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Crossed products of UHF algebras by some amenable groups

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Abstract. Let A be a UHF C^*-algebra. It is shown that for every homomorphism $\alpha : \mathbb{Z}^n \to \mathrm{Aut}(A)$ there exists an AF embedding $\rho : A \rtimes_\alpha \mathbb{Z}^n \hookrightarrow B$ such that $\rho_* : K_0(A \rtimes_\alpha \mathbb{Z}^n) \to K_0(B)$ is also injective.

Using Green's imprimitivity theorem it will follow that if A is UHF and $\alpha : G \to \mathrm{Aut}(A)$ is a homomorphism then $A \rtimes_\alpha G$ is always quasidiagonal for a large class of amenable groups including all extensions of discrete abelian groups by compact (not necessarily discrete or abelian) groups.

Key words: quasidiagonality, crossed products, K-theory, amenable groups.

1. Introduction

Quasidiagonal C^*-algebras are those which enjoy a certain local finite dimensional approximation property (cf. [Vo2]). In light of some remarkable recent results (cf. [DE], [Li]), it appears that quasidiagonality will play an increasingly important role in Elliott's classification program. A C^*-algebra is called AF embeddable if it is isomorphic to a subalgebra of an AF algebra. Blackadar and Kirchberg asked if the notions of quasidiagonality and AF embeddability agree for nuclear C^*-algebras and there is reason to believe that they do (cf. [BK]). Unfortunately, neither of these notions behave well under taking crossed products (unless the group is compact). Indeed, Voiculescu has asked when $C(X) \rtimes_\varphi \mathbb{Z}^2$ is AF embeddable ([Vo3]) which illustrates how much we have yet to learn in this direction. (Recall that Pimsner characterized the AF embeddability (and quasidiagonality) of $C(X) \rtimes_\varphi \mathbb{Z}$ more than 15 years ago in [Pi].)

In this note we study the AF embeddability and quasidiagonality of crossed products of UHF algebras by certain amenable groups. Our main result is the following.

Theorem 1 *If A is a UHF algebra and $\alpha : \mathbb{Z}^n \to \mathrm{Aut}(A)$ is a homomor-*

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phism then there always exists a $*$-monomorphism $\rho : A \rtimes_\alpha \mathbb{Z}^n \to B$ where B is AF and $\rho_* : K_0(A \rtimes_\alpha \mathbb{Z}^n) \to K_0(B)$ is injective.

Understanding crossed products by $\mathbb{Z}^n$ is enough to gain valuable information about crossed products by much more general groups. Let Γ (resp. Γ_{fg}) denote the smallest class of separable locally compact groups such that every countable (resp. finitely generated) discrete abelian group is in Γ (resp. Γ_{fg}) and such that Γ (resp. Γ_{fg}) is closed under taking extensions by separable compact groups. (See also Definition 3.4.) As a consequence of Theorem 1 and an imprimitivity theorem of P. Green we will show:

Theorem 2 *If A is a UHF algebra, $G \in \Gamma$ (resp. $G \in \Gamma_{fg}$) and $\alpha : G \to \mathrm{Aut}(A)$ is a homomorphism then $A \rtimes_\alpha G$ is always quasidiagonal (resp. AF embeddable).*

As in [Br], the main tools will be K-theory and the Rohlin property (cf. [Ki1,2,3]) for automorphisms. Our results depend in an essential way on the recent work [Ki2] of A. Kishimoto, which builds on the previous work [Ki3] and [KK1,2] (see also [Ki4]).

In Section 2 we will collect all of the facts that we need and derive some AF embedding results which illustrate the techniques to be used in proving Theorem 1. In Section 3 we will prove the theorems stated above.

2. Preliminaries

Let A_1, A_2 be unital C^*-algebras, $\alpha_i \in \mathrm{Aut}(A_i)$ be automorphisms and $\rho : (A_1, \alpha_1) \hookrightarrow (A_2, \alpha_2)$ be a *covariant embedding*; i.e. $\rho : A_1 \hookrightarrow A_2$ is a unital $*$-monomorphism such that

$$\begin{array}{ccc} A_1 & \xrightarrow{\rho} & A_2 \\ {\scriptstyle \alpha_1}\downarrow & & \downarrow{\scriptstyle \alpha_2} \\ A_1 & \xrightarrow{\rho} & A_2 \end{array}$$

is a commutative diagram. In this case, there always exists an induced $*$-monomorphism $\tilde{\rho} : A_1 \rtimes_{\alpha_1} \mathbb{Z} \hookrightarrow A_2 \rtimes_{\alpha_2} \mathbb{Z}$.

