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On pairs of regular foliations in the plane

R.D.S. OLIVEIRA and F. TARI

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Abstract. We classify in this paper germs of pairs of regular foliations in the plane. We show that under some conditions, the topological type of a pair is completely determined by the discriminant, which is the locus of points where the foliations are tangential. We also classify all the $\mathcal{K}^*$-singularities of the discriminant that lie in the $\mathcal{K}$-simple singularities.

Key words: discriminants, pairs of foliations, singularities, topological equivalence.

1. Introduction

In this paper we study germs of pairs of regular foliations in the plane. Pairs of foliations have been studied by several authors with applications to implicit differential equations ([2], [3], [5]) and control theory ([8]). See also [11]–[15], [17], [18]. We shall assume here that the foliations are the leaves of differential 1-forms α and β. An important feature of the pair is its discriminant, that is the locus of points where the form α is a multiple of β. If we are only interested in the singularities of the discriminant and the way this curve bifurcates as the pair (α, β) is deformed, we proceed as in [6] as follows. Let $(\alpha, \beta) = (a(x, y)dx + b(x, y)dy,\ c(x, y)dx + d(x, y)dy)$, where a, b, c, d are germs of smooth functions. We associate to the pair the family of matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}(x, y)$, which is a map-germ $A : \mathbb{R}^2, 0 \to M(2, \mathbb{R})$, where $M(2, \mathbb{R})$ is the set of 2×2 matrices. Then the discriminant of (α, β) is the zero set of the determinant of the matrix A. Given the matrix A above, and two other 2×2 matrices X and Y, whose entries are smooth functions in (x, y), and which are invertible at the origin, we can consider the matrix valued function XAY. Clearly its determinant vanishes at precisely the set of points where the discriminant of A vanishes. Similarly it is not hard to show that any smooth change of coordinates in the source of A, via a diffeomorphism ϕ, takes the discriminant of A to that of $A \circ \phi$. All these changes of coordinates form a subgroup $\mathcal{G}$ of the contact group

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$\mathcal{K}$ acting on the space of families of matrices. So one can classify the $\mathcal{G}$-singularities of A and obtain the singularities of the discriminant as well as their versal deformations. This approach deals well with the singularities of the discriminant, and in the case of regular foliations reduces to the $\mathcal{K}$-classification of function ([6]). However, the above changes of coordinates do not preserve the foliations of (α, β). For example the pairs $(dy, d(y-x^2))$ and $(dy, d(y+xy-x^3))$ are not topologically equivalent but have smooth discriminants which are thus $\mathcal{K}$-equivalent.

We start in this paper by obtaining another classification of the singularities of the discriminant that preserves some geometric information about the pair of foliations. Without loss of generality, we can assume that, locally at the origin, one foliation is given by the 1-form $\alpha = dy$ and the other as the level sets of a regular function f, that is $\beta = df$. A topologically stable singularity of the pair occurs when the discriminant is smooth and the leaf of β has 3-point contact with the leaf of α at the origin. This is modelled by $(dy, d(y+xy-x^3))$ (denoted by 3_1 in [14]) and occurs at isolated points ([8], [14]). Given a pair (dy, df) as above, we can calculate the maximum number of 3_1-singularities that occur in a deformation of the pair. This number is given by $\#3_1 = \dim_{\mathbb{C}} \mathcal{O}_2/\langle f_x, f_{xx} \rangle$, where $\mathcal{O}_2$ denotes the set of holomorphic germs $\mathbb{C}^2, 0 \to \mathbb{C}$. The singularity 3_1 can also be characterized by the discriminant being smooth and having ordinary tangency with the leaf of $\alpha = dy$ (and β). If we wish to model the singularities of the discriminant where its tangency with the foliation of α is preserved, we need to classify germs of functions $\mathbb{R}^2, 0 \to \mathbb{R}, 0$ up to a subgroup of the contact group $\mathcal{K}$, denoted by $\mathcal{K}^*$, where the changes of coordinates in the source preserve the horizontal lines (the foliation of α). This we do in Section 2 where we list all the $\mathcal{K}^*$-singularities of the discriminant that are inside the $\mathcal{K}$-simple singularities.

We prove in Section 2 that for a large class of pairs of regular foliations, the discriminant determines the topological type of the pair. More precisely, pairs of regular foliations with a discriminant having at most two branches in each half region delimited by the leaf of α (or β) at the origin can be classified topologically. We comment on the cases when the number of branches is greater than two.

2. $\mathcal{K}^*$-classification of the discriminant function

Let (α, β) denote a germ at the origin of a pair of smooth 1-forms in the plane. When the 1-forms are regular we can assume that the foliation of α is given by the horizontal lines and the foliation of β is the level sets of a regular function $f : \mathbb{R}^2, 0 \to \mathbb{R}, 0$. The discriminant of (α, β) is the set of points p where $\alpha(p)$ is a multiple of $\beta(p)$, that is, points where $\alpha \wedge \beta$ vanishes. This is given by the zero set of $f_x(x, y)$, where the subscript denotes the partial derivative with respect to x.

