the zero set V of an homogeneous ideal is equal to V itself. We use this fact and Proposition 3.2 to prove the invariance of $e_{\mathbf{i}}(f)$ for weighted homogeneous map germs.

Corollary 3.3 *Let $\mathbf{i} \in \mathcal{B}(n,p)$ be a Boardman symbol. Suppose that there is a Zariski open subset $\Omega \subseteq \mathcal{H}_n(\mathbf{d})$ such that for all $f \in \Omega$, the germ $V(\Delta^{\mathbf{i}})_\sigma$ is Cohen-Macaulay and $\operatorname{codim} V(J_{\mathbf{i}}(f)) = \operatorname{codim}_\sigma V(\Delta^{\mathbf{i}})$, where $\sigma = j^k f(0)$. Then, the multiplicity $e_{\mathbf{i}}(f)$ is constant on Ω.*

Proof. Let $f \in \Omega$ and consider a one parametric unfolding $F(u,x) = (u, f_u(x))$ such that $f_u \in \Omega$, for all u. Since f_u is homogeneous for all u, we have that $\mathbb{C}^r \times \{0\} \subseteq V(J_{\mathbf{i}}(F))$ and by Proposition 3.2 it suffices to prove the equimultiplicity of $\mathbb{C}^r \times \{0\}$ along $V(J_{\mathbf{i}}(F))$. We define $\phi : V(J_{\mathbf{i}}(F)) \to (\mathbb{C}^r \times \mathbb{C}^d, 0)$ as in the proof of Proposition 3.2. The fact that $J_{\mathbf{i}}(f_u) + \langle g_1, \ldots, g_d \rangle$ is homogeneous of finite colength implies, for all u in a neighbourhood of 0 in $\mathbb{C}^r$, the relation

$$\phi^{-1}(\phi(u,0)) \cap V(J_{\mathbf{i}}(F)) = (\{u\} \times L) \cap V(J_{\mathbf{i}}(f_u)) = \{(u,0)\}.$$

Moreover, it is easy to see that the tangent cone of $V(J_{\mathbf{i}}(F))$ at $(u,0)$ is equal to $\mathbb{C}^r \times V(J_{\mathbf{i}}(f_u))$. Therefore, the equimultiplicity of $V(J_{\mathbf{i}}(F))$ along $\mathbb{C}^r \times \{0\}$ follows from the first equivalence of equimultiplicity (see Definition 3.1). □

Corollary 3.4 *Let $\mathbf{i} \in \mathcal{B}(n,p)$ be a Boardman symbol. Suppose that there is a Zariski open subset $\Omega \subseteq \mathcal{H}(\mathbf{w},\mathbf{d})$ such that for all $f \in \Omega$, the germ $V(\Delta^{\mathbf{i}})_\sigma$ is Cohen-Macaulay and* $\operatorname{codim} V(J_{\mathbf{i}}(f)) = \operatorname{codim}_\sigma V(\Delta^{\mathbf{i}}) = n$, *where $\sigma = j^k f(0)$. Then, the multiplicity $e_{\mathbf{i}}(f)$ is constant on Ω.*

Proof. Following the same approach as in the above corollary, we now consider a one parametric unfolding $F(u,x) = (u, f_u(x))$ of a given weighted homogeneous map germ $f \in \Omega$, such that $f_u \in \Omega$, for all u. In this case, we find that $V(J_{\mathbf{i}}(F)) = \mathbb{C}^r \times \{0\}$ and therefore the result follows again from the first equivalence of equimultiplicity and Proposition 3.2. □

For general weights $w_1, \dots, w_n$ and codimension $\nu < n$, the same result is not true. For instance, consider in $\mathcal{H}((2,1);(2,3))$ the Zariski open subset Ω given by the map germs $f : (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ such that the Jacobian determinant is not identically zero. It follows that for any $f \in \Omega$ and for any unfolding F of f, the ring $\mathcal{O}_3/J_1(F)$ is Cohen-Macaulay of codimension 1, since this ring defines a hypersurface in this case. However, if we consider $f,g \in \Omega$ given by $f(x,y) = (x,y^3)$ and $g(x,y) = (x, y^3 + xy)$, we get $e_1(f) = 2$ and $e_1(g) = 1$.

Note also that the result is true in the special case that $\mathbf{i} = (1)$, $n = p$ and $f : (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$ is $\mathcal{A}$-finitely determined. In this case, the ideal $J_1(f)$ defines a hypersurface with isolated singularity and it follows that its multiplicity is determined by the weights and the degree of the generator of $J_1(f)$ (see [24], Lemma 6).

In view of Corollaries 3.3 ans 3.4 it is interesting to provide expressions for the multiplicity of weighted homogeneous map germs depending only on the degrees and weights of the germs.

There are some situations where it is possible to ensure that there is a Zariski open subset Ω which verifies the hypothesis of Corollaries 3.3 and 3.4, that is, for all $f \in \Omega$, the ring $\mathcal{O}_{n+1}/J_{\mathbf{i}}(F)$ associated to any unfolding F of f is Cohen-Macaulay of codimension $\nu \leq n$. That condition holds in each one of the following cases [10]:

1. $\mathbf{i} = (1)$, $n \leq p$ and $\Omega = \{f : \operatorname{codim} V(J_{\mathbf{i}}(f)) = \nu(\mathbf{i})\}$.
2. $\mathbf{i} = (n-p+1)$, $n > p$ and $\Omega = \{f : \operatorname{codim} V(J_{\mathbf{i}}(f)) = \nu(\mathbf{i})\}$.
3. $\mathbf{i} = (2)$, $n = p$ and $\Omega = \{f : \operatorname{codim} V(J_{\mathbf{i}}(f)) = \nu(\mathbf{i})\}$.
4. $\mathbf{i} = (1,1)$, $n = p$ and $\Omega = \{f : \operatorname{codim} V(J_{\mathbf{i}}(f)) = \nu(\mathbf{i})\}$.
5. $\mathbf{i} = (i_1, \dots, i_k)$ and $\Omega = \{f : \operatorname{codim} V(J_{\mathbf{i}}(f)) = \nu(\mathbf{i})$ and f has type $\Sigma^{\mathbf{i}}\}$.
6. $\mathbf{i} = (i_1, \dots, i_k, 1)$ and $\Omega = \{f : \operatorname{codim} V(J_{\mathbf{i}}(f)) = \nu(\mathbf{i})$ and f has type $\Sigma^{i_1,\dots,i_k}\}$.
7. $\mathbf{i} = (n-p+1, 2)$ and $\Omega = \{f : \operatorname{codim} V(J_{\mathbf{i}}(f)) = \nu(\mathbf{i})$ and f has type $\Sigma^{n-p+1}\}$.

