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On solutions of $x'' = t^{\alpha\lambda-2}x^{1+\alpha}$ starting at some positive t

Ichiro TSUKAMOTO

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Abstract. In this paper we shall consider an initial value problem of the second order nonlinear differential equation written in the title and get asymptotic behavior of the solution in terms of obtaining its analytical expressions valid in neighborhoods of both end points of a domain of the solution. Since we shall treat all initial conditions, all solutions will be investigated.

Key words: the Thomas-Fermi differential equation, asymptotic behavior, an initial value problem, an autonomous system of dimension 2, the Briot-Bouquet differential equation.

1. Introduction

As in the papers [2], [3], let us consider a second order nonlinear differential equation

$$x'' = t^{\beta}x^{1+\alpha} \quad (' = d/dt) \tag{1.1}$$

which is worth solving, because this contains the famous Thomas-Fermi differential equation

$$x'' = t^{-1/2}x^{3/2}$$

in atomic physics. In (1.1) we suppose

$$0 < t < \infty, \quad 0 < x < \infty$$

and that α, β are real parameters and $\alpha > 0$. Throughout this paper, we also suppose any real power of a positive variable takes its positive branch.

First let us review contents of [2], [3] briefly. In [2], it was concluded that (1.1) had a bounded solution with its bounded first derivative defined for $0 \leq t < \infty$ if and only if $\beta > -1$, and analytical expressions of the bounded solution valid near $t = 0$ and $t = \infty$ were obtained. The analytical expression valid near $t = 0$ was got in [2] under an assumption that $\alpha\lambda$ was not an integer, but in [3] this was obtained without supposing so. Moreover

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in [3], $\beta > -1$ was supposed and under this, existence and asymptotic behavior of the solution satisfying "an initial condition"

$$\lim_{t \to 0} x = a, \quad \lim_{t \to 0} x' = b$$

was shown. Namely the solutions starting at $t = 0$ were obtained.

So in this paper we shall consider the solutions starting at some positive t. For this purpose we shall treat an initial value problem of (1.1) whose initial condition is

$$x(T) = A, \quad x'(T) = B \quad (0 < T < \infty) \tag{1.2}$$

and investigate asymptotic behavior of the solutions. This will enable us to get new solutions we did not discover in the previous papers [2], [3].

As was done in [2], [3], in addition to $\alpha > 0$ we put

$$\beta = \alpha\lambda - 2, \quad \lambda > 0,$$

for convenience. In [3] we assumed $\alpha\lambda > 1$ but does not here, since the discussions will be carried out in the same way whether $\alpha\lambda > 1$ or $0 < \alpha\lambda \leq 1$. However it will be found later that if $0 < \alpha\lambda \leq 1$, then there exists no bounded solution of (1.1) with its bounded first derivative, since the first derivative of the solution loses the boundedness at $t = 0$.

Now as a beginning of discussing the initial value problem (1.1), (1.2) we use a transformation

$$y = \psi(t)^{-\alpha}\phi(t)^{\alpha}, \quad z = ty' \tag{1.3}$$

used in [2], [3]. Here $\phi(t)$ is an arbitrary solution of (1.1) and

$$\psi(t) = \{\lambda(\lambda + 1)\}^{1/\alpha} t^{-\lambda}$$

is a particular solution of (1.1). Note that we always have $y > 0$, since $\phi(t) > 0$. The transformation (1.3) transforms (1.1) into a first order rational differential equation

$$\frac{dz}{dy} = \frac{-\lambda(\lambda + 1)\alpha^2 y^2 + (2\lambda + 1)\alpha yz - (1 - \alpha)z^2 + \lambda(\lambda + 1)\alpha^2 y^3}{\alpha yz}. \tag{1.4}$$

Here the case $y' = 0$ corresponds to the singularities of (1.4) with $z = 0$. Using a parameter s, we rewrite (1.4) as an autonomous system of dimension 2

$$\begin{aligned}\frac{dy}{ds} &= \alpha yz \\ \frac{dz}{ds} &= -\lambda(\lambda+1)\alpha^2y^2 + (2\lambda+1)\alpha yz - (1-\alpha)z^2 + \lambda(\lambda+1)\alpha^2y^3.\end{aligned} \tag{1.5}$$

A solution $z = z(y)$ of (1.4) represents an orbit of a solution $(y(s), z(s))$ of (1.5).

From (1.3) we get

$$z = \alpha y\left(\lambda + \frac{t\phi'(t)}{\phi(t)}\right). \tag{1.6}$$

Therefore from (1.2), an initial condition

$$z(y_0) = z_0 \tag{1.7}$$

where

$$y_0 = \psi(T)^{-\alpha}A^{\alpha}, \quad z_0 = \alpha y_0\left(\lambda + \frac{TB}{A}\right) \tag{1.8}$$

is naturally given to (1.4) and from a solution $x = \phi(t)$ of (1.1), (1.2) we obtain a solution $z = z(y)$ of (1.4), (1.7) or a solution $(y(s), z(s))$ of (1.5) passing (y_0, z_0) and vice versa.