We first recall the *naturality of the PV-sequence* for the K-theory of crossed products by $\mathbb{Z}$. The following theorem is well known and follows easily, for example, from the original proof of the PV-sequence ([PV]). This fundamental result will be used repeatedly in what is to follow.

Theorem 2.1 *If $\rho : (A_1, \alpha_1) \hookrightarrow (A_2, \alpha_2)$ is a covariant embedding then the following diagram is commutative with exact rows $(i \in \mathbb{Z}_2)$:*

$$\begin{array}{ccccccc}
\cdots K_i(A_1) & \xrightarrow{id-\alpha_{1*}} & K_i(A_1) & \xrightarrow{\iota_*} & K_i(A_1 \rtimes_{\alpha_1} \mathbb{Z}) & \xrightarrow{\delta} & K_{i+1}(A_1) \cdots \\
\rho_* \downarrow & & \rho_* \downarrow & & \tilde{\rho}_* \downarrow & & \rho_* \downarrow \\
\cdots K_i(A_2) & \xrightarrow{id-\alpha_{2*}} & K_i(A_2) & \xrightarrow{\iota_*} & K_i(A_2 \rtimes_{\alpha_2} \mathbb{Z}) & \xrightarrow{\delta} & K_{i+1}(A_2) \cdots
\end{array}$$

where δ is an index map and $\iota : A \hookrightarrow A \rtimes_\alpha \mathbb{Z}$ will always denote the natural inclusion.

Note that if $\alpha, \beta \in \mathrm{Aut}(A)$ and $\alpha \circ \beta = \beta \circ \alpha$ then (the embedding) $\beta : A \to A$ defines a covariant embedding $\beta : (A, \alpha) \to (A, \alpha)$. In this case, the induced map $\tilde{\beta} : A \rtimes_\alpha \mathbb{Z} \to A \rtimes_\alpha \mathbb{Z}$ is just the natural automorphism of $A \rtimes_\alpha \mathbb{Z}$ induced by β.

The following observation will prove quite useful.

Proposition 2.2 *Let (A_1, α_1), (A_2, α_2) be C^*-dynamical systems (where $\alpha_i \in \mathrm{Aut}(A_i)$) and $\rho : (A_1, \alpha_1) \hookrightarrow (A_2, \alpha_2)$ be a covariant embedding. Assume that $\rho_* : K_*(A_1) \to K_*(A_2)$ is injective and that $\alpha_{2*} = id : K_*(A_2) \to K_*(A_2)$. Then $\alpha_{1*} = id : K_*(A_1) \to K_*(A_1)$ and $\tilde{\rho}_* : K_*(A_1 \rtimes_{\alpha_1} \mathbb{Z}) \to K_*(A_2 \rtimes_{\alpha_2} \mathbb{Z})$ is injective.*

Proof. Our hypotheses easily imply the triviality of α_{1*} on K-theory. By Theorem 2.1, the following diagram is commutative with exact rows $(i \in \mathbb{Z}_2)$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_i(A_1) & \xrightarrow{\iota_*} & K_i(A_1 \rtimes_{\alpha_1} \mathbb{Z}) & \xrightarrow{\delta} & K_{i+1}(A_1) & \longrightarrow & 0 \\
 & & \rho_* \downarrow & & \rho_* \downarrow & & \rho_* \downarrow & & \\
0 & \longrightarrow & K_i(A_2) & \xrightarrow{\iota_*} & K_i(A_2 \rtimes_{\alpha_2} \mathbb{Z}) & \xrightarrow{\delta} & K_{i+1}(A_2) & \longrightarrow & 0.
\end{array}$$

But we have assumed the vertical arrows on the left and right to be injective which easily implies injectivity in the middle. □

The following well known facts will be used repeatedly without reference. (See [LOP] for a vast generalization of the first, while the second can be deduced from the first using the universal property of (full) crossed products.) In the facts below, A is a unital C^*-algebra, G is a discrete amenable group (hence the full and reduced crossed products agree) and $\alpha : G \to \mathrm{Aut}(A)$ is a homomorphism.

Fact: If $K \subset G$ is a subgroup then there is a natural inclusion $A \rtimes_{\alpha|_K} K \hookrightarrow A \rtimes_\alpha G$.

