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On singular solutions of implicit second-order ordinary differential equations

Mohan BHUPAL*

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Abstract. In this note we discuss the notion of singular solutions of completely integrable implicit second-order ordinary differential equations. After restricting the class of admissible equations we give conditions under which singular solutions occur in 1-parameter families and as isolated objects.

Key words: ordinary differential equations, implicit equations, completely integrable, geometric solutions, singular solutions.

1. Introduction

Consider an implicit second-order ordinary differential equation

$$F(x, y, y', y'') = 0, \tag{1}$$

where F is a smooth function of the independent variable x and of the "unknown" function y and its first and second derivatives $y' = dy/dx$, $y'' = d^2y/dx^2$. Replacing y' by p and y'' by q, it is natural to consider F as being defined on an open subset $\mathcal{O} \subset J^2(\mathbb{R}, \mathbb{R}) \cong \mathbb{R}^4$ of the space of 2-jets of functions of one variable. We will assume that $F\colon \mathcal{O} \to \mathbb{R}$ is a submersion. It follows that the set $S = F^{-1}(0)$ is a hypersurface of $\mathcal{O}$. We shall denote by $\xi \subset TJ^2(\mathbb{R}, \mathbb{R})$ the canonical second-order contact structure on $J^2(\mathbb{R}, \mathbb{R})$. This, by definition, is the tangent 2-plane field given as the common zero set of the two 1-forms

$$\alpha_1 = dy - p\,dx, \qquad \alpha_2 = dp - q\,dx.$$

Let $z_0 = (x_0, y_0, p_0, q_0)$ be a point in S. A solution of (1) from the jet bundle point of view corresponds to a regular integral curve $\gamma\colon ((a, b), t_0) \to (S, z_0)$ of ξ that can be parametrised by x. By a *geometric solution* of (1) we shall mean any regular integral curve $\gamma\colon ((a, b), t_0) \to (S, z_0)$ of ξ. We say that

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(1) is *completely integrable around* z_0 if there exists a diffeomorphism

$$\Gamma\colon ((\alpha_1, \beta_1) \times (\alpha_2, \beta_2) \times (a, b), (0, 0, t_0)) \to (S, z_0)$$

such that for each pair $(c_1, c_2) \in (\alpha_1, \beta_1) \times (\alpha_2, \beta_2)$, $\Gamma(c_1, c_2, \cdot)\colon ((a, b), t_0) \to (S, z_0)$ is a geometric solution of (1). We call such a diffeomorphism Γ a *complete solution around* z_0. We say that a geometric solution $\gamma\colon ((a, b), t_0) \to (S, z_0)$ is a *singular solution of* (1) *around* z_0 if for any open subinterval $(c, d) \subset (a, b)$, $\gamma|_{(c,d)}$ is never contained in a leaf of a complete solution (*c.f.* Izumiya [2], Izumiya and Yu [3], M. and T. Fukuda [1]).

Around points $z \in S$ such that the contact plane ξ_z intersects T_zS transversally, it is easy to see that a complete solution exists simply by integrating the line field $\xi \cap TS$. Around points where transversality fails the situation is more complicated. As we shall see, there may not be a complete solution around such points. We call points where transversality fails to hold *contact singular points* and denote by $\Sigma_c = \Sigma_c(F)$ the set of contact singular points. It is easy to check that the set of contact singular points is given by

$$\Sigma_c = \{z \in \mathcal{O} \mid F(z) = 0,\ F_x(z) + pF_y(z) + qF_p(z) = 0,\ F_q(z) = 0\}\,.$$

From the definition of singular solutions, it is easy to see that a geometric solution $\gamma\colon ((a, b), t_0) \to (S, z_0)$ is a singular solution only if it is contained in $\Sigma_c(F)$.

We present an example illustrating the notions of complete integrability and singular solutions. This example was observed by Izumiya. Consider the second-order Clairaut equation $F(x, y, p, q) = p - qx - f(q) = 0$, where f is a smooth function of one variable. In this example $F_x + pF_y + qF_p \equiv 0$ and $F_q = -x - f'(q)$. Thus the contact singular set is given by

$$\Sigma_c = \{(x, y, p, q) \mid x = -f'(q),\ p = -qf'(q) + f(q)\}.$$

Notice that $F(x, y, p, q) = 0$ admits the solution

$$y' = c_1x + f(c_1)$$

for each $c_1 \in \mathbb{R}$ and thus

$$y = \frac{1}{2}c_1x^2 + f(c_1)x + c_2$$

for $c_1, c_2 \in \mathbb{R}$ is a general solution which gives rise to the complete solution

$\Gamma\colon \mathbb{R}\times\mathbb{R}\times\mathbb{R}\to S$ given by

$$\Gamma(c_1,c_2,t)=\left(t,\,\frac{1}{2}c_1t^2+f(c_1)t+c_2,\;c_1t+f(c_1),\;c_1\right).$$

Also observe that the map

$$\Phi\colon \mathbb{R}\times\mathbb{R}\to\Sigma_c$$

given by

$$\Phi(c,t)=\left(-f'(t),\;\int(tf'(t)f''(t)-f(t)f''(t))\,dt+c,\;-tf'(t)+f(t),\;t\right)$$

gives a 1-parameter family of geometric solutions (depending on c) lying in Σ_c. Clearly each member of this family is not a member of the complete solution and thus we have a 1-parameter family of singular solutions foliating Σ_c.

We will also need to consider the subset $\Delta=\Delta(F)\subset\Sigma_c$ which is defined to be the set of points $z\in\Sigma_c$ such that $T_z(F^{-1}(0))$ coincides with the kernel of $\alpha_1(z)$. Explicitly, this set is given by $\Delta=\{z\in\Sigma_c\mid F_p(z)=0\}$. Around points $z\in\Delta$, assuming that Δ is nonempty, the presence of a complete solution is not sufficient to ensure that the set Σ_c is a manifold (see Section 2 for examples). To exclude this possibility, for simplicity, we make assumption that 0 is regular value of $F_q|_S$. We can now state our results regarding the relation between complete solutions and the set Σ_c.

Theorem 1.1 *Suppose that* 0 *is a regular value of* $F_q|_S$. *Then* (1) *is completely integrable around a point* $z_0\in S$ *if and only if* $z_0\notin\Sigma_c$ *or* Σ_c *is a* 2*-dimensional manifold around* z_0.