In fact, you can find in [10] or [12] longer lists of Boardman symbols and map germs which satisfy the required property. Here, we have selected only those in which computations involve determinantal ideals defined by maximal or submaximal minors. In next section we will provide formulae for $e_{\mathbf{i}}(f)$ in terms of weights and degrees of f in these cases.

4. Expressions for the multiplicity $e_{\mathbf{i}}(f)$

The results we show here will be consequence of certain more general formulae about multiplicities of determinantal rings given in the Section 5.

In the case $\mathbf{i} = (1)$, here we give a formula which generalizes the number of cross-caps of a weighted homogeneous map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ [22].

Proposition 4.1 *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a weighted homogeneous map germ of degrees $d_1, \dots, d_p$ and weights $w_1, \dots, w_n$. Consider the Boardman symbol $\mathbf{i} = (1)$ when $n \leq p$ or $\mathbf{i} = (n-p+1)$ when $n > p$. Suppose that $\dim \mathcal{O}_n / J_{\mathbf{i}}(f) = n - \nu(\mathbf{i})$ and that either $w_1 = \cdots = w_n = 1$, or $\nu(\mathbf{i}) = n$. Then*

$$e_{\mathbf{i}}(f) = \frac{1}{w_1 \cdots w_n} \sum_{1 \leq i_1 < \cdots < i_{r+1} \leq \max\{p,n\}} d_{i_1,i_1} \cdots d_{i_{r+1}, i_{r+1}-r},$$

where $r = |p-n|$ and

$$d_{ij} = \begin{cases} d_i - w_j, & \textit{if} \quad n \leq p, \\ d_j - w_i, & \textit{if} \quad n > p. \end{cases}$$

Proof. Note that $J_{\mathbf{i}}(f)$ has grade $\nu(\mathbf{i}) = p-n+1$, if $n \geq p$, or $\nu(\mathbf{i}) = n-p+1$, otherwise. Moreover, it is generated by the maximal minors of a matrix of size $n \times p$, whose entries are weighted homogeneous polynomials of degree d_{ij}. Thus, the result follows from Lemma 5.5 in the case $w_1 = \cdots = w_n = 1$, or Lemma 5.6 in the case $\nu(\mathbf{i}) = n$. $\square$

We can apply the same argument to obtain the following formula, which generalizes the number of cusps of a weighted homogeneous map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ [14].

Proposition 4.2 *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a weighted homogeneous map germ of degrees $d_1, \ldots, d_n$ and weights $w_1, \ldots, w_n$. Suppose that* $\dim \mathcal{O}_n / J_{1,1}(f) = n - 2$ *and that either $w_1 = \cdots = w_n = 1$, or $n = 2$. Then*

$$e_{1,1}(f) = \frac{1}{w_1 \cdots w_n} \sum_{1 \leq i < j \leq n+1} d_{i,i} d_{j,j-1},$$

where $d_{n+1} = d_1 + \cdots + d_n - w_1 - \cdots - w_n$ and $d_{ij} = d_i - w_j$.

For a general Boardman symbol $\mathbf{i} = (i_1, \ldots, i_k)$, if $\operatorname{codim} V(J_{\mathbf{i}}(f)) = \nu(\mathbf{i})$ and f has type $\Sigma^{\mathbf{i}}$, it follows that $\mathcal{O}_n / J_{\mathbf{i}}(f)$ is a complete intersection (see [23]), that is, the ideal $J_{\mathbf{i}}(f)$ can be generated by $g_1, \ldots, g_{\nu(\mathbf{i})} \in \mathcal{O}_n$. Hence, it follows from Bézout Theorem that

$$e_{\mathbf{i}}(f) = \frac{D_1 \cdots D_{\nu(\mathbf{i})}}{w_1 \cdots w_n},$$

where D_j is the degree of each g_i. However, it is very difficult in general, to obtain an expression of the D_j in terms of the degrees d_j of the map germ f. In the following proposition, we do that for the special case $i_1 = \cdots = i_k$.

From now on, we will denote by $\mathbf{i} = (i_{(k)})$ the Boardman symbol in which the integer i is repeated k times.

Proposition 4.3 *Let $\mathbf{i} = (i_{(k)})$ and let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a map germ given by $f(x) = (g(x), x_{i+1}, \ldots, x_n)$, where g is weighted homogeneous of degrees $d_1, \ldots, d_{p-n+i}$ and weights $w_1, \ldots, w_n$. Suppose that* $\dim \mathcal{O}_n / J_{\mathbf{i}}(f) = n - \nu(\mathbf{i})$ *and that either $w_1 = \cdots = w_n = 1$, or $\nu(\mathbf{i}) = n$. Then*

$$e_{\mathbf{i}}(f) = \frac{1}{w_1 \cdots w_n} \prod_{\substack{\alpha = 1, \ldots, p-n+i \\ \beta = 1, \ldots, k \\ 1 \leq \ell_1 \leq \cdots \leq \ell_\beta \leq i}} (d_\alpha - w_{\ell_1} - \cdots - w_{\ell_\beta}).$$

Proof. Since f is written in the form $f(x) = (g(x), x_{i+1}, \ldots, x_n)$, then the Jacobian extension $J_{\mathbf{i}}(f)$ is equal to $J_{\mathbf{i}}(g; x_1, \ldots, x_i)$. Using this we find that:

$$\begin{aligned}
J_i(f) &= \left\langle \frac{\partial g_1}{\partial x_\ell}, \ldots, \frac{\partial g_{p-n+i}}{\partial x_\ell} : \ell = 1, \ldots, i \right\rangle, \\
J_{i,i}(f) &= J_i(f) + \left\langle \frac{\partial^2 g_1}{\partial x_{\ell_1} \partial x_{\ell_2}}, \ldots, \frac{\partial^2 g_{p-n+i}}{\partial x_{\ell_1} \partial x_{\ell_2}} : 1 \leq \ell_1 \leq \ell_2 \leq i \right\rangle, \\
&\cdots \\
J_{i_{(k)}}(f) &= J_{i_{(k-1)}}(f) + \left\langle \frac{\partial^k g_1}{\partial x_{\ell_1} \cdots \partial x_{\ell_k}}, \ldots, \frac{\partial^k g_{p-n+i}}{\partial x_{\ell_1} \cdots \partial x_{\ell_k}} : \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. 1 \leq \ell_1 \leq \cdots \leq \ell_k \leq i \right\rangle.
\end{aligned}$$

