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On polarized surfaces (X, L) with $h^0(L) \geq 2$, $\kappa(X) \geq 0$, and $g(L) = q(X) + 1$

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Abstract. Let (X, L) be a polarized surface defined over the complex number field $\mathbb{C}$. If $h^0(L) > 0$, then $g(L) \geq q(X)$ holds, where $g(L)$ is the sectional genus of (X, L) and $q(X)$ is the irregularity of X. In previous papers, we have studied polarized surfaces (X, L) with $h^0(L) > 0$ and $g(L) = q(X)$. In this paper, we study a classification of (X, L) with $\kappa(X) \geq 0$, $h^0(L) \geq 2$, and $g(L) = q(X) + 1$.

Key words: polarized surface, sectional genus, irregularity.

Introduction

Let X be a smooth projective variety over the complex number field $\mathbb{C}$, and let L be an ample line bundle on X. Then we call the pair (X, L) a polarized manifold. The sectional genus $g(L)$ of (X, L) is defined as follows:

$$g(L) = 1 + \frac{1}{2}(K_X + (n-1)L)L^{n-1},$$

where K_X is the canonical line bundle of X and $n = \dim X$. A classification of (X, L) with small value of sectional genus was obtained by several authors. On the other hand, Fujita proved the following theorem (see Theorem (2.13.1) in [Fj5]).

Theorem *Let (X, L) be a polarized manifold. Then for any fixed $n = \dim X$ and $g(L)$, there are only finitely many deformation type of (X, L) unless (X, L) is a scroll over a smooth curve.*

(For a definition of the deformation type of (X, L), see §13 of Chapter II in [Fj5].) By this theorem, Fujita proposed the following conjecture (see (II.13.7) in [Fj5], Question 7.2.11 in [BS]);

Conjecture (Fujita) Let (X, L) be a polarized manifold. Then $g(L) \geq q(X)$, where $q(X) = h^1(\mathcal{O}_X)$: the irregularity of X.

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This conjecture is very difficult and it is unsolved even for the case in which X is a surface.

In this paper we consider the case in which $\dim X = 2$. If $\dim X = 2$, then we can prove that $g(L) \geq q(X)$ under one of the following conditions;

(A) $\kappa(X) \leq 1$,
(B) $\kappa(X) = 2$ and $h^0(L) > 0$.

In [Fk2] and [Fk3], we obtained a classification of (X, L) with $g(L) = q(X)$ under one of the following cases;

(0-1) $\kappa(X) \leq 1$,
(0-2) $\kappa(X) = 2$ and $h^0(L) > 0$.

Furthermore in [Fk4], we obtained a classification of (X, L) with $g(L) = q(X) + 1$ under one of the following cases;

(1-1) $\kappa(X) = -\infty$,
(1-2) $\kappa(X) \geq 0$ and $h^0(L) > 0$.

In particular if (X, L) is the case (1-2) above, then we determined a type of the divisor L, but we were not able to classify the type (X, L) in detail. So in this paper, we consider this case under the condition that $h^0(L) \geq 2$, and we get the following theorem;

Theorem 2.1 *Let (X, L) be a polarized surface with $\kappa(X) \geq 0$ and $h^0(L) \geq 2$. If $g(L) = q(X) + 1$, then (X, L) is one of the following types.*

(1) *X is birationally equivalent to an Abelian surface and $g(L) = 3$.*
(2) *$X = F \times C$, $L \equiv 2F + C$, where F and C are smooth curves such that $g(F) = 1$ and $g(C) \geq 2$. (Here $\equiv$ denotes the numerical equivalence of divisors.)*
(3) *X is the relatively minimal elliptic fibration which is a quasi-bundle and has two multiple fibers, and $L \equiv F_1 + N$ with $g(L) = 3$ and $q(X) = 2$, where F_1 is an irreducible component of some multiple fiber and N is a smooth irreducible curve of $g(N) = 2$ with $NF_1 = 1$.*
(4) *X is minimal with $\kappa(X) = 2$, $L^2 = 1$, $g(L) = 3$, and $q(X) = 2$. Then there exists a smooth projective surface X', a birational morphism $\mu : X' \to X$ and a fiber space $f : X' \to \mathbb{P}^1$ such that μ is a one point blowing up of X and the (-1)-curve E is a section of f, any fiber of f is irreducible and reduced, $L = \mu_*(F)$, and $g(F) = 3$ for a general fiber F of f.*