Fact: If $G = K \rtimes H$ is a semidirect product then there is a natural decomposition $A \rtimes_\alpha G \cong (A \rtimes_{\alpha|_K} K) \rtimes H$.

Finally, we recall the results from [Ki2] that we need. (See also [Ki4] for some very recent generalizations.) The following theorems build upon the previous work [Ki3] and [KK1,2]. For convenience, we let $\mathcal{C}$ denote the class of unital simple A$\mathbb{T}$ algebras with real rank zero and unique tracial state. We also let $\overline{Inn(A)}$ and $HInn(A)$ denote the groups of approximately inner automorphisms and automorphisms which are homotopic to inner automorphisms, respectively.

Theorem 2.3 (Cor. 2.3, [Ki2]) *If $A \in \mathcal{C}$ and $K_1(A) \neq \mathbb{Z}$ then the quotient group $\overline{Inn(A)}/HInn(A)$ is isomorphic to*

$$\mathrm{Ext}(K_1(A),\, K_0(A)) \oplus \mathrm{Ext}(K_0(A),\, K_1(A)).$$

Theorem 2.4 (Cor. 6.7, [Ki2]) *If $A \in \mathcal{C}$, $\alpha \in HInn(A)$, α has the Rohlin property and B is a UHF algebra then $(A \rtimes_\alpha \mathbb{Z}) \otimes B \in \mathcal{C}$.*

Our results can actually be derived from Theorem 2.4 above, but the following corollary will make the proofs a little easier. Throughout this paper, we will let $\mathcal{U}$ denote the Universal UHF algebra ($\mathcal{U} = \otimes_{n \geq 1} M_n(\mathbb{C})$).

Corollary 2.5 *If $A \in \mathcal{C}$ and $\alpha \in \overline{Inn(A)}$ is an approximately inner automorphism with the Rohlin property then $(A \rtimes_\alpha \mathbb{Z}) \otimes \mathcal{U} \in \mathcal{C}$.*

Proof. First note that $(A \rtimes_\alpha \mathbb{Z}) \otimes \mathcal{U} \cong A \otimes \mathcal{U} \rtimes_{\alpha \otimes id} \mathbb{Z}$. Note also that $K_i(A \otimes \mathcal{U}) = K_i(A) \otimes \mathbb{Q}$. In particular, both K-groups are divisible and hence $\mathrm{Ext}(K_i(A \otimes \mathcal{U}), K_{i+1}(A \otimes \mathcal{U})) = 0$ for $i \in \mathbb{Z}_2$. Thus from Theorem 2.3 we have that $\alpha \otimes id$ is homotopic to an inner automorphism of $A \otimes \mathcal{U}$. Hence from Theorem 2.4 we see that $(A \otimes \mathcal{U} \rtimes_{\alpha \otimes id} \mathbb{Z}) \otimes \mathcal{U} \cong (A \rtimes_\alpha \mathbb{Z}) \otimes \mathcal{U} \otimes \mathcal{U} \cong (A \rtimes_\alpha \mathbb{Z}) \otimes \mathcal{U}$ is again in $\mathcal{C}$. □

Corollary 2.6 *If $A \in \mathcal{C}$ and $\alpha \in \mathrm{Aut}(A)$ is an automorphism such that $\alpha_* = id : K_i(A) \to K_i(A)$ ($i = 0, 1$) then there exists a $*$-monomorphism $\rho : A \rtimes_\alpha \mathbb{Z} \to B$ where B is AF and $\rho_* : K_0(A \rtimes_\alpha \mathbb{Z}) \to K_0(B)$ is injective.*

Proof. Let $\mathcal{U}$ be the Universal UHF algebra and $\sigma \in \mathrm{Aut}(\mathcal{U})$ have the

Rohlin property (see Example 2.2 in [Br] for an explicit construction). Then we have an obvious covariant embedding $(A, \alpha) \hookrightarrow (A \otimes \mathcal{U}, \alpha \otimes \sigma)$. Since $(\alpha \otimes \sigma)_*$ is trivial on K-theory (recall that all automorphisms of UHF algebras are approximately inner [Da]) Proposition 2.2 implies that the induced embedding $A \rtimes_\alpha \mathbb{Z} \hookrightarrow A \otimes \mathcal{U} \rtimes_{\alpha\otimes\sigma} \mathbb{Z}$ yields an injective map on K-theory and hence it suffices to prove the corollary for $A \otimes \mathcal{U} \rtimes_{\alpha\otimes\sigma} \mathbb{Z}$.