2. Solutions of (1.1) obtained from solutions of (1.5) tending to its singularities

In the yz plane, singularities of (1.5) are points $(0,0)$, $(1,0)$ if $\alpha \neq 1$, and all points of the z axis and a point $(1,0)$ if $\alpha = 1$.

Owing to [2], an orbit $z = z(y)$ of a solution $(y(s), z(s))$ of (1.5) tending to $(1,0)$ is given as either

$$z = (\mu/\alpha)(y-1) + \cdots \tag{2.1}$$

or

$$z = (\mu'/\alpha)(y-1) + \cdots \tag{2.2}$$

in the neighborhood of $y = 1$ where $\cdots$ denotes a uniquely determined convergent power series starting from a term whose degree is greater than that of the previous term and

$$\mu = \left\{(2\lambda+1)\alpha - \sqrt{(2\lambda+1)^2\alpha^2 + 4\lambda(\lambda+1)\alpha^3}\right\}/2 < 0,$$

$$\mu' = \left\{(2\lambda+1)\alpha + \sqrt{(2\lambda+1)^2\alpha^2 + 4\lambda(\lambda+1)\alpha^3}\right\}/2 > 0.$$

Moreover through (1.3) we get a solution $x = \phi(t)$ of (1.1) represented as

$$\phi(t) = \{\lambda(\lambda+1)\}^{1/\alpha} t^{-\lambda}\left\{1 + \sum_{n=1}^{\infty} \widehat{a}_n (Ct^{\mu/\alpha})^n\right\} \tag{2.3}$$

converging in the neighborhood of $t = \infty$ from (2.1) and

$$\phi(t) = \{\lambda(\lambda+1)\}^{1/\alpha} t^{-\lambda}\left\{1 + \sum_{n=1}^{\infty} \widehat{a}_n (Ct^{\mu'/\alpha})^n\right\} \tag{2.4}$$

converging in the neighborhood of $t = 0$ from (2.2). Here $\widehat{a}_1 = 1/\alpha$, $\widehat{a}_n$ are constants and C is an arbitrary constant.

Now let $z = z_1(y)$ be an orbit of a solution of (1.5) situated in $0 < y < 1$, $z > 0$ and represented as (2.1) in the neighborhood of $y = 1$. Then $z = z_1(y)$ is a solution of (1.4). If $(y(s), z(s))$ is a solution of (1.5) whose orbit is $z = z_1(y)$, then from [2], [3] we get

$$\lim_{s\to-\infty} (y(s), z(s)) = (0,0), \qquad \lim_{s\to-\infty} z(s)/y(s) = \alpha\lambda$$

respectively and hence

$$\lim_{y\to 0} z_1(y)/y = \alpha\lambda.$$

Let us consider solutions of (1.5) tending to $(0,0)$. Then we conclude

Lemma 1 *If $z = z(y)$ is an orbit of a solution $(y(s), z(s))$ of (1.5) and*

$$\lim_{y\to 0} z(y) = 0,$$

then we get

$$\lim_{y\to 0} z(y)/y = \alpha\lambda, \quad \alpha(\lambda+1), \quad \pm\infty.$$

Proof. When $(y(s), z(s))$ passes a line $z = \sigma y$, we have

$$(d/ds)(z - \sigma y) = -y^2\{(\sigma - \alpha\lambda)(\sigma - \alpha(\lambda+1)) - \lambda(\lambda+1)\alpha^2 y\}$$

which implies

$$(d/ds)(z-\sigma y) < 0 \quad \text{if} \quad \sigma < \alpha\lambda, \ \sigma > \alpha(\lambda+1)$$
$$(d/ds)(z-\sigma y) > 0 \quad \text{if} \quad \alpha\lambda \le \sigma \le \alpha(\lambda+1)$$

in the neighborhood of $y = 0$. However if $z(y)/y$ accumulated to two values d_1, d_2 ($d_1 < d_2$) as $y \to 0$, then in the case $d_1 < \sigma < d_2$ the sign of $(d/ds)(z-\sigma y)$ would not be definite in contradiction to the above inequalities. Hence we may put

$$d = \lim_{y\to 0} z(y)/y.$$

On the other hand, we get

$$\frac{dz}{dy} = -\lambda(\lambda+1)\alpha(1-y)\frac{y}{z} + 2\lambda + 1 - \frac{1-\alpha}{\alpha}\frac{z}{y}$$

from (1.4). Therefore letting y tend to 0, the limit of dz/dy exists and is equal to d from l'Hospital's theorem. Hence we conclude

$$d = \alpha\lambda, \quad \alpha(\lambda+1)$$

if d is finite. Now the proof is completed. □

Following the discussion for obtaining (3.20) of [8] now, from an orbit $z = z(y)$ of a solution of (1.5) such that