At present unfortunately we don't know whether an example of the case (4) in Theorem 2.1 exists or not. But the above result is very useful to classify (X, L) of $\dim X = n \geq 3$ with $g(L) = q(X) + 1$ and $\dim \mathrm{Bs}\,|L| = 0$. We study these (X, L) in a forthcoming paper.

Here we remark that the method in this paper is different from that in [Fk4].

We use the customary notation in algebraic geometry.

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1. Preliminaries

Lemma 1.1 (Debarre) *Let X be a minimal surface of general type with $q(X) \geq 1$. Then $K_X^2 \geq 2p_g(X)$. (Hence $K_X^2 \geq 2q(X)$ for any minimal surface of general type.)*

Proof. See Théorème 6.1 in [De]. □

Theorem 1.2 ([Fk1]) *Let (X, L) be an L-minimal quasi-polarized surface with $\kappa(X) \geq 0$. If $h^0(L) \geq 2$, then (X, L) satisfies one of the following conditions:*

(1) *$g(L) \geq 2q(X) - 1$.*

(2) *For any linear pencil $\Lambda \subseteq |L|$, the fixed part $Z(\Lambda)$ of Λ is not zero and $\mathrm{Bs}\,\Lambda_M = \phi$, where Λ_M is movable part of Λ. Let $f : X \to C$ be the fiber space induced by Λ_M. Then $g(L) \geq g(C) + 2g(F) \geq q(X) + g(F)$, $g(C) \geq 2$ and $LF = 1$ for a general fiber F of f.*

Proof. See Theorem 3.1 in [Fk1]. □

Definition 1.3 (See Definition 1.1 in [Se1].) Let X (resp. C) be a smooth projective surface (resp. a smooth curve). A fibration $f : X \to C$ is called a *quasi-bundle* if all smooth fibers are pairwise isomorphic, and the only singular fibers are multiples of smooth curves.

Proposition 1.4 *Let X be a smooth projective surface. Assume that X is minimal and X has an elliptic fibration $f : X \to C$ over a smooth curve C. If $q(X) = g(C) + 1$, then $\chi(\mathcal{O}_X) = 0$ and f is a quasi-bundle.*

Proof. See Lemma 1.5 and Lemma 1.6 in [Se1]. □

Lemma 1.5 *Let $f : X \to C$ be a relatively minimal elliptic fibration with $q(X) = g(C) + 1$. If f has a section, then $X \cong F \times C$, where F is a general fiber of f.*

Proof. (See [Fj4].) By Proposition 1.4, f is a quasi-bundle. Let C' be a section of f. Let $\Sigma \subset C$ be the singular locus of f and $U = C - \Sigma$. We fix an elliptic curve $E \cong f^{-1}(x)$ for $x \in U$. Then by [Fj4], we have a map $\varphi : \pi_1(U) \to \mathrm{Aut}(E, C'_E)$. Since $q(X) = g(C) + 1$, $\pi_1(U)$ acts on E as translations. Since $\deg C'_E = 1$, the translation part of $\mathrm{Aut}(E, C'_E)$ is trivial. Hence φ is trivial and we get the assertion. □

Lemma 1.6 *Let (X, L) be a polarized surface with $\kappa(X) \geq 0$, $L^2 = 1$, $g(L) = 2$, and $\Delta(L) = 1$, where $\Delta(L)$ is the delta genus of (X, L) (See* [Fj1], [Fj2] *or* [Fj4]*). Then $q(X) = 0$.*