Theorem 1.2 *Suppose that* 0 *is a regular value of* $F_q|_S$ *and* (1) *is completely integrable.*

(i) *Leaves of the complete solution which meet* Σ_c *away from* Δ *intersect* Σ_c *transversally.*

(ii) *Leaves of the complete solution which meet* Δ *meet* Σ_c *tangentially.*

Assume now that $\Sigma_c\neq\emptyset$. As mentioned above, singular solutions, if they exist, necessarily lie in Σ_c. Assuming that 0 is a regular value of F_q, if (1) is completely integrable around a point $z_0\in\Sigma_c$, Σ_c is locally a 2-dimensional manifold around z_0, and thus we may consider geometric solutions $\gamma\colon((a,b),t_0)\to(S,z_0)$ such that $\mathrm{Image}(\gamma)\subset\Sigma_c$. It follows from

Theorem 1.2 that if the image of γ is not contained in Δ, then such solutions, if they exist, constitute singular solutions of (1). We call a diffeomorphism

$$\Phi\colon ((\alpha,\beta)\times(a,b),(0,t_0)) \to (\Sigma_c, z_0)$$

such that for each $c \in (\alpha,\beta)$, $\Phi(c,\cdot)\colon ((a,b),t_0) \to (\Sigma_c, z_0)$ is a singular solution, a *complete singular solution around* z_0.

As before, around points $z \in \Sigma_c$ such that ξ_z intersects $T_z\Sigma_c$ transversally in $T_z(F^{-1}(0))$, it is easy to see that a complete singular solution exists by integrating the line field $\xi \cap T\Sigma_c$. Around points where transversality does not hold a complete singular solution need not exist. We call such points *second-order contact singular points* and denote by $\Sigma_{cc} = \Sigma_{cc}(F)$ the set of second-order contact singular points. The following result, concerning the relation between complete singular solutions and the set Σ_{cc}, is similar to the first-order case considered by Izumiya and Yu [3].

Theorem 1.3 *Suppose that* 0 *is a regular value of* $F_q|_S$*,* (1) *is completely integrable and* $\Sigma_c \neq \emptyset$*.*

(i) *Equation* (1) *admits a complete singular solution around a point* $z_0 \in \Sigma_c$ *if and only if* $z_0 \notin \Sigma_{cc}$ *or* Σ_{cc} *is a* 1*-dimensional manifold around* z_0*.*
(ii) *Suppose that* (1) *admits a complete singular solution, then each leaf of the complete singular solution intersects* Σ_{cc} *transversely.*

If Σ_{cc} is a 1-dimensional manifold, it is necessarily a geometric solution of (1). Also, as we shall see later (Lemma 3.7), Σ_{cc} is contained in Δ. Thus, in view of Theorem 1.2 (ii), it is not clear *a priori* whether Σ_{cc} is a singular solution of (1) or not. However we have the following result.

Proposition 1.4 *Suppose that* 0 *is a regular value of* $F_q|_S$*,* (1) *is completely integrable and* Σ_{cc} *is a* 1*-dimensional manifold. Then* Σ_{cc} *is an isolated singular solution of* (1)*.*

2. Geometrical interpretation of implicit second-order ordinary differential equations

In this section we give a brief introduction to the concepts involved in the geometric interpretation of second-order ordinary differential equations. Further details and examples may be found in, for example, Komrakov and Lychagin [4]. Let

$$F(x, y, y', y'') = 0 \tag{2}$$

be an implicit second-order ordinary differential equation. Then a solution of (2) is a function $h\colon (a,b) \to \mathbb{R}$, defined on an interval $(a,b) \subset \mathbb{R}$, such that $F(x, h(x), h'(x), h''(x)) = 0$ for all $x \in (a,b)$. This can naturally be interpreted in the language of jet-bundles as follows.

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a smooth function and $x_0 \in \mathbb{R}$. The 2-jet of f at x_0 is, by definition, the 4-tuple

$$[f]^2_{x_0} = (x_0, f(x_0), f'(x_0), f''(x_0)).$$

The space of all 2-jets of smooth functions, $J^2(\mathbb{R}, \mathbb{R})$, can naturally be identified with $\mathbb{R}^4$. The differential equation (2) can now be regarded as a hypersurface

$$S = \{(x, y, p, q) \in J^2(\mathbb{R}, \mathbb{R}) \mid F(x, y, p, q) = 0\}$$

in $J^2(\mathbb{R}, \mathbb{R})$. In this language a solution of (2) is a curve lying on S having the form

$$\gamma_h = \{(x, y, p, q) \mid y = h(x),\ p = h'(x),\ q = h''(x)\}$$

for some real valued function $h\colon (a,b) \to \mathbb{R}$. If we fix a point $z_0 = (x_0, y_0, p_0, q_0)$ in $J^2(\mathbb{R}, \mathbb{R})$, then the space spanned by the tangent vectors to all curves of the form γ_h through z_0 has the form

$$\xi_{z_0} = \{(X, Y, P, Q) \mid Y = p_0 X,\ P = q_0 X\}.$$

Alternatively, ξ_{z_0} is given as the common zero set of the two 1-forms

$$\alpha_1 = dy - p_0\, dx, \qquad \alpha_2 = dp - q_0\, dx.$$

We call the family of 2-planes $\xi_z \subset T_z J^2(\mathbb{R}, \mathbb{R})$, as z varies, the canonical second-order contact structure on $J^2(\mathbb{R}, \mathbb{R})$. One can now easily check that a smooth curve $\gamma\colon (a,b) \to S$, which is regular in the sense that $\dot\gamma(t) \neq 0$ for every $t \in (a,b)$, is a solution of (2) if and only if the following two conditions hold:

(i) γ is an integral curve of ξ;
(ii) γ can be parametrised by x.

Dropping condition (ii) we arrive at the notion of a geometric solution. These can be thought of as multivalued solutions of the original differential equation. We now show how to construct a geometric solution through a general point z_0 in S.

