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Biharmonic classification of Riemannian spaces

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Abstract. R is a Riemann surface or a Riemannian manifold of dimension ≥ 2. We carry out the biharmonic classification of R based on the extension property that given a biharmonic function b outside a compact set in R there exists a biharmonic function B on R such that $b - B$ is bounded near infinity.

Key words: biharmonic extension, flux property.

1. Introduction

Let R be a Riemannian space, a term that denotes here either a Riemann surface R or a Riemannian manifold R of dimension ≥ 2. In the harmonic classification of R, we know that R is hyperbolic if and only if given any harmonic function h outside a compact set in R there exists a harmonic function H in R such that $h - H$ is bounded near infinity. In this note we consider a similar result in the biharmonic classification of R.

In a Riemannian manifold R, if Δ is the Laplace-Beltrami operator, b is said to be biharmonic if $\Delta^2 b = 0$ and a positive function $Q_y(x)$ is said to be the biharmonic Green potential with pole $\{y\}$ if $\Delta^2 Q_y = \delta_y$. We prove here that if R is a radial space, that is a Riemannian manifold like the n-spaces and the Poincaré n-balls endowed with a radial metric $ds = \lambda(|x|)|dx|$, and if there exists a biharmonic Green potential on R, then given any biharmonic function b outside a compact set, there exists a biharmonic function B on R such that $b - B$ is bounded near infinity. Actually we show that in an arbitrary Riemannian manifold R, biharmonic Green potentials exist if and only if given any biharmonic function b outside a compact set, there exist on R a biharmonic function B and two potentials g and p such that near infinity $|b - B| \leq g$ and $|\Delta(b - B)| \leq p$.

Also, making precise the definitions of biharmonic functions and biharmonic Green potentials in a general Riemannian space R, we investigate this biharmonic-extension property in R whether it is hyperbolic or parabolic.

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This note studies principally three categories of Riemannian spaces: bipotential spaces, biharmonic-extension spaces and tapered spaces.

2. Preliminaries

In this note, R denotes a Riemann surface or an oriented C^∞-Riemannian manifold of dimension $n \geq 2$; dx denotes the volume element. If μ is a Radon measure on an open set ω in R, we know that (see, for example, Theorem 4.2 [1]) there exists a superharmonic function s on ω with μ as its associated measure in a local Riesz representation. This relation is denoted by $Lu = -\mu$. If f is a locally dx-integrable function on ω, we write $Lu = -f$ if u is a δ-superharmonic function on ω with $d\lambda = f(x)dx$ as its associated signed measure.

If h is a harmonic function on an open set ω in R, a function b such that $Lb = -h$ is said to be *biharmonic* on ω. For $y \in R$, if $G_y(x) = G(x, y)$ is the Green potential on R (resp. $E_y(x) = E(x, y)$ is the Evans function on R) if R is hyperbolic (resp. if R is parabolic), we write $LQ_y = -G_y$ (resp. $LQ_y = -E_y$). If $Q_y(x)$ is a potential on R, we say that $Q_y(x)$ is the *biharmonic Green potential* on R with pole $\{y\}$. R is termed a *bipotential space* if there are biharmonic Green potentials on R.

We remark that if R is a Riemannian manifold and if Δ is the Laplace-Beltrami operator on R (see Sario et al. [8]) then $L \equiv -\Delta$. We use the symbol L to cover the case when R is a Riemann surface also where the Laplacian is not invariant under a parametric change.

3. Biharmonic extension in hyperbolic spaces

In this section, R stands for a hyperbolic Riemann surface or a hyperbolic Riemannian manifold of dimension ≥ 2.

Definition 3.1 In a hyperbolic Riemannian space R, a function b that is biharmonic outside a compact set is said to have the biharmonic-extension property if there exists a biharmonic function B on R such that $b - B$ is bounded near infinity. R itself is said to be a biharmonic-extension space if every b on R that is biharmonic outside a compact set has the biharmonic-extension property.