Since $(\alpha \otimes \sigma)_*$ is trivial on K-theory we have that $\alpha \otimes \sigma$ is an approximately inner automorphism of $A \otimes \mathcal{U}$ ([Ell]). Also, it follows easily from the definition that $\alpha \otimes \sigma$ has the Rohlin property. Thus by Corollary 2.5 we see that $(A \otimes \mathcal{U} \rtimes_{\alpha\otimes\sigma} \mathbb{Z}) \otimes \mathcal{U} \in \mathcal{C}$. Hence (since $\mathcal{C}$ consists of A$\mathbb{T}$ algebras of real rank zero) there exists a $*$-homomorphism (injective by simplicity) $\rho : (A \otimes \mathcal{U} \rtimes_{\alpha\otimes\sigma} \mathbb{Z}) \otimes \mathcal{U} \hookrightarrow B$ where B is AF and ρ_* is an isomorphism on K_0 (cf. [Ell]). □

Corollary 2.7 *If $A \in \mathcal{C}$ is AF and $\alpha : \mathbb{Z}^2 \to \overline{Inn(A)}$ is a homomorphism then $A \rtimes_\alpha \mathbb{Z}^2$ is AF embeddable with an injective map on $K_0(A \rtimes_\alpha \mathbb{Z}^2)$.*

Proof. Let e_1, e_2 denote the canonical generators of $\mathbb{Z}^2$ and $\widetilde{\alpha(e_2)}$ denote the induced automorphism of $A \rtimes_{\alpha(e_1)} \mathbb{Z}$. Theorem 2.1 implies commutativity of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_i(A) & \xrightarrow{\iota_*} & K_i(A \rtimes_{\alpha(e_1)} \mathbb{Z}) & \xrightarrow{\delta} & K_{i+1}(A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha(e_2)_*} & & \downarrow{\scriptstyle \widetilde{\alpha(e_2)}_*} & & \downarrow{\scriptstyle \alpha(e_2)_*} & & \\ 0 & \longrightarrow & K_i(A) & \xrightarrow{\iota_*} & K_i(A \rtimes_{\alpha(e_1)} \mathbb{Z}) & \xrightarrow{\delta} & K_{i+1}(A) & \longrightarrow & 0. \end{array}$$

Since $K_1(A) = 0$ and the rows above are exact, we see that $\widetilde{\alpha(e_2)}_*$ is trivial on both K-groups of $A \rtimes_{\alpha(e_1)} \mathbb{Z}$.

As in the proof of the previous corollary, the obvious embedding $A \rtimes_{\alpha(e_1)} \mathbb{Z} \hookrightarrow A \otimes \mathcal{U} \rtimes_{\alpha(e_1)\otimes\sigma} \mathbb{Z}$ is injective at the level of K-theory. Note that $\alpha(e_2) \otimes id$ commutes with $\alpha(e_1) \otimes \sigma$ and hence there is an induced automorphism $\widetilde{\alpha(e_2) \otimes id} \in \mathrm{Aut}(A \otimes \mathcal{U} \rtimes_{\alpha(e_1)\otimes\sigma} \mathbb{Z})$. Note also that $\widetilde{\alpha(e_2) \otimes id}$ is trivial on K-theory by the first paragraph of this proof. It is easy to see the commutativity of the diagram

$$\begin{array}{ccc} A \rtimes_{\alpha(e_1)} \mathbb{Z} & \longrightarrow & A \otimes \mathcal{U} \rtimes_{\alpha(e_1)\otimes\sigma} \mathbb{Z} \\ \downarrow{\scriptstyle \widetilde{\alpha(e_2)}} & & \downarrow{\scriptstyle \widetilde{\alpha(e_2)\otimes id}} \\ A \rtimes_{\alpha(e_1)} \mathbb{Z} & \longrightarrow & A \otimes \mathcal{U} \rtimes_{\alpha(e_1)\otimes\sigma} \mathbb{Z}. \end{array}$$

Thus by Proposition 2.2, the induced embedding $A \rtimes_\alpha \mathbb{Z}^2 \cong (A \rtimes_{\alpha(e_1)} \mathbb{Z}) \rtimes_{\widetilde{\alpha(e_2)}} \mathbb{Z} \hookrightarrow (A \otimes \mathcal{U} \rtimes_{\alpha(e_1)\otimes\sigma} \mathbb{Z}) \rtimes_{\widetilde{\alpha(e_2)\otimes id}} \mathbb{Z}$ is injective on K-theory. Finally, it is known that $A \otimes \mathcal{U} \rtimes_{\alpha(e_1)\otimes\sigma} \mathbb{Z} \in \mathcal{C}$ (cf. Thm. 6.4 in [Ki2]) and hence we are done by the previous corollary. □