$$\lim_{y\to 0} z(y)/y = \alpha\lambda,$$

we obtain a solution $x = \phi(t)$ of (1.1) represented as

$$\phi(t) = a\left[1 + \sum_{m+n>0} \gamma_{mn}\left\{\frac{a^\alpha}{\lambda(\lambda+1)}t^{\alpha\lambda}\right\}^m \left(\frac{\alpha b}{a}t\right)^n\right] \quad \text{if} \quad \alpha\lambda > 1$$
$$\phi(t) = a\left[1 + \sum_{m+n>0} \gamma_{mn}(Dt)^{\alpha\lambda m}\{Dt(\widehat{h}\log t + \widehat{\Gamma})\}^n\right] \quad \text{if} \quad 0 < \alpha\lambda \le 1 \tag{2.5}$$

converging in the neighborhoods of $t = 0$. Here γ_{mn}, a, b, D, $\widehat{h}$, $\widehat{\Gamma}$ are constants and

$$\gamma_{01} = 1/\alpha, \quad \gamma_{0n} = 0 \ (n = 2,3,\ldots), \quad a = \phi(0) > 0, \quad b = \phi'(0),$$
$$D = a^{1/\lambda}/\{\lambda(\lambda+1)\}^{1/\alpha\lambda}.$$

Note that a, b can be taken arbitrarily and that from the final discussion of [8], $\phi'(t)$ is unbounded if $0 < \alpha\lambda \leq 1$. According to the above we get (2.5) from $z_1(y)$.

Moreover owing to [8], if $z = z(y)$ is an orbit of a solution of (1.5) and

$$\lim_{y \to 0} z(y)/y = \alpha(\lambda + 1),$$

then $z(y)$ exists uniquely so that in the neighborhood of $y = 0$ we get

$$z = \alpha(\lambda + 1)y \left\{ 1 + \frac{\lambda}{(\lambda + 1)\alpha + 1} y + \cdots \right\}. \tag{2.6}$$

So we denote $z(y)$ by $z_2(y)$. Furthermore from (2.6) or $z_2(y)$ we get a solution $x = \phi(t)$ of (1.1) represented as

$$\phi(t) = bt \sum_{n=0}^{\infty} \widetilde{a}_n \left\{ \frac{b^\alpha}{\lambda(\lambda + 1)} t^{\alpha(\lambda+1)} \right\}^n, \quad \widetilde{a}_0 = 1 \tag{2.7}$$

converging in the neighborhood of $t = 0$. Here b, $\widetilde{a}_n$ are constants.

Finally we show the following:

Lemma 2 *If an interval* (ω_-, ω_+) *is a domain of a solution* $x = \phi(t)$ *of* (1.1), *then as* $t \to \omega_+$ *or* ω_-, (y, z) *defined as* (1.3) *does not tend to a nonsingular finite point of* (1.5) *situated in* $y > 0$.

Proof. Suppose that (y, z) tends to a nonsingular finite point $(\widehat{y}, \widehat{z})$ $(\widehat{y} > 0)$ as $t \to \omega_+$. Then if $\omega_+ < \infty$, $\phi(t)$ can exist even for $t > \omega_+$. Hence $\omega_+ = \infty$. If $\widehat{z} \neq 0$, then we immediately get a contradiction, since from (1.3) we have

$$\int (1/z(y))\, dy = \log t + C$$

where C is a constant. If $\widehat{z} = 0$, then considering reciprocals of both sides of (1.4) we get

$$y - \widehat{y} = b_N z^N + \cdots \quad (b_N \neq 0)$$

where $N \geq 2$, since $dy/dz = 0$ if $y = \widehat{y}$. Hence we obtain

$$z = \widetilde{b}_N (y - \widehat{y})^{1/N} + \cdots \quad (\widetilde{b}_N \neq 0)$$

and from (1.3)

$$\frac{(y-\widehat{y})^{1-(1/N)}}{1-(1/N)}(1+\cdots) = \widetilde{b}_N \log t + C$$

which deduces a contradiction as $t \to \omega_+$.

As $t \to \omega_-$, the proof is similar. □

3. The case $0 < A \leq \psi(T)$

In what follows, let (ω_-, ω_+) be a domain of a solution $x = \phi(t)$ of (1.1), (1.2).

In this section we first fix A in (1.2) so that $0 < A < \psi(T)$. Then from (1.8), y_0 is fixed so that $0 < y_0 < 1$. Moreover let B_1 denote B satisfying (1.8) if $z_0 = z_1(y_0)$.