Theorem 3.1 *In a hyperbolic Riemannian space R, the following are equivalent:*

1. *R is a biharmonic-extension space.*
2. *For any finite continuous potential p with compact harmonic support, if $Lg = -p$ then g has the biharmonic-extension property.*

Proof. 1) $\Rightarrow$ 2) : This is obvious since g is a biharmonic function outside a compact set in R.

2) $\Rightarrow$ 1) : Let b be a function on R, that is biharmonic outside a compact set A. Let $Lb = -h$ on $R \backslash A$. Since h is harmonic, there exist finite continuous potentials p_1 and p_2 with compact harmonic support and a harmonic function H on R such that $h = p_1 - p_2 + H$ near infinity. Let $Lg_1 = -p_1, Lg_2 = -p_2$ and $LB = -H$, so that $b = g_1 - g_2 + B +$ (a harmonic function v) near infinity. Then $v = H_1 + v_1$ near infinity where H_1 is harmonic on R and v_1 is bounded harmonic near infinity.

Now by the assumption (2), there exist biharmonic functions B_1 and B_2 such that $g_1 - B_1$ and $g_2 - B_2$ are bounded near infinity. Let $B_0 = B_1 - B_2 + B + H_1$. Then B_0 is biharmonic on R and $b - B_0$ is bounded near infinity. □

Search for examples: We have not succeeded in constructing Riemann surfaces R or Riemannian manifolds R of dimension ≥ 2 to establish that there is no inclusion relation between the two properties of R having biharmonic Green potentials and R being a biharmonic-extension space. However, if we carry out similar classifications of the second order elliptic differential operators on a domain Ω in $\mathbb{R}^n$, we can notice from the following two examples that the two properties are independent.

1. Let $\Omega = (0, \infty)$. Let $L = \frac{d^2}{dx^2}$ define the harmonic sheaf on Ω. Then, considering Ω as a biharmonic space, we can easily show that Ω is a biharmonic-extension space but there is no biharmonic Green potential on Ω.
2. Let $\Omega = \mathbb{R}$. Let $L = \frac{d^2}{dx^2} - \frac{d}{dx}$ define the harmonic sheaf on Ω. Then Ω has biharmonic Green potentials on Ω, but it is not a biharmonic-extension space.

In this context, it is of interest to know that if R is a radial Riemannian manifold having biharmonic Green potentials, then R is a biharmonic extension space (see the paragraph after Definition 3.2 and also Theorem 3.6); and if R is an arbitrary Riemannian space with biharmonic Green potentials, then given any biharmonic function b outside a compact set there exist on R, a unique biharmonic function B and a potential p such that $|b-B| \leq p$ near infinity as the following theorem shows.

Theorem 3.2 *In a Riemannian space R, the following are equivalent:*

1) *There are biharmonic Green potentials on R.*

2) *Given any biharmonic function b outside a compact set, there exist on R a biharmonic function B and two potentials s_1 and s_2 such that near infinity, $|b-B| \leq s_1$ and $|L(b-B)| \leq s_2$.*

Proof. $1) \Rightarrow 2)$: Let b be a biharmonic function defined outside a compact set in R. Then, as in Remark 2 (a) in [6], b can be represented near infinity in the form $b = g_1 - g_2 + p_1 - p_2 + B$ where B is a uniquely determined biharmonic function on R; g_1 and g_2 are potentials on R; and $-Lg_1$, $-Lg_2$, p_1 and p_2 are finite continuous potentials on R with compact harmonic support. Set $s_1 = g_1 + g_2 + p_1 + p_2$ and $s_2 = -Lg_1 - Lg_2$. Then s_1 and s_2 are two potentials on R such that near infinity, $|b-B| \leq s_1$ and $|L(b-B)| \leq s_2$.

$2) \Rightarrow 1)$: It is enough to show that for a potential $p > 0$ on R with compact harmonic support, there exists a potential g such that $Lg = -p$ on R. Let s be a superharmonic function on R such that $Ls = -p$. Since s is biharmonic outside a compact set, by the assumption (2), there exist a biharmonic function B on R and two potentials s_1 and s_2 such that near infinity $|s-B| \leq s_1$ and $|L(s-B)| \leq s_2$. This implies that the subharmonic function $|LB|$ on R is majorized by the potential $p + s_2$ outside a compact set and hence $LB \equiv 0$; that is B is harmonic on R.

