Hokkaido Mathematical Journal Vol. 32 (2003) p. 661–671

Hamiltonian dynamics of a charged particle

Osamu IKAWA*

(Received May 7, 2002; Revised June 19, 2002)

Abstract. We study the Hamiltonian dynamics of a charged particle using a noncanonical symplectic structure on the tangent bundle. We show that if the motion of a charged particle in a homogeneous space satisfying a certain condition intersects itself, then it is simply closed.

Key words: charged particle, Hamiltonian dynamics, symplectic structure.

Introduction

Let F be a closed 2-form and U a function on a connected semi-Riemannian manifold $(M, \langle\ ,\ \rangle)$. We denote by $\iota(X) : \bigwedge^m(M) \to \bigwedge^{m-1}(M)$ the interior product operator induced from X, and by $\mathcal{L} : T(M) \to T^*(M)$, the Legendre transformation defined by

$$\mathcal{L} : T(M) \to T^*(M); u \mapsto \mathcal{L}(u), \quad \mathcal{L}(u)(v) = \langle u, v\rangle \quad (v \in T(M)).$$

A curve $x(t)$ in M is called the *motion of a charged particle under electromagnetic field F and potential energy U*, if it satisfies the following differential equation

$$\nabla_{\dot{x}}\dot{x} = -\operatorname{grad} U - \mathcal{L}^{-1}(\iota(\dot{x})F),$$

where ∇ is the Levi-Civita connection of M. This equation originated in the theory of general relativity (see [6, § 1] or [10, p. 112, (19.15)]). When $F = 0$ and $U = 0$, then $x(t)$ is merely a geodesic. If $x(t)$ is the motion of a charged particle under F and U, then the total energy

$$\frac{1}{2}\langle \dot{x}, \dot{x}\rangle + U(x(t)) \tag{0.1}$$

is a constant. If F has an *electromagnetic potential* A, that is $F = dA$, then we define a functional E by

---

2000 Mathematics Subject Classification : 83C10, 53C30, 83C50.

*Partially supported by Grant-in-aid for Scientific research No. 14740055.

$$E(x) = \int_0^1 \left(\frac{1}{2}\langle \dot{x}, \dot{x} \rangle + \frac{1}{2} A(\dot{x}) - U(x(t)) \right) dt.$$

The Euler-Lagrange equation of E is the motion of a charged particle under F and U.

We denote by $\pi : T(M) \to M$ the tangent bundle over M. Based on (0.1), we define a function H on $T(M)$ as

$$H(u) = \frac{1}{2}\langle u, u \rangle + U(\pi(u)) \quad (u \in T(M)).$$

In this paper, we show that, even if F does not have an electromagnetic potential, the motion of a charged particle is a Hamiltonian system using H and a noncanonical symplectic structure on $T(M)$ (Theorem 2.1). We here mention some fundamental definitions concerning symplectic geometry. A *symplectic structure* on a manifold is a closed 2-form which is nondegenerate at each point. A *symplectic manifold* is a manifold possessing a symplectic structure. A symplectic manifold is even-dimensional and orientable. A diffeomorphism on a symplectic manifold is called a *symplectic transformation* if it preserves the symplectic structure, though, in old literatures, a symplectic transformation was called a canonical transformation.

In general it is an interesting question whether a given equation of motion has a periodic solution or not. In relation to this problem, we study the simpleness of the motion of a charged particle under electromagnetic field F and $U = 0$ in a homogeneous space satisfying a certain condition. Here a curve in a manifold is *simple* if it is either a simply closed periodic curve or if it does not intersect itself. Our main purpose in this paper is to show that every motion of a charged particle under an electromagnetic field associated with G-homogeneous semi-Riemannian manifold is simple, if the Lie algebra $\mathfrak{g}$ of G satisfies $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ (see Theorem 2.4). Other sufficient conditions for the simpleness of the motion of a charged particle in a homogeneous space and its application are found in [6, Th. 2.3, Cor. 2.4, Cor 2.5, Th. 3.9]. We refer to [1], [2], [3] and their references for studies of the motion of charged particles.

The author would like to thank the referee for his useful suggestions.