We say that a pair of 1-forms have k-point contact at the origin if the leaves through the origin of the two forms have order of contact k at that point.

Two pairs of foliations are smoothly (resp. topologically) equivalent if there exists a diffeomorphism (resp. homeomorphism) taking the leaves of one pair to the other. (Note that we can multiply a 1-form by non-vanishing functions as this leaves its foliation unchanged.)

As highlighted in the introduction, we seek a classification of germs of functions $\mathbb{R}^2, 0 \to \mathbb{R}, 0$ under the action of the subgroup $\mathcal{K}^*$ of the contact group $\mathcal{K}$, where the changes of coordinates in the source preserve the horizontal lines, i.e. they are of the form $(\phi(x, y), \psi(y))$. The group $\mathcal{K}^*$ is a geometric subgroup of $\mathcal{K}$ ([7]) so all the results on determinacy of germs apply here.

It is not hard to show that any smoothly equivalent pairs yield $\mathcal{K}^*$-equivalent discriminants. Also, if δ denotes the discriminant function of the pair, then $\#3_1 = \dim_{\mathbb{C}} \mathcal{O}_2 / \langle \delta, \delta_x \rangle$ and this number is $\mathcal{K}^*$-invariant.

The $\mathcal{K}^*$ classification of germs $\delta : \mathbb{R}^2, 0 \to \mathbb{R}, 0$ is carried out inductively on the jet levels of δ. We use the complete transversal results in [4] to obtain all the finitely $\mathcal{K}^*$-germs that are inside the $\mathcal{K}$-simple singularities A_k, D_k, E_6, E_7, E_8. We also calculate the $\mathcal{K}^*$-codimension of δ, the invariant $\#3_1$ and the number of branches of $\Delta = \delta^{-1}(0)$ in each half plane $y > 0$ and $y < 0$. This (unordered) pair of numbers, which is $\mathcal{K}^*$-invariant, is key in determining the topological class of a pair of regular foliations.

Theorem 2.1 *The finitely $\mathcal{K}^*$-determined germs $\delta : \mathbb{R}^2, 0 \to \mathbb{R}, 0$ that lie in the $\mathcal{K}$-simple singularities are listed below.*

Normal form	$\mathcal{K}$-*class*	$\mathcal{K}^*$-*codim*	$\#3_1$	*#branches*
x	A_0	0	0	$0,0$
$y+x^k,\ 2\leq k$	A_0	$k-2$	$k-1$	$0,2$ $(+)$ $1,1$ $(-)$
$x^2+y^k,\ 2\leq k$	A_{k-1}	$k-1$	k	$0,0$ $(+)$ $0,2$ $(-)$
$x^2-y^{2k},\ 1\leq k$	A_{2k-1}	$2k-1$	$2k$	$2,2$
$xy+x^k,\ 3\leq k$	A_1	$k-1$	k	$2,2$ $(+)$ $1,3$ $(-)$
$y^2+x^k,\ 2\leq k$	A_{k-1}	$2k-3$	$2k-2$	$0,0$ $(+)$ $1,1$ $(-)$
$y^2-x^{2k},\ 2\leq k$	A_{2k-1}	$4k-3$	$4k-2$	$2,2$
$y^2+x^ky+\mathbf{a}x^l,\ 2\leq k<l<2k$	A_{l-1}	$2l-4$	$2l-2$	$0,0$ $(+,\pm)$ $1,1$ $(-,\pm)$
$y^2+x^ky+\mathbf{b}x^l,\ 4\leq 2k<l$	A_{2k-1}	$l+2k-4$	$l+2k-2$	$0,4$ $(+,+)$ $1,3$ $(-,\pm)$ $2,2$ $(+,-)$
$y^2+x^ky-\mathbf{a}x^l,\ 2\leq k<l<2k$	A_{l-1}	$2l-4$	$2l-2$	$2,2$ $(+,\pm)$ $1,1$ $(-,\pm)$
$y^2+x^ky-\mathbf{b}x^l,\ 4\leq 2k<l$	A_{2k-1}	$l+2k-4$	$l+2k-2$	$2,2$ $(+,\pm)$ $1,3$ $(-,\pm)$
$y^2+x^ky+ax^{2k}+\mathbf{c}x^l,$ $4\leq 2k<l\leq 3k-1,\ a\neq 0,\ \frac{1}{4}$	A_{2k-1}	$4k-3$	$4k-2$	$0,0$ (1) $0,4$ (2) $2,2$ (3) $2,2$ (4)
$y^2+x^ky+ax^{2k},\ 2\leq k,$ $a\neq 0,\ \frac{1}{4}$	A_{2k-1}	$4k-2$	$4k-2$	$0,0$ (1) $0,4$ (2) $2,2$ (3) $2,2$ (4)