As we can see, the number of generators which are necessary to obtain the Jacobian extension $J_{i_{(s)}}(f)$ from the previous one is equal to $(p-n+1)\#\{(\ell_1, \ldots, \ell_s) : 1 \leq \ell_1 \leq \cdots \leq \ell_s \leq i\} = (p-n+1)\binom{i+s-1}{s}$. It's easy to see that this number coincides with $\nu(i_{(s)}) - \nu(i_{(s-1)})$. Therefore, we have an explicit presentation of $\nu(\mathbf{i})$ generators for $J_{\mathbf{i}}(f)$ and thus, the result follows from Bézout Theorem. □

In a Cohen-Macaulay ring R the notions of height and grade of an ideal coincide (see [3] Corollary 2.1.4). Moreover, if R is local we can also use the relation $\text{height}(I) + \dim R/I = \dim R$, for any ideal I of R. Then, it is interesting to have a method to determine the Cohen-Macaulay property on a given ring. Next lemma provides a useful method in our case.

Lemma 4.4 [16] *Let R be a Noetherian ring of dimension d. Let M be a p by q matrix with entries in R and let I_r be the ideal generated by the r-minors of M. Then,*

1. $\dim(R/I_r) \geq d - (p-r+1)(q-r+1)$;
2. *if R is Cohen-Macaulay and* $\dim(R/I_r) = d - (p-r+1)(q-r+1)$ *then, the ring R/I_r is Cohen-Macaulay (in this case R/I_r is said to be a determinantal ring).*

Proposition 4.5 *Let f be a weighted homogeneous germ expressed as in Proposition* 4.3 *and consider the Boardman symbol* $\mathbf{i} = (i_{(k)}, 1)$. *Suppose that* $\dim \mathcal{O}_n/J_{\mathbf{i}}(f) = n - \nu(\mathbf{i})$ *and that either* $w_1 = \cdots = w_n = 1$, *or*

$\nu(\mathbf{i}) = n$. *Then*

$$e_{i_{(k)},1}(f) = e_{i_{(k)}}(f) \sum_{1\leq\alpha_1<\cdots<\alpha_{r+1}\leq t} (D_{\alpha_1} - w_{\alpha_1})\cdots(D_{\alpha_{r+1}} - w_{\alpha_{r+1}-r}),$$

where $t = (p-n+1)\binom{i+k-1}{k}$, $r = t - i$ *and* $D_1, \dots, D_t$ *are the elements of the set*

$$\{d_\alpha - w_{\ell_1} - \cdots - w_{\ell_k} : \alpha = 1, \dots, p-n+i,\ 1 \leq \ell_1 \leq \cdots \leq \ell_k \leq i\}.$$

Proof. We proceed as in the proof of Proposition 4.3. Just observe that the Jacobian extension $J_{i_{(k)},1}(f)$ is expressed as

$$J_{i_{(k)},1}(f) = J_{i_{(k)}}(f) + I_i(A),$$

where A is the differential matrix with respect to the variables $x_1, \dots, x_i$ of the map whose coordinates are given by all k-th partial derivatives of the coordinate functions $g_1, \dots, g_{p-n+i}$ with respect to such variables. Hence we have, by Lemma 4.4, that:

$$\begin{aligned} n - \nu(i_{(k)}, 1) &= \dim \frac{\mathcal{O}_n}{J_{i_{(k)},1}(f)} \\ &\geq \dim \frac{\mathcal{O}_n}{J_{i_{(k)}}(f)} - \left(\nu(i_{(k)}) - \nu(i_{(k-1)}) - i + 1\right). \end{aligned}$$

On the other hand, it is easy to see that

$$\nu(i_{(k)}, 1) = 2\nu(i_{(k)}) - \nu(i_{(k-1)}) - i + 1,$$

which gives

$$n - \nu(i_{(k)}) \geq \dim \frac{\mathcal{O}_n}{J_{i_{(k)}}(f)}.$$

Note that the other inequality is always true, since $J_{i_{(k)}}(f)$ is generated by $\nu(i_{(k)})$ elements, as we have seen in the proof of Proposition 4.3. Then it is actually an equality and we deduce that it is a Cohen-Macaulay ring. Therefore, the ideal $J_{i_{(k)},1}(f)/J_{i_{(k)}}(f)$ has the right grade in $\mathcal{O}_n/J_{i_{(k)}}(f)$ and it only remains to apply lemmas 5.5 and 5.6. $\square$

Next, we give a pair of formulas for the number $e_{\mathbf{i}}(f)$, when $\mathbf{i}$ involves computations with submaximal minors of squared matrices. These formulas are consequences of a pair of results also shown in Section 5..

Proposition 4.6 *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ a weighted homogeneous map germ with degrees $d_1, \ldots, d_n$ and weights $w_1, \ldots, w_n$, and consider the Boardman symbol $\mathbf{i} = (2)$. Suppose that $\dim \mathcal{O}_n/J_2(f) = n - \nu(2) = n - 4$ and that either $w_1 = \cdots = w_n = 1$, or $n = 4$. Then*

$$e_2(f) = \frac{1}{w_1 \cdots w_n}\Big(\sum_{i<j} d_{ii}d_{jj}d_{ij}d_{ji} + \sum_{i<j<k} d_{ii}d_{jj}d_{kk}(d_{ii} + d_{jj} + d_{kk}) + \sum_{i<j<k<l} d_{ii}d_{jj}d_{kk}d_{ll}\Big),$$

where $d_{ij} = d_i - w_j$.