1. Hamiltonian dynamics of a geodesic

In this section, we review the Hamiltonian dynamics of a geodesic, which is defined by $\nabla_{\dot{x}}\dot{x} = 0$, in a semi-Riemannian manifold $(M, \langle\ ,\ \rangle)$, in order to contrast it with the Hamiltonian dynamics of a charged particle discussed in the next section. The results obtained in this section will be used in the next section. Define a function H on $T(M)$ by

$$H(u) = \frac{1}{2}\langle u, u\rangle \quad (u \in T(M)),$$

which corresponds to the kinetic energy. We denote by X_H the Hamiltonian vector field of the Hamiltonian H with respect to the canonical symplectic structure ω on $T(M)$, that is, $dH = \iota(X_H)\omega$. We denote by $\{\ ,\ \}$ the Poisson bracket on $C^\infty(T(M))$ with respect to ω, which is defined by

$$\{f, g\} = X_f(g) = \omega(X_g, X_f) \quad \text{for} \quad f, g \in C^\infty(T(M)).$$

It is known that each orbit of the geodesic flow on $T(M)$ coincides with the integral curve of X_H ([4]). We define a mapping

$$P : \mathfrak{X}(M) \to (C^\infty(T(M)), \{\ ,\ \});\ Y \mapsto P_Y$$

by $P_Y(u) = \langle u, Y\rangle$. It is clear that P is injective. It is known that if Y is a Killing vector field, then P_Y is a conservative constant for geodesics ([7, Lemma 9.26]). In other words,

$$\{H, P_Y\} = 0 \tag{1.1}$$

for any Killing vector field Y.

Proposition 1.1 ([4, p. 222]) $\{P_Y, P_Z\} = P_{[Y,Z]}$ $(Y, Z \in \mathfrak{X}(M))$.

Proof. This result is well-known. But we give a proof for completeness. Let $(x^1, \dots, x^n)$ be a local coordinate system in M. The components g_{ij} of $\langle\ ,\ \rangle$ with respect to $(x^1, \dots, x^n)$ are given by $g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle$. We denote by (g^{ij}) the inverse matrix of (g_{ij}). We introduce a local coordinate system $(x^1, \dots, x^n, u^1, \dots, u^n)$ in $T(M)$ by setting

$$u = \sum_{i=1}^{n} u^i(u)\frac{\partial}{\partial x^i} \quad (u \in T(M)).$$

The local expression for the canonical symplectic structure ω is then given by

$$\omega = \sum_{i,j,k} \frac{\partial g_{ij}}{\partial x^k} u^j dx^i \wedge dx^k + \sum_{i,j} g_{ij} dx^i \wedge du^j = -d\left(\sum g_{ij} u^j dx^i\right).$$

We use these notations throughout this paper. The vector fields Y and Z can be written as $Y=\sum Y^i\frac{\partial}{\partial x^i}$, $Z=\sum Z^i\frac{\partial}{\partial x^i}$, so

$$P_Z=\sum g_{ij}Z^iu^j, \quad \text{and} \quad P_{[Y,Z]}=\sum g_{jk}\left(Y^i\frac{\partial Z^j}{\partial x^i}-Z^i\frac{\partial Y^j}{\partial x^i}\right)u^k.$$

Since $dP_Y=\iota(X_{P_Y})\omega$, we have

$$X_{P_Y}=\sum Y^i\frac{\partial}{\partial x^i}-\sum\left(Y^k\frac{\partial g_{ij}}{\partial x^k}+\frac{\partial Y^k}{\partial x^i}g_{jk}\right)g^{il}u^j\frac{\partial}{\partial u^l}. \tag{1.2}$$

Hence we obtain

$$\begin{aligned}\{P_Y,P_Z\}&=X_{P_Y}(P_Z)\\&=\sum Y^i\frac{\partial(g_{jk}Z^j)}{\partial x^i}u^k-\sum\left(Y^k\frac{\partial g_{ij}}{\partial x_k}+\frac{\partial Y^k}{\partial x^i}g_{jk}\right)g^{il}u^jg_{pl}Z^p\\&=\sum Y^i\frac{\partial(g_{jk}Z^j)}{\partial x^i}u^k-\sum\left(Y^j\frac{\partial g_{ik}}{\partial x_j}+\frac{\partial Y^j}{\partial x^i}g_{jk}\right)Z^iu^k\\&=\sum g_{jk}\left(Y^i\frac{\partial Z^j}{\partial x^i}-Z^i\frac{\partial Y^j}{\partial x^i}\right)u^k\\&=P_{[Y,Z]}.\end{aligned}$$