Normal form	$\mathcal{K}$*-class*	$\mathcal{K}^*$*-codim*	$\#3_1$	*#branches*
$x^3+axy^2+y^3$, $\eta=4a^3+27\neq 0$	D_4	5	6	$1,1\quad \eta>0$ $3,3\quad \eta<0$
$xy^2+x^k,\ 3\le k$	D_{k+1}	$2k-1$	$2k$	$2,2\quad (+)$ $1,1\quad (-)$
$xy^2-x^{2k+1},\ 2\le k$	D_{2k+2}	$4k+1$	$4k+2$	$3,3$
$x^2y+x^k+y^l,\ 4\le k,\ 3\le l$	D_{l+1}	$l+k-2$	$l+k$	$0,2\quad (-,+)$ $1,1\quad (-,-)$ $0,4\quad (+,+)$ $1,3\quad (+,-)$
$x^2y+x^k-y^l,\ 4\le k,l$	D_{l+1}	$l+k-2$	$l+k$	$2,4\quad (-,+)$ $3,3\quad (-,-)$ $2,2\quad (+,+)$ $1,3\quad (+,-)$
$xy^2+x^ky+\mathbf{a}x^l$, $3\le k<l<2k-1$	D_{l+1}	$2l-3$	$2l$	$2,2\quad (+,\pm)$ $1,1\quad (-,\pm)$
$xy^2+x^ky+\mathbf{d}x^l$, $5\le 2k-1<l$	D_{2k}	$l+2k-4$	$l+2k-1$	$2,4\quad (+,\pm)$ $3,3\quad (-,+)$ $1,5\quad (-,-)$
$xy^2+x^ky-\mathbf{a}x^l$, $3\le k<l<2k-1$	D_{l+1}	$2l-3$	$2l$	$2,2\quad (+,\pm)$ $3,3\quad (-,\pm)$
$xy^2+x^ky-\mathbf{d}x^l$, $5\le 2k-1<l$	D_{2k}	$l+2k-4$	$l+2k-1$	$2,4\quad (+,\pm)$ $3,3\quad (-,\pm)$
$xy^2+x^ky+ax^{2k-1}+\mathbf{e}x^l$, $5\le 2k-1<l\le 3k-3$, $a\neq 0,\ \frac{1}{4}$	D_{2k}	$4k-5$	$4k-2$	$1,1\quad (1)$ $3,3\quad (2)$ $1,5\quad (3)$ $3,3\quad (4)$
$xy^2+x^ky+ax^{2k-1},\ 3\le k$, $a\neq 0,\ \frac{1}{4}$	D_{2k}	$4k-4$	$4k-2$	$1,1\quad (1)$ $3,3\quad (2)$ $1,5\quad (3)$ $3,3\quad (4)$

Normal form	$\mathcal{K}$*-class*	$\mathcal{K}^*$*-codim*	$\#3_1$	*#branches*
$x^3+y^4 \pm xy^k,\ 3 \le k$	E_6	$k+3$	8	$1,1$
$y^3 \pm x^2y^2+x^4$	E_6	7	9	$0,2$
y^3+x^4	E_6	8	9	$0,2$
$x^3+xy^3+y^5$	E_7	7	9	$1,3$
x^3+xy^3	E_7	8	9	$1,3$
$y^3+x^3y+x^k+ax^{k+1},\ 5 \le k$	E_7	$k+4$	$k+6$	$2,2\ (+)$ $1,3\ (-)$
$x^3+y^5 \pm xy^k,\ 4 \le k$	E_8	$k+4$	10	$1,1$
$y^3+x^2y^2+x^5+ax^6$	E_8	11	12	$1,1$
y^3+x^5	E_8	11	12	$1,1$

$\mathbf{a} = 1+a_1x+\cdots+a_{l-k-2}x^{l-k-2}$
$\mathbf{b} = 1+b_1x+\cdots+b_{k-2}x^{k-2}$
$\mathbf{c} = 1+c_1x+\cdots+b_{3k-l-1}x^{3k-l-1}$
$\mathbf{d} = 1+d_1x+\cdots+b_{k-3}x^{k-3}$
$\mathbf{e} = 1+e_1x+\cdots+e_{3k-l-4}x^{3k-l-4}$

l, k even (resp. odd) is represented by the sign $+$ (resp. $-$), so for example $(l,k)=(+,+)$ means both l and k are even.

$(\mathbf{1}): a > \frac{1}{4}$; $(\mathbf{2}): 0 < a < \frac{1}{4}$, $(\pm,+)$; $(\mathbf{3}): 0 < a < \frac{1}{4}$, $(\pm,-)$;
$(\mathbf{4}): a < 0$.

Proof. The classification follows by relatively straight forward calculations that we omit here. We observe that the first three classes are known [10]. To calculate the codimension of the ideal $\langle\, \delta, \delta_x \,\rangle$ in $\mathcal{O}_2$ we use the results in [1] for the non-trivial cases. □

3. Topological classification of pairs of regular foliations

We show in this section that the discriminant determines the topological type of a large number of pairs of regular foliations in the plane.