Remark 2.8 The reason that we can deduce this corollary from Corollary 2.5 (via Corollary 2.6) is that $\widetilde{\alpha(e_2)}_*$ is trivial on the K-theory of $A \rtimes_{\alpha(e_1)} \mathbb{Z}$ (since A is AF). Which leads to a natural question. If $A \in \mathcal{C}$ and $\alpha, \beta \in \overline{Inn(A)}$ are commuting automorphisms then is $\tilde{\beta}_*$ always trivial on $K_*(A \rtimes_\alpha \mathbb{Z})$? While we were unable to construct a specific example, A. Kishimoto has shown us a result (unpublished) which leads one to believe that $\tilde{\beta}_*$ need not always be trivial.

3. Main Results

Recall that $C^*(\mathbb{Z}^n) = C(\mathbb{T}^n)$, where $\mathbb{T}^n$ is the n-torus, and whenever A is unital there is a natural inclusion $\eta : C(\mathbb{T}^n) \hookrightarrow A \rtimes_\alpha \mathbb{Z}^n$. If $\alpha : \mathbb{Z}^n \to \mathrm{Aut}(A)$ is given, we will let $\alpha_j : \mathbb{Z}^j \to \mathrm{Aut}(A)$ $(1 \le j \le n)$ denote the restriction of α to the first j coordinates of $\mathbb{Z}^n$. We will also let $\{e_i\}_{1\le i\le n}$ be the canonical generators of $\mathbb{Z}^n$ and $\widetilde{\alpha(e_j)}$ denote the automorphism of $A \rtimes_{\alpha_{j-1}} \mathbb{Z}^{j-1}$ induced by $\alpha(e_j)$ $(1 \le j \le n)$.

Proposition 3.1 *Let $\mathcal{U}$ be the Universal UHF algebra ($\mathcal{U} = \otimes_{n\ge1} M_n(\mathbb{C})$) and $\alpha : \mathbb{Z}^n \to \mathrm{Aut}(\mathcal{U})$ be a homomorphism. Then the following assertions hold.*

1. $\widetilde{\alpha(e_n)}_* = id : K_i(\mathcal{U} \rtimes_{\alpha_{n-1}} \mathbb{Z}^{n-1}) \to K_i(\mathcal{U} \rtimes_{\alpha_{n-1}} \mathbb{Z}^{n-1})$ *for* $i = 0, 1$.
2. $K_i(\mathcal{U} \rtimes_\alpha \mathbb{Z}^n) = \mathbb{Q}^{2^{n-1}}$ *for* $i = 0, 1$.
3. $\eta_* : K_i(C(\mathbb{T}^n)) \to K_i(\mathcal{U} \rtimes_\alpha \mathbb{Z}^n)$ *is injective for* $i = 0, 1$.

Proof. The PV sequence implies that the proposition holds when $n = 1$ and hence proceeding by induction we may assume the proposition to hold for $n = 1, \ldots, k$. Assume now that we have a homomorphism $\alpha : \mathbb{Z}^{k+1} \to \mathrm{Aut}(\mathcal{U})$. By the induction hypotheses we have that the inclusion $\eta : C(\mathbb{T}^k) \hookrightarrow A \rtimes_{\alpha_k} \mathbb{Z}^k$ gives an injective map on K-theory and $K_i(\mathcal{U} \rtimes_{\alpha_k} \mathbb{Z}^k) = \mathbb{Q}^{2^{k-1}}$ for $i = 0, 1$. However, the induced automorphism $\widetilde{\alpha(e_{k+1})}$ of $\mathcal{U} \rtimes_{\alpha_k} \mathbb{Z}^k$ restricts to the identity on $\eta(C(\mathbb{T}^k))$ and hence induces the identity map on $\mathbb{Z}^{2^{k-1}} = K_i(C(\mathbb{T}^k)) = \eta_*(K_i(C(\mathbb{T}^k)))$. But since $K_i(\mathcal{U} \rtimes_{\alpha_k} \mathbb{Z}^k) = \mathbb{Q}^{2^{k-1}}$ for $i = 0, 1$ we see that $\widetilde{\alpha(e_{k+1})}_* = id : K_i(\mathcal{U} \rtimes_{\alpha_k} \mathbb{Z}^k) \to K_i(\mathcal{U} \rtimes_{\alpha_k} \mathbb{Z}^k)$.