Proof. By hypothesis, $h^0(L) = 2$. Since $L^2 = 1$, any element of $|L|$ is an irreducible reduced curves and $\mathrm{Bs}\,|L| = \{p\}$. Let $\varphi : X \dashrightarrow \mathbb{P}^1$ be a rational map defined by $|L|$. Then by blowing up at p, we have a fiber space $f : X' \to \mathbb{P}^1$ where $\mu : X' \to X$ is blowing up at p. Let $D \in |L|$ be an irreducible reduced smooth curve. (This D exists since $L^2 = 1$ and $\mathrm{Bs}\,|L| = \{p\}$.) Then $\mu^*D - E$ is a fiber of f, where E is a (-1)-curve with $\mu(E) = p$. On the other hand $K_{X'} = \mu^*K_X + E$. Hence

$$\begin{aligned} g(F) &= 1 + \frac{1}{2}(K_{X'} + F)F \\ &= 1 + \frac{1}{2}(\mu^*(K_X + D))(\mu^*D - E) \\ &= 1 + \frac{1}{2}(K_X + D)D \\ &= g(L) = 2 \end{aligned}$$

Therefore F is a hyperelliptic curve. Hence $q(X) = 0$ by Fujita's result [Fj1] (see also (I.6.18) in [Fj5]). This completes the proof of Lemma 1.6. □

2. Polarized surfaces with $h^0(L) \geq 2$, $\kappa(X) \geq 0$ and $g(L) = q(X) + 1$

In this section, we classify polarized surfaces with $h^0(L) \geq 2$, $\kappa(X) \geq 0$ and $g(L) = q(X) + 1$.

Theorem 2.1 *Let (X, L) be a polarized surface with $\kappa(X) \geq 0$ and $h^0(L) \geq 2$. If $g(L) = q(X) + 1$, then (X, L) is one of the following types.*

(1) *X is birationally equivalent to an Abelian surface and $g(L) = 3$.*

(2) *$X = F \times C$, $L \equiv 2F + C$, where F and C are smooth curves such that $g(F) = 1$ and $g(C) \geq 2$. (Here $\equiv$ denotes the numerical equivalence of divisors.)*

(3) *X is the relatively minimal elliptic fibration which is a quasi-bundle and has two multiple fibers, and $L \equiv F_1 + N$ with $g(L) = 3$ and $q(X) = 2$, where F_1 is an irreducible component of some multiple fiber and N is a smooth irreducible curve of $g(N) = 2$ with $NF_1 = 1$.*

(4) *X is minimal with $\kappa(X) = 2$, $L^2 = 1$, $g(L) = 3$, and $q(X) = 2$. Then there exists a smooth projective surface X', a birational morphism $\mu : X' \to X$ and a fiber space $f : X' \to \mathbb{P}^1$ such that μ is a one point blowing up of X and the (-1)-curve E is a section of f, any fiber of f is irreducible and reduced, $L = \mu_*(F)$, and $g(F) = 3$ for a general fiber F of f.*

Proof. By Theorem 1.2, we have the following two cases. (We use the notation in Theorem 1.2.)

Case (A) $g(L) \geq 2q(X) - 1$.

Case (B) $Z(\Lambda) \neq 0$, $\operatorname{Bs} \Lambda_M = \phi$, $g(C) \geq 2$ and $LF = 1$. Then $g(L) \geq g(C) + 2g(F)$.

