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Smooth unique solutions for a modified Mullins-Sekerka model arising in diblock copolymer melts

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Abstract. Of concern is a modified Mullins-Sekerka model arising in diblock copolymer melts. As the new feature of this system a nonlocal inhomogeneous term is introduced. It is shown that the corresponding moving boundary problem is classically well posed.

Key words: Mullins-Sekerka flow, Hele-Shaw flow, Cahn-Hilliard equation, free boundary problem, diblock copolymer melt, convexity, curvature.

1. Introduction

In [18] a modified Cahn-Hilliard equation is proposed to study microphase separation of diblock copolymer. Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ and consider the following parabolic initial boundary value problem

$$\left\{\begin{aligned} u_t+\Delta(\varepsilon^2\Delta u+W'(u))-\sigma(u-\overline{u}_0)=0 &\quad \text{in } \Omega\times(0,\infty)\\ \partial_\nu u=\partial_\nu\Delta u=0 &\quad \text{on } \partial\Omega\times[0,\infty)\\ u(0,\cdot)=u_0 &\quad \text{in } \Omega, \end{aligned}\right. \tag{1.1}$$

where ε and σ are positive contants and W stands for a double-well potential with global minima at ± 1. Moreover, $\overline{u}_0:=\frac{1}{|\Omega|}\int_\Omega u_0\,dx$, with $|\Omega|$ being the Lebesgue measure of Ω, and $\partial_\nu u$ stands for the derivative of u with respect to the outer unit normal ν on $\partial\Omega$. In the case $\sigma=0$ system (1.1) reduces to the usual Cahn-Hilliard model, cf. [21]. However, if one considers separation of diblock copolymer, the effect of nonlocality should be taken into account, which stems from a long-range interaction of diblock copolymer. The third term of the left-hand side of the first equation above comes from the nonlocal term associated to Gibbs energy and the parameter σ is inversely proportional to the square of the total chain length of the

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copolymer, cf. [20, 4, 18]. The effect of this term has a strong influence on the manner of phase separation, in fact there are a variety of stable patterns of microphase with scale $(\sigma/\varepsilon)^{1/3}$ which makes a strong contrast with the usual macrophase separation realized by the Cahn-Hilliard equation. It was proven rigorously in [19] for the one-dimensional case that the global minimizer has such a microphase order.

Introducing the scaling $x \mapsto (\sigma/\varepsilon)^{1/3}x$ and $t \mapsto \sigma t$ the formal singular limit of (1.1) as $\varepsilon \to 0$ and $\sigma \to 0$ leads to the following moving boundary problem, cf. [18]: Given a compact embedded hypersurface Γ_0 in Ω that is the boundary of an open set Ω_0^- such that its closure $cl\,(\Omega_0^-)$ is contained in Ω, find a family $\Gamma = \{\Gamma(t);\, t \geq 0\}$ of embedded hypersurfaces and a family of functions $v_\pm(t) : \Omega^\pm(t) \to \mathbb{R}$ satisfying

$$\begin{cases} -\Delta v_\pm(t) = \pm 1 - f(t) & \text{in } \Omega^\pm(t), \quad t \geq 0 \\ v_\pm(t) = C\kappa(t) & \text{on } \Gamma(t), \quad t \geq 0 \\ V(t) = \dfrac{1}{2}[\partial_\nu v(t)] & \text{on } \Gamma(t), \quad t > 0 \\ \partial_\nu v_+(t) = 0 & \text{on } \partial\Omega, \quad t \geq 0 \\ \Gamma(0) = \Gamma_0. \end{cases} \tag{1.2}$$

Here $\Omega^-(t)$ and $\Omega^+(t)$ denote the regions in Ω separated by $\Gamma(t)$ and being diffeomorphic to Ω_0^- and Ω_0^+, respectively. Furthermore we use the notation

$$f(t) := \frac{1}{|\Omega|}(|\Omega^+(t)| - |\Omega^-(t)|), \quad t \geq 0.$$

We write $V(t)$ for the normal velocity of Γ at time t and $\kappa(t)$ stands for the mean curvature of $\Gamma(t)$, with the sign convention that $V(t) \geq 0$ if Γ is expanding locally $\Omega_-(t)$ and $\kappa(t) \geq 0$ for a surface $\Gamma(t)$ being locally convex with respect to $\Omega_-(t)$. Finally,

$$[\partial_\nu v(t)] := \partial_\nu v_+(t) - \partial_\nu v_-(t)$$

denotes the jump of the normal derivatives of $v_\pm(t)$ across $\Gamma(t)$, where ν denotes the outer normal with respect to $\Omega_-(t)$ and C is a positive constant.