By Theorem 2.1, the following diagram is commutative with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_i(C(\mathbb{T}^k)) & \longrightarrow & K_i(C(\mathbb{T}^{k+1})) & \longrightarrow & K_{i+1}(C(\mathbb{T}^k)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Q}^{2^{k-1}} & \longrightarrow & K_i(\mathcal{U} \rtimes_\alpha \mathbb{Z}^{k+1}) & \longrightarrow & \mathbb{Q}^{2^{k-1}} & \longrightarrow & 0. \end{array}$$

However, this diagram implies the remaining two assertions. □

Proof of Theorem 1. Let A be a UHF algebra and $\alpha : \mathbb{Z}^n \to \mathrm{Aut}(A)$ be a homomorphism. We first observe that any covariant embedding $\rho : (A, \mathbb{Z}^n, \alpha) \hookrightarrow (\mathcal{U}, \mathbb{Z}^n, \beta)$ (i.e. a unital $*$-homomorphism $\rho : A \to \mathcal{U}$ such that $\beta(g) \circ \rho = \rho \circ \alpha(g)$ for all $g \in \mathbb{Z}^n$) induces an embedding $\tilde{\rho} : A \rtimes_\alpha \mathbb{Z}^n \hookrightarrow \mathcal{U} \rtimes_\beta \mathbb{Z}^n$ which is injective on K-theory.

If $n = 1$ this follows from Proposition 2.2 (it is well known that any embedding $A \hookrightarrow \mathcal{U}$ is injective on K-theory), so we assume it to be true for $1, \ldots, n-1$. Any covariant embedding $\rho : (A, \mathbb{Z}^n, \alpha) \hookrightarrow (\mathcal{U}, \mathbb{Z}^n, \beta)$ restricts to an covariant embedding $\rho_{n-1} : (A, \mathbb{Z}^{n-1}, \alpha_{n-1}) \hookrightarrow (\mathcal{U}, \mathbb{Z}^{n-1}, \beta_{n-1})$ which induces an embedding $\tilde{\rho}_{n-1} : A \rtimes_{\alpha_{n-1}} \mathbb{Z}^{n-1} \hookrightarrow \mathcal{U} \rtimes_{\beta_{n-1}} \mathbb{Z}^{n-1}$ with $(\tilde{\rho}_{n-1})_*$ injective on K-theory (by induction hypothesis). One easily checks commutativity in the diagram

$$\begin{array}{ccc} A \rtimes_{\alpha_{n-1}} \mathbb{Z}^{n-1} & \xrightarrow{\tilde{\rho}_{n-1}} & \mathcal{U} \rtimes_{\beta_{n-1}} \mathbb{Z}^{n-1} \\ \widetilde{\alpha(e_n)} \Big\downarrow & & \Big\downarrow \widetilde{\beta(e_n)} \\ A \rtimes_{\alpha_{n-1}} \mathbb{Z}^{n-1} & \xrightarrow{\tilde{\rho}_{n-1}} & \mathcal{U} \rtimes_{\beta_{n-1}} \mathbb{Z}^{n-1}, \end{array}$$

where $\widetilde{\alpha(e_n)}$ and $\widetilde{\beta(e_n)}$ denote the induced automorphisms of $A \rtimes_{\alpha_{n-1}} \mathbb{Z}^{n-1}$ and $\mathcal{U} \rtimes_{\beta_{n-1}} \mathbb{Z}^{n-1}$, respectively. Thus from Propositions 3.1.1 and 2.2 we see that the induced embedding $\tilde{\rho} : A \rtimes_\alpha \mathbb{Z}^n \hookrightarrow \mathcal{U} \rtimes_\beta \mathbb{Z}^n$ is also injective at the level of K-theory. (Note that this also shows that $\widetilde{\alpha(e_n)}$ is trivial on the K-theory of $A \rtimes_{\alpha_{n-1}} \mathbb{Z}^{n-1}$ and hence $K_i(A \rtimes_\alpha \mathbb{Z}^n)$ defines an element of $\mathrm{Ext}(K_i(A \rtimes_{\alpha_{n-1}} \mathbb{Z}^{n-1}), K_{i+1}(A \rtimes_{\alpha_{n-1}} \mathbb{Z}^{n-1})$ for $i \in \mathbb{Z}_2$.)