(I) First we consider the case (B). Then $q(X) + 1 = g(L) \geq g(C) + 2g(F) \geq q(X) + g(F)$. Hence $g(F) \leq 1$. Since $\kappa(X) \geq 0$, then $g(F) = 1$ and $q(X) = g(C) + 1$, that is, $f : X \to C$ is an elliptic fibration with $\chi(\mathcal{O}_X) = 0$ by Proposition 1.4. Since $LF = 1$, we obtain that any fiber of f is irreducible, f has no multiple fiber, and $K_X L = (2g(C) - 2)LF = 2g(C) - 2$. Hence $g(C) + 2 = q(X) + 1 = g(L) = 1 + \frac{1}{2}L^2 + g(C) - 1$. So we have $L^2 = 4$. Furthermore since $h^0(L) \geq 2$ and $LF = 1$, there exists an irreducible and reduced curve C' such that $h^0(L - C') > 0$ and C' is a section of f. Hence $X \cong F \times C$ by Lemma 1.5, and there exists an effective divisor D' on X such that $C' + D' \in |L|$. Since $K_X C' = (2g(C) - 2)C'F = 2g(C) - 2 = 2g(C') - 2$, we have $(C')^2 = 0$. Since $LF = 1$ for any fiber F of f, we obtain that f has no multiple fiber, any fiber of f is irreducible, and D' is a sum of fibers of f. Because $L^2 = 4$ and $(C')^2 = 0$, we get that $D' = F_1 + F_2$, where F_1 and F_2 are fibers of f. For any $t \in C$, we put $(u(t), t) = C' \cap f^{-1}(t) \subset F \times C$. Next we consider the following morphism

$\theta : F \times C \to F \times C$; $\theta(x,t) = (x - u(t), t)$. Then θ is an isomorphism and $L \cong C + F_1 + F_2$ by this isomorphism. Therefore $L \equiv 2F + C$. This is the type (2) in Theorem 2.1.

(II) Next we consider the case (A). Then $q(X)+1 = g(L) \geq 2q(X)-1$. Hence $q(X) \leq 2$. On the other hand $2 \leq g(L) = q(X)+1$. So we have $q(X) = 1$ or 2. Hence we get the following two types:

Case (A-1) The case in which $q(X) = 1$ and $g(L) = 2$.

Case (A-2) The case in which $q(X) = 2$ and $g(L) = 3$.

We study these cases by the value of the Kodaira dimension of X.

(II.1) First we consider the case in which $\kappa(X) = 0$.

Claim 2.2 If $\kappa(X) = 0$, then (X, L) is the type (1) in Theorem 2.1.

Proof. (i) First we consider the case (A-1).

(i-1) If X is minimal, then $K_X L = 0$ and $L^2 = 2$. Furthermore since $q(X) = 1$, by the classification theory of surfaces we have $\chi(\mathcal{O}_X) = 0$. But by the Riemann-Roch theorem and the Kodaira vanishing theorem, $h^0(L) = \chi(\mathcal{O}_X) + \frac{1}{2}(L^2 - K_X L) = 1$. So this case cannot occur.

(i-2) If X is not minimal, then $K_X L = 1$ and $L^2 = 1$. Let $\mu : X \to X_1$ be the minimal model of X and $L_1 = \mu_* L$. Then $L_1^2 = 2$ and $K_{X_1} L_1 = 0$. But this (X_1, L_1) does not occur by the same argument as the case (i-1).

(ii) Next we consider the case (A-2). Then X is birationally equivalent to an Abelian surface X and $g(L) = 3$. If X is an Abelian surface, then this (X, L) is known (See (1.2) in [Ba] or [LB] P. 317 Ex (1)). This completes the proof of Claim 2.2. □

(II.2) Next we consider the case in which $\kappa(X) \geq 1$.

Claim 2.3 If $\kappa(X) \geq 1$, then the case (A-1) cannot occur.

Proof. Assume that $(g(L), q(X)) = (2, 1)$. Then we get that $L^2 = 1$ and $K_X L = 1$. Since $h^0(L) \geq 2$ and $\kappa(X) \geq 1$, we get that $\Delta(L) = 1$. By Lemma 1.6, we obtain that $q(X) = 0$ and this is a contradiction. □

By Claim 2.3 we assume that (X, L) is the case (A-2).

(II.2.1) We consider the case in which $\kappa(X) = 2$.

Claim 2.4 If $\kappa(X) = 2$, then (X, L) is the type (4) in Theorem 2.1.

Proof. Let $\mu : X \to X'$ be the minimal model of X and $L' = \mu_* L$. Then $K_X L \geq K_{X'} L'$. Since $q(X) = q(X') = 2$, we have $K_{X'}^2 \geq 4$ by Lemma 1.1.