If the first two equations in (1.2) are replaced by $\Delta v_\pm(t) = 0$ the resulting system

$$\begin{cases} \Delta v_\pm(t) = 0 & \text{in } \Omega^\pm(t), \ t \geq 0 \\ v_\pm(t) = C\kappa(t) & \text{on } \Gamma(t), \ t \geq 0 \\ V(t) = \frac{1}{2}[\partial_\nu v(t)] & \text{on } \Gamma(t), \ t > 0 \\ \partial_\nu v_+(t) = 0 & \text{on } \partial\Omega, \ t \geq 0 \\ \Gamma(0) = \Gamma_0 \end{cases} \tag{1.3}$$

is known as the two-phase Mullins-Sekerka problem, cf. [5]–[10], [16], [17], [21]. The Mullins-Sekerka system (1.3) is a widely used model for phase separation and coarsening phenomena in a melted binary alloy. The non-local inhomogeneities $\pm 1 - f(t)$ in (1.2) issued from the additional term $-\sigma(u - \overline{u}_0)$ in the first equation of (1.1) and takes care of the fact that we are dealing with diblock polymer.

As for the Mullins-Sekerka model, system (1.2) preserves the volume. More precisely, assume that (1.2) possesses smooth solutions and let

$$\mathrm{vol}(t) := |\Omega_-(t)| \quad \text{for } \ t > 0,$$

be the volume inclosed by $\Gamma(t)$. Then the function vol is smooth with

$$\frac{d}{dt}\,\mathrm{vol}(t) = \int_{\Gamma(t)} V(t)\, d\sigma(t),$$

where $\sigma(t)$ denotes the surface measure on $\Gamma(t)$. Using (1.2) and Gauss' theorem we obtain

$$\begin{aligned} 2\int_\Gamma V d\sigma &= \int_\Gamma \partial_\nu v_+ d\sigma - \int_\Gamma \partial_\nu v_- d\sigma \\ &= -\int_{\Omega^+} \Delta v_+ dx - \int_{\Omega^-} \Delta v_- dx \\ &= \int_{\Omega^+} (1 - f(t))dx + \int_{\Omega^-} (-1 - f(t))dx = 0, \end{aligned}$$

so that the flow induced by (1.2) preserves the volume of $\Omega^-(t)$ and of $\Omega^+(t)$. This particularly implies that the term f, which depends a priori on the time variable, is (at least for smooth solutions) in fact constant in time.

In contrast to the classical Mullins-Sekerka model, it cannot be expected that (1.2) decreases the area $A(t) := \int_{\Gamma(t)} d\sigma(t)$ of $\Gamma(t)$. Indeed, one has

$$\frac{d}{dt}A(t) = (n-1)\int_{\Gamma(t)} \kappa(t)V(t)\,d\sigma(t),$$

see [14], [9], and we find

$$\begin{aligned}\frac{C}{2}\int_\Gamma \kappa V\,d\sigma &= \int_\Gamma v_+\partial_\nu v_+ d\sigma - \int_\Gamma v_-\partial_\nu v_- d\sigma \\ &= -\int_{\Omega^+} \operatorname{div}(v_+\nabla v_+)dx - \int_{\Omega^-} \operatorname{div}(v_-\nabla v_-)dx \\ &= -\int_{\Omega^+} |\nabla v_+|^2 dx - \int_{\Omega^-} |\nabla v_-|^2 dx \\ &\quad + (1-f)\int_{\Omega^+} v_+ dx - (1+f)\int_{\Omega^-} v_- dx.\end{aligned}$$

so that there is no reason to expect that $dA(t)/dt$ is non-positive.