If $B > B_1$, then a solution $z = z(y)$ of (1.4), (1.7) satisfies $z(y) > z_1(y)$. Indeed from (1.8), z_0 is increasing in B and uniqueness of a solution holds for (1.4), (1.7). As was shown in [3] we get

$$\lim_{y\to\infty} z(y) = \infty, \qquad \lim_{y\to\infty} y^{-3/2} z(y) = \alpha\sqrt{\frac{2\lambda(\lambda+1)}{\alpha+2}}$$

and from $z(y)$ and (1.3) a solution $x = \phi(t)$ of (1.1), (1.2) represented as

$$\phi(t) = \left(\frac{2(\alpha+2)}{\alpha^2\omega_+^{\alpha\lambda-2}}\right)^{1/\alpha}(\omega_+ - t)^{-2/\alpha}$$
$$\left[1 + \sum_{m+n>0} c_{mn}(\omega_+ - t)^m\{(\omega_+ - t)^{2+4/\alpha}\}^n\right] \text{ if } 4/\alpha \notin \boldsymbol{N},$$

$$\phi(t) = \left(\frac{2(\alpha+2)}{\alpha^2\omega_+^{\alpha\lambda-2}}\right)^{1/\alpha}(\omega_+ - t)^{-2/\alpha}$$
$$\left\{1 + \sum_{m>0}(\omega_+ - t)^m p_m(\log(\omega_+ - t))\right\} \text{ if } 4/\alpha \in \boldsymbol{N}, \tag{3.1}$$

which converge in the neighborhoods of $t = \omega_+$. Here $0 < \omega_+ < \infty$, c_{mn} are constants and $p_m(\xi)$ are polynomials of ξ whose degrees are at most $[m\alpha/(2\alpha+4)]$, [ ] denoting Gaussian symbol.

Since $z_2(y) > z_1(y)$ from $\alpha\lambda < \alpha(\lambda+1)$, the above statements are valid for $z_2(y)$ and we have a solution of (1.1), (1.2) represented as (3.1) in the neighborhood of $t = \omega_+$ from $z_2(y)$. Now let B_2 denote B satisfying (1.8) if $z_0 = z_2(y_0)$ and let us consider the case $t \to \omega_-$.

If $B_1 < B < B_2$, then a solution $z = z(y)$ of (1.4), (1.7) satisfies $z_1(y) < z(y) < z_2(y)$ and from Lemma 1

$$\lim_{y\to 0} y^{-1}z = \alpha\lambda,$$

since $z_2(y)$ is a unique solution of (1.4) satisfying

$$\lim_{y\to 0} y^{-1}z = \alpha(\lambda+1).$$

Hence from $z(y)$ we obtain a solution of (1.1), (1.2) represented as (2.5) in the neighborhood of $t = 0$.

If $B > B_2$, a solution $z(y)$ of (1.4), (1.7) can be continued up to $y = 0$ because if this were not true, putting $z = 1/\zeta$ in (1.4) we would get

$$\begin{aligned} d\zeta/dy = -\{&-\lambda(\lambda+1)\alpha^2y^2\zeta^2 + (2\lambda+1)\alpha y\zeta \\ &- (1-\alpha) + \lambda(\lambda+1)\alpha^2y^3\zeta^2\}\zeta/\alpha y \end{aligned} \tag{3.2}$$

which implied a contradiciton $\zeta \equiv 0$. Therefore it follows from the uniqueness of $z_2(y)$ that concerning a solution $z = z(y)$ of (1.4), (1.7) there exist the following possibilities:

(i) $\lim_{y\to 0} z = 0, \quad \lim_{y\to 0} y^{-1}z = \infty$

(ii) $\lim_{y\to 0} z = c, \quad 0 < c < \infty$

(iii) $\lim_{y\to 0} z = \infty.$

In every case of these we get

$$\lim_{y\to 0} y^{-1}z = \infty.$$

Therefore as in [3] we put $w = yz^{-1}$ and get a Briot-Bouquet differential equation

$$y\frac{dw}{dy} = \frac{w}{\alpha} - (2\lambda+1)w^2 + \lambda(\lambda+1)\alpha w^3 - \lambda(\lambda+1)\alpha y w^3.$$

Since $1/\alpha > 0$ and the righthand side of this is divisible by w, we have

$$w = \sum_{m+n>0} w_{mn}y^m(Cy^{1/\alpha})^n, \quad w_{01} = 1 \tag{3.3}$$

where w_{mn} are constants and C is an arbitrary constant. As $w \equiv 0$ is also a solution, for some C we obtain

$$w = \sum_{m+n>0} w_{mn} y^m (C y^{1/\alpha})^n \equiv 0.$$

This implies

$$w_{m0} = 0 \quad (m = 1, 2, \ldots), \quad C = 0.$$

Moreover in the same way as in [3], we get from (3.3) a solution $x = \phi(t)$ of (1.1), (1.2) represented as

$$\phi(t) = \Gamma(t - \omega_-)\left\{1 + \sum_{m+n>0} b_{mn}(t - \omega_-)^m (t - \omega_-)^{\alpha n}\right\}, \tag{3.4}$$

which converges in the neighborhood of $t = \omega_-$. Here Γ is an arbitrary constant, $0 < \omega_- < \infty$ and b_{mn} are constants.