□

A diffeomorphism φ of M induces a transformation φ_* of $T(M)$. Thus a vector field Y of M induces vector fields of $T(M)$ in the following two ways: One is the Hamiltonian vector field X_{P_Y} of P_Y, and the other is $\frac{d\varphi_{t*}(u)}{dt}_{|t=0}$ ($u\in T(M)$), where φ_t is the one parameter transformation group of M generated by Y.

When Y is a Killing vector field, by (1.1) Noether's theorem tells us that the one-parameter transformation group of $T(M)$ generated by X_{P_Y} is a symplectic transformation which preserves H.

Lemma 1.2 *Let φ_t be the one-parameter transformation group of M generated by a vector field $Y=\sum Y^i\frac{\partial}{\partial x^i}$. Then the vector field $\frac{d\varphi_{t*}}{dt}_{|t=0}$ can be expressed as*

$$\frac{d\varphi_{t*}}{dt}_{|t=0}=\sum Y^i\frac{\partial}{\partial x^i}+\sum\frac{\partial Y^l}{\partial x^j}u^j\frac{\partial}{\partial u^l}.$$

Proof. For $u\in T(M)$, set $x=\pi(u)\in M$.

Take a curve $x(s) = (x^1(s), \ldots, x^n(s))$ in M such that $\dot{x}(0) = u = \sum u^i \frac{\partial}{\partial x^i}$. Then

$$\frac{d\varphi_{t*}(u)}{dt}_{|t=0} = \frac{d}{dt}\left(\varphi_t(x), \frac{d}{ds}\varphi_t(x(s))_{|s=0}\right)_{|t=0}$$
$$= \sum Y^i \frac{\partial}{\partial x^i} + \frac{d}{ds}(Y^1(x(s)), \ldots, Y^n(x(s)))_{|s=0}$$
$$= \sum Y^i \frac{\partial}{\partial x^i} + \sum \frac{\partial Y^i}{\partial x^j} u^j \frac{\partial}{\partial u^i}.$$

□

Proposition 1.3 *Let φ_{t*} be the one-parameter transformation group of $T(M)$ induced from the one parameter transformation group φ_t of M generated by a Killing vector field Y. Then φ_{t*} coincides with the one-parameter transformation group generated by the Hamiltonian vector field of P_Y.*

Proof. Since Y is a Killing vector field,

$$\sum_k Y^k \frac{\partial g_{ij}}{\partial x^k} = Y\left(\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right\rangle\right)$$
$$= \left\langle \left[Y, \frac{\partial}{\partial x^i}\right], \frac{\partial}{\partial x^j}\right\rangle + \left\langle \frac{\partial}{\partial x^i}, \left[Y, \frac{\partial}{\partial x^j}\right]\right\rangle$$
$$= -\sum_k \left(\frac{\partial Y^k}{\partial x^i} g_{kj} + \frac{\partial Y^k}{\partial x^j} g_{ki}\right).$$

Applying $\sum_i g^{il}$ to the equation above, we have

$$\frac{\partial Y^l}{\partial x^j} = -\sum \left(\frac{\partial Y^k}{\partial x^i} g_{kj} + Y^k \frac{\partial g_{ij}}{\partial x^k}\right) g^{il}.$$

Using (1.2) and Lemma 1.2, we obtain

$$X_{P_Y} = \sum Y^i \frac{\partial}{\partial x^i} + \sum \frac{\partial Y^l}{\partial x^j} u^j \frac{\partial}{\partial u^l} = \frac{d\varphi_{t*}}{dt}_{|t=0}.$$