It can be shown that at a general point z_0 in S the tangent space to S, $T_{z_0}S$, intersects ξ_{z_0} transversally. This is obvious for a general smooth equation F; in fact, this is true for every smooth F, this follows from the complete nonintegrability of contact structures on $\mathbb{R}^3$ (as this fact will not be needed later we omit the proof). Thus in a neighbourhood of z_0 the tangent spaces to S intersect the contact planes transversally and thus the intersections define a tangent line field in a neighbourhood of z_0. Finding an integral curve of this line field that passes through z_0 now gives our geometric solution, which is obviously unique up to reparametrisation. Thus, from the point of view of constructing geometric solutions, the points where something interesting may occur are those points $z \in S$ where T_zS does not intersect ξ_z transversally. The set of such points is the set $\Sigma_c = \Sigma_c(F)$ referred to in the previous section. In this note we discuss ideas related to the existence of certain geometric solutions contained entirely in the set Σ_c, namely singular solutions, under the assumption of complete integrability.

3. Preliminary results

We begin with the following elementary necessary and sufficient condition for the existence of a local complete solution.

Lemma 3.1 *Equation* (1) *is completely integrable around a point* $z_0 \in S$ *if and only if there exists a neighbourhood* $\Omega \subset S$ *of* z_0 *and functions* $\alpha, \beta \colon \Omega \to \mathbb{R}$, *which do not vanish simultaneously, such that*

$$\alpha(F_x + pF_y + qF_p)|_\Omega + \beta F_q|_\Omega \equiv 0.$$

Proof. Suppose that (1) is completely integrable around z_0 and let

$$\Gamma \colon ((\alpha_1, \beta_1) \times (\alpha_2, \beta_2) \times (a, b), (0, 0, t_0)) \to (S, z_0)$$

be a complete solution of (1) around z_0. Then differentiating Γ with respect to t yields a vector field $Z \colon \Omega \to TS$, where $\Omega = \mathrm{Image}(\Gamma)$, given by

$$Z(\Gamma(c_1, c_2, t)) = \Gamma_t(c_1, c_2, t).$$

Since $Z(z)$ lies in the contact plane ξ_z for each $z \in \Omega$ it has the form

$$Z = (\alpha, p\alpha, q\alpha, \beta)$$

for some functions $\alpha, \beta \colon \Omega \to \mathbb{R}$ which do not vanish simultaneously. But

$Z(z)$ also lies in $T_z(F^{-1}(0))$ for each $z \in \Omega$. It follows that the identity

$$\alpha(F_x + pF_y + qF_p)|_\Omega + \beta F_q|_\Omega \equiv 0$$

holds. Reversing the above argument yields the converse. □

Corollary 3.2 *Suppose that* (1) *is completely integrable around a point* $z_0 \in S$. *Then either* $z_0 \notin \Sigma_c$ *or* Σ_c *is a codimension* 1 *variety around* z_0 *in* S.

In addition to the contact singular set $\Sigma_c(F)$, it will also be useful to think of the the π-singular set $\Sigma_\pi(F)$. This is defined as follows. Let $\pi\colon J^2(\mathbb{R},\mathbb{R}) \to J^1(\mathbb{R},\mathbb{R})$ denote the canonical projection of $J^2(\mathbb{R},\mathbb{R})$ onto the space of 1-jets of functions of one variable, given by $(x,y,p,q) \mapsto (x,y,p)$. We say that a point $z \in S$ is a *π-singular point* of (1) if $\pi|_S$ is not a diffeomorphism at z, that is, $F_q(z) = 0$, and denote by $\Sigma_\pi = \Sigma_\pi(F)$ the set of π-singular points. In most of our examples this set coincides with the contact singular set Σ_c.

We now give some examples of completely integrable equations together with a description of their contact singular sets. In the first example we also explicitly describe the singular solutions. The details of the first example were already substantially known to Izumiya.

Example 3.3 (First-order Clairaut equation) Let $F(x,y,p,q) = px + f(p) - y$. Then $F_x + pF_y + qF_p = q(x + f'(p))$, $F_q = 0$. Thus, by Lemma 3.1, $F(x,y,p,q) = 0$ is completely integrable with the complete solution being given by

$$\Gamma(c_1,c_2,t) = (c_2, c_1c_2 + f(c_1), c_1, t).$$

In this example, the π-singular set Σ_π is all of S and the contact singular set Σ_c decomposes as a union $\Sigma_1 \cup \Sigma_2$ of two 2-dimensional manifolds intersecting transversely in S, where

$$\Sigma_1 = \{(x,y,p,q) \mid y = px + f(p),\, q = 0\},$$
$$\Sigma_2 = \big\{(x,y,p,q) \mid x = -f'(p),\, y = -pf'(p) + f(p)\big\}.$$

Notice that Σ_1 is foliated by a 1-parameter family of geometric solutions

$$\Phi_1(c,t) = (t, ct + f(c), c, 0).$$

This family is not contained in the complete solution and thus constitutes a complete singular solution. The 1-parameter family of geometric solutions

$$\Phi_2(c,t) = (-f'(c), -cf'(c) + f(c), c, t)$$

foliates Σ_2, however this family is contained in the complete solution and thus its members are not singular solutions. This failure is related to the fact that Σ_2 coincides with Δ. Also notice that the second-order contact singular set Σ_{cc} is contained in Σ_2 and is given by

$$\begin{aligned}\Sigma_{cc} = \big\{(x,y,p,q) \mid x = -f'(p),\ y = -pf'(p) + f(p),\\ q = -(f''(p))^{-1},\ f''(p) \neq 0\big\}.\end{aligned}$$

Away from values t such that $f''(t) = 0$,

$$\sigma(t) = (-f'(t), -tf'(t) + f(t), t, -(f''(p))^{-1})$$

defines a geometric solution contained in Σ_{cc}. This is an isolated singular solution and corresponds to the geometric solution arising as the envelope of the family Φ_1.

The next example shows that even for genuine second-order equations which are completely integrable the set Σ_c can fail to be a manifold.

Example 3.4 Let $F(x,y,p,q) = \frac{2}{3}q^3 + q^2x + px - y$. In this case $F_x + pF_y + qF_p = q^2 + qx$, $F_q = 2q^2 + 2qx$. Thus, again, by Lemma 3.1, $F(x,y,p,q) = 0$ is completely integrable. In this example, the contact singular set Σ_c coincides with the π-singular set Σ_π and is given by

$$\begin{aligned}\Sigma_c = &\{(x,y,p,q) \mid y = px, q = 0\}\\ &\cup \left\{(x,y,p,q) \mid y = \frac{1}{3}x^3 + px, q = -x\right\}.\end{aligned}$$

That is, Σ_c consists of two 2-dimensional manifolds intersecting transversely in S. Notice that the intersection of these two manifolds is Δ which in this case is a 1-dimensional manifold.