Proof. The ideal $J_2(f)$ is generated by the submaximal minors of the Jacobian matrix of f and has height equal to 4. Then, we only have to apply Lemma 5.11. □

Lemma 4.7 [18] *Let R be a ring of dimension n and M an $m \times m$ symmetric matrix with entries in R. Let I_t the ideal generated by the minors of order $m - t + 1$ of M. Then*

1. $\dim R/I_t(U) \geq d - t(t + 1)/2$;
2. *if* $\dim R/I_t(U) = d - t(t + 1)/2$ *and R is Cohen-Macaulay, then R/I_t is Cohen-Macaulay.*

Proposition 4.8 *Let $n \geq p \geq 4$ and let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a map germ given by $f(x) = (g(x), x_{n-p+2}, \ldots, x_n)$, where g is weighted homogeneous of degree d and weights $w_1, \ldots, w_n$. Consider the Boardman symbol $\mathbf{i} = (n - p + 1, 2)$, suppose that $\dim \mathcal{O}_n/J_{\mathbf{i}}(f) = n - \nu(\mathbf{i}) = p - 4$ and that either $w_1 = \cdots = w_n = 1$, or $p = 4$. Then*

$$e_{n-p+1,2}(f) = \frac{1}{w_1 \cdots w_n}(d - w_1) \cdots \cdots (d - w_{n-p+1})\Big(\sum_{i<j} d_{ii}d_{jj}d_{ij} + \sum_{i<j<k} d_{ii}d_{jj}d_{kk}\Big),$$

where $d_{ij} = d - w_i - w_j$, $i, j = 1, \ldots, n - p + 1$.

Proof. If G denotes the map germ whose components are the partial derivatives $G_i = \frac{\partial g}{\partial x_i}$, for $i = 1, \ldots, n - p + 1$, then

$$J_{\mathbf{i}}(f) = J_{\mathbf{i}}(g; x_1, \ldots, x_{n-p+1}) = \langle G_1, \ldots, G_{n-p+1}\rangle + I_{n-p}(D(g, G)).$$

Let's denote by I_1 and I_2, the two ideals in this sum. If we now apply the first part of Lemma 4.7 and the hypotheses, we find that

$$p - 4 = \dim \frac{\mathcal{O}_n}{J_1(f)} \geq \dim \frac{\mathcal{O}_n}{I_1} - 3,$$

which implies that $\mathcal{O}_n/I_1$ is a complete intersection. In particular, it is Cohen-Macaulay and the image of I_2 in $\mathcal{O}_n/I_1$ has height 3. Then it only remains to apply Lemma 5.12. □

5. Multiplicity of some determinantal rings

In this section we show the necessary results to obtain the expressions of Propositions 4.1, 4.2, 4.5, 4.6 and 4.8. They could also provide other expressions for the multiplicity of map germs in other situations. We will develop these results in the context of graded rings.

If U is an $n \times m$ matrix, $n \geq m$, with homogeneous entries in a graded ring $R = \oplus_{n \geq 0} R_n$, we look at the multiplicity of the ideal $I_m(U)$ generated by the maximal order minors of U. The main tool we have used in order to compute this multiplicity, under some restrictions that we will make precise, is the Eagon-Northcott complex. This complex gives a free resolution of $R/I_m(U)$ when $I_m(U)$ has grade $n-m+1$, that is, when it is a determinantal ring. It was used by Eagon and Northcott to compute the normalized leading coefficient of the Hilbert series of $R/I_m(U)$ but they do not give an explicit formula in terms of the degrees of the entries of the matrix. As a corollary we give a formula for the length of $R/I_m(U)$ that was also obtained by Damon in [5] for the case $R = \mathbb{C}[x_1, \dots x_n]$ using a totally different approach.

5.1. The Hilbert series of determinantal rings defined by maximal minors

Let $R = \oplus_{n \geq 0} R_n$ be a Noetherian graded ring with R_0 Artinian, and let M be a finitely generated graded R-module. The *Hilbert series* of M is defined as

$$P(M, t) = \sum_{n=0}^{\infty} \ell(M_n) t^n \in \mathbb{Z}[[t]],$$

where $\ell(M_n)$ denotes the length of each piece M_n as R_0-module (since M_n is finitely generated as R_0-module and R_0 is Artinian, this length is always finite).

It is very common to denote by $M[\lambda]$ the same module M but with its grading shifted by λ, that is, $M[\lambda]_n = M_{n-\lambda}$. Its Hilbert series is then given by

$$P(M[\lambda], t) = t^\lambda P(M, t). \tag{1}$$

Suppose now that $U = (u_{ij})$ is an $n \times m$ matrix with entries in R, with $n \geq m$ and so that each u_{ij} is homogeneous of degree d_{ij}. In order to ensure that all the minors of U are also homogeneous, we have to put the following condition:

$$d_{ij} + d_{kl} = d_{il} + d_{kj}, \quad \forall i, j, k, l. \tag{2}$$

Then the ideal $I_m(U)$ defined by the maximal minors of U is homogeneous an the quotient ring $R_m(U) = R/I_m(U)$ is a finitely generated graded R-module.

Now, we introduce some notation in order to construct the Eagon-Northcott complex. This will lead us to the Hilbert series of $R_m(U)$, when the ideal $I_m(U)$ has grade $n - m + 1$.

Definition 5.1 Let $r = n - m$. For each $k = 0, \ldots, r$ we consider the following free R-module,

$$F_k = \bigoplus_{\substack{1 \leq j_1 \leq \cdots \leq j_k \leq m \\ 1 \leq i_1 < \cdots < i_{m+k} \leq n}} R,$$

which has rank $\binom{n}{m+k}\binom{m+k-1}{m-1}$. We denote the corresponding canonical basis of F_k by

$$\left\{ e_{i_1, \ldots, i_{m+k}, j_1, \ldots, j_k} \right\}_{\substack{1 \leq j_1 \leq \cdots \leq j_k \leq m \\ 1 \leq i_1 < \cdots < i_{m+k} \leq n}}.$$

Moreover, we define the R-homomorphism $\partial : F_{k+1} \to F_k$ by

$$\partial(e_{i_1, \ldots, i_{m+k+1}, j_1, \ldots, j_{k+1}}) = \sum_{\alpha=1}^{m+k+1} \sum_{\beta=1}^{k+1} (-1)^{\alpha+1} u_{i_\alpha j_\beta} e_{i_1, \ldots, \widehat{i_\alpha}, \ldots, i_{m+k+1}, j_1, \ldots, \widehat{j_\beta}, \ldots, j_{k+1}},$$

and the R-homomorphism $\nu : F_0 \to R$ by

$$\nu(e_{i_1,\ldots,i_m}) = \begin{vmatrix} u_{i_1 1} & \cdots & u_{i_1 m} \\ \cdots\cdot & \cdot\cdot & \cdots\cdot\cdot \\ u_{i_m 1} & \cdots & u_{i_m m} \end{vmatrix}.$$