Theorem 3.1 *Let (α,β) be a pair of germs, at the origin, of regular 1-forms in the plane. Suppose that the discriminant Δ of the pair has at most two branches in each semi-region defined by the leaf of α (or β) through the origin.*

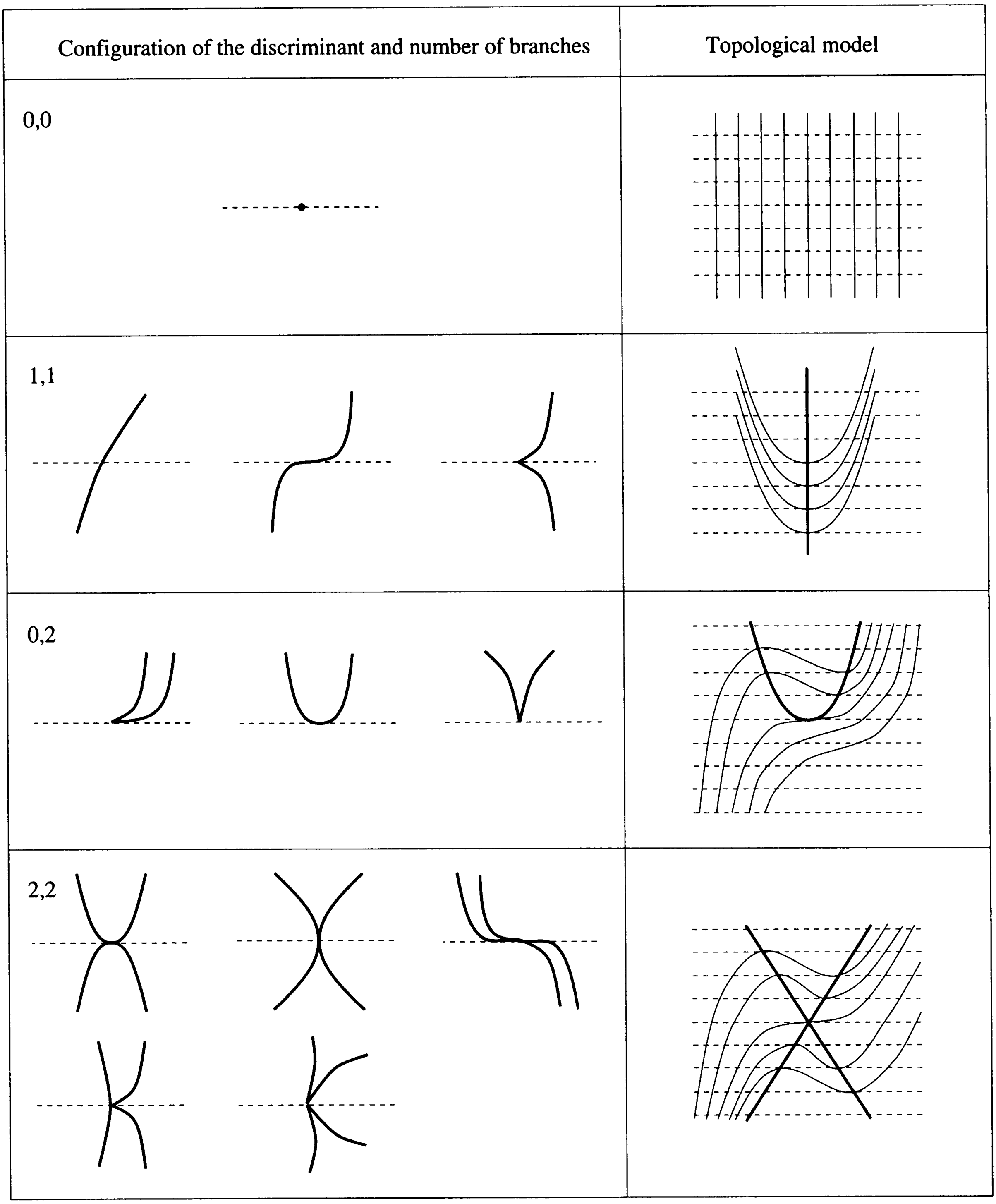

Fig. 1. Topological models of pairs of regular foliations (discriminant in thick line).