A further significant difference between (1.2) and (1.3) is concerned with the equilibria of these flows. Indeed, it follows from Alexandrovs characterization of Euclidean spheres (cf. [1]) that (1.3) admits only spheres as embedded equilibria. In contrast, spheres are in general not equilibria to (1.2). To see this, let Γ_0 be a sphere of radius R and assume that $\Gamma(t) = \Gamma_0$, $t > 0$ is a stationary solution to (1.2). Then the corresponding chemical potentials $-v_\pm$ have to satisfy the following elliptic boundary value problems

$$\begin{aligned} -\Delta v_\pm(t) &= \pm 1 - f(t) && \text{in } \Omega^\pm \\ v_\pm(t) &= C/R && \text{on } \Gamma_0 \\ \partial_\nu v_+(t) &= 0 && \text{on } \partial\Omega, \quad t \geq 0. \end{aligned}$$

Recall that $|f| < 1$. Hence the strong maximum principle and the symmetry of v_- imply that there is a positive constant c such that $\partial_\nu v_-(x) = c$ for all $x \in \Gamma_0$. But Γ_0 is an equilibrium. Thus $V = (\partial_\nu v_+ - \partial_\nu v_-)/2$ vanishes on Γ_0, so that

$$\partial_\nu v_+(x) = c \qquad \text{for all} \quad x \in \Gamma_0. \tag{1.4}$$

Observe that (1.4) is independent of the shape and position of $\partial\Omega$, which is not possible. Finally, in the forthcoming paper [12] it is shown that the flow induced by (1.2) does not preserve convexity, in agreement with the usual Mullins-Sekerka flow, cf. [15].

The unknowns Γ and $v_\pm$ are coupled through the system (1.2). However, if the position and the regularity of the moving boundary $\Gamma = \{\Gamma(t);\, t \in [0,T)\}$ is known, the chemical potentials $-v_\pm$ are obtained by solving at each time $t \in [0,T)$ the elliptic boundary value problems

$$\begin{aligned} -\Delta v_\pm(t) &= \pm 1 - f(t) && \text{in } \ \Omega^\pm(t) \\ v_\pm(t) &= C\kappa(t) && \text{on } \ \Gamma(t) \\ \partial_\nu v_+(t) &= 0 && \text{on } \ \partial\Omega, \quad t \geq 0. \end{aligned}$$

In this sense we call a family $\Gamma = \{\Gamma(t);\, t \in [0,T)\}$ of hypersurfaces a solution of (1.2).

To give a precise statement of our results, we have to introduce some notation. Given $\alpha \in (0,1)$, $m \in \mathbb{N}$, and an open bounded subset U of $\mathbb{R}^n$, let $h^{m+\alpha}(U)$ denote the little Hölder space of order $m+\alpha$, i.e. the closure of $C^\infty(U)$ in the norm of the usual Banach space $C^{m+\alpha}(\overline{U})$. Given a sufficiently smooth manifold M, the space $h^{m+\alpha}(M)$ is defined by means of local coordinates.

Theorem 1 *Let Γ_0 be a compact, closed, embedded hypersurface in Ω of class $h^{2+\alpha}$. Then there exists a unique classical solution $\Gamma = \{\Gamma(t);\, t \in [0,T)\}$ of problem (1.2) emerging from Γ_0. The mapping $[t \mapsto \Gamma(t)]$ is smooth on $(0,T)$ with respect to the C^∞-topology and continuous on $[0,T)$ with respect to the $h^{2+\alpha}$-topology. Moreover, if Γ_0 is the $h^{2+\alpha}$-graph in normal direction over a smooth hyperface Σ, then the mapping $[(t,\Gamma_0) \mapsto \Gamma(t)]$ defines a local smooth semiflow on some open subset of $h^{2+\alpha}(\Sigma)$.*

There is a different approach to moving boundary problems of type (1.3) which is based on introducing a regularizing term to get approximate solutions for these motions. Using energy estimates it is possible to pass to the limit in the regularized problem and to get the existence of a local weak solution to (1.3). In certain cases it is possible to prove a posteriori additional regularity of these weak solutions, cf. [5, 7]. However, for the modified Mullins-Sekerka model (1.2) the area functional fails to be a Ljapunov function for the corresponding flow, which is an essential task to get powerful energy estimates. In addition, the approach followed in [5, 7] does not produce any information about the uniqueness of solutions.

2. Existence and uniquness of classical solutions

In this section we transform the original problem to a nonlinear problem on the fixed domains $\Omega^{\pm}$. After a natural reduction of this transformed problem we are left to solve a quasi-linear parabolic evolution equation for the moving boundary involving a nonlocal pseudo-differential operator of third order. We shall work out a quasi-linear structure of this propagator and we will see that the corresponding linear part can be represented as an elliptic pseudo-differential operator of third order. This rather precise linear analysis allows us then to apply the general theory of quasi-linear parabolic evolution equation due to H. Amann.