Lemma 5.2 [6, 4] *Let R be a Noetherian ring and let $U = (u_{ij})$ be an $n \times m$ matrix with entries in R, with $n \geq m$. If $I_m(U)$ has grade $n-m+1$, then the following sequence is exact:*

$$0 \longrightarrow F_r \xrightarrow{\partial} F_{r-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} F_0 \xrightarrow{\nu} R \longrightarrow R_m(U) \longrightarrow 0.$$

Suppose now that R is a Noetherian graded ring, and that each u_{ij} is homogeneous of degree d_{ij} and condition (2) is satisfied. Then, all the R-modules in the Eagon-Northcott complex are graded. However, the homomorphisms ∂, ν are not graded homomorphisms, since they do not preserve homogeneous elements. We have to modify the grading of these R-modules in order to get an exact sequence of graded R-modules.

For instance, we consider in R the original grading and then we redefine

$$\widetilde{F_0} = \bigoplus_{1\leq i_1<\cdots<i_m\leq n} R[d_{i_1 1} + \cdots + d_{i_m m}],$$

so that now $\nu : \widetilde{F_0} \to R$ is a graded homomorphism of degree 0. For $k = 1$, we have to modify the grading in the following way:

$$\widetilde{F_1} = \bigoplus_{\substack{1\leq j_1\leq m \\ 1\leq i_1<\cdots<i_{m+1}\leq n}} R[d_{i_1 1} + \cdots + d_{i_m m} + d_{i_{m+1} j_1}].$$

Then, $\partial : \widetilde{F_1} \to \widetilde{F_0}$ is a graded homomorphism of degree 0. In general, we define

$$\widetilde{F_k} = \bigoplus_{\substack{1\leq j_1\leq\cdots\leq j_k\leq m \\ 1\leq i_1<\cdots<i_{m+k}\leq n}} R[d_{i_1 1} + \cdots + d_{i_m m} + d_{i_{m+1} j_1} + \cdots + d_{i_{m+k} j_k}],$$

and we get the following graded version of the Eagon-Northcott complex.

Lemma 5.3 *Let R be a Noetherian graded ring and let $U = (u_{ij})$ be an $n \times m$ matrix with entries in R, with $n \geq m$ and so that each u_{ij} is homogeneous of degree d_{ij} and condition (2) is satisfied. If $I_m(U)$ has grade*

$n-m+1$, then the following sequence of graded R-modules is exact (with all the homomorphisms of degree 0) :

$$0 \longrightarrow \widetilde{F}_r \xrightarrow{\partial} \widetilde{F}_{r-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \widetilde{F}_0 \xrightarrow{\nu} R \longrightarrow R_m(U) \longrightarrow 0.$$

This graded free resolution of $R_m(U)$ allows us to compute immediately its Hilbert series (see Lemma 4.1.13 of [3]).

Theorem 5.4 *Let $R = \oplus_{n\geq 0} R_n$ be a Noetherian graded ring with R_0 Artinian and let $U = (u_{ij})$ be an $n \times m$ matrix with entries in R, with $n \geq m$ and so that each u_{ij} is homogeneous of degree d_{ij} and condition (2) is satisfied. If $I_m(U)$ has grade $n-m+1$, then the Hilbert series of $R_m(U)$, $P(R_m(U), t)$, is given by*

$$P(R,t)\left(1+\sum_{k=0}^{r}(-1)^{k+1} \sum_{\substack{1\leq j_1\leq \cdots \leq j_k \leq m \\ 1\leq i_1 < \cdots < i_{m+k}\leq n}} t^{d_{i_1 1}+\cdots+d_{i_m m}+d_{i_{m+1} j_1}+\cdots+d_{i_{m+k} j_k}}\right),$$

where $r = n-m$.

A special case is when R is a polynomial ring, $R = R_0[X_1, \ldots, X_s]$, over an Artinian ring R_0. Thus, Theorem 5.4 generalizes the results of [7].

5.2. The length and the multiplicity

In this section we use the Hilbert series of $R_m(U)$ in order to obtain some consequences on the length and the multiplicity of $R_m(U)$ in the case that these concepts have sense. We shall use a combinatorial result that will be proved in next subsection.

Let $R = \oplus_{n\geq 0} R_n$ graded ring with R_0 Artinian. We set $R_+ = \oplus_{n>0} R_n$. This is a homogeneous ideal of R which verifies $R/R_+ \cong R_0$. Moreover, if R is Noetherian and R_+ is generated by homogeneous elements $\zeta_1, \ldots, \zeta_s$, we have that $R = R_0[\zeta_1, \ldots, \zeta_s]$. Suppose that each ζ_i belong to R_1, for $i = 1, \ldots, s$. In this situation, we have that the Hilbert series of a finitely generated graded R-module M can be written uniquely in the form:

$$P(M,t) = \frac{f(t)}{(1-t)^d},$$

where $f(t) \in \mathbb{Z}[t, t^{-1}]$, $f(1) \neq 0$ and $d = \dim(M)$ (see Corollary 4.1.8 of [3]). Once we have expressed the Hilbert series of M in this form, the multiplicity of M as a graded R-module can be computed as $f(1)$.

Lemma 5.5 *Let $R = \oplus_{n\geq 0} R_n$ be a Noetherian graded ring with R_0 Artinian and such that R can be generated over R_0 by elements of degree 1. Let $U = (u_{ij})$ be an $n \times m$ matrix with entries in R, so that each u_{ij} is homogeneous of degree d_{ij} and condition (2) is satisfied. If $I_m(U)$ has grade $n-m+1$, we have that the multiplicity of $R_m(U)$ is given by*

$$e(R_m(U)) = e(R) \sum_{1\leq i_1<\cdots<i_{r+1}\leq n} d_{i_1 i_1} d_{i_2 i_2 -1} \cdots d_{i_{r+1} i_{r+1}-r},$$

where $r = n-m$.