Now let us discuss when (i)–(iii) occur. Let $(y(s), z(s))$ be a solution of (1.5) whose orbit is $z = z(y)$. Then we have

$$\lim_{s \to s_0} y(s)^{-1} z(s) = \infty$$

for some s_0 $(\geq -\infty)$. Hence from (1.5) we obtain

$$\begin{aligned}\frac{dz}{ds} = -(1-\alpha)z^2\left\{1 + \frac{\lambda(\lambda+1)\alpha^2}{1-\alpha}\left(\frac{y}{z}\right)^2 - \frac{(2\lambda+1)\alpha}{1-\alpha}\frac{y}{z}\right. \\ \left. - \frac{\lambda(\lambda+1)\alpha^2}{1-\alpha}\left(\frac{y}{z}\right)^2 y\right\} \sim (\alpha-1)z^2\end{aligned}$$

as s $\to s_0$. Therefore for s sufficiently close to s_0 we conclude that if $\alpha > 1$, then $dz/ds > 0$ and if $0 < \alpha < 1$, then $dz/ds < 0$. Namely if $\alpha > 1$, then (i) occurs and if $0 < \alpha < 1$, then (iii) occurs. Indeed we get $dy(s)/ds > 0$ from (1.5) and $B > B_2$, and y tends to 0 as s decreases. If $\alpha = 1$, then from (3.3) we get

$$w = Cy(1 + \cdots),$$

that is

$$z = C^{-1}(1 + \cdots),$$

since $C = 0$ implies $w \equiv 0$. Hence (ii) occurs.

Next let us consider the case $B < B_1$. In this case, a solution $z = z(y)$ of (1.4), (1.7) satisfies $z(y) < z_1(y)$. Let $z = z_3(y)$ be a solution of (1.4) represented as (2.2) in the neighborhood of $y = 1$ and situated in $0 < y < 1$, $z < 0$. Then from the same discussion as in the case $B > B_2$, this can be continued up to $y = 0$ and satisfies

$$\lim_{y \to 0} y^{-1} z = -\infty.$$

Let B_3 be B satisfying (1.8) if $z_0 = z_3(y_0)$.

If $B_3 < B < B_1$, then from Lemma 1 and the uniqueness of $z_1(y)$ a solution $(y(s), z(s))$ of (1.5) whose orbit is a solution $z = z(y)$ of (1.4), (1.7) satisfies

$$\lim_{s \to -\infty} (y(s), z(s)) = (0{,}0), \quad \lim_{s \to -\infty} \frac{z(s)}{y(s)} = \alpha\lambda,$$
$$\lim_{s \to s_0} y(s) = 0, \quad \lim_{s \to s_0} \frac{z(s)}{y(s)} = -\infty$$

for some s_0 ($\leq \infty$). Therefore from the above discussions we get a solution $x = \phi(t)$ of (1.1), (1.2) defined for $0 < t < \omega_+$ where $\omega_+ < \infty$, represented as (2.5) in the neighborhood of $t = 0$ and

$$\phi(t) = \Gamma(\omega_+ - t)\left\{1 + \sum_{m+n>0} b_{mn}(\omega_+ - t)^m(\omega_+ - t)^{\alpha n}\right\} \tag{3.5}$$

in the neighborhood of $t = \omega_+$. Here Γ is an arbitrary constant and b_{mn} are constants. In fact, we have (3.5) in the same way as was used for obtaining (3.4).

If $B = B_3$, then from the above discussions we get a solution $x = \phi(t)$ of (1.1), (1.2) defined for $0 < t < \omega_+$ where $\omega_+ < \infty$, represented as (2.4) in the neighborhood of $t = 0$ and (3.5) in the neighborhood of $t = \omega_+$.

Now let us suppose $B < B_3$. Then as in the case $B_3 < B < B_1$ the solution $(y(s), z(s))$ of (1.5) satisfies

$$\lim_{s \to s_0} y(s) = 0, \quad \lim_{s \to s_0} \frac{z(s)}{y(s)} = -\infty$$

for some s_0 ($\leq \infty$) and hence we get a solution $x = \phi(t)$ of (1.1), (1.2) represented as (3.5) in the neighborhood of $t = \omega_+$.

Before considering the case $t \to \omega_-$, we note that as in the case $B > B_2$ there exist the following possibilities concerning a solution $z = z(y)$ of (1.4), (1.7) in the case $B < B_1$:

(i) $$\lim_{y\to 0} z = 0, \quad \lim_{y\to 0} y^{-1}z = -\infty$$

(ii) $$\lim_{y\to 0} z = c, \quad -\infty < c < 0$$

(iii) $$\lim_{y\to 0} z = -\infty$$

which occur respectively if $\alpha > 1$, $\alpha = 1$, $0 < \alpha < 1$.