□

2. Hamiltonian dynamics of a charged particle

In this section, we study the Hamiltonian dynamics of the motion of a charged particle in a connected semi-Riemannian manifold $(M, \langle\ ,\ \rangle)$, which is defined as

$$\nabla_{\dot{x}}\dot{x} = -\operatorname{grad} U - \mathcal{L}^{-1}(\iota(\dot{x})F). \tag{2.1}$$

We define a function H on $T(M)$ by

$$H(u) = \frac{1}{2}\langle u, u\rangle + U(\pi(u)) \quad (u \in T(M)),$$

corresponding to the total energy. We define a closed 2-form ω_F on $T(M)$ by

$$\omega_F = \omega - \pi^* F.$$

For each tangent vector $u \in T(M)$, we denote by x_u the motion of a charged particle (2.1) with the initial vector u. The electromagnetic flow $\Phi_t : T(M) \to T(M)$ is defined by $\Phi_t(u) = \dot{x}_u(t)$.

Theorem 2.1 (1) *The closed 2-form ω_F is a symplectic structure on $T(M)$.*

(2) *We denote by X_H^F the Hamiltonian vector field of the Hamiltonian H with respect to ω_F. Each orbit of the electromagnetic flow on $T(M)$ coincides with the integral curve of X_H^F.*

Remark This theorem is well-known when $F = 0$. The theorem is also well-known when $M = \boldsymbol{R}_1^4$ and $U = 0$ ([8], [4, § 20] and [5, § 4]).

Proof. (1) The components F_{ij} of F with respect to $(x^1, \ldots, x^n)$ are given by $F_{ij} = F\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$. The local expression for the closed 2-form ω_F is given by

$$\omega_F = \sum_{i,j,k} \frac{\partial g_{ij}}{\partial x^k} u^j dx^i \wedge dx^k + \sum_{i,j} g_{ij} dx^i \wedge du^j - \frac{1}{2}\sum_{i,j} F_{ij} dx^i \wedge dx^j.$$

Hence ω_F is nondegenerate at each point; that is, ω_F is a symplectic structure on $T(M)$.

(2) We denote by Γ_{ij}^k the Christoffel symbols. Let $x(t) = (x^1(t), \ldots, x^n(t))$ be a curve in M. Then

$$\nabla_{\dot{x}}\dot{x} = \sum_k \Big(\ddot{x}^k + \sum_{i,j} \dot{x}^i \dot{x}^j \Gamma_{ij}^k\Big) \frac{\partial}{\partial x^k}.$$

Since

$$\operatorname{grad} U = \sum_{i,j} g^{ij} \frac{\partial U}{\partial x^i} \frac{\partial}{\partial x^j} \quad \text{and} \quad \mathcal{L}^{-1}(\iota(\dot{x})F) = \sum_{i,j,k} \dot{x}^k F_{ki} g^{ij} \frac{\partial}{\partial x^j},$$

the equation of motion (2.1) of a charged particle is equivalent to

$$\ddot{x}^k + \sum_{i,j} \dot{x}^i \dot{x}^j \Gamma^k_{ij} = -\sum_i g^{ik} \frac{\partial U}{\partial x^i} - \sum_{i,j} \dot{x}^j F_{ji} g^{ik}. \tag{2.2}$$

Since the local expression for the Hamiltonian H is given by

$$H(x^1, \ldots, x^n, u^1, \ldots, u^n) = \frac{1}{2} \sum_{i,j} u^i u^j g_{ij} + U(x^1, \ldots, x^n),$$

we have

$$dH = \frac{1}{2} \sum_{i,j,k} \frac{\partial g_{ij}}{\partial x^k} u^i u^j dx^k + \sum_{i,j} g_{ij} u^i du^j + \sum_k \frac{\partial U}{\partial x^k} dx^k.$$

Since $dH = \iota(X^F_H)\omega_F$, we obtain

$$X^F_H = \sum_i u^i \frac{\partial}{\partial x^i} - \sum \left(\Gamma^l_{ji} u^j u^i + g^{kl} \frac{\partial U}{\partial x^k} + g^{kl} F_{ik} u^i \right) \frac{\partial}{\partial u^l}. \tag{2.3}$$