Proof. Suppose that the Hilbert series of R is written as $P(R,t) = f(t)/(1-t)^d$, where $f \in \mathbb{Z}[t,t^{-1}]$ and d is the dimension of R. Then, by Theorem 5.4, the Hilbert series of $R_m(U)$ is given by

$$P(R_m(U),t) = \frac{f(t)\Delta(t)}{(1-t)^d},$$

where

$$\Delta(t) = 1 + \sum_{k=0}^{r} (-1)^{k+1} \sum_{\substack{1\leq j_1\leq\cdots\leq j_k\leq m \\ 1\leq i_1<\cdots<i_{m+k}\leq n}} t^{d_{i_1 1}+\cdots+d_{i_m m}+d_{i_{m+1} j_1}+\cdots+d_{i_{m+k} j_k}}.$$

Since height $I_m(U) \geq$ grade $I_m(U)$ we have that $\dim R_m(U) \leq d-(n-m+1)$. Moreover, the other inequality is always true by 4.4. Then, the dimension of $R_m(U)$ must be equal to $d-(n-m+1)$ and its multiplicity can be expressed as:

$$e(R_m(U)) = \lim_{t\to 1} \frac{f(t)\Delta(t)}{(1-t)^{r+1}},$$

where $r = n-m$.

Now we compute the derivatives of $\Delta(t)$ at $t=1$. It is not very difficult to see that $\Delta(1) = 0$. By using Lemma 5.9 and induction on ℓ, we have that for $\ell = 1,\ldots,r$,

$$\Delta^{(\ell)}(1) = \sum_{k=0}^{r} (-1)^{k+1} \sum_{\substack{1\leq j_1\leq\cdots\leq j_k\leq m \\ 1\leq i_1<\cdots<i_{m+k}\leq n}} (d_{i_1 1}+\cdots+d_{i_m m}+d_{i_{m+1} j_1}+\cdots+d_{i_{m+k} j_k})^{\ell} = 0.$$

However, for $\ell = r+1$, we have:

$$\Delta^{(r+1)}(1) = \sum_{k=0}^{r}(-1)^{k+1} \sum_{\substack{1\le j_1\le\cdots\le j_k\le m\\ 1\le i_1<\cdots<i_{m+k}\le n}} (d_{i_1 1}+\cdots+d_{i_m m}+d_{i_{m+1}j_1}+\cdots+d_{i_{m+k}j_k})^{r+1}.$$

And by Proposition 5.7 it is possible to simplify this last expression to

$$\Delta^{(r+1)}(1) = (-1)^{r+1}(r+1)! \sum_{1\le i_1<\cdots<i_{r+1}\le n} d_{i_1 i_1} d_{i_2 i_2-1}\cdots d_{i_{r+1} i_{r+1}-r}.$$

Now, we use the Leibniz formula to compute the derivatives of $f(t)\Delta(t)$ at $t=1$ and we get that

$$\begin{aligned} e(R_m(U)) &= \frac{(f\Delta)^{(r+1)}(1)}{(-1)^{r+1}(r+1)!} = \frac{f(1)\Delta^{(r+1)}(1)}{(-1)^{r+1}(r+1)!} \\ &= e(R) \sum_{1\le i_1<\cdots<i_{r+1}\le n} d_{i_1 i_1} d_{i_2 i_2-1}\cdots d_{i_{r+1} i_{r+1}-r}. \end{aligned}$$

□

Finally, note that the same kind of computations can be done when R is not generated over R_0 by elements of degree 1, but the quotient ring $R_m(U)$ has finite length. The next result is a formula which has been obtained by Damon [5] in the case $R = \mathbb{C}[X_1,\ldots,X_s]$ by using different methods.

Lemma 5.6 *Let $R = k[X_1,\ldots,X_s]$ be a polynomial ring over a field k and let $U = (u_{ij})$ be an $(s+m-1)\times m$ matrix with entries in R, so that each u_{ij} is weighted homogeneous of degree d_{ij} and condition (2) is satisfied. If the length of $R_m(U)$ is finite, it is given by*

$$\ell(R_m(U)) = \frac{1}{w_1\cdots w_s}\sum_{1\le i_1<\cdots<i_s\le s+m-1} d_{i_1 i_1} d_{i_2 i_2-1}\cdots d_{i_s i_s-s+1},$$

where w_i is the degree of X_i in R.

Proof. Just observe that if the length of $R_m(U)$ is finite, then $R_m(U)$ has dimension zero and thus $I_m(U)$ has grade $s = (s+m-1)-m+1$. Moreover, the fact that the length of $R_m(U)$ is finite also implies that $P(R_m(U),t)$ is a polynomial, so that the length is equal to $P(R_m(U),1)$. Hence, since the Hilbert series of R can be written as $P(R,t) = 1/\prod_{i=1}^{s}(1-t^{w_i})$, we have that

$$\begin{aligned}\ell(R_m(U)) &= P(R_m(U),1) = \lim_{t\to 1}\frac{\Delta(t)}{\prod_{i=1}^{s}(1-t^{w_i})}\\ &= \frac{1}{w_1\cdots w_s}\sum_{1\le i_1<\cdots<i_s\le n} d_{i_1 i_1}d_{i_2 i_2-1}\cdots d_{i_s i_s-s+1},\end{aligned}$$

where the last equality is obtained by applying l'Hôpital rule s times and Proposition 5.7. □

5.3. The combinatorial part

Here we prove a combinatorial formula necessary to simplify the expressions obtained in subsection 5.2.

Proposition 5.7 *Let (d_{ij}) be an $n\times m$ matrix of rational numbers satisfying condition (2). Then,*

$$\begin{aligned}&\sum_{k=0}^{r}(-1)^{k+1}\sum_{\substack{1\le j_1\le\cdots\le j_k\le m\\ 1\le i_1<\cdots<i_{m+k}\le n}}(d_{i_1 1}+\cdots+d_{i_m m}+d_{i_{m+1}j_1}+\cdots+d_{i_{m+k}j_k})^{r+1}\\ &\qquad= (-1)^{r+1}(r+1)!\sum_{1\le i_1<\cdots<i_{r+1}\le n} d_{i_1 i_1}d_{i_2 i_2-1}\cdots d_{i_{r+1}i_{r+1}-r},\end{aligned}$$

where $r=n-m$.

In order to simplify the computations, we introduce the following notation, which reduces the nm variables d_{ij} to $n+m$ variables, a_i, b_j.