Return to the case $B < B_3$. Then as t $\to \omega_-$, there exist the following possibilities:

(i) $$\omega_- > 0, \quad \lim_{t\to\omega_-} \phi(t) = 0$$

(ii) $$\omega_- > 0, \quad \lim_{t\to\omega_-} \phi(t) = \infty$$

(iii) $$\omega_- = 0, \quad \lim_{t\to\omega_-} \phi(t) = 0$$

(iv) $$\omega_- = 0, \quad 0 < \lim_{t\to\omega_-} \phi(t) < \infty$$

(v) $$\omega_- = 0, \quad \lim_{t\to\omega_-} \phi(t) = \infty.$$

Define y, z as (1.3). Then in the cases (i), (iii), (iv) we have a contradiction

$$\lim_{t\to\omega_-} y = 0.$$

Indeed $dy/dt = z < 0$ in this case. If (ii) occurs, we obtain

$$\lim_{t\to\omega_-} y = \infty.$$

Suppose that (v) occurs. Since from $B < B_3$ we get $z_0 < 0$ and from (1.5),

$$\frac{dz}{ds} = \lambda(\lambda+1)\alpha^2 y^2(y-1) > 0$$

on $y > 1$, $z = 0$, the orbit of $(y(s), z(s))$ cannot cross the y axis as s decreases. Therefore $z(s) < 0$ and from (1.3) we have $y' < 0$. Hence there exists

$$\lim_{t\to 0} y = \lim_{t\to 0}\{\lambda(\lambda+1)\}^{-1}\left(\frac{\phi(t)}{t^{-\lambda}}\right)^{\alpha}.$$

This implies that there exists

$$c = \lim_{t\to 0} \frac{\phi(t)}{t^{-\lambda}}.$$

However owing to l'Hospital's theorem we have

$$c = \frac{c^{1+\alpha}}{\lambda(\lambda+1)}$$

since $\lim_{t\to 0} \phi(t) = \infty$ implies $\lim_{t\to 0} \phi'(t) = -\infty$ and $\phi''(t) = t^{\alpha\lambda-2}\phi(t)^{1+\alpha}$. Hence we obtain

$$c = 0, \quad \{\lambda(\lambda+1)\}^{1/\alpha},\ \infty$$

and

$$\lim_{t\to 0} y = 0,\ 1,\ \infty$$

respectively. But we now get $dy/dt = z < 0$ and $\lim_{t\to 0} y = 0$ is impossible. If $\lim_{t\to 0} y = 1$, then from Lemma 2 we get $\lim_{t\to 0} z = 0, -\infty$. If $\lim_{t\to 0} z = 0$, then we have a contradiction $B = B_3$ and if $\lim_{t\to 0} z = -\infty$, then putting $z = 1/\zeta$ and getting (3.2) we obtain a contradiction $\zeta \equiv 0$. Hence we conclude

$$\lim_{t\to 0} y = \infty.$$

Now we suppose (ii) or (v). Then we put $y = 1/\eta$ so that $\lim_{t\to\omega_-} \eta = 0$. If z accumulates to a finite value as $t \to \omega_-$, then we get a contradiction $\eta \equiv 0$ from

$$dz/d\eta = \{\lambda(\lambda+1)\alpha^2\eta - (2\lambda+1)\alpha\eta^2 z + (1-\alpha)\eta^3 z^2 - \lambda(\lambda+1)\alpha^2\}/\alpha\eta^4 z.$$

Thus

$$\lim_{t\to\omega_-} z = \lim_{\eta\to 0} z = -\infty.$$

So we put $\zeta = 1/z$. Then we have

$$d\zeta/d\eta = \{-\lambda(\lambda+1)\alpha^2\eta\zeta^3 + (2\lambda+1)\alpha\eta^2\zeta^2 - (1-\alpha)\eta^3\zeta + \lambda(\lambda+1)\alpha^2\zeta^3\}/\alpha\eta^4.$$

Moreover if we put $w = \eta^{-3/2}\zeta$, $\xi = \eta^{1/2}$, then we obtain

$$dw/d\xi = \{-(\alpha+2)w + 2(2\lambda+1)\alpha\xi w^2 + 2\lambda(\lambda+1)\alpha^2 w^3 - 2\lambda(\lambda+1)\alpha^2\xi^2 w^3\}/\alpha\xi. \quad (3.6)$$

In case of $\xi = 0$, the numerator of the righthand side vanishes if and only if $w = 0$, γ where

$$\gamma = -\frac{1}{\alpha}\sqrt{\frac{\alpha+2}{2\lambda(\lambda+1)}},$$

since $w \leq 0$.

Here we consider a solution of (3.6) satisfying $w(0) = c$. If $c \neq 0$, γ, then from (3.6) we get

$$d\xi/dw = \alpha\xi/\{-(\alpha+2)w + 2(2\lambda+1)\alpha\xi w^2 + 2\lambda(\lambda+1)\alpha^2 w^3 - 2\lambda(\lambda+1)\alpha^2\xi^2 w^3\}$$

which implies a contradiction $\xi \equiv 0$.