Then the pair is topologically equivalent to one of the following:

(i) (dx, dy), *if* Δ *is empty or is an isolated point*;

(ii) $(dy, d(y - x^2))$, *if* Δ *has one branch in each half region*;

(iii) $(dy, d(y+xy-x^3))$, *if* Δ *has two branches in one half region and none in the other*;

(iv) $(dy, d(y+xy^2-x^3))$, *if* Δ *has two branches in each half region.*

See Figure 1. *In particular, if* Δ *is a regular curve then* (α,β) *is topologically equivalent to*

(a) $(dy, d(y-x^2))$, *if the contact of* Δ *with the leaf of* α *(and* β*) at the origin is odd*;

(b) $(dy, d(y+xy-x^3))$, *if the contact of* Δ *with the leaf of* α *(and* β*) at the origin is even.*

Proof. Case (i) is not difficult and shall be omitted. We can suppose that $\alpha = dy$ and $\beta = df$, where f is a germ of a regular function. Note that away from the origin the foliations have ordinary tangency on the discriminant, and the pair is locally smoothly equivalent to $(dy, d(y-x^2))$ ([8]). Let (α,β') denote the pair of 1-forms that yield the foliations of the model.

(ii) The curve Δ separates the plane into two regions. In one (open) region the function f is increasing and in the other decreasing (Figure 2). Choose a region, say region 1 as in Figure 2 and assume without loss of generality that f is decreasing there. We orient the discriminant upwards. Then a point q in region 1 is uniquely determined as the intersection of the leaf of α starting from a point q_α on Δ and the leaf of β starting from a point q_β on Δ. Observe that $q_\beta \leq q_\alpha$. Let $\beta' = d(y-x^2)$ and Δ' be the discriminant of (α,β') (oriented upwards). This discriminant also separates the plane into two regions. Choose an increasing homeomorphism l from Δ to Δ'.

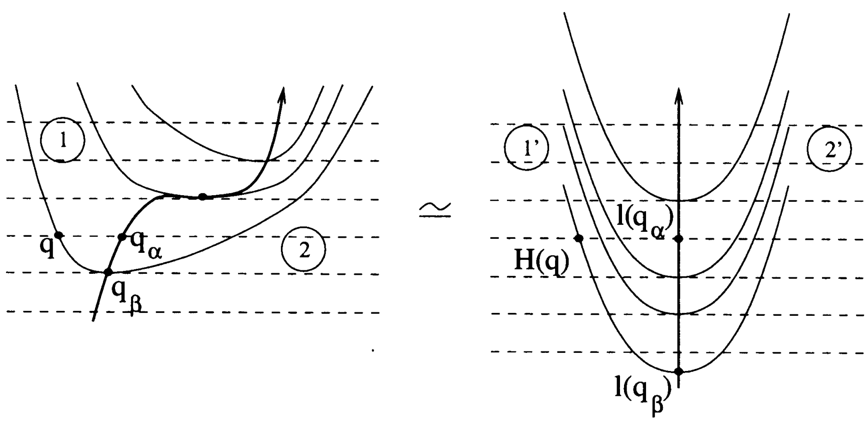

Fig. 2. Discriminant with one branch in each half plane.

Since l is increasing $l(q_\beta) \leq l(q_\alpha)$, so the leaf of α starting from $l(q_\alpha)$ and the leaf of β' starting from $l(q_\beta)$ intersect at a unique point in region $1'$. We define $H(q)$ to be the intersection point. It is not hard to see that H is a homeomorphism and takes the foliations of (α, β) to those of (α, β') in the chosen region. The same homeomorphism l defines H similarly from region 2 to region $2'$. The obtained map is the required homeomorphism that takes the foliations of (α, β) to those of the model. Note that H is completely determined by its restriction to the discriminant.

(iii) Given the hypothesis on the discriminant, the contact of the leaf of β with the x-axis at the origin is odd. The function f is decreasing (or increasing) in the open region inside the discriminant and increasing (or decreasing) in the region outside the discriminant. We assume here, without loss of generality that f is decreasing inside the region delimited by the discriminant. We shall split a neighbourhood of the origin into four regions delimited by the discriminant and the leaf of β through the origin (see Figure 4). We construct a homeomorphism from each region to its corresponding one in the model in such a way that these coincide on the boundary.

Region 1: Let p_0 be a point on Δ distinct from the origin (see Figure 3) and denote by I_1 the segment of Δ between p_0 and the origin oriented as in Figure 3. Denote by I_2 the other segment of Δ from the origin to q_0, where q_0 is the other point of intersection of the leaf of β from p_0 with the discriminant. We also give an orientation to I_2 as in Figure 3.

The pair (α, β) determines a decreasing homeomorphism $h : I_1 \to I_2$ by sliding along the leaves of β. It also determines an increasing homeomorphism $k : I_1 \to I_1$ as follows. Given $p \in I_1$, we slide along the leaf of β through p until reaching I_2 and come back to $k(p) \in I_1$ along the leaf of α through $h(p)$ (Figure 3). Similarly, we obtain homeomorphisms h' and k' associated to the model pair (α, β'). Suppose that there is a homeomorphism H_1 from region 1 to region $1'$ in the model. Then its restriction to I_1 determines an increasing homeomorphism $l_1 : I_1 \to I_1'$. As H_1 preserves the foliations, it is not hard to show that $l_1 \circ k = k' \circ l_1$.