Lemma 5.8 *Let (d_{ij}) be an $n\times m$ matrix of rational numbers. Then, it satisfies condition (2) if and only if there exist $a_1,\dots,a_n,b_1,\dots,b_m\in\mathbb{Q}$ such that $d_{ij}=a_i+b_j$, for any i, j.*

Now we can rewrite Proposition 5.7 in terms of the new variables a_i and b_j. Let $a_1,\dots,a_n$ and $b_1,\dots,b_m$ be rational numbers and let $r=n-m\ge 0$. Then, we have to prove the following identity:

$$\begin{aligned}&\sum_{k=0}^{r}(-1)^{k+1}\sum_{\substack{1\le j_1\le\cdots\le j_k\le m\\ 1\le i_1<\cdots<i_{m+k}\le n}}(a_{i_1}+\cdots+a_{i_{m+k}}+b_{j_1}+\cdots+b_{j_k}+b_1+\cdots+b_m)^{r+1}\\ &= (-1)^{r+1}(r+1)!\sum_{1\le i_1<\cdots<i_{r+1}\le n}(a_{i_1}+b_{i_1})(a_{i_2}+b_{i_2-1})\cdots(a_{i_{r+1}}+b_{i_{r+1}-r}).\end{aligned}$$

We will denote the left hand side of the above identity by A_{nm} and the other one by B_{nm}.

Lemma 5.9 *For all $p \in \{1, \dots, r\}$, we have:*

$$\sum_{k=0}^{r}(-1)^{k+1} \sum_{\substack{1 \le j_1 \le \cdots \le j_k \le m \\ 1 \le i_1 < \cdots < i_{m+k} \le n}} (a_{i_1} + \cdots + a_{i_{m+k}} + b_{j_1} + \cdots + b_{j_k} + b_1 + \cdots + b_m)^p = 0.$$

The proof of the previous lemma can be done by induction on p. Moreover, for each fixed p the proof of the corresponding identity requires the same kind of induction argument as the one of Proposition 5.7.

Lemma 5.10 *Let M be a number and suppose that $b_1 + \cdots + b_m = 0$. Then, for any $p \in \{1, \dots, r+1\}$,*

$$\sum_{k=0}^{r}(-1)^{k+1} \sum_{\substack{1 \le j_1 \le \cdots \le j_k \le m \\ 1 \le i_1 < \cdots < i_{m+k} \le n}} (a_{i_1} + \cdots + a_{i_{m+k}} + b_{j_1} + \cdots + b_{j_k} + M)^p$$

$$= \begin{cases} (-1)^{r+1}(r+1)!A_{nm} - M^{r+1}, & \text{if } \ p = r+1, \\ -M^p, & \text{if } \ p \in \{1, \dots, r\}. \end{cases}$$

Proof of Proposition 5.7. We have to prove that for each $m \ge 1$ the relation $A_{nm} = B_{nm}$ is satisfied for all $n \ge m$. For this purpose, we proceed by induction on m. The case $m = 1$ is equivalent to prove that for all $n \ge 1$:

$$\sum_{k=1}^{n}(-1)^k \sum_{1 \le i_1 < \cdots < i_k \le n} (a_{i_1} + \cdots + a_{i_k})^n = (-1)^n n!\, a_1 \cdots a_n$$

which follows by applying induction on n and Lemma 5.9.

Suppose that $m > 1$ and that the result is true for $m - 1$. As in the preceding case we apply induction on n. It is obvious that the proposition is true when $n = m$. We will prove that $A_{n+1,m} = B_{n+1,m}$ from the relations $A_{nm} = B_{nm}$ and $A_{n,m-1} = B_{n,m-1}$.

Let $a_1, \dots, a_{n+1}, b_1, \dots, b_m$ be numbers and let $r = n - m$. Since both $A_{n+1,m}$ and $B_{n+1,m}$ are invariants if we add a constant to all the a_i and subtract the same constant to all the b_j, we can suppose without loss of generality that $b_1 + \cdots + b_m = 0$. Then, we have that

$$A_{n+1,m} = \sum_{k=0}^{r+1} (-1)^{k+1} \sum_{\substack{1 \le j_1 \le \cdots \le j_k \le m \\ 1 \le i_1 < \cdots < i_{m+k} \le n+1}} (a_{i_1} + \cdots + a_{i_{m+k}} + b_{j_1} + \cdots + b_{j_k})^{r+2}.$$

Now we split $A_{n+1,m}$ into two terms, $A_{n+1,m} = S_1 + S_2$, where

$$S_1 = \sum_{k=0}^{r+1} (-1)^{k+1} \sum_{\substack{1 \le j_1 \le \cdots \le j_k \le m \\ 1 \le i_1 < \cdots < i_{m+k-1} \le n}} (a_{i_1} + \cdots + a_{i_{m+k-1}} + a_{n+1} + b_{j_1} + \cdots + b_{j_k})^{r+2},$$

$$S_2 = \sum_{k=0}^{r} (-1)^{k+1} \sum_{\substack{1 \le j_1 \le \cdots \le j_k \le m \\ 1 \le i_1 < \cdots < i_{m+k} \le n}} (a_{i_1} + \cdots + a_{i_{m+k}} + b_{j_1} + \cdots + b_{j_k})^{r+2}.$$

After applying the Newton's formula to S_1, we can separate the variables $j_1, \ldots, j_k$ in order to use the Newton's formula again and write the resulting expressions in such a way that we can apply Lemma 5.10. By a hard calculation we get:

$$\begin{aligned} A_{n+1,m} &= A_{n,m-1} - (r+2)(a_{n+1} + b_m) A_{nm} \\ &\quad + (a_{n+1} + b_m)^{r+2} - a_{n+1}^{r+2} - S_{11} + S_{12}, \end{aligned}$$

where

$$S_{11} = \sum_{\ell=1}^{r+2} \binom{r+2}{\ell} (-b_m)^{\ell} p_{\ell}, \quad S_{12} = \sum_{\ell=1}^{r+1} \binom{r+2}{\ell} a_{n+1}^{\ell} p_{\ell}$$

and p_ℓ is the expression given by

$$\sum_{k=0}^{r+1} (-1)^{k+1} \sum_{\substack{1 \le j_1 \le \cdots \le j_k \le m-1 \\ 1 \le i_1 < \cdots < i_{m-1+k} \le n}} (a_{i_1} + \cdots + a_{i_{m+k-1}} + b_{j_1} + \cdots + b_{j_k})^{r+2-\ell},$$

for all $\ell \in \{1, \ldots, r+2\}$. The last expression given for $A_{n+1,m}$ simplifies to:

$$A_{n+1,m} = A_{n,m-1} - (r+2)(a_{n+1} + b_m) A_{nm}.$$

Moreover, it is easy to see that:

$$B_{n+1,m} = B_{n,m-1} - (r+2)(a_{n+1} + b_m)B_{nm},$$

therefore, we only have to apply the induction hypotheses to obtain that $A_{n+1,m} = B_{n+1,m}$. □