If $c = 0$, then from (3.6) we have a Briot-Bouquet differential equation

$$\xi\frac{dw}{d\xi} = -\frac{\alpha+2}{\alpha}w + 2(2\lambda+1)\xi w^2 + 2\lambda(\lambda+1)\alpha w^3 - 2\lambda(\lambda+1)\alpha\xi^2 w^3.$$

If w is a solution of this which accumulates to 0, then applying Painlevé's theorem (cf. Theorem 3.2.1 of [1]) to reciprocals of both sides we conclude w converges to 0 as $\xi \to 0$. However if w were not identically zero, then solving this asymptotically we would get

$$\log w = -\frac{\alpha+2}{\alpha}(\log\xi)(1+o(1)) \quad \text{as} \quad \xi \to 0$$

which is a contradiction, since $-(\alpha+2)/\alpha < 0$ (cf. a lemma of [9] or [10]). Suppose $c = \gamma$. Then if we put

$$\theta = w - \gamma,$$

we get the same transformation as

$$y^{-1/2} = \eta, \quad z^{-1} = \eta^3(\gamma+u).$$

In [3] we used this transformation and got a differential equation from which we obtained an analytical expression of $z(y)$ ($> z_1(y)$) in the neighborhood of $y = \infty$. Since we get the same differential equation, we follow the calculation done in [3] and have analytical expressions similar to (3.1) such as

$$\phi(t) = \left(\frac{2(\alpha+2)}{\alpha^2\omega_-^{\alpha\lambda-2}}\right)^{1/\alpha}(t-\omega_-)^{-2/\alpha}$$
$$\left[1 + \sum_{m+n>0} c_{mn}(t-\omega_-)^m\{(t-\omega_-)^{2+4/\alpha}\}^n\right] \quad \text{if } 4/\alpha \notin N$$
$$\phi(t) = \left(\frac{2(\alpha+2)}{\alpha^2\omega_-^{\alpha\lambda-2}}\right)^{1/\alpha}(t-\omega_-)^{-2/\alpha}$$
$$\left\{1 + \sum_{m>0}(t-\omega_-)^m p_m(\log(t-\omega_-))\right\} \quad \text{if } 4/\alpha \in N \tag{3.7}$$

which converge in the neighborhoods of $t = \omega_-$. Here $0 < \omega_- < \infty$ and c_{mn}, $p_m(\xi)$ are the same as in (3.1). If $c = -\infty$, then putting $w = 1/\theta$ in (3.6) we have

$$d\xi/d\theta = \alpha\xi\theta/\{(\alpha+2)\theta^2 - 2(2\lambda+1)\alpha\xi\theta - 2\lambda(\lambda+1)\alpha^2 + 2\lambda(\lambda+1)\alpha^2\xi^2\}$$

which implies a contradiction $\xi \equiv 0$. Consequently only (ii) occurs.

Summarizing the above conclusions, we obtain

Theorem I *Let $x = \phi(t)$ be a solution of* (1.1), (1.2). *Then if $0 < A < \psi(T)$, there exist numbers B_1, B_2, B_3 such that*

(i) *if $B < B_3$, then $\phi(t)$ is defined for $\omega_- < t < \omega_+$, represented as* (3.7) *in the neighborhood of $t = \omega_-$ and* (3.5) *in the neighborhood of $t = \omega_+$,*

(ii) *if $B = B_3$, then $\phi(t)$ is defined for $0 < t < \omega_+$, represented as* (2.4) *in the neighborhood of $t = 0$ and* (3.5) *in the neighborhood of $t = \omega_+$,*

(iii) *if $B_3 < B < B_1$, then $\phi(t)$ is defined for $0 < t < \omega_+$, represented as* (2.5) *in the neighborhood of $t = 0$ and* (3.5) *in the neighborhood of $t = \omega_+$,*

(iv) *if $B = B_1$, then $\phi(t)$ is defined for $0 < t < \infty$, represented as* (2.5) *in the neighborhood of $t = 0$ and* (2.3) *in the neighborhood of $t = \infty$,*

(v) *if $B_1 < B < B_2$, then $\phi(t)$ is defined for $0 < t < \omega_+$, represented as* (2.5) *in the neighborhood of $t = 0$ and* (3.1) *in the neighborhood of $t = \omega_+$,*

(vi) *if* $B = B_2$, *then* $\phi(t)$ *is defined for* $0 < t < \omega_+$, *represented as* (2.7) *in the neighborhood of* $t = 0$ *and* (3.1) *in the neighborhood of* $t = \omega_+$,

(vii) *if* $B > B_2$, *then* $\phi(t)$ *is defined for* $\omega_- < t < \omega_+$, *represented as* (3.4) *in the neighborhood of* $t = \omega_-$ *and* (3.1) *in the neighborhood of* $t = \omega_+$.

Here $0 < \omega_- < \omega_+ < \infty$.

Similarly we conclude

Theorem II *If* $A = \psi(T)$, *then there exists a number* B_2 *such that*

(i) *if* $B < -\lambda A/T$, *then* (i) *of Theorem* I *is valid,*

(ii) *if* $B = -\lambda A/T$, *then* $\phi(t) = \psi(t)$,

(iii) *if* $-\lambda A/T < B < B_2$, *then* (v) *of Theorem* I *is valid,*

(iv) *if* $B = B_2$, *then* (vi) *of Theorem* I *is valid,*

(v) *if* $B > B_2$, *then* (vii) *of Theorem* I *is valid.*

4. The case $A > \psi(T)$

In this section we suppose $A > \psi(T)$ in (1.2). Then from (1.8) we get $y_0 > 1$.