Conversely, any increasing homeomorphism l_1 that conjugates k and k' determines a homeomorphism H_1 that sends the pair of foliations in region 1 to those in region $1'$. For, any point q in region 1 is uniquely determined as the intersection of a leaf of β starting from $q_\beta \in I_1$ and a leaf of α starting from $q_\alpha \in I_1$ (Figure 4). We also have $q_\beta \leq q_\alpha \leq k(q_\beta)$ (Figure 4). As l_1 is increasing, $l_1(q_\beta) \leq l_1(q_\alpha) \leq l_1(k(q_\beta)) = k'(l_1(q_\beta))$, so the leaf of β' through $l_1(q_\beta)$ and that of α through $l_1(q_\alpha)$ intersect at a unique point, denoted by $H_1(q)$. It is clear that H_1 is a homeomorphism taking the pair of foliations to those of the model. Now the homeomorphism l_1 exists as k and k' are increasing (see for example [16], pp. 19–20).

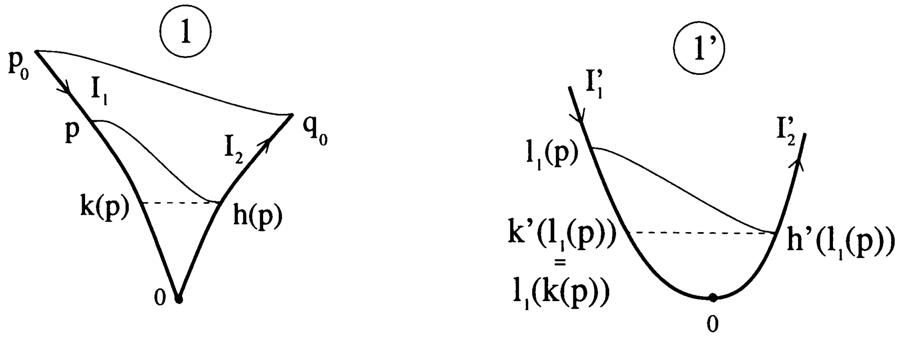

Fig. 3. Homeomorphisms on a branch of the discriminant.

Region 2: (See Figure 4) Let J_2 denote the segment of the leaf of β through the origin that lives in the complement of the half plane where the discriminant lies. We orient J_2 as in Figure 4 and denote by $I = I_1 \cup J_2$. Let $l_2 : J_2 \to J_2'$ be any increasing homeomorphism and $l : I \to I$ such that $l|_{I_1} = l_1$ and $l|_{I_2} = l_2$. Any point in region 2 is uniquely determined as the intersection of a leaf of β starting from $q_\beta \in I_1$ and a leaf of α starting from $q_\alpha \in I$. We also have $q_\beta \leq q_\alpha$. As l is increasing, $l(q_\beta) \leq l(q_\alpha)$, so the leaf of β' through $l(q_\beta)$ and that of α through $l(q_\alpha)$ intersect at a unique point $H_2(q)$. It is clear that H_2 is a homeomorphism taking the pair of foliations in region 2 to the pair in region $2'$ and coincides with H_1 on I_1.

Region 4: (See Figure 4) The homeomorphism l_1 induces an increasing homeomorphism $g : I_2 \to I_2'$. A point in region 4 is determined by a pair of points (q_α, q_β), where q_β (resp. q_α) is the intersection of a leaf of β (resp. q_α) through q with I_2. We have $q_\beta \leq q_\alpha$. As g is increasing and $g(q_\beta) \leq g(q_\alpha)$, the leaf of β' through $g(q_\beta)$ and that of α through $g(q_\alpha)$ intersect at a unique point $H_4(q)$. It is clear that H_4 is a homeomorphism taking the pair of foliations in region 4 to the pair in region $4'$ and agreeing with H_1 on I_2.

Region 3: (See Figure 4) Let $J = J_1 \cup J_2$, then we have a homeomorphism $m : J \to J'$ defined by the restrictions of H_2 to J_2 in region 2 and H_4

to J_1 in region 4. Now any other homeomorphism from a segment of the horizontal line (say, J_3) to J_3' produces a homeomorphism taking the pair in region 3 to that in region $3'$ (see Figure 4) and this map coincides with H_2 on J_2 and H_4 on J_1.

We can thus define a homeomorphism from a neighbourhood of the origin taking the pair of foliations (α, β) to the model, whose restriction to each region are the homeomorphisms exhibited above. We observe that the homeomorphism is completely determined by its restriction to a finite number of curves in the neighbourhood of the origin, namely I_1, J_2 and J_3.