In the next proposition we will assume that the contact singular set Σ_c is nonempty.

Proposition 3.5 *Suppose that* (1) *is completely integrable around a point* $z_0 \in \Sigma_c$.

(i) *If* $z_0 \in \Sigma_c \setminus \Delta$, *then* Σ_c *is a 2-dimensional manifold around* z_0.

(ii) *If* $z_0 \in \Delta$, *then* Σ_c *is locally the zero set of a function on* S *which has nonzero 2-jet at this point.*

This proposition gives us some restriction on the topology of the set Σ_c in the completely integrable case. For instance, Σ_c cannot consist of three or more 2-dimensional manifolds intersecting at a point.

Proof of Proposition 3.5. (i) It is sufficient to show that one of the functions $(F_x + pF_y + qF_p)|_S$, $F_q|_S$ has nonzero gradient at z_0. Since $z_0 \notin \Delta$, we have $F_p(z_0) \neq 0$. Thus, by the implicit function theorem, there exists a function $g\colon U \to \mathbb{R}$, defined on an open set $U \subset \mathbb{R}^3$, such that, in a neighbourhood of z_0, a point $(x, y, p, q) \in \mathcal{O}$ is in S if and only if $p = g(x, y, q)$. Thus, without loss of generality, we may assume that $F(x, y, p, q) = g(x, y, q) - p$. Let $\varphi\colon U \to S$ denote the map $(x, y, q) \mapsto (x, y, g(x, y, q), q)$. Then it is sufficient to check that one of the functions $g_x + gg_y - q$, g_q has nonzero gradient at $u_0 = \varphi^{-1}(z_0)$. Now we have either

$$\frac{\partial}{\partial q}(g_x + gg_y - q)(u_0) = g_{xq}(u_0) + g(u_0)g_{yq}(u_0) - 1 \neq 0$$

or one of $g_{xq}(u_0) = \partial_x g_q(u_0)$, $g_{yq}(u_0) = \partial_y g_q(u_0)$ is nonzero. This proves (i).

(ii) Since $F_p(z_0) = 0$ and since $\nabla F(z_0) \neq 0$, from the definition of Σ_c we have $F_y(z_0) \neq 0$. Thus, again, by the implicit function theorem, there exists a function $h\colon V \to \mathbb{R}$, defined on an open set $V \subset \mathbb{R}^3$, such that, in a neighbourhood of z_0, a point $(x, y, p, q) \in \mathcal{O}$ is in S if and only if $y = h(x, p, q)$. Hence, without loss of generality, we may now assume that $F(x, y, p, q) = h(x, p, q) - y$. Let $\psi\colon V \to S$ denote the map $(x, p, q) \mapsto (x, h(x, p, q), p, q)$. Then it sufficient to check that one of the functions $h_x - p + qh_p$, h_q has nonzero first or second derivatives at $v_0 = \psi^{-1}(z_0)$. Since $F_p(z_0) = 0$, we have $h_p(v_0) = 0$. Now it may happen that all first derivatives of $h_x - p + qh_p$ and h_q vanish at v_0. Suppose this is the case, then, in particular,

$$\frac{\partial}{\partial p}(h_x - p + qh_p)(v_0) = h_{xp}(v_0) - 1 + qh_{pp}(v_0) = 0$$

and thus one of $h_{xp}(v_0)$, $h_{pp}(v_0)$ is nonzero. Suppose that $h_{xp}(v_0) \neq 0$. Then either

$$\frac{\partial}{\partial q}\frac{\partial}{\partial x}(h_x - p + qh_p)(v_0) = h_{xxq}(v_0) + h_{px}(v_0) + qh_{pxq}(v_0) \neq 0$$

or one of $h_{xxq}(v_0) = \partial_x\partial_x h_q(v_0)$, $h_{pxq}(v_0) = \partial_x\partial_p h_q(v_0)$ is nonzero, as required. The case $h_{pp}(v_0) \neq 0$ is similar. This proves Proposition 3.5. □

We continue to assume that $\Sigma_c \neq 0$. When 0 is a regular value of $F_q|_S$ we can obtain more precise information about the sets Σ_c and Δ in the completely integrable case.

Proposition 3.6 *Suppose that* 0 *is a regular value of* $F_q|_S$ *and* (1) *is completely integrable around a point* $z_0 \in \Sigma_c$.
(i) Σ_c *is a* 2*-dimensional manifold around* z_0.
(ii) *If* $z \in \Delta$, *then* Δ *is a* 1*-dimensional manifold around* z_0.

We point out that, in case Δ is a 1-dimensional manifold it need not be a geometric solution (see Example 5.2).

Proof of Proposition 3.6. (i) Follows immediately from Lemma 3.1.

(ii) As in the proof of Proposition 3.5, we may assume, without loss of generality, that F has the form $F(x, y, p, q) = h(x, p, q) - y$ for some function $h\colon V \to \mathbb{R}$, where V is a open subset of $\mathbb{R}^3$. Now, by assumption, 0 is a regular value of $F_q|_S$ and hence 0 is also a regular value of h_q. It follows that $\psi^{-1}(\Sigma_c) = h_q^{-1}(0)$, where $\psi\colon V \to F^{-1}(0)$ is defined in the proof of Proposition 3.5, and hence $\psi^{-1}(\Delta) = h_q^{-1}(0) \cap h_p^{-1}(0)$. Let $A \in \mathbb{R}^{2\times 3}$ be the matrix with rows $\nabla h_p(v_0)$, $\nabla h_q(v_0)$:

$$A = \begin{pmatrix} h_{px}(v_0) & h_{pp}(v_0) & h_{pq}(v_0) \\ h_{qx}(v_0) & h_{qp}(v_0) & h_{qq}(v_0) \end{pmatrix}, \tag{3}$$

where $v_0 = \psi^{-1}(z_0)$. To show that Δ is a 1-dimensional manifold, it is sufficient to show that A has rank 2. Now since $\psi^{-1}(\Sigma_c)$ is a 2-dimensional manifold around z_0 and $\nabla h_q(v_0)$ is nonzero, shrinking V if necessary, there exists a function $\rho\colon V \to \mathbb{R}$ such that

$$h_x - p + qh_p \equiv \rho h_q. \tag{4}$$

Now differentiating (4) with respect to p and q and evaluating at $v_0 = (x_0, p_0, q_0)$ gives

$$\begin{aligned} h_{xp}(v_0) &= \rho(v_0)h_{qp}(v_0) + 1 - q_0 h_{pp}(v_0) \\ h_{xq}(v_0) &= \rho(v_0)h_{qq}(v_0) - q_0 h_{pq}(v_0). \end{aligned} \tag{5}$$