Here we mention the meaning of the right-hand side of the above equation. The vector field $X_{H_0} = \sum_i u^i \frac{\partial}{\partial x^i} - \sum \Gamma^l_{ji} u^j u^i \frac{\partial}{\partial u^l}$ is the Hamiltonian vector field of $H_0(u) = \frac{1}{2}\langle u, u \rangle$ with respect to ω, the vector field $-\sum g^{kl} \frac{\partial U}{\partial x^k} \frac{\partial}{\partial u^l}$ is the Hamiltonian vector field of $U \circ \pi$ with respect to ω, and $Y = -\sum g^{kl} F_{ik} u^i \frac{\partial}{\partial u^l}$ is characterized by the equation $\iota(Y)\omega = \iota(X_{H_0})\pi^* F$. The integral curve $(x^1(t), \ldots, x^n(t), u^1(t), \ldots, u^n(t))$ of X^F_H satisfies

$$\dot{x}^l = u^l, \quad \dot{u}^l = -\left(\sum \Gamma^l_{ji} u^j u^i + \sum g^{kl} \frac{\partial U}{\partial x^k} + \sum g^{kl} F_{ik} u^i \right)$$

by (2.3), which, together with (2.2), yields the assertion. □

Henceforth, we set $U = 0$. We define a tensor field ϕ of type $(1,1)$ by

$$\phi X = -\mathcal{L}^{-1}(\iota(X)F), \quad F(X, Y) = \langle X, \phi Y \rangle,$$

which is skew-symmetric with respect to $\langle \ , \ \rangle$. We consider the motion of a charged particle

$$\nabla_{\dot{x}} \dot{x} = \phi \dot{x} \tag{2.4}$$

under electromagnetic field F. We define a Lie subalgebra $\mathcal{I}_\phi(M)$ in $\mathfrak{X}(M)$ by

$$\mathcal{I}_\phi(M) = \{X \in \mathfrak{X}(M) \mid L_X\langle\ ,\ \rangle = 0,\ L_X\phi = 0\}.$$

For $X \in \mathcal{I}_\phi(M)$, we have $d(\iota(X)F) = 0$.

Proposition 2.2 *Let X and Y be in $\mathcal{I}_\phi(M)$. Then*

$$\iota([X,Y])F = -d(F(X,Y)).$$

Proof. Let Z be any vector field of M. Since $\langle\ ,\ \rangle$ is parallel,

$$\begin{aligned} Z(F(X,Y)) &= Z(\langle X, \phi Y\rangle) \\ &= \langle \nabla_Z X, \phi Y\rangle - \langle \phi X, \nabla_Z Y\rangle + \langle X, (\nabla_Z\phi)(Y)\rangle. \end{aligned}$$

Since X and Y are Killing vector fields,

$$\langle \nabla_Z X, \phi Y\rangle - \langle \phi X, \nabla_Z Y\rangle = \langle Z, \nabla_{\phi X} Y - \nabla_{\phi Y} X\rangle.$$

Since X and Y are infinitesimal automorphisms of ϕ,

$$\begin{aligned} \nabla_{\phi X} Y - \nabla_{\phi Y} X &= \nabla_Y(\phi X) + [\phi X, Y] - \nabla_X(\phi Y) - [\phi Y, X] \\ &= \phi[X,Y] + (\nabla_Y\phi)(X) - (\nabla_X\phi)(Y). \end{aligned}$$

Combining these equations above,

$$\begin{aligned} Z(F(X,Y)) &= \langle Z, \phi[X,Y]\rangle + \mathfrak{S}_{X,Y,Z}\langle X, (\nabla_Z\phi)(Y)\rangle \\ &= -F([X,Y],Z), \end{aligned}$$

where the last equality derives from $dF = 0$. □

We apply the above proposition to the study on simpleness of motions of charged particles (2.4).

Definition 2.3 Let $(M, \langle\ ,\ \rangle)$ be a semi-Riemannian manifold and ϕ a tensor field of type $(1,1)$ on M such that $\langle \phi X, Y\rangle + \langle X, \phi Y\rangle = 0$. A manifold $(M, \langle\ ,\ \rangle, \phi)$ of this type is called *G-homogeneous* (or simply *homogeneous*) if a Lie transformation group G of isometries acts transitively and effectively on M, and ϕ is invariant under the action of G.