Assume that Γ_0 is a compact, closed hypersurface in Ω of class $h^{2+\alpha}$ and let $a_0 := \operatorname{dist}(\Gamma_0, \partial\Omega)$. Then we find a smooth hypersurface Σ, a positive constant $r > 0$, and a function $\rho_0 \in h^{2+\alpha}(\Sigma)$ such that

$$X : \Sigma \times (-r, r) \to \mathbb{R}^n, \quad X(s, \lambda) := s + \lambda\nu(s)$$

is a smooth diffeomorphism onto its image $Y := \operatorname{im}(X)$ and such that $\theta_{\rho_0}(s) := X(s, \rho_0(s))$ is a $C^{2+\alpha}$-diffeomorphism mapping Σ onto Γ_0. Here, ν denotes the outer normal at Σ. Of course Σ separates Ω in two domains Ω^- and Ω^+, with Ω^- being enclosed by Σ.

Let $T > 0$ be fixed. We are looking for $\Gamma = \{\Gamma(t);\, t \in [0, T]\}$ in the form

$$\Gamma(t) := \{x \in \mathbb{R}^n;\, x = X(s, \rho(s,t)),\, s \in \Sigma\},$$

with a function $\rho : \Sigma \times [0, T] \to \mathbb{R}$ to be determined. More precisely, let

$$\mathcal{A} := \{\hat\rho \in h^{2+\alpha}(\Sigma);\, \|\hat\rho\|_{C^1} < a\}$$

denote a set of admissible parametrizations, where $a \in (0, r)$ will be chosen later. Given $\hat\rho \in \mathcal{A}$, let $\theta_{\hat\rho} := \mathrm{id}_\Sigma + \hat\rho\nu$ and write $\Gamma_{\hat\rho} := \operatorname{im}(\theta_{\hat\rho})$. Obviously,

$$\theta_{\hat\rho} \in \operatorname{Diff}^{2+\alpha}(\Sigma, \Gamma_{\hat\rho}), \quad \hat\rho \in \mathcal{A},$$

provided $r > 0$ is chosen sufficiently small. With this notation we try to find $\rho \in C([0, T], \mathcal{A})$ such that $\Gamma(t) = \Gamma_{\rho(t)}$ for $t \in [0, T]$. Clearly, each surface $\Gamma_{\rho(t)}$ separates Ω into an interior domain $\Omega^-_{\rho(t)}$ and an exterior domain $\Omega^+_{\rho(t)}$, $t \in [0, T]$. In the following we fix $t \in [0, T]$ and supress it in our notation. It is convenient to express Γ_ρ as the level set of an appropriate function on $\mathbb{R}^n$. For this we decompose the inverse of X into $X^{-1} = (S, \Lambda)$, where

$S \in C^\infty(Y, \Sigma)$ denotes the metric projection of Y onto Σ and Λ stands for the signed distance function with respect to Σ. Obviously, we have $\Gamma_\rho = N_\rho^{-1}(0)$ with $N_\rho = \Lambda - \rho \circ S$. Since we have to deal with elliptic boundary value problems in the domains $\Omega_\rho^\pm$ we need appropriate extensions of the diffeomorphism θ_ρ to $\Omega^\pm$. For this we introduce the following construction which was first proposed in [13] to transform Stefan problems on fixed domains. Fix now $a \in (0, r/4)$ and pick $\varphi \in C^\infty(\mathbb{R}, [0,1])$ such that $\varphi(\lambda) = 1$ if $|\lambda| \le a$ and $\varphi(\lambda) = 0$ if $|\lambda| \ge 3a$, and such that $\sup|\varphi'(\lambda)| < 1/a$. Given $\rho \in \mathcal{A}$, let

$$\Theta_\rho(x) := \begin{cases} X\big(S(x), \Lambda(x) + \varphi(\Lambda(x))\rho(S(x))\big) & \text{if } x \in Y, \\ x & \text{if } x \notin Y. \end{cases}$$

Observe that $[\lambda \mapsto \lambda + \varphi(\lambda)\rho]$ is strictly increasing since $|\varphi'(\lambda)\rho| < 1$. Therefore we conclude that

$$\begin{aligned} &\Theta_\rho \in \mathrm{Diff}^{2+\alpha}(\Omega, \Omega) \cap \mathrm{Diff}^{2+\alpha}(\Omega^\pm, \Omega_\rho^\pm), \\ &\Theta_\rho|\Sigma = \theta_\rho, \quad \Theta_\rho|U = \mathrm{id}_U, \end{aligned}$$

for some neighbourhood $U \subset \mathbb{R}^n$ of $\partial\Omega$. In order to economize our notation we use the same symbol θ_ρ for both diffeomorphims θ_ρ and Θ_ρ.