Substituting in (3) for $h_{px}(v_0) = h_{xp}(v_0)$ and $h_{qx}(u_0) = h_{xq}(u_0)$ now gives

$$A = \begin{pmatrix} \rho(v_0)h_{qp}(v_0) + 1 - q_0 h_{pp}(v_0) & h_{pp}(v_0) & h_{pq}(v_0) \\ \rho(v_0)h_{qq}(v_0) - q_0 h_{pq}(v_0) & h_{qp}(v_0) & h_{qq}(v_0) \end{pmatrix}.$$

Now using column operations it follows that $\operatorname{rank} A = 2$ if and only if

$$\operatorname{rank} \begin{pmatrix} 1 & h_{pp}(v_0) & h_{pq}(v_0) \\ 0 & h_{qp}(v_0) & h_{qq}(v_0) \end{pmatrix} = 2.$$

Suppose now for a contradiction that $\operatorname{rank} A = 1$. Then $h_{qp}(v_0) = h_{qq}(v_0) = 0$. Also, from (5) it follows that $h_{qx}(v_0) = h_{xq}(v_0) = 0$. But this contradicts our assumption that $\nabla h_q(v_0)$ is nonzero. Thus $\operatorname{rank} A = 2$ and Δ is a 1-dimensional manifold around z_0 as required. □

Under the same assumptions as those of Proposition 3.6 we can also obtain the following information about the second-order contact singular set Σ_{cc}.

Lemma 3.7 *Suppose that* 0 *is a regular value of* $F_q|_S$ *and* (1) *is completely integrable. Then* Σ_{cc} *is contained in* Δ.

Proof. Assume that $\Sigma_{cc} \neq \emptyset$ and let $z_0 \in \Sigma_{cc}$. We show that $z_0 \in \Delta$. Since $\nabla F(z_0) \neq 0$ and $z_0 \in \Sigma_c$, either $F_y(z_0) \neq 0$ or $F_p(z_0) \neq 0$. First suppose that $F_y(z_0) \neq 0$. Due to the implicit function theorem, we may assume that F has the form $F(x, y, p, q) = h(x, p, q) - y$ for some function $h\colon V \to \mathbb{R}$, where V is a open subset of $\mathbb{R}^3$. Also, it follows from our assumptions that $\psi^{-1}(\Sigma_c(F)) = h_q^{-1}(0)$ and, as in the proof of Proposition 3.6, shrinking V if necessary, there exists a function $\rho\colon V \to \mathbb{R}$ such that the identity (4) holds. Now since $z_0 \in \Sigma_{cc}$, from the definition of Σ_{cc} we have

$$h_{qx}(v_0) + q_0 h_{qp}(v_0) = 0, \qquad h_{qq}(v_0) = 0. \tag{6}$$

Here $(x_0, p_0, q_0) = v_0 = \psi^{-1}(z_0)$. On the other hand, differentiating (4) with respect to q and evaluating at v_0 gives

$$h_{xq}(v_0) + h_p(v_0) + q_0 h_{qp}(v_0) = \rho(v_0)h_{qq}(v_0). \tag{7}$$

Comparing (6) and (7) now shows that $h_p(v_0) = 0$ and hence $z_0 \in \Delta$, as required.

Now suppose that $F_y(z_0) = 0$ and hence $F_p(z_0) \neq 0$. Again, due to the implicit function theorem, we may assume that F has the form

$F(x,y,p,q) = g(x,y,q) - p$ for some function $g\colon U \to \mathbb{R}$, where U is an open subset of $\mathbb{R}^3$. Also, it follows from our assumptions that $\varphi^{-1}(\Sigma_c) = g_q^{-1}(0)$. As before, shrinking U if necessary, there exists a function $\mu\colon U \to \mathbb{R}$ such that

$$g_x + gg_y - q \equiv \mu g_q. \tag{8}$$

In this case, from the definition of Σ_{cc} we have

$$g_{qx}(u_0) + g(u_0)g_{qy}(u_0) = 0, \qquad g_{qq}(u_0) = 0, \tag{9}$$

where $u_0 = \varphi^{-1}(z_0)$. On the other hand, differentiating (8) with respect to q and evaluating at u_0 gives

$$g_{qx}(u_0) + g(u_0)g_{qy}(u_0) - 1 = g_{qq}(u_0). \tag{10}$$

The incompatibility of (9) and (10) now shows that this case cannot occur. This proves Lemma 3.7 $\square$

Our final example in this section shows that even when the contact singular set Σ_c is a 2-dimensional manifold the equation $F(x,y,p,q) = 0$ need not be completely integrable.

Example 3.8 Let $F(x,y,p,q) = q^3 + px - y$. In this case $F_x + pF_y + qF_p = qx$, $F_q = 3q^2$. By Lemma 3.1, $F(x,y,p,q) = 0$ does not admit a complete solution in a neighbourhood of the contact singular point $z_0 = (0,0,0,0)$. Note that the contact singular set Σ_c coincides with the π-singular set Σ_π and is given by $\Sigma_c = \{(x,y,p,q) \mid y = px,\ q = 0\}$ and is thus a 2-dimensional manifold.