Since $A \otimes \mathcal{U} \cong \mathcal{U}$ one readily verifies that covariant embeddings $\rho : (A, \mathbb{Z}^n, \alpha) \to (\mathcal{U}, \mathbb{Z}^n, \beta)$ always exist and hence it suffices to prove the theorem for all the C^*-dynamical systems $(\mathcal{U}, \mathbb{Z}^n, \beta)$.

We now further reduce to the case that each $\widetilde{\beta(e_i)}$ has the Rohlin property as an automorphism of $\mathcal{U} \rtimes_{\beta_{i-1}} \mathbb{Z}^{i-1}$. To arrange this we simply define an embedding $\rho : \mathcal{U} \to \otimes_1^{n+1} \mathcal{U}$ by $\rho(x) = x \otimes 1 \otimes \cdots \otimes 1$ and an action γ

of $\mathbb{Z}^n$ on $\otimes_1^{n+1}\mathcal{U}$ ($\cong \mathcal{U}$) by $\gamma(e_i) = \beta(e_i) \otimes id \otimes \cdots \otimes id \otimes \sigma \otimes id \otimes \cdots \otimes id$ (where $\sigma \in \mathrm{Aut}(\mathcal{U})$ has the Rohlin property and appears in the $i+1^{st}$ coordinate above). It is easy to check that each $\widetilde{\gamma(e_i)}$ has the Rohlin property as an automorphism of $\mathcal{U} \rtimes_{\gamma_{i-1}} \mathbb{Z}^{i-1}$ and hence we have reduced to this case. (Note that the K-theory of $\mathcal{U} \rtimes_\beta \mathbb{Z}^n$ sits injectively inside that of $\mathcal{U} \rtimes_\gamma \mathbb{Z}^n$ by the first part of the proof.)

But we are now done since an easy induction argument shows that $(\mathcal{U} \rtimes_\gamma \mathbb{Z}^n) \otimes \mathcal{U} \in \mathcal{C}$. (Use Corollary 2.5 and the isomorphisms $(\mathcal{U} \rtimes_\gamma \mathbb{Z}^n) \otimes \mathcal{U} \cong (\mathcal{U} \rtimes_\gamma \mathbb{Z}^n) \otimes \mathcal{U} \otimes \mathcal{U} \cong ((\mathcal{U} \rtimes_{\gamma_{n-1}} \mathbb{Z}^{n-1}) \otimes \mathcal{U} \rtimes_{\widetilde{\gamma(e_n)} \otimes id} \mathbb{Z}) \otimes \mathcal{U}$.) □

Remark 3.2 T. Loring first discovered that there exist AF embeddings $\rho : C(\mathbb{T}^2) \hookrightarrow B$ such that ρ_* was injective on K_0 ([Lo]). Theorem 1 together with Proposition 3.1.3 also provides AF embeddings of $C(\mathbb{T}^n)$ with injective maps on K_0 (see [DL], [EL] for more general results though).

We now recall an imprimitivity theorem which is due to P. Green. If G is a separable locally compact group and $H \subset G$ is a closed subgroup then G/H (the space of left cosets) is a separable locally compact space. There is a natural action γ of G on $C_0(G/H)$ defined by $\gamma_g(f)(xH) = g^{-1}xH$ for all $xH \in G/H$ and $f \in C_0(G/H)$. The crossed products below are the full crossed products (as G is not required to be amenable).

Theorem 3.3 (Cor. 2.8, [Gr]) *Let $\alpha : G \to \mathrm{Aut}(A)$ be a homomorphism from the separable locally compact group G. For each closed subgroup $H \subset G$ there is an isomorphism $A \otimes C_0(G/H) \rtimes_{\alpha\otimes\gamma} G \cong (A \rtimes_{\alpha|_H} H) \otimes \mathcal{K}$, where $\mathcal{K}$ denotes the compact operators on a separable (possibly finite dimensional) Hilbert space.*

Note that if G/H is compact and G is amenable then there is a natural inclusion $A \rtimes_\alpha G \hookrightarrow A \otimes C(G/H) \rtimes_{\alpha\otimes\gamma} G$ (cf. 7.7.9 in [Pe]).

We are now in a position to prove Theorem 2. However, we first give a precise definition of the classes Γ and Γ_{fg} described in the introduction.