We are now prepared to transform problem (1.2) into a problem on the fixed domains $\Omega^\pm$. Given $v_\pm \in C(\overline{\Omega}_\rho^\pm)$ and $u_\pm \in C(\overline{\Omega}^\pm)$, we write

$$\theta_\rho^* v_\pm := v_\pm \circ \theta_\rho \quad \text{and} \quad \theta_*^\rho u_\pm := u_\pm \circ \theta_\rho^{-1}$$

for the pull-back and push-forward operator, respectively, induced by θ_ρ. We now set

$$\begin{aligned} A_\pm(\rho)u_\pm &:= -\theta_\rho^*(\Delta(\theta_*^\rho u_\pm)) \\ B_\pm(\rho)u_\pm &:= \frac{1}{2}\gamma_\pm \theta_\rho^*(\nabla(\theta_*^\rho u_\pm)|\nabla N_\rho), \end{aligned}$$

for $u_\pm \in C^2(\overline{\Omega}^\pm)$, where $\gamma_\pm$ stands for the restriction operator of C^1-functions on $\overline{\Omega}^\pm$ to Σ. Furthermore, let

$$f(\rho) := \frac{1}{|\Omega|}(|\Omega_\rho^+| - |\Omega_\rho^-|) \quad \text{and} \quad K(\rho) := C\theta_\rho^* \kappa_\rho,$$

where κ_ρ denotes the mean curvature of Γ_ρ. Finally, the normal velocity V

of $\Gamma = \{\Gamma_{\rho(t)};\ t \in [0,T]\}$ can be expressed as

$$V(s,t) = -\frac{\partial_t N_\rho(x,t)}{|\nabla N_\rho(x,t)|}\Big|_{x=\theta_{\rho(t)}(s)} = \frac{\partial_t \rho(s,t)}{|\nabla N_\rho(x,t)|}\Big|_{x=\theta_{\rho(t)}(s)},$$

for $(s,t) \in \Sigma \times (0,T]$. Observe that the outer unit normal ν at Γ_ρ is given as $\nabla_x N_\rho / |\nabla_x N_\rho|$. Hence, writing $u_\pm := \theta_\rho^* v_\pm$, problem (1.2) is transformed into

$$\begin{cases} A_\pm(\rho)u_\pm = \pm 1 - f(\rho) & \text{in } \Omega^\pm, \quad t \geq 0 \\ u_\pm = K(\rho) & \text{on } \Sigma, \quad t \geq 0 \\ \partial_t \rho = B_+(\rho)u_+ - B_-(\rho)u_- & \text{on } \Sigma, \quad t > 0 \\ \partial_\nu u_+ = 0 & \text{on } \partial\Omega, \quad t \geq 0 \\ \rho(\cdot,0) = \rho_0. \end{cases} \tag{2.1}$$

It is not difficult to verify that problem (1.2) and problem (2.1) are equivalent. Note that the unknowns ρ and $u_\pm$ are still coupled through (2.1). To obtain an equation for ρ only, we need the following result. Fix $0 < \gamma < \beta < \alpha < 1$ and let $U := h^{2+\beta}(\Sigma) \cap \mathcal{A}$.

Lemma 2.1 *Let $\sigma \in [\gamma, \beta]$ be fixed. Then*

a) $(A_\pm, B_\pm) \in C^\infty(U, \mathcal{L}(h^{1+\sigma}(\Omega^\pm), h^{\sigma-1}(\Omega^\pm) \times h^\sigma(\Sigma)))$.

b) *Given $\rho \in U$, we have*

$$(A_\pm(\rho), \gamma_\pm, \partial_\nu) \in \mathrm{Isom}(h^{1+\sigma}(\Omega^\pm), h^{\sigma-1}(\Omega^\pm) \times h^{1+\sigma}(\Sigma) \times h^\sigma(\partial\Omega))$$

c) $f \in C^\infty(U, \mathbb{R})$.