4. Proofs of main results

Theorem 1.1 *Suppose that 0 is a regular value of $F_q|_S$. Then (1) is completely integrable around a point $z_0 \in S$ if and only if $z_0 \notin \Sigma_c$ or Σ_c is a 2-dimensional manifold around z_0.*

Proof. Suppose that (1) is completely integrable around z_0. Then, by Proposition 3.6 (i), if $z_0 \in \Sigma_c$, then Σ_c is a 2-dimensional manifold around z_0. Now suppose that Σ_c is a 2-dimensional manifold around z_0. Since $\nabla F(z_0) \neq 0$ and $z_0 \in \Sigma_c$, either $F_y(z_0) \neq 0$ or $F_p(z_0) \neq 0$. First suppose that $F_y(z_0) \neq 0$. Then, due to the implicit function theorem, we may assume, without loss of generality, that F has the form $F(x,y,p,q) = h(x,p,q) - y$

for some function $h\colon V \to \mathbb{R}$, where V is an open subset of $\mathbb{R}^3$. Now, by assumption, 0 is a regular value of $F_q|_S$ and hence 0 is also a regular value of h_q. Thus $\psi^{-1}(\Sigma_c) = h_q^{-1}(0)$, where $\psi\colon V \to F^{-1}(0)$ is defined in the proof of Proposition 3.5. Also, as in the proof of Proposition 3.6, shrinking V if necessary, there exists a function $\rho\colon V \to \mathbb{R}$ such that the identity (4) holds. A complete solution of (1) in a neighbourhood of z_0 is now given by integrating the vector field $\psi_* X$, where $X\colon V \to TV$ is given by

$$X = (1, q, -\rho).$$

Now suppose that $F_p(z_0) \neq 0$. Again, due to the implicit function theorem, we may assume, without loss of generality, that F has the form $F(x,y,p,q) = g(x,y,q) - p$ for some function $g\colon U \to \mathbb{R}$, where U is an open subset of $\mathbb{R}^3$. Also, by assumption, 0 is also a regular value of g_q. Thus $\varphi^{-1}(\Sigma_c) = g_q^{-1}(0)$, where $\varphi\colon U \to F^{-1}(0)$ is defined in the proof Proposition 3.5. Also, as before, shrinking U if necessary, there exists a function $\mu\colon U \to \mathbb{R}$ such that the identity (8) holds. A complete solution of (1) in a neighbourhood of z_0 is now given by integrating the vector field $\varphi_* Y$, where $Y\colon U \to TU$ is given by

$$Y = (1, g, -\mu).$$

This proves Theorem 1.1. □

Theorem 1.2 *Suppose that* 0 *is a regular value of* $F_q|_S$ *and* (1) *is completely integrable.*

(i) *Leaves of the complete solution which meet* Σ_c *away from* Δ *intersect* Σ_c *transversally.*

(ii) *Leaves of the complete solution which meet* Δ *meet* Σ_c *tangentially.*

Proof. (i) Fix a point z_0 in $\Sigma_c \backslash \Delta$, which we assume is nonempty. We show that the leaf of the complete solution which passes through z_0 intersects Σ_c transversely. Since $F_p(z_0) \neq 0$, we may assume that F has the form $F(x,y,p,q) = g(x,y,q) - p$ for some function $g\colon U \to \mathbb{R}$, where $U \subset \mathbb{R}^3$. Also, we may assume that $\varphi^{-1}(\Sigma_c) = g_q^{-1}(0)$, where $\varphi\colon U \to S$ is defined in the proof of Proposition 3.5. Let $u_0 = \varphi^{-1}(z_0)$. Since $\nabla g_q(u_0)$ is normal to $\varphi^{-1}(\Sigma_c)$ at u_0 and the vector $(\varphi_* Y)(z_0)$, where $Y\colon U \to TU$ is defined in the proof of Theorem 1.1, is tangent to the leaf of the complete solution passing through z_0, it is sufficient to check that the scalar product of $\nabla g_q(u_0)$ and

$Y(u_0)$ is nonzero. Now

$$\langle \nabla g_q(u_0), Y(u_0) \rangle = g_{qx}(u_0) + g(u_0) g_{qy}(u_0) - \mu(u_0) g_{qq}(u_0). \tag{11}$$

On the other hand, differentiating (8) with respect to q at evaluating at u_0 gives

$$g_{xq}(u_0) + g(u_0) g_{yq}(u_0) - 1 = \mu(u_0) g_{qq}(u_0).$$

Substituting for $\mu(u_0) g_{qq}(u_0)$ on the right hand side of (11) we find that the scalar product of $\nabla g_q(u_0)$ and $Y(u_0)$ is nonzero as required.

(ii) We now assume that $\Delta \neq \emptyset$. Let $z_0 \in \Delta$. We show that the leaf of the complete solution passing through z_0 meets Σ_c tangentially. Since $F_y(z_0) \neq 0$, we may now assume that F has the form $F(x,y,p,q) = h(x,p,q) - y$ for some function $h\colon V \to \mathbb{R}$, where $V \subset \mathbb{R}^3$. Also, we may assume that $\psi^{-1}(\Sigma_c) = h_q^{-1}(0)$, where $\psi\colon V \to S$ is defined in the proof of Proposition 3.5. Let $v_0 = \psi^{-1}(z_0)$. In this case, since $\nabla h_q(v_0)$ is normal to $\psi^{-1}(\Sigma_c)$ at v_0 and the vector $(\psi_* X)(z_0)$, where $X\colon V \to TV$ is defined in the proof of Theorem 1.1, is tangent to the leaf of the complete solution passing through z_0, it is sufficient to check that the scalar product of $\nabla h_q(v_0)$ and $X(v_0)$ is 0. Now

$$\langle \nabla h_q(v_0), X(v_0) \rangle = h_{qx}(v_0) + q h_{qp}(v_0) - \rho(v_0) h_{qq}(v_0). \tag{12}$$

On the other hand, differentiating (4) with respect to q at u_0 gives

$$h_{xq}(v_0) + q h_{pq}(v_0) = \rho(v_0) h_{qp}(v_0).$$

It follows that the right hand side of (12) is 0 as required. □

Theorem 1.3 *Suppose that* 0 *is a regular value of* $F_q|_S$, (1) *is completely integrable and* $\Sigma_c \neq \emptyset$.

(i) *Equation* (1) *admits a complete singular solution around a point* $z_0 \in \Sigma_c$ *if and only if* $z_0 \notin \Sigma_{cc}$ *or* Σ_{cc} *is a 1-dimensional manifold around* z_0.