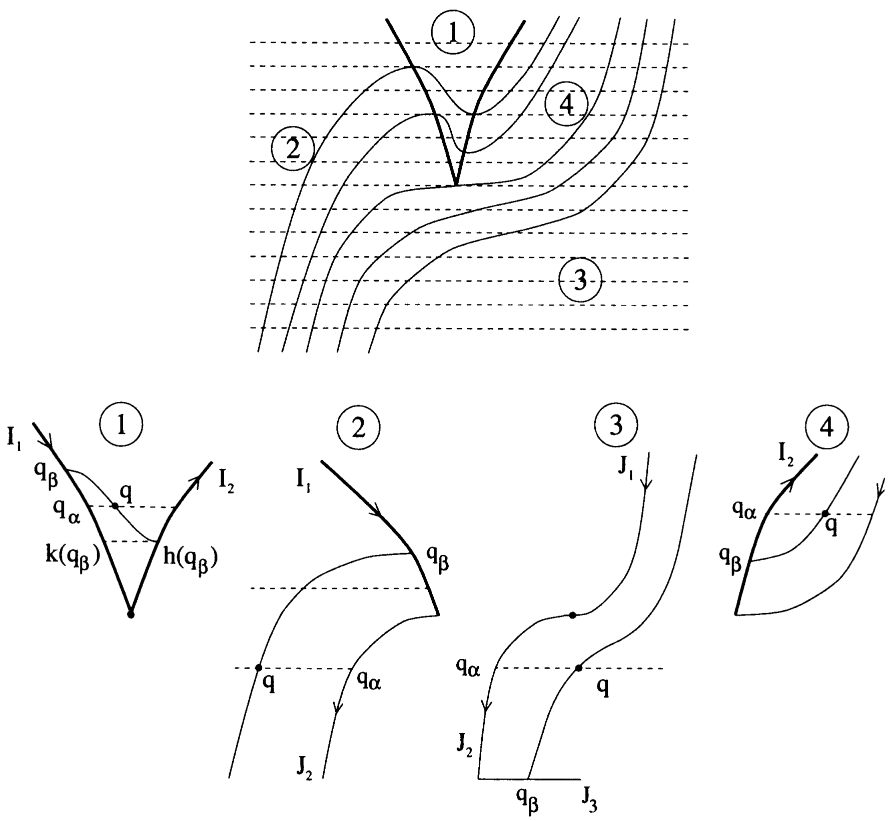

Fig. 4. Discriminant with two branches in one half plane and none in the other.

(iv) We follow the same approach as in case (iii). Here too we have 4 regions and in region 1 we proceed as in case (iii) for region 1. We then construct a homeomorphism l_2 in region 3 in the same way as for region 1. Now we have an increasing homeomorphism $l : I \to I'$ with $l|_{I_1} = l_1$ and $l|_{I_2} = l_2$. A point in region 2 is determined by a pair of points (q_α, q_β), where q_β (resp. q_α) is the intersection of a leaf of β (resp. α) through q

with I. We have $q_\beta \leq q_\alpha$. As $l(q_\beta) \leq l(q_\alpha)$, the leaf of β' through $l(q_\beta)$ and that of α through $l(q_\alpha)$ intersect at a unique point $H_2(q)$. The map H_2 is a homeomorphism taking the pair of foliations in region 2 to those in region $2'$ and coincides with H_1 on I_1 and H_3 on I_2. We construct in the same way the homeomorphism H_4. Note that the resulting homeomorphism taking the pair to the model is completely determined by its restriction to the component $I = I_1 \cup I_2$ of the discriminant.

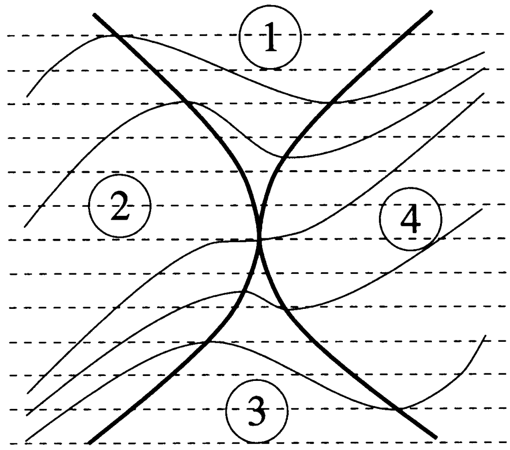

Fig. 5. Discriminant with two branches in each half plane.

□

3.1. Applications

(1) Suppose that the leaf of α and β have $k+1$-point contact at the origin. If we set $\alpha = dy$ and $\beta = df$, this means that $f(x,0)$ is $\mathcal{R}$-equivalent to $\pm x^{k+1}$, which is an A_k-singularity and has a versal unfolding given by $F = \pm x^{k+1} + u_{k-1}x^{k-2} + \cdots + u_1 x + u_0$. (We can multiply f by constants and remove the sign $\pm$.) As $f(x,y)$ can be considered as an unfolding of $f(x,0)$, there exist a change of coordinates of the form $\Psi = (\psi(x,y), y)$, i.e. that preserves the foliation of α, and a map $h(y) = (\phi_{k-1}(y), \ldots, \phi_1(y), \phi_0(y))$, such that $f \circ \Psi = h^*.F$, that is,

$$f(\psi(x,y),y) = x^{k+1} + \phi_{k-1}(y)x^{k-1} + \cdots + \phi_1(y)x + \phi_0(y).$$

As f is regular we can set $\phi_0(y) = y$. The discriminant is regular if $\phi_1'(0) \neq 0$. Theorem 3.1 asserts that all pairs of foliations with f as above and $\phi_1'(0) \neq 0$ can be classified topologically and the models are those in Theorem 3.1 (a) and (b).