Consequently, $s - B$ is superharmonic on R and majorizes the subharmonic function-s_1 near infinity. Hence we can write $s - B = g + h$ where g is a potential and h is harmonic on R. Then $Lg = -p$ on R. □

The following theorem gives a sufficient condition for the hyperbolic R to be a biharmonic-extension space. Let $LQ = -G$ where $G = G(x, y)$ is the Green potential on R with pole at $\{y\}$.

Theorem 3.3 *If there exists a biharmonic function B on R such that $LB \geq 0$ and $Q - B$ is bounded near infinity, then R is a biharmonic-extension space.*

Proof. Let p be a finite continuous potential on R with compact harmonic support A and let $Lg = -p$. By Theorem 3.1, it is enough to show that g has the biharmonic-extension property.

Let k be an outerregular compact set such that $A \subset \overset{0}{k}$. Then $p = B_k p$ on $R \backslash k$ where $B_k p$ denotes the Dirichlet solution on $R \backslash k$ with boundary values p on ∂k and 0 at infinity. Since for the assumption in the theorem the position of the pole $\{y\}$ in $G_y(x) = G$ is immaterial, we can assume $y \in A$; as a result, we can find some $\lambda > 0$ such that $B_k p \leq \lambda G$ on $R \backslash A$. Since p is finite continuous on R, we can suppose $p \leq \lambda G$ on R.

Let $Ls = -(\lambda G - p)$. Then $L(g + s) = \lambda LQ$. Hence $g + s = \lambda Q +$ (a harmonic function h) on R. Since $Q - B$ is, by assumption, a superharmonic fucntion on R, bounded near infinity, $Q - B =$ (a potential Q_1)$+$(a bounded harmonic function h_1) on R. Then, the superharmonic function $g - \lambda B = -s + \lambda(Q_1 + h_1) + h$ majorizes a subharmonic function on R and hence $g - \lambda B =$ (a potential g_1) $+$ (a harmonic function H) on R.

Similarly, s has a subharmonic minorant outside a compact set and hence $s =$ (a potential s_1) $+$ (a harmonic function h_3) on R. Consequently, $(g_1 + H) + (s_1 + h_3) = g - \lambda B + s = \lambda(Q - B) + h = \lambda(Q_1 + h_1) + h$. Equating the potential parts, we have $g_1 + s_1 = \lambda Q_1$. Since Q_1 is bounded near infinity, so is $g_1 = g - \lambda B - H$. Hence g has the biharmonic-extension property. □

Definition 3.2 A hyperbolic Riemannian space R is said to be tapered if Q (where $LQ = -G$) is bounded outside a compact set. [6]

At this point, it is useful to remark that from the results in [3] and [6] we can deduce that there exist biharmonic Green potentials on a Riemannian space R if and only if there exist potentials p and g on R such that $Lg = -p < 0$ [3]; and that R is tapered if and only if there exist potentials p and g on R, g being bounded outside a compact set such that $Lg = -p < 0$ [6]. We have yet to construct a Riemannian manifold R that has biharmonic Green potentials but is not tapered. However, if R is a radial space like the n-spaces and the Poincaré n-balls, each endowed with a radial metric $ds = \lambda(|x|)|dx|$ (see p. 311 Sario et al. [8]), it can be proved using Theorem 6 [6] that R is tapered if and only if R has biharmonic Green potentials. Thus the Euclidean space $\mathbb{R}^n$ is tapered if and only if $n \geq 5$; also, using Theorem 2.4, p. 316 Sario et al. [8] we conclude that the Poincaré n-ball $B^n_\alpha = \{x \in \mathbb{R}^n,\ |x| = r,\ r < 1,\ ds = (1 - r^2)^\alpha |dx|,\ \alpha \in \mathbb{R}\}$ is tapered if and only if $\alpha > -\frac{3}{2}$ when $n = 2$, $-3 < \alpha < 1$ when $n = 3$ and $\alpha < \frac{1}{n-2}$ when $n > 3$.