(ii) *Suppose that* (1) *admits a complete singular solution, then each leaf of the complete singular solution intersects* Σ_{cc} *transversely.*

Proof. We assume that Σ_{cc} is nonempty and fix a point $z_0 \in \Sigma_{cc}$. We first suppose that Σ_{cc} is a 1-dimensional manifold around z_0 and show that (1) admits a complete singular solution around z_0 such that each leaf of this complete singular solution intersects Σ_{cc} transversely. As before, since

$F_y(z_0) \neq 0$, we may assume that F has the form $F(x,y,p,q) = h(x,p,q) - y$ for some function $h\colon V \to \mathbb{R}$, where $V \subset \mathbb{R}^3$. Also, we may assume that $\psi^{-1}(\Sigma_c) = h_q^{-1}(0)$, where $\psi\colon V \to S$ is defined in Proposition 3.5. Now since $\nabla h_q(v_0)$ is nonzero, from (6) we have $h_{qp}(v_0) \neq 0$, where $v_0 = \psi^{-1}(z_0)$. Thus, by the implicit function theorem, there exists a function $f\colon W \to \mathbb{R}$, defined on some open set $W \subset \mathbb{R}^2$, such that, in a neighbourhood of v_0, a point $(x,p,q) \in V$ is in $\psi^{-1}(\Sigma_c)$ if and only if $p = f(x,q)$. Thus, without loss of generality, we may assume that $h_q(x,p,q) = f(x,q) - p$ and hence

$$\psi^{-1}(\Sigma_{cc}) = \{\vartheta(w) \mid w = (x,q) \in W,\ f_x(w) - q = 0,\ f_q(w) = 0\},$$

where $\vartheta\colon W \to \Sigma_c$ is the map $(x,q) \mapsto (x, f(x,q), q)$. Let $w_0 = \vartheta^{-1}(v_0)$. There are two cases to consider: (a) $f_{xq}(w_0) - 1 \neq 0$ and (b) $f_{xq}(w_0) - 1 = 0$. First suppose that $f_{xq}(w_0) - 1 \neq 0$. Then, since Σ_{cc} is 1-dimensional and $\nabla(f_x - q)(w_0)$ is nonzero, $\vartheta^{-1}(\Sigma_{cc}) = (f_x - q)^{-1}(0)$. Also, shrinking W if necessary, there exists a function $\delta\colon W \to \mathbb{R}$ such that

$$f_q = \delta(f_x - q). \tag{13}$$

The required foliation of Σ_c is now given by integrating the vector field $(\psi \circ \vartheta)_* S$, where $S\colon W \to TW$ is given by

$$S = (\delta, -1).$$

To show that each leaf of this foliation is transverse to Σ_{cc} it is sufficient to check that the scalar product of $\nabla(f_x - q)(w_0)$ and $S(w_0)$ is nonzero. Now

$$\langle \nabla(f_x - q)(w_0), S(w_0)\rangle = f_{xx}(w_0)\delta(w_0) - (f_{xq}(w_0) - 1) = 1,$$

where the second equality follows from differentiating (13) with respect to x and evaluating at w_0.

Now suppose that $f_{xq}(w_0) - 1 = 0$. In this case $\vartheta^{-1}(\Sigma_{cc}) = f_q^{-1}(0)$. Now, shrinking W if necessary, there exists a function $\gamma\colon W \to \mathbb{R}$ such that

$$f_x - q = \gamma f_q.$$

The required foliation of Σ_c in this case is given by integrating the vector field $(\psi \circ \vartheta)_* T$, where $T\colon W \to TW$ is given by

$$T = (1, -\gamma).$$

Now

$$\langle \nabla f_q(w_0), T(w_0)\rangle = f_{qx}(w_0) - f_{qq}(w_0)\gamma(w_0) = 1$$

shows that each leaf of this foliation intersects Σ_{cc} transversely.

Now suppose that (1) admits a complete singular solution around z_0. We show that Σ_{cc} is a 1-dimensional manifold around z_0. Let

$$\Phi\colon (\alpha,\beta)\times(a,b) \to \Sigma_c$$

be a complete singular solution around z_0. Then, by definition, for each $c \in (\alpha,\beta)$, $\Phi(c,\cdot)\colon (a,b) \to \Sigma_c$ is a geometric solution of (1) and for each $(c,t) \in (\alpha,\beta)\times(a,b)$

$$\operatorname{rank}\begin{pmatrix} x_t & y_t & p_t & q_t \\ x_c & y_c & p_c & q_c \end{pmatrix} = 2. \tag{14}$$

Also

$$\Phi^{-1}(\Sigma_{cc}) = \{(c,t) \mid y_c = px_c, p_c = qx_c\}.$$

Since we are assuming $F(x,y,p,q) = h(x,p,q) - y$, at $(c,t) \in \Phi^{-1}(\Sigma_c)$ we have

$$\begin{aligned} y_c &= h_x x_c + h_p p_c + h_q q_c \\ &= (-qh_p + p)x_c + h_p p_c. \end{aligned}$$

Thus if $p_c = qx_c$, then $y_c = px_c$ holds automatically. Thus

$$\Phi^{-1}(\Sigma_{cc}) = \{(c,t) \mid p_c = qx_c\}.$$

Now let $\lambda(c,t) = p_c - qx_c$. We claim that $\lambda_t(c_0,t_0) \neq 0$, where $(c_0,t_0) = \Phi^{-1}(z_0)$. Now

$$\lambda_t = p_{ct} - q_t x_c - qx_{ct}. \tag{15}$$

Also $p_t = qx_t$, since $\Phi(c,\cdot)$ is a geometric solution. Thus

$$p_{tc} = q_c x_t + qx_{tc}. \tag{16}$$

Substituting (16) into (15) now gives

$$\lambda_t = q_c x_t - q_t x_c. \tag{17}$$

On the other hand, since $z_0 \in \Sigma_{cc}$, $p_t = qx_t$, $y_t = px_t$, $p_c = qx_c$, $y_c = px_c$.

Thus (14) holds if and only if

$$\operatorname{rank}\begin{pmatrix} x_t & q_t \\ x_c & q_c \end{pmatrix} = 2.$$

That is, $0 \neq x_t q_c - x_c q_t = \lambda_t$. Thus Σ_{cc} is a 1-dimensional manifold around z_0 as required. □

Proposition 1.4 *Suppose that* 0 *is a regular value of* $F_q|_S$, (1) *is completely integrable and* Σ_{cc} *is a* 1*-dimensional manifold. Then* Σ_{cc} *is an isolated singular solution of* (1).