(2) When $k = 2$ above, the discriminant has an A_s-singularity, and when $s < \infty$ it is shown in [14] that for any large integer N, the N-jet of the pair of 1-forms is smoothly equivalent to

$$(dy, d(y + xy^{2p+1} - x^3)), \quad p \geq 0,$$
$$(dy, d(y + xy^2 - x^3)) \quad \text{or}$$
$$(dy, d(y + x(\pm y^{2p} + \lambda y^{5p-1}) - x^3)), \quad p \geq 2$$

where λ is a smooth modulus. It follows from Theorem 3.1 that the pairs whose N-jets are equivalent to one of the above are respectively topologically equivalent to $(dy, d(y + xy - x^3))$, $(dy, d(y + xy^2 - x^3))$ or (dx, dy).

(3) We can also deduce from Theorem 3.1 the topological class of a pair whose discriminant is $\mathcal{K}^*$-equivalent to one of the normal forms in Theorem 2.1 where the number of branches (last column) is either $(0, 0)$, $(1, 1)$, $(0, 2)$ or $(2, 2)$.

3.2. The general case

When there are two branches of the discriminant in one half plane, we have a homeomorphism $k : I_1 \to I_1$ defined by sliding along the leaves of β until reaching I_2 and coming back to I_1 along the leaves of α (Figure 3). We have seen that two pairs of foliations in this region are homeomorphic if and only if the homeomorphism k and k' are conjugate.

Suppose there are more then two branches of the discriminant in one half plane. For simplicity, assume that there are three. Let $k_2 : I_1 \to I_1$ denote the homeomorphism defined by sliding along the leaf of β until reaching the branch I_2 and coming back to I_1 along the leaves of α. Let $k_1 : I_1 \to I_1$ denote the homeomorphism defined by sliding along the leaves of β until reaching I_3 and coming back to I_1 along the leaves of α (see Figure 6, right). If there exists a homeomorphism that sends the pair of foliations to a model it must induce a homeomorphism $l : I_1 \to I_1$ such that $k_1' \circ l = l \circ k_1$ and $k_2' \circ l = l \circ k_2$, where k_1' and k_2' are the homeomorphisms defined on the model in the same way as k_1 and k_2. That is, l must conjugate simultaneously the pairs (k_1, k_1') and (k_2, k_2'). Such an l does not exist in general.

Remark 3.2 When dealing with deformations of vector fields in the plane one can adopt the notion of fibre topological equivalence for such families ([9]). Two families X_t and Y_s are fibre topologically equivalent if there exist

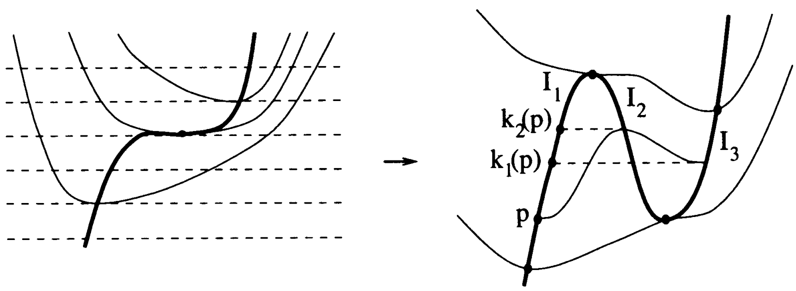

Fig. 6.

a homeomorphism $s = \psi(t)$ between the parameter space and a family of homeomorphisms of $\mathbb{R}^2$ depending on the parameter t, say h_t, such that for all t, h_t is a topological equivalence between X_t and $Y_{\psi(t)}$. The map h_t is not required to be continuous in t. This definition of equivalence can be used for pairs of foliations in the plane. However, it follows from above that for a pair of regular foliations with contact ≥ 4 at the origin any two families of the pair are in general not fibre topologically equivalent (see Figure 6). As the discriminant is a key feature of the configuration of the pair, one could introduce a weak equivalence relation for the deformations using the discriminant and define two deformations to be equivalent if their discriminants are $\mathcal{K}^*$-equivalent as deformations.

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F. Tari
Instituto de Ciências Matemáticas
e de Computação
Universidade de São Paulo
Avenida Trabalhador São-Carlense
400-Centro, Caixa Postal 668
CEP 13560-970, São Carlos (SP)
Brazil

R.D.S. Oliveira
Departamento de Matemática
Universidade Estadual de Maringá
Avenida Colombo, 5790
CEP 87020-900, Maringá (PR)
Brazil
E-mail: rdsoliveira@uem.br