Now from Theorem 3.3, it is evident that a tapered Riemannian space R is a biharmonic-extension space. Conversely, the following two propositions discuss some of the additional conditions required so that a biharmonic-extension space becomes a tapered space. Recall (Theorem 10 [7]) that if b is a bounded biharmonic function outside a compact set in $\mathbb{R}^n$, $n \geq 2$, then Δb tends to 0 at infinity; hence if $n \geq 3$, there exists a potential p on $\mathbb{R}^n$ such that $|\Delta b| \leq p$ near infinity.

Proposition 3.4 *Let R be a biharmonic-extension space on which if b is a bounded biharmonic function outside a compact set, there is a potential p on R such that $|Lb| \leq p$ near infinity. Then R is tapered.*

Proof. To show that R is tapered, we have to show that if G is the Green function on R, then there exists a potential Q on R, bounded near infinity and $LQ = -G$. Let now $LQ_1 = -G$ for some superharmonic function Q_1 on R. Since Q_1 is biharmonic near infinity, there exists a biharmonic function B on R such that $Q_1 - B$ is bounded near infinity. Then by the assumption, there exists a potential p on R such that $|L(Q_1 - B)| \leq p$ near infinity. Hence $|LB| \leq p + G$ near infinity.

Consequently, $LB \equiv 0$ and B is harmonic on R. Since $Q_1 - B$ is superharmonic on R and bounded near infinity, its g.h.m. h is bounded on R. If we write $Q_1 - B = Q + h$, Q is a potential on R, bounded near infinity and $LQ = -G$. Hence R is tapered. □

Again, a tapered Riemannian space R is a biharmonic-extension space with biharmonic Green potentials. We do not expect the converse to be true without any additional hypothesis. We show below that if R is a biharmonic-extension space of Almansi-type with biharmonic Green potentials, then R is tapered; we define the Almansi-type (see [7]) as follows: Let ω_n be a regular exhaustion of R with ρ_x^n denoting the harmonic measure on $\partial\omega_n$. R is said to be of Almansi-type if every biharmonic fucntion B on R, for which $h_n(x) = \int B d\rho_x^n$ is locally uniformly bounded on R for large n, is necessarily harmonic on R.

Proposition 3.5 *Let R be a biharmonic-extension space having biharmonic Green potentials. Suppose R is of Almansi-type. Then R is tapered.*

Proof. Let G be a Green potential with point support and $LQ = -G$. We have to show that Q is bounded near infinity.

Since R has biharmonic Green potentials, Q can be assumed to be a potential on R [3]. Since R is a biharmonic-extension space, there exists a biharmonic function B on R such that $Q-B$ is bounded near infinity. Hence $|B| \leq Q+c$ near infinity where $c \geq 0$ is a constant. Since $\int Q d\rho_x^n$ tends to 0 uniformly on compact sets on R which is assumed to be of Almansi-type, we conclude that $\int B d\rho_x^n$ is bounded on compact sets for large n. Hence B is harmonic on R.

Thus, $Q-B$ is superharmonic on R and bounded near infinity. Hence if h is the g.h.m. of $Q-B$ on R, h is bounded on R. Now, in the equation $Q-B=(Q-B-h)+h$, equating the harmonic parts, we obtain $B=-h$ is bounded harmonic on R. Hence $Q=(Q-B)+B$ is bounded near infinity. This proves that R is tapered. □

Remark 3.1 The assumptions in the above two propositions imply that every bounded biharmonic fucntion on R is harmonic. But in some tapered spaces we can find nonharmonic bounded biharmonic functions, as in the case of the Poincaré ball B_α^n for $n>4$ and $-1<\alpha<\frac{1}{n-2}$. Other simple examples of tapered spaces which are not of Almansi-type include $\mathbb{R}^n \backslash 0$ for $n \geq 5$ (see [7]).

We give now a necessary and sufficient condition for a hyperbolic space R to be tapered.