First let $z = z_4(y)$ be a solution of (1.4), (1,7) represented as (2.2) in the neighborhood of $y = 1$ and situated in $y > 1$, $z > 0$. Then from $z_4(y)$ we have a solution $x = \phi(t)$ of (1.1), (1.2) represented as (2.4) in the neighborhood of $t = 0$. Moreover using the discussion of [3] also here, we conclude that $z_4(y)$ can be continued up to $y = \infty$ and get (3.1) in the neighborhood of $t = \omega_+$ from $z_4(y)$. Let B_4 denote B satisfying (1.8) if $z_0 = z_4(y_0)$.

Next let $z = z_5(y)$ be a solution of (1.4), (1.7) represented as (2.1) in the neighborhood of $y = 1$ and situated in $y > 1$, $z < 0$. From $z_5(y)$ we then obtain a solution $x = \phi(t)$ of (1.1), (1.2) represented as (2.3) in the neighborhood of $t = \infty$. Furthermore from the discussion of Section 3, $z_5(y)$ can be continued up to $y = \infty$ and get (3.7) in the neighborhood of $t = \omega_-$. Now let B_5 denote B satisfying (1.8) if $z_0 = z_5(y_0)$.

If $B > B_4$, then a solution $z = z(y)$ of (1.4), (1.7) satisfies $z(y) > z_4(y)$. In the region $z > z_4(y)$, there exists $z_2(y)$ appearing in Section 2. Therefore if B_2 denotes B satisfying (1.8) for $z_0 = z_2(y_0)$ also here, then $B_2 > B_4$.

Discussing as in Section 3, we now get

Theorem III *If $A > \psi(T)$, then there exist numbers B_2, B_4, B_5 such that*

(i) *if $B < B_5$, then* (i) *of Theorem* I *is valid,*

(ii) *if $B = B_5$, then $\phi(t)$ is defined for $\omega_- < t < \infty$, represented as* (3.7) *in the neighborhood of $t = \omega_-$ and* (2.3) *in the neighborhood of $t = \infty$,*

(iii) *if $B_5 < B < B_4$, then $\phi(t)$ is defined for $\omega_- < t < \omega_+$, represented as* (3.7) *in the neighborhood of $t = \omega_-$ and* (3.1) *in the neighborhood of $t = \omega_+$,*

(iv) *if $B = B_4$, then $\phi(t)$ is defined for $0 < t < \omega_+$, represented as* (2.4) *in the neighborhood of $t = 0$ and* (3.1) *in the neighborhood of $t = \omega_+$,*

(v) *if $B_4 < B < B_2$, then* (v) *of Theorem* I *is valid,*

(vi) *if $B = B_2$, then* (vi) *of Theorem* I *is valid,*

(vii) *if $B > B_2$, then* (vii) *of Theorem* I *is valid.*

Here $0 < \omega_- < \omega_+ < \infty$.

References

[1] Hille E., *Ordinary differential equations in the complex domain*. Wiley, 1976.

[2] Saito T., *On bounded solutions of $x'' = t^\beta x^{1+\alpha}$*. Tokyo J. Math. **1** (1978), 57–75.

[3] Saito T., *Solutions of $x'' = t^{\alpha\lambda-2}x^{1+\alpha}$ with movable singularity*. Tokyo J. Math. **2** (1979), 261–283.

[4] Smith R.A., *Singularities of solutions of certain plane autonomous systems*. Proc. Roy. Soc. Edinburgh, Sect. A, **72** (1973/74), 307–315.

[5] Tsukamoto I., Mishina T. and Ono M., *On solutions of $x'' = e^{\alpha\lambda t}x^{1+\alpha}$*. Keio Sci. Tech. Rep. **35** (1982), 1–36.

[6] Tsukamoto I., *On solutions of $x'' = -e^{\alpha\lambda t}x^{1+\alpha}$*. Tokyo J. Math. **12** (1989), 181–203.

[7] Tsukamoto I., *Remarks on the papers "On solutions of $x'' = \pm e^{\alpha\lambda t}x^{1+\alpha}$"*. Funkcial. Ekvac. **37** (1994), 363–371.

[8] Tsukamoto I., *On the generalized Thomas-Fermi differential equations and applicability of Saito's transformation*. Tokyo J. Math. **20** (1997), 107–121.

[9] Tsukamoto I., *Asymptotic behavior of solutions of $x'' = e^{\alpha\lambda t}x^{1+\alpha}$ where $-1 < \alpha < 0$*. To appear in Osaka J. Math.

[10] Tsukamoto I., *Asymptotic behavior of solutions of $x'' = e^{\alpha\lambda t}x^{1+\alpha}$ where $\alpha < -1$*. To appear.

3-10-38 Higashi-kamagaya
Kamagaya-shi, Chiba 273-0104
Japan
E-mail: itkmt@k9.dion.ne.jp