Definition 3.4 Let Γ (resp. Γ_{fg}) denote the class of separable locally compact groups G with the following property: There exist subgroups $H_1 \subset H_2 \subset \cdots \subset H_n = G$ such that (1) H_i is a closed normal subgroup of H_{i+1} for $i = 1, \ldots, n-1$, (2) H_{i+1}/H_i is compact for $i = 1, \ldots, n-1$ and (3) H_1 is discrete and abelian (resp. finitely generated discrete and abelian).

Note that since extensions of amenable groups are again amenable, the classes Γ and Γ_{fg} consist of amenable groups.

Proof of Theorem 2. First, let $G \in \Gamma_{fg}$ and $\alpha : G \to \mathrm{Aut}(A)$ be a homomorphism. Let $H_1 \subset H_2 \subset \cdots \subset H_n = G$ be subgroups satisfying the above definition. Then for each $1 \leq i \leq n-1$ we appeal to Theorem 3.3 to get embeddings

$$A \rtimes_{\alpha|_{H_{i+1}}} H_{i+1} \hookrightarrow (A \rtimes_{\alpha|_{H_i}} H_i) \otimes \mathcal{K}.$$

Composing these embeddings we get

$$A \rtimes_\alpha G \hookrightarrow (A \rtimes_{\alpha|_{H_1}} H_1) \otimes \mathcal{K},$$

where H_1 is a finitely generated discrete abelian group. Thus $H_1 \cong \mathbb{Z}^k \oplus F$ for some $k \in \mathbb{Z}$ and finite group F. Applying Theorem 3.3 one more time to the subgroup $\mathbb{Z}^k \subset H_1$ we get an embedding

$$A \rtimes_{\alpha|_{H_1}} H_1 \hookrightarrow (A \rtimes_{\alpha|_{\mathbb{Z}^k}} \mathbb{Z}^k) \otimes \mathcal{K}.$$

Thus from Theorem 1 we get an AF embedding of $A \rtimes_\alpha G$. In particular, we see that $A \rtimes_\alpha H$ is quasidiagonal for all finitely generated discrete abelian groups.

In the case that $G \in \Gamma$, the above argument still provides an embedding

$$A \rtimes_\alpha G \hookrightarrow (A \rtimes_{\alpha|_H} H) \otimes \mathcal{K},$$

for some discrete abelian group H. But then we simply write $H = \bigcup_{\lambda \in \Lambda} H_\lambda$, where each H_λ is finitely generated, and observe that $\bigcup_{\lambda \in \Lambda} A \rtimes_{\alpha|_{H_\lambda}} H_\lambda$ is dense in $A \rtimes_{\alpha|_H} H$. But since locally quasidiagonal algebras are quasidiagonal, this proves the theorem. □

Remark 3.5 Theorem 2 holds for more general groups than just the class Γ defined above. Note that Theorem 3.3 does not require H to be a normal subgroup of G. One may also drop the separability hypothesis on G if one assumes the existence of a 'measurable cross section' (see the introduction of [Gr] for a precise definition). In particular, Theorem 2 holds for arbitrary compact or discrete abelian groups.

Note also that since quotients of compact groups are again compact, one readily verifies that Γ is closed under taking quotients. Hence if G is a discrete group which is isomorphic to an inductive limit of discrete elements of Γ then $A \rtimes_\alpha G$ is always quasidiagonal (when A is UHF).

Remark 3.6 M. Izumi first showed us a proof that $A \rtimes_\alpha \mathbb{R}$ is always AF embeddable whenever A is an AF algebra. In fact, it was trying to understand his proof that eventually led us to Green's imprimitivity theorem above. Indeed, what Theorem 3.3 above implies (which is also what Izumi first proved) is that if A is a C^*-algebra with the property that for every $\alpha \in \mathrm{Aut}(A)$ which is homotopic to the identity, the crossed product $A \rtimes_\alpha \mathbb{Z}$ is AF embeddable then it follows that *every* crossed product of A by $\mathbb{R}$ is also AF embeddable.

In particular, since it was first proved in [Vol] that all crossed products of AF algebras by approximately inner automorphisms are AF embeddable (see also [Br]) and Corollary 2.6 asserts the same thing for algebras in the class $\mathcal{C}$ (defined in Section 2) we conclude that $A \rtimes_\alpha \mathbb{R}$ is always AF embeddable whenever A is AF or if $A \in \mathcal{C}$. Similarly, Theorem 3.3 and Corollary 2.7 imply that $A \rtimes_\alpha \mathbb{R}^2$ is always AF embeddable whenever $A \in \mathcal{C}$ is AF.

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