Theorem 3.6 *Let R be a hyperbolic Riemannian space. Then R is tapered if and only if for any biharmonic function b outside a compact set there exist potentials p and g on R, both bounded near infinity, and a (unique) biharmonic function B on R such that near infinity $|b-B| \leq g$ and $|L(b-B)| \leq p$.*

Proof. If R is tapered, any biharmonic function b outside a compact set has a representation of the form $b=g_1-g_2+p_1-p_2+B$ near infinity (see Theorem 5 and Lemma 8 in [6]) where g_1 and g_2 are potentials on R bounded near infinity; $-Lg_1$, $-Lg_2$, p_1 and p_2 are finite continuous potentials with compact harmonic support; and B is biharmonic on R. Consequently, with $g=g_1+g_2+p_1+p_2$ and $p=-Lg_1-Lg_2$, we have near infinity, $|b-B| \leq g$ and $|L(b-B)| \leq p$. Note that under these conditions, B is uniquely determined.

Conversely, let $LQ_1=-G$. Then Q_1 being biharmonic near infinity, by the hypothesis, there exist a biharmonic function B and potentials p and g

on R such that p and g are bounded near infinity and $|Q_1 - B| \leq g$ and $|L(Q_1 - B)| \leq p$ outside a compact set. Since the harmonic function LB on R satisfies the condition $|LB| \leq p + G$ near infinity, $LB \equiv 0$; that is, B is harmonic. Hence $Q_1 - B$ is a superharmonic function on R bounded near infinity. Write $Q_1 - B = Q + h$ where Q is a potential on R bounded near infinity and h is harmonic on R. Moreover $LQ = L(Q_1 - B) = LQ_1 = -G$. Hence R is tapered. □

Corollary 3.7 *Let R be a tapered Riemannian space on which every bounded biharmonic function is harmonic. Let h be a harmonic function defined outside a compact set in R. Then the following are equivalent:*

1) *There exists a bounded biharmonic fucntion u near infinity such that $Lu = -h$.*

2) *There is a potential p on R such that $|h| \leq p$ near infinity.*

Proof. 1) $\Rightarrow$ 2): Let u be a bounded biharmonic function near infinity such that $Lu = -h$. Write $h = p_1 - p_2 + H$ where p_1 and p_2 are finite continuous potentials with compact support and H is harmonic on R. Let $Lg_1 = -p_1$, $Lg_2 = -p_2$ and $LB = -H$. Then $u = g_1 - g_2 + B +$ (a harmonic function v) near infinity. Now v can be written as $v = f + v_1$ near infinity where f is harmonic on R and v_1 is bounded harmonic near infinity; moreover, since R is tapered, g_1 and g_2 can be assumed to be bounded near infinity (Theorem 9 [6]). Consequently, the biharmonic function $B + f$ on R is bounded near infinity and hence is harmonic by the hypothesis. This implies that $H = -LB = -L(B + f) = 0$ and that $h = p_1 - p_2$. Hence $|h| \leq p_1 + p_2$ near infinity.

2) $\Rightarrow$ 1): Conversely, let h be a harmonic function outside a compact set such that $|h| \leq p$ near infinity, for a potential p on R. Writing $h = p_1 - p_2 + H$ near infinity, we note that $|H| \leq p + p_1 + p_2$ near infinity and hence $H \equiv 0$. Since R is tapered, we can choose g_1 and g_2 on R, both bounded near infinity, such that $Lg_1 = -p_1$ and $Lg_2 = -p_2$. Thus, near infinity, $u = g_1 - g_2$ is a bounded biharmonic function such that $Lu = -h$. (For the implication (2) $\Rightarrow$ (1), the assumption that every bounded biharmonic function on R is harmonic is redundant.) □

Remark 3.2 1) Suppose there is a potential > 0 on R tending to 0 at infinity. Let h be harmonic outside a compact set. Then $|h| \leq p$ near infinity for a potential p on R if and only if h tends to 0 at infinity. This is so, since in this case every potential with compact harmonic support tends to 0 at infinity.