Hence $K_X L \geq K_{X'} L' \geq 2$ by the Hodge index theorem. Because $g(L) = 3$, we have the following two types.
(a) $L^2 = 2$, $K_X L = 2$.
(b) $L^2 = 1$, $K_X L = 3$.

(a) The case in which $L^2 = 2$, $K_X L = 2$.
Then X is minimal by the above argument. So we have $(LK_X)^2 \geq L^2 K_X^2 \geq 8$ by the Hodge index theorem. But this case cannot occur.

(b) The case in which $L^2 = 1$, $K_X L = 3$.
First we prove that X is minimal. Assume that X is not minimal. Let $\delta : X \to Y$ be its minimalization and $A = \delta_*(L)$, where Y is a smooth projective surface. Here we remark that A is ample. Then $1 = L^2 < A^2$ and $0 < K_Y A < K_X L = 3$. In this case $(A^2, K_Y A) = (2,2)$, $(3,1)$ or $(5,1)$. By the Hodge index theorem, we get that $(A^2, K_Y A) = (2,2)$. In this case $g(A) = 3 = q(Y) + 1$. But this is impossible by the same argument as in the case (a). Therefore X is minimal.

Next we prove that $\dim H^0(L) = 2$ and $\dim \mathrm{Bs}\,|L| = 0$. If $h^0(L) \geq 3$, then $\Delta(L) = 0$ and this is impossible because $\kappa(X) = 2$. So we get that $h^0(L) = 2$ and $\Delta(L) = 1$. By Δ-genus inequality, we get that $\dim \mathrm{Bs}\,|L| < \Delta(L) \leq 1$. Since L is ample with $h^0(L) = 2$, we get that $\dim \mathrm{Bs}\,|L| = 0$. Let $\Phi_{|L|} : X \dashrightarrow \mathbb{P}^1$ be a rational map which is defined by $|L|$. Since $L^2 = 1$ and $\dim \mathrm{Bs}\,|L| = 0$, we get that there exist a smooth surface X' and a birational morphism $\mu : X' \to X$ such that μ is one point blowing up of X at the base point of $|L|$. Then there exists a fiber space $f : X' \to \mathbb{P}^1$ and f has a section E of f such that E is the (-1)-curve with $\dim \mu(E) = 0$. Here we remark that any fiber of f is irreducible and reduced since L is ample with $L^2 = 1$. For a general fiber F of f, $g(F) = 3$ and $q(X') = 2$. This is the type (4) of this theorem. This completes the proof of Claim 2.4. □

(II.2.2) We consider the case in which $\kappa(X) = 1$.

Claim 2.5 If $\kappa(X) = 1$, then (X, L) is the type (3) in Theorem 2.1.

Proof. Let $f : X \to C$ be an elliptic fibration. Then $g(C) \leq q(X) \leq g(C) + 1$. So $g(C) = 1$ or 2.

(α) The case in which f is relatively minimal.

(α.1) The case in which $g(C) = 1$.
Then $q(X) = g(C) + 1$. Hence $\chi(X) = 0$ and f is a quasi-bundle by Proposition 1.4. By Lemma 1.6 (ii) in [Se1] we get that $(R^1 f_* \mathcal{O}_X)^\vee \cong \mathcal{O}_C$,

so by the canonical bundle formula for an elliptic fibration,

$$\begin{aligned} K_X &= f^*(K_C \otimes (R^1 f_* \mathcal{O}_X)^\vee) \otimes \mathcal{O}\Big(\sum_i (m_i - 1)F_i\Big) \\ &= \mathcal{O}\Big(\sum_i (m_i - 1)F_i\Big), \end{aligned}$$

where $m_i F_i$ is a multiple fiber of f for any i.

Since $\kappa(X) = 1$, by Proposition 1.3 in [Se2], f has at least 2 multiple fibers. Since L is ample and $g(L) = 3$, we have the following two types.

(α.1.1) $L^2 = 1$, $LK_X = 3$.

(α.1.2) $L^2 = 2$, $LK_X = 2$.