Proof. Assertions a) and b) follow from Lemma 2.2 in [9].
To see c), observe that

$$f(\rho) = \frac{1}{|\Omega|}\left(\int_{\Omega^+} |\det D\theta_\rho|\, dx - \int_{\Omega^-} |\det D\theta_\rho|\, dx\right)$$

for $\rho \in U$. □

Given $\rho \in U$, define

$$S_\pm(\rho) := (A_\pm(\rho), \gamma_\pm, \partial_\nu)^{-1}(\cdot, 0, 0) \in \mathcal{L}(h^{\sigma-1}(\Omega^\pm), h^{1+\sigma}(\Omega^\pm))$$

and

$$T_\pm(\rho) := (A_\pm(\rho), \gamma_\pm, \partial_\nu)^{-1}(0,\cdot,0) \in \mathcal{L}(h^{1+\sigma}(\Sigma), h^{1+\sigma}(\Omega^\pm)).$$

Observe that, given $\hat{f}_\pm \in h^{\sigma-1}(\Omega^\pm)$ and $\hat{\rho} \in h^{1+\sigma}(\Sigma)$, the functions $w_\pm(\rho) := S_\pm(\rho)\hat{f}_\pm + T_\pm(\rho)\hat{\rho}$ are the unique solutions in $h^{1+\sigma}(\Omega^\pm)$ of

$$\begin{aligned} A_\pm(\rho)w_\pm &= \hat{f}_\pm \quad \text{in } \Omega^\pm \\ \gamma_\pm w_\pm &= \hat{\rho} \quad \text{on } \Sigma \\ \partial_\nu w_+ &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Let us now introduce the operator $\Phi : U \cap h^{3+\alpha}(\Sigma) \to h^\alpha(\Sigma)$, defined by

$$\begin{aligned} \Phi(\rho) := {} & B_+(\rho)[T_+(\rho)K(\rho) + S_+(\rho)(1 - f(\rho))] \\ & - B_-(\rho)[T_-(\rho)K(\rho) - S_-(\rho)(1 + f(\rho)]. \end{aligned}$$

Then problem (2.1) and the abstract evolution equation

$$\frac{d}{dt}\rho = \Phi(\rho), \quad \rho(0) = \rho_0 \tag{2.2}$$

are equivalent in the following sense: Let $\rho_0 \in h^{3+\alpha}(\Sigma)$ be given and assume that

$$\rho \in C([0,T], h^{3+\alpha}(\Sigma) \cap U) \cap C^1([0,T], h^{2+\alpha}(\Sigma))$$

is a solution to (2.2). Then the triple $(\rho, u_\pm)$ with

$$u_\pm := T_\pm(\rho)K(\rho) + S_\pm(\rho)(\pm 1 - f(\rho))$$

is a solution to (2.1); and vice-versa: if $(\rho, u_\pm)$ is a solution to (2.1) then the above construction shows that ρ is a solution to (2.2).

Although the nonlocal and nonlinear operator Φ consists already in a sum of four terms, we shall introduce a further splitting of Φ. This splitting is motivated by the fact that the mean curvature operator K carries a quasi-linear structure in the following sense:

Lemma 2.2 *There exist functions*

$$P \in C^\infty(U, \mathcal{L}(h^{3+\gamma}(\Sigma), h^{1+\gamma}(\Sigma))) \quad \textit{and} \quad Q \in C^\infty(U, h^{1+\beta}(\Sigma))$$

such that

$$K(\rho) = P(\rho)\rho + Q(\rho) \quad \textit{for} \ \ \rho \in U \cap h^{3+\gamma}(\Sigma).$$

A proof of Lemma 2.2 can be found in [9], Lemma 3.1.

We now introduce the quasi-linear principal part Π of $-\Phi$ by setting

$$\Pi(\rho)\rho := [B_-(\rho)T_-(\rho) - B_+(\rho)T_+(\rho)]P(\rho)\rho$$

and (correspondingly) the lower order part

$$\begin{aligned} F(\rho) := &[B_+(\rho)T_+(\rho) - B_-(\rho)T_-(\rho)]Q(\rho) \\ &+ B_+(\rho)S_+(\rho)(1 - f(\rho)) + B_-(\rho)S_-(\rho)(1 + f(\rho)). \end{aligned}$$

Clearly, we have $\Phi(\rho) = -\Pi(\rho)\rho + F(\rho)$, so that problem (2.2) is equivalent to

$$\frac{d}{dt}\rho + \Pi(\rho)\rho = F(\rho), \quad \rho(0) = \rho_0. \tag{2.3}$$