Proof. Let $z_0 \in \Sigma_{cc}$. As before, we can assume that F has the form $F(x,y,p,q) = h(x,p,q) - y$ for some function $h\colon V \to \mathbb{R}$, where $V \subset \mathbb{R}^3$. Also, by our assumptions, $\psi^{-1}(\Sigma_c) = h_q^{-1}(0)$,

$$\psi^{-1}(\Sigma_{cc}) = \left\{ v = (x,p,q) \in \psi^{-1}(\Sigma_c) \mid h_{qx}(v) + q h_{qp}(v) = 0,\ h_{qq}(v) = 0 \right\},$$

where $\psi\colon V \to S$ is defined in Proposition 3.5. Also, a complete solution of (1) in a neighbourhood of z_0 is given by integrating the vector field $\psi_* X$, where $X\colon V \to TV$ is defined in the proof of Theorem 1.1. Now since $\nabla h_q(v_0)$ is nonzero we have $h_{qp}(v_0) \neq 0$, where $v_0 = \psi^{-1}(z_0)$. It follows that one of $\nabla h_{qq}(v_0)$, $\nabla(h_{qx} + q h_{qp})(v_0)$ is nonzero. Suppose first that $\nabla h_{qq}(v_0)$ is nonzero. Then $\psi^{-1}(\Sigma_{cc}) = h_q^{-1}(0) \cap h_{qq}^{-1}(0)$. To show that Σ_{cc} is not a leaf of the complete solution around z_0 it is sufficient to check that the scalar product of $\nabla h_{qq}(v_0)$ and $X(v_0)$ is nonzero. Now

$$\langle \nabla h_{qq}(v_0), X(v_0) \rangle = h_{qqx}(v_0) + q_0 h_{qqp}(v_0) - \rho(v_0) h_{qqq}(v_0), \tag{18}$$

where $v_0 = (x_0, p_0, q_0)$. On the other hand, differentiating the identity (4) twice with respect to q and evaluating at v_0 gives

$$h_{xqq}(v_0) + q_0 h_{pqq}(v_0) + 2 h_{pq}(v_0) = \rho(v_0) h_{qqq}(v_0).$$

Thus, since $h_{pq}(v_0) \neq 0$, the right hand side of (18) is nonzero as required.

Now suppose that $\nabla(h_{qx} + q h_{qp})(v_0)$ is nonzero. Then $\psi^{-1}(\Sigma_{cc}) = h_q^{-1}(0) \cap (h_{qx} + q h_{qp})^{-1}(0)$. In this case it is sufficient to check that the scalar product of $\nabla(h_{qx} + q h_{qp})(v_0)$ and $X(v_0)$ is nonzero. Now

$$\begin{aligned}
&\langle \nabla(h_{qx} + q h_{qp})(v_0), X(v_0) \rangle \\
&\quad = h_{qxx}(v_0) + q_0 h_{qpx}(v_0) + q_0 (h_{qxp}(v_0) + q_0 h_{qpp}(v_0)) \\
&\qquad - \rho(v_0)(h_{qxq}(v_0) + h_{qp}(v_0) + q_0 h_{qpq}(v_0)).
\end{aligned} \tag{19}$$

On the other hand, differentiating (4) with respect to x and then q and evaluating at v_0 gives

$$h_{xxq}(v_0) + q_0 h_{pxq}(v_0) + h_{px}(v_0) = \rho(v_0) h_{qxq}(v_0) + \rho_q(v_0) h_{qx}(v_0). \tag{20}$$

Also, differentiating (4) with respect to p and then q and evaluating at v_0 gives

$$h_{xpq}(v_0) + q_0 h_{ppq}(v_0) + h_{pp}(v_0) = \rho(v_0) h_{qpq}(v_0) + \rho_q(v_0) h_{qp}(v_0). \tag{21}$$

Comparing (19) with the equality obtained by adding (21) multiplied by q_0 to (20), it is now sufficient to check that $h_{px}(v_0) + q_0 h_{pp}(v_0) - \rho(v_0) h_{qp}(v_0)$ is nonzero. This can be seen to be the case by differentiating (4) with respect to p and evaluating at v_0. This proves Proposition 1.4. □

5. Further examples

Example 5.1 Let $F(x,y,p,q) = -\frac{1}{3}q^3 + qp - y$. In this case $F_x + pF_y + qF_p = q^2 - p$, $F_q = -q^2 + p$, thus, by Lemma 3.1, $F(x,y,p,q) = 0$ is completely integrable. Also,

$$\Sigma_c = \Sigma_\pi = \left\{(x,y,p,q) \mid y = \frac{2}{3}q^3,\ p = q^2\right\}$$

$$\Sigma_{cc} = \Delta = \{(x,y,p,q) \mid y = p = q = 0\}.$$

Thus by Theorem 1.1 and Theorem 1.2, the complete solution of $F(x,y,p,q) = 0$ intersects Σ_c transversely away from Δ and is tangential to Σ_c at points in Δ. In addition, by Theorem 1.3, $F(x,y,p,q) = 0$ admits a complete singular solution. By Theorem 1.4, Σ_{cc} is an isolated singular solution.

Example 5.2 Let $F(x,y,p,q) = \frac{2}{3}q^3 + q^2x + qp + 2xp - y$. In this case $F_x + pF_y + qF_p = F_q = 2q^2 + 2qx + p$, thus, by Lemma 3.1, $F(x,y,p,q) = 0$ is completely integrable. Also,

$$\Sigma_c = \Sigma_\pi = \left\{(x,y,p,q) \mid y = -\frac{4}{3}q^3 - 5q^2x - 4qx^2,\ p = -2q^2 - 2qx\right\}$$

$$\Delta = \left\{(x,y,p,q) \mid y = -\frac{4}{3}x^3,\ p = -4x^2,\ q = -2x\right\},$$

$$\Sigma_{cc} = (0,0,0,0).$$

Again, by Theorem 1.1 and Theorem 1.2, the complete solution of $F(x, y, p, q) = 0$ intersects Σ_c transversely away from Δ and is tangential to Σ_c at points in Δ. Note in this example, however, that, by Theorem 1.3, there is no complete singular solution around the second-order contact singular point $(0, 0, 0, 0)$.

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References

[1] Fukuda M. and T., *Singular solutions of ordinary differential equations*. Yokohama Math. J. **15** (1977), 41–58.

[2] Izumiya S., *Singular solutions of first-order differential equations*. Bull. London Math. Soc. **26** (1994), 69–74.

[3] Izumiya S. and Yu J., *How to define singular solutions*. Kodai Math. J. **16**, No.2 (1993), 227–234.

[4] Komrakov B. and Lychagin V., *Symmetries and Integals*. (English translation), Preprint series - Matematisk istitutt, Universitetet i Oslo (1993).

Department of Mathematics
Middle East Technical University
06531 Ankara
Turkey