5.4. Submaximal minors of a squared matrix

Here we give the algebraic results that allows us to write Propositions 4.6 and 4.8. We consider the same setup as in the beginning of this section. Now, suppose that the matrix U has size $m \times m$, where m is any positive integer. Let $I_{m-1}(U)$ be the ideal generated by the submaximal minors of U. When this ideal has grade 4, then Gulliksen-Negård complex (see [4]) provides a free resolution of the quotient ring $R_{m-1}(U) = R/I_{m-1}(U)$. Moreover, it is possible to regraduate such complex so that all the homomorphisms have degree 0 in the following way:

$$0 \longrightarrow \widetilde{G}_4 \longrightarrow \widetilde{G}_3 \longrightarrow \widetilde{G}_2 \longrightarrow \widetilde{G}_1 \longrightarrow R \longrightarrow R_{m-1}(U) \longrightarrow 0,$$

where

$$\begin{aligned}
&\widetilde{G}_1 = \oplus_{i,j} R[L - d_{ij}],\\
&\widetilde{G}_2 = \left(\oplus_{(i,j)\neq(1,1)} R[L - d_{ij} + d_{ii}]\right) \oplus \left(\oplus_{(i,j)\neq(1,1)} R[L - d_{ij} + d_{jj}]\right),\\
&\widetilde{G}_3 = \oplus_{i,j} R[L + d_{ji}], \quad \widetilde{G}_4 = R[2L],
\end{aligned}$$

(d_{ij}) is the degree matrix of U and $L = d_{11} + \cdots + d_{mm}$. For the sake of simplicity, we do not give here the explicit expression of the homomorphisms. Following the same argument as in 4.1 we can express the Hilbert series of $R_{m-1}(U)$ as $P(R_{m-1}(U), t) = P(R, t)\Delta(t)$, where $\Delta(t)$ is the polynomial

$$\begin{aligned}
\Delta(t) = {} & 1 - \sum_{i,j} t^{L-d_{ij}} + \sum_{(i,j)\neq(1,1)} t^{L-d_{ij}+d_{jj}}\\
& + \sum_{(i,j)\neq(1,1)} t^{L-d_{ij}+d_{ii}} - \sum_{i,j} t^{L+d_{ji}} + t^{2L}.
\end{aligned}$$

Lemma 5.11 *Let $R = \oplus_{n\geq 0} R_n$ be a Noetherian graded ring with R_0 Artinian and such that R can be generated over R_0 by elements of degree 1. Let $U = (u_{ij})$ be an $m \times m$ matrix with entries in R, so that each u_{ij} is homogeneous of degree d_{ij} and condition (2) is satisfied. If $I_{m-1}(U)$ has* *grade* 4, *we have that the multiplicity of* $R_{m-1}(U)$ *is given by*

$$e(R_{m-1}(U)) = e(R)\Big(\sum_{i<j} d_{ii}d_{jj}d_{ij}d_{ji} + \sum_{i<j<k} d_{ii}d_{jj}d_{kk}(d_{ii}+d_{jj}+d_{kk}) + \sum_{i<j<k<l} d_{ii}d_{jj}d_{kk}d_{ll}\Big).$$

As we did in Lemma 5.6, we can obtain an analogous result when $R = k[X_1,\dots,X_n]$, the ideal $I_{m-1}(U)$ has finite colength and we consider some weights $w_1,\dots,w_n$ on the variables $X_1,\dots,X_n$. We just have to fix $n=4$ and change $e(R)$ by $1/w_1w_2w_3w_4$ in the above formula.

If we suppose that $I_{m-1}(U)$ has grade 3 and that U is a symmetric matrix, then we can use the complex of Józefiak (see [17]) to obtain the Hilbert series of $R_{m-1}(U)$. In this case, the graded version of such complex is given by

$$0 \longrightarrow \widetilde{J}_3 \longrightarrow \widetilde{J}_2 \longrightarrow \widetilde{J}_1 \longrightarrow R \longrightarrow R_{m-1}(U) \longrightarrow 0,$$

where

$$\begin{aligned}\widetilde{J}_1 &= \oplus_{i\le j}R[L-d_{ij}],\\ \widetilde{J}_2 &= \left(\oplus_{i\le j,\ (i,j)\ne(1,1)}R[L-d_{ij}+d_{ii}]\right)\oplus\left(\oplus_{i<j}R[L-d_{ij}+d_{jj}]\right),\\ \widetilde{J}_3 &= \oplus_{i<j}R[L+d_{ij}].\end{aligned}$$

Then we have that $P(R_{m-1}(U),t) = P(R,t)\Delta(t)$, where now $\Delta(t)$ is the polynomial

$$\begin{aligned}\Delta(t) &= 1-\sum_{i\le j}t^{L-d_{ij}} + \sum_{i\le j,\ (i,j)\ne(1,1)} t^{L-d_{ij}+d_{ii}}\\ &\qquad + \sum_{i<j}t^{L-d_{ij}+d_{jj}} - \sum_{i<j}t^{L+d_{ij}}.\end{aligned}$$

Lemma 5.12 *Under the same hypotheses as in the previous lemma, if we now suppose that* U *is a symmetric matrix and that* $I_{m-1}(U)$ *has grade* 3, *the multiplicity of* $R_{m-1}(U)$ *is given by*

$$e(R_{m-1}(U)) = e(R)\Big(\sum_{i<j} d_{ii}d_{jj}d_{ij} + \sum_{i<j<k} d_{ii}d_{jj}d_{kk}\Big).$$

It is worth noting that in the above situation we can also state a result when R is a graded ring of polynomials and $I_{m-1}(U)$ has finite colength in R.

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