Using Lemma 2.1 and Lemma 2.2 it is not difficult to verify that the mappings

$$\Pi : U \to \mathcal{L}(h^{3+\gamma}(\Sigma), h^\gamma(\Sigma)) \quad \text{and} \quad F : U \to h^\beta(\Sigma)$$

are well-defined and smooth. In order to solve equation (2.3) we need parameter dependent a priori estimates for the principal part Π of $-\Phi$. In order to formulate these crucial estimates, let us introduce the following notation. Given two Banach spaces E and F such that E is dense and continuously embedded in F, let $\mathcal{H}(E, F)$ consist of all $A \in \mathcal{L}(E, F)$ such that $-A$, viewed as an unbounded operator in F, generates a strongly continuous analytic semigroup on F. It is known that a linear operator $A : E \subset F \to F$ belongs to $\mathcal{H}(E, F)$ if and only if there exist $\omega \in \mathbb{R}$ and $\kappa \geq 1$ such that $\omega + A \in \mathrm{Isom}(E, F)$ and the following parameter dependent a priori estimate

$$|\lambda| \|x\|_F \leq \kappa \|(\lambda + A)x\|_F, \quad x \in E, \quad \lambda \in \mathbb{C} \quad \text{with} \quad \mathrm{Re}\, \lambda \geq \omega,$$

holds true. Based on the Mikhlin-Hörmander Fourier multiplier theorem, representation formulas of Poisson operators and subtle perturbation techniques the following result can be shown, cf. [9], p.641.

Theorem 2.3 *Given $\rho \in U$, we have*

$$\Pi(\rho) \in \mathcal{H}(h^{3+\gamma}(\Sigma), h^\gamma(\Sigma)).$$

We are now prepared to prove our main result.

Proof of Theorem 1. Let Γ_0 satisfy the hypotheses and choose Σ and ρ_0 as above. Recall that $\mathcal{A}$ is open in $h^{2+\gamma}(\Sigma)$.

a) We first show that equation (2.3) has a unique solution ρ belonging to

$$C([0,T),\mathcal{A})\cap C^\infty((0,T),C^\infty(\Sigma)),$$

with $T = T(\rho_0) > 0$ being the maximal interval of existence. Indeed, set $E_0 := h^\gamma(\Sigma)$ and $E_1 := h^{3+\gamma}(\Sigma)$, and let $E_\theta := (E_0, E_1)^0_{\theta,\infty}$, $\theta \in (0,1)$ denote the continuous interpolation spaces between E_0 and E_1, cf. [3]. It is known (cf. [22]) that the scale of small Hölder spaces is stable under continuous interpolation. Hence letting $\theta_0 := (2+\beta-\gamma)/3$, $\theta_1 := (2+\alpha-\gamma)/3$, and $\theta := (\beta-\gamma)/3$, we find

$$E_{\theta_1} = h^{2+\alpha}(\Sigma), \quad E_{\theta_0} = h^{2+\beta}(\Sigma), \quad E_\theta = h^\beta(\Sigma).$$

Consequently, Lemma 2.1 and Lemma 2.2 yields $(\Pi, F)\in C^\infty(U, \mathcal{L}(E_1, E_0)\times E_\theta)$. Due to Theorem 2.3 we can now apply Theorem 12.1 in [2] to obtain a unique $T = T(\rho_0) > 0$ and a unique solution

$$C([0,T),\mathcal{A})\cap C((0,T),h^{3+\gamma}(\Sigma))\cap C^1((0,T),h^\gamma(\Sigma))$$

of the evolution equation (2.3). The fact that this solution actually belongs to $C^\infty((0,T),C^\infty(\Sigma))$ follows from the same bootstrapping argument presented in [9], p.634.

b) Let ρ be the above constructed solution to (2.3) and define

$$\Gamma(t) := \Gamma_{\rho(t)} = \{x\in\mathbb{R}^n;\, x = X(s,\rho(t)(s)),\, s\in\Sigma\}, \quad t\in[0,T),$$

and

$$v_\pm(t) := \theta_*^{\rho(t)}[S_\pm(\rho(t))(\pm 1 - f(\rho(t))) + T_\pm(\rho(t))K(\rho(t))], \qquad t\in[0,T).$$

Then it is not difficult to verify that $\Gamma = \{\Gamma(t);\, t\in[0,T)\}$ together with $v_\pm$ form the unique solution to (1.2). □

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