First we consider the case (α.1.1). Then $3 = K_X L = \sum_i (m_i - 1)LF_i$, where $m_i F_i$ is a multiple fiber of f and m_i is its multiplicity. If $m_k \geq 3$ for some k, then $m_k = 3$ because f has at least two multiple fibers. But then $LF_k = 1$ because $(m_k - 1)LF_k = 2LF_k \geq 2$ and $LK_X = 3$. Therefore $LF = 3$ for any fiber F. Let $m_j F_j$ be another multiple fiber. Since $K_X L = 3$, we obtain that $(m_j - 1)F_j L = 1$. Therefore $m_j = 2$ and $LF_j = 1$. But this is a contradiction because $LF = 3$ for any fiber F. Therefore $m_i = 2$ for any i. Moreover $LF_i = LF_j$ for any $i \neq j$ and $(m_i - 1)LF_i = LF_i$ for any i. Hence f has three multiple fibers, each multiplicity is 2, and $LF = 2$ for a general fiber F of f because $K_X L = 3$. Then $L^2 = 1$ and $K_X = F_1 + F_2 + F_3$. Since $L^2 = 1$, then $\mathrm{Bs}\,|L|$ is one point. Therefore for some i, $\mathrm{Bs}\,|L_{F_i}| = \phi$. But since $LF_i = 1$, we obtain $F_i \cong \mathbb{P}^1$. This is a contradiction.

Next we consider the case (α.1.2). Since $K_X L = 2$, we obtain that f has just two multiple fibers whose multiplicities are 2. We write these multiple fibers as $2F_1$ and $2F_2$. Here we remark that F_1 and F_2 are smooth elliptic curves because f is a quasi-bundle. Since $2 = K_X L = (F_1 + F_2)L$, we get that $LF_1 = 1$. Hence $h^0(L|_{F_1}) = 1$. By the following exact sequence

$$0 \to L - F_1 \to L \to L|_{F_1} \to 0,$$

we get that

$$0 \to H^0(L - F_1) \to H^0(L) \to H^0(L|_{F_1})$$

is exact. Because $h^0(L) \geq 2$ and $h^0(L|_{F_1}) = 1$, we get that $h^0(L - F_1) \neq 0$. Since F_1 is not ample, there exists an effective divisor N such that $N + F_1 \in |L|$. Since $L^2 = 2$ and $LF_1 = 1$, N is an irreducible and reduced curve with $N^2 = 0$ and N is a double covering of C. Here we remark that

$K_XN=K_XL=2$. Hence $g(N)=2$. Let $\tilde{N}$ be a normalization of N and let $\pi:\tilde{N}\to N$ be its morphism. We note that $f|_N\circ\pi:\tilde{N}\to N\to C$ is a surjective morphism. Let $x_j=N\cap F_j$ for $j=1,2$. Since $NF_j=1$ for $j=1,2$, N is smooth at x_j. On the other hand by construction $f|_N\circ\pi$ is ramified at x_j for $j=1,2$. Since $g(C)=1$, we get that $g(\tilde{N})\geq 2$. Therefore $\tilde{N}\cong N$ because $g(N)=2$. Namely N is a smooth curve. Therefore we get the type (3) in Theorem 2.1.

(α.2) The case in which $g(C)=2$.

Then $K_XL\geq(2g(C)-2+\chi(\mathcal{O}_X))LF\geq 2LF$, where F is a general fiber of f. Since $g(L)=3$, we have $K_XL\leq 3$. So $LF=1$. If $L^2=1$, then any element D of $|L|$ is irreducible and smooth. But since $g(L)=3$, D is not a section of $f:X\to C$. Hence $LF=DF\geq 2$. This is a contradiction. If $L^2\geq 2$, then $L^2=2$ because $K_XL\geq 2$. Since $h^0(L)\geq 2$, we have $\Delta(L)\leq 2$ and $\dim\mathrm{Bs}|L|\leq 1$. If $\dim\mathrm{Bs}|L|\leq 0$, then $|L|$ has an irreducible and reduced curve. But this is impossible by the same argument as above. If $\dim\mathrm{Bs}|L|=1$, then $\Delta(L)=2$ and the fixed part Z of $|L|$ is $\mathbb{P}^1$. (See (1.14) in [Fj3]. See also (I.10.4) in [Fj5].) Hence Z is contained in a fiber of f because $g(C)=2$. Therefore L_F is free for a general fiber F of f, and $LF\geq 2$. But this is a contradiction.