2) $\mathbb{R}^n$, $n \geq 3$, and B^n_α with $\alpha \leq -1$ and $n \geq 2$ are some of the hyperbolic Riemannian manifolds in which there exist potentials > 0 tending to 0 at infinity and every bounded biharmonic function is harmonic. Here $\mathbb{R}^3$ and $\mathbb{R}^4$ are not tapered and for every bounded biharmonic function u outside a compact set in $\mathbb{R}^n$, $n \geq 2$, $\Delta u \to 0$ at infinity, but there is no bounded biharmonic function u near infinity such that $\Delta u = \frac{1}{r^{n-2}}$ ($n = 3, 4$). However the above corollary is valid in $\mathbb{R}^n$, $n \geq 5$, which are tapered and in these cases we have a variant of this corollary as follows:

Theorem 3.8 *Let h be a harmonic function defined outside a compact set in $\mathbb{R}^n$, $n \geq 5$. Then $h \in L^p(\omega)$ for some p, $1 \leq p < \infty$ and some neighbourhood ω of infinity if and only if there exists a bounded biharmonic function b near infinity such that $\Delta b = h$.*

To prove this theorem, we need the following lemmas.

Lemma 3.9 *If s is a subharmonic function in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$ and $n \geq 2$, then $s \leq 0$ on $\mathbb{R}^n$.*

Proof. For $x_0 \in \mathbb{R}^n$, let $S_n = \{x : |x - x_0| = 1\}$ and σ_n be the surface area of S_n. Since $t = s^+ \geq 0$, t^p is subharmonic and using the polar coordinates for $x = (r, \omega)$, $|x - x_0| = r$, we have $t^p(x_0) \leq \frac{1}{\sigma_n} \int_{S_n} t^p(r, \omega)\, d\omega$.

Since $t \in L^p(\mathbb{R}^n)$ by hypothesis,

$$\infty > \int_0^\infty \int_{S_n} t^p(r, \omega) r^{n-1} dr\, d\omega \geq \int_0^\infty \sigma_n t^p(x_0)\, r^{n-1} dr.$$

This is possible if and only if $t^p(x_0) = 0$. Since x_0 is arbitrary $t^p \equiv 0$ on $\mathbb{R}^n$ and hence $s \leq 0$ on $\mathbb{R}^n$. $\square$

For the statement of the following lemma, we shall say that a subharmonic function f defined outside a compact set on $\mathbb{R}^n$ *extends subharmonically* on $\mathbb{R}^n$, if there exists a subharmonic function g on $\mathbb{R}^n$ such that g is not majorized by a harmonic function on $\mathbb{R}^n$ and $f = g$ outside a compact set.

Lemma 3.10 *Let u be an L^p-subharmonic function, $1 \leq p < \infty$, defined outside a compact set in $\mathbb{R}^n$, $n \geq 2$. Then, u cannot be extended subhar-*

monically on $\mathbb{R}^n$.

Proof. Suppose there exists a subharmonic function v not majorized by a harmonic function on $\mathbb{R}^n$ such that $u = v$ outside a compact set. Then, for large r, the function s defined as u on $|x| \geq r$ and $D_r u$ on $|x| < r$ where $D_r u$ is the Dirichlet solution on $|x| < r$ with boundary values u, is subharmonic on $\mathbb{R}^n$ and $s \geq v$.

If $u(x) \in L^p$ on $|x| \geq r$, $s(x)$ is in the harmonic Hardy class on $|x| < r$ (Axler et al. p. 103 [5]) and hence there exists a harmonic function $H(x)$ on $|x| < r$ such that $|s|^p < H$. Then $\int_{|x|<r} |s(x)|^p dx \leq c_n H(0)$ for a constant c_n. That is, s belongs to L^p on $|x| < r$, which implies that $s \in L^p(\mathbb{R}^n)$ since $s(x) = u(x)$ on $|x| \geq r$. Then, by Lemma 3.9, $s \leq 0$ and hence $v \leq 0$ on $\mathbb{R}^n$, a contradiction. □

Lemma 3.11 *Let u be a subharmonic function on an open set ω containing $|x| \geq r$ on $\mathbb{R}^n$, $n \geq 2$. Suppose $u \in L^p(\omega)$ for some p, $1 \leq p < \infty$. Then u is upper bounded on $|x| \geq r$.*

Proof. By hypothesis, $u^+(x)$ is an L^p-subharmonic function on the open set ω containing $|x| \geq r$.