We refer to T. Adachi [1], T. Sunada [9] and their references for studies on electromagnetic flows associated with a scalar multiple of Kähler forms in a complex projective space and a complex hyperbolic space. These spaces are typical examples of G-homogeneous Riemannian manifolds.

In a manner similar to the proof of [6, Theorem 2.3], we have the following.

Theorem 2.4 *Let* $(M, \langle\ ,\ \rangle, \phi)$ *be a* G*-homogeneous semi-Riemannian manifold. Assume that the* 2*-form* F *defined by* $F(X, Y) = \langle X, \phi Y\rangle$ *is closed. If* $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$*, then every motion* (2.4) *of a charged particle is simple, where* $\mathfrak{g}$ *is the Lie algebra of* G*.*

Proof. Let $x(t)$ be the motion of a charged particle. For $X \in \mathfrak{g}$, we also denote by X the induced Killing vector field of M, which is an infinitesimal automorphism of ϕ. Since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, it follows from Proposition 2.2 that there exixts a function f_X such that $\iota(X)F = df_X$. Using Proposition 2.5, (2) and Theorem 2.1, (2), we have

$$\langle \dot{x}(t), X_{x(t)}\rangle - f_X(x(t)) = \text{a constant}.$$

More precisely, the left-hand side of the equation above is independent of t. (See [6], for a more direct proof of the fact using (2.4).) Assuming that $x(0) = x(1) = o$, we then have

$$\langle \dot{x}(0), X_o\rangle = \langle \dot{x}(1), X_o\rangle.$$

Since M is homogeneous,

$$T_o(M) = \{X_o \mid L_X\langle\ ,\ \rangle = 0,\ L_X\phi = 0\}.$$

Hence we have $\dot{x}(0) = \dot{x}(1)$. Since (2.4) is an ordinary differential equation of second order, the theorem is proved. □

Let Y be in $\mathcal{I}_\phi(M)$. Assume that there exists a function f_Y such that

$$\iota(Y)F = df_Y.$$

(For instance, if $H^1(M) = \{0\}$, such a function f_Y exists. When M is an almost α-Sasakian manifold $(M, \langle\ ,\ \rangle, \phi, \eta, \xi)$, where α is a nonzero constant, then such a function exists. See [6, Prop. 3.8] for details.) We define a function P_Y^F on $T(M)$ by

$$P_Y^F(u) = \langle u, Y\rangle - f_Y(\pi(u)) = (P_Y - f_Y \circ \pi)(u) \quad (u \in T(M)).$$

We denote by $\{\ ,\ \}_F$ the Poisson bracket with respect to ω_F.

Proposition 2.5 *Let Y be in $\mathcal{I}_\phi(M)$. Assume that there exists a function f_Y such that $\iota(Y)F = df_Y$. Then*

(1) *The Hamiltonian vector field of P_Y^F with respect to ω_F coincides with the Hamiltonian vector field X_{P_Y} of P_Y with respect to ω.*

(2) $\{H, P_Y^F\}_F = 0$, *where* $H(u) = \frac{1}{2}\langle u, u\rangle$.

Proof. (1) Using $\iota(Y)F = df_Y$ and (1.2), we have $d(f_Y \circ \pi) = \iota(X_{P_Y})\pi^*F$. Thus

$$dP_Y^F = dP_Y - d(f_Y \circ \pi) = \iota(X_{P_Y})\omega - \iota(X_{P_Y})\pi^*F = \iota(X_{P_Y})\omega_F.$$

(2) $\{H, P_Y^F\}_F = -X_{P_Y}(H) = \{H, P_Y\} = 0$,
where (1) guarantees the first equality, and the last follows from (1.1). □

Using Noether's theorem, Propositions 1.3 and 2.5, we see the following: The one-parameter transformation group of $T(M)$ which is induced from the one-parameter transformation group of M generated by $X \in \mathcal{I}_\phi(M)$ is a symplectic transformation that preserves H.