(β) The case in which f is not relatively minimal.
Let $\mu:X\to X_1$ be the relatively minimal model of $f:X\to C$. Then we have a fiber space $f_1:X_1\to C$ such that f_1 is relatively minimal. Let $L_1=\mu_*L$. Then $K_XL>K_{X_1}L_1\geq 1$. Therefore we have the following two types.

(β.1) $L^2=2$, $LK_X=2$

(β.2) $L^2=1$, $LK_X=3$

If (X,L) satisfy (β.1), then $L_1^2=3$ and $L_1K_{X_1}=1$. (We remark that $h^0(L_1)\geq 2$.) But by the same argument as the cases (α.1) and (α.2), we get that $K_{X_1}L_1\geq 2$ and this (X,L) does not exist.

We assume that (X,L) satisfy (β.2). If $L_1^2=2$ and $L_1K_{X_1}=2$, then μ is one point blowing up and $L=\mu^*(L_1)-E_1$, where E_1 is the (-1)-curve. Moreover (X_1,L_1) is the type (3) in Theorem 2.1 by the above argument. If $h^0(L)<h^0(L_1)$, then $\Delta(L_1)\leq 1$ because $h^0(L)\geq 2$. By a Fujita's result (see e.g. Corollary 6.13 in [Fj5]) we get that $q(X)=0$. Hence $g(L)=1$ and this is impossible because $\kappa(X)\geq 0$. Hence $h^0(L)=h^0(L_1)$ and $\mu(E_1)\in\mathrm{Bs}|L_1|$. But this is impossible because L_1 is the type (3) in Theorem 2.1 and L does not become ample. Therefore $L_1^2=3$ or 5 and $L_1K_{X_1}=1$.

Then $g(C)=1$ or 2, because $q(X)=g(L)-1=2$.

If $g(C)=2$, then $K_{X_1}L_1 \geq (2g(C)-2+\chi(\mathcal{O}_X))LF \geq 2$. So this case cannot occur.

If $g(C)=1$, then $q(X)=g(C)+1$ and $\chi(\mathcal{O}_X)=0$ by Proposition 1.4. By the canonical bundle formula, $K_{X_1} \equiv \sum(m_i-1)F_i$, where m_iF_i is a multiple fiber of f_1 with $m_i \geq 2$. But since $\chi(\mathcal{O}_X)=0$ and $q(X)=g(C)+1$, f_1 is locally trivial or the number of multiple fiber of f_1 is greater than one (Proposition 1.3 in [Se2]). But this case cannot occur since $L_1K_{X_1}=1$ and L_1 is ample. This completes the proof of Claim 2.5. □

By the above, we get the assertion of Theorem 2.1. □

Remark 2.6 We consider the type (4) in Theorem 2.1. If there exist a smooth projective surface X' and a fiber space $f: X' \to \mathbb{P}^1$ such that $q(X')=2$, $g(F)=3$ for a general fiber F of f, f has a section E of f, any fiber of f is irreducible and reduced, and E is a (-1)-curve, then there exist a smooth projective surface X and a birational morphism $\mu: X' \to X$ such that μ is the blowing down of E. We put $L=\mu_*(F+E)$. Then L is ample with $L^2=1$, $g(L)=3$, $q(X)=2$, and $h^0(L)=2$. In [C] Cai gets a classification of a fiber space $f: X \to C$ with $g(F)=3$ and $q(X)-g(C)=2$. But it is unknown whether there is an example of $f: X \to \mathbb{P}^1$ with $\kappa(X)=2$, $g(F)=3$, $q(X)=2$, any fiber of f is irreducible and reduced, and f has a section E such that E is a (-1)-curve.

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