Assume that there exists a function f_Y such that $df_Y = \iota(Y)F$ for any $Y \in \mathcal{I}_\phi(M)$. We examine the relation between $\{P_Y^F, P_Z^F\}_F$ and $P_{[Y,Z]}^F$ for $Y, Z \in \mathcal{I}_\phi(M)$. In order to formulate this, we define an equivalence relation $\sim$ on $C^\infty(T(M))$ by

$$f_1 \sim f_2 \Leftrightarrow f_2 - f_1 = \text{a constant function} \quad (f_1, f_2 \in C^\infty(T(M))).$$

We denote by $C^\infty(T(M))/\boldsymbol{R}$ the set of equivalence classes of $C^\infty(T(M))$. If we set

$$\{[f_1], [f_2]\}_F = [\{f_1, f_2\}_F] \quad (f_1, f_2 \in C^\infty(T(M))),$$

then the induced Poisson bracket $\{\ ,\ \}_F$ on $C^\infty(T(M))/\boldsymbol{R}$ is well-defined, where we denote by $[f]$ the equivalence class of $f \in C^\infty(T(M))$.

Proposition 2.6 *Assume that there exists a function f_Y such that $df_Y = \iota(Y)F$ for any $Y \in \mathcal{I}_\phi(M)$. Then the mapping*

$$[P^F] : (\mathcal{I}_\phi(M), [\ ,\ ]) \to (C^\infty(T(M))/\boldsymbol{R}, \{\ ,\ \}_F);\ Y \mapsto [P_Y^F]$$

is a Lie homomorphism, that is,

$$\{[P_Y^F], [P_Z^F]\}_F = [P_{[Y,Z]}^F] \qquad (Y, Z \in \mathcal{I}_\phi(M)).$$

Proof.

$$\begin{aligned}\{P_Y^F, P_Z^F\}_F &= -\omega_F(X_{P_Y}, X_{P_Z}) \\ &= -(\omega(X_{P_Y}, X_{P_Z}) - (\pi^* F)(X_{P_Y}, X_{P_Z})) \\ &= \{P_Y, P_Z\} + F(Y, Z)\circ\pi \\ &= P_{[Y,Z]} + F(Y, Z)\circ\pi \\ &= P_{[Y,Z]} - f_{[Y,Z]}\circ\pi + f_{[Y,Z]}\circ\pi + F(Y, Z)\circ\pi \\ &= P^F_{[Y,Z]} + (f_{[Y,Z]} + F(Y, Z))\circ\pi,\end{aligned}$$

where the first equality comes from Proposition 2.5, the third from (1.2) and the fourth from Propositon 1.1. Taking into account Proposition 2.2 completes the proof. □

References

[1] Adachi T., *Kähler magnetic fields on a complex hyperbolic space*. A report on Korea-Japan joint workshop in Mathematics 2000, 9–20.

[2] Adachi T., Maeda S. and Udagawa S., *Simpleness and closedness of circles in compact Hermitian symmetric spaces*. Tsukuba J. Math. **24** (2000), 1–13.

[3] Comtet A., *On the Landau levels on the hyperbolic plane*. Annals of physics **173** (1987), 185–209.

[4] Guillemin S. and Sternberg S., *Symplectic techniques in physics*. Cambridge Univ. Press, New York (1984).

[5] Kheyfets A. and Norris L.K., $P(4)$ *affine and superhamiltonian formulations of charged particle dynamics*. International Journ. of Theoretical Physics **27**, no.2, (1988), 159–182.

[6] Ikawa O., *Motion of charged particles in homogeneous Kähler and homogeneous Sasakian manifolds*. Preprint.

[7] O'Neill B., *Semi-Riemannian geometry*. Academic Press, New York (1983).

[8] Sternberg S., *On the role of field theories in our physical conception of geometry*. in Differential geometric methods in mathematical physics II, Springer Lecture Notes in Mathematics, No. 676, Springer-Verlag, New York, 1978.

[9] Sunada T., *Magnetic flows on a Riemannian surface*. Proc. KAIST Math. Workshop (Analysis and geometry) **8** (1993), 93–108.

[10] Utiyama R., *Theory of general relativity*. (in Japanese) Syokabo, Tokyo (1991).

Department of General Education
Fukushima National College of Technology
Iwaki, Fukushima 970-8034, Japan
E-mail: ikawa@fukushima-nct.ac.jp