1) In $\mathbb{R}^2$, if u^+ is not upper bounded on $|x| \geq r$, it can be extended subharmonically on $\mathbb{R}^2$ (Corollary 1 [2]). This is a contradiction (Lemma 3.10) since $u^+ \in L^p$ on $|x| \geq r$. This means that u^+ and hence u is upper bounded on $|x| \geq r$.

2) In $\mathbb{R}^n$, $n \geq 3$, there exists a subharmonic function $s(x)$ on $\mathbb{R}^n$ and some $\alpha \leq 0$ such that $u^+(x) = s(x) - \alpha|x|^{2-n}$ on $|x| \geq r$ (Theorem $1'$ [2]). Hence $s(x) \geq \alpha|x|^{2-n}$.

Denoting by $M(R, s)$ the mean-value of $s(x)$ on $|x| = R$, suppose $\lim_{R\to\infty} M(R, s) = \infty$. Then $\lim_{R\to\infty} M(R, u^+) = \infty$. Hence u^+ can be extended subharmonically on $\mathbb{R}^n$ (Theorem $2'$ [2]), a contradiction (Lemma 3.10); thus $\lim_{R\to\infty} M(R, s) < \infty$, in which case s has a harmonic majorant h on $\mathbb{R}^n$. Since h is lower bounded, it is a constant c and $c \geq 0$. Hence u^+ is bounded on $|x| \geq r$ and u is upper bounded by $c - \alpha|x|^{2-n}$ on $|x| \geq r$.

Thus, for all $n \geq 2$, u is upper bounded on $|x| \geq r$ in $\mathbb{R}^n$. □

Lemma 3.12 *Let h be a harmonic function defined outside a compact set in $\mathbb{R}^n$, $n \geq 2$. Then h tends to 0 at infinity if and only if $h \in L^p(\omega)$ for some p, $1 \leq p < \infty$, and some neighbourhood ω of infinity.*

Proof. Suppose $h \in L^p(\omega)$. Then by Lemma 3.11, h is bounded near infinity and hence tends to a limit l at infinity; l should be 0 since $h \in L^p(\omega)$.

Conversely, let h tend to 0 at infinity.

1) In $\mathbb{R}^n$, $n \geq 3$, write $h = p_1 - p_2 + H$ near infnity where p_i $(i = 1, 2)$ is a finite continuous potential with compact support on $\mathbb{R}^n$ and H is harmonic on $\mathbb{R}^n$. Since p_i and h tend to 0 at infinity, $H \equiv 0$. Now, for sufficiently large r, $p_i(x) = B_r p_i(x)$ for $|x| > r$ where $B_r p_i$ denotes the Dirichlet solution on $|x| > r$ with boundary values $p_i(x)$ on $|x| = r$ and 0 at infinity. Consequently $p_i(x) \leq \alpha_i |x|^{2-n}$ in $|x| > r$ where $\alpha_i = \max_{|x|=r} p_i(x)$ and hence if ω is the open set $|x| > r$, $p_i \in L^p(\omega)$ for $p > \frac{n}{n-2}$. Hence $h \in L^p(\omega)$.

2) In $\mathbb{R}^2$, write $h = s_1 - s_2 + H$ near infinity where s_i $(i = 1, 2)$ is a finite continuous logarithmic potential with compact harmonic support on $\mathbb{R}^2$ and H is harmonic on $\mathbb{R}^2$. Suppose flux s_i at infinty is α_i. Then $\alpha_1 = \alpha_2 = \alpha$ since flux h and flux H at infinity are 0; also since h and $s_i - \alpha \log|x|$ tend to 0 at infinity, $H \equiv 0$. Now, for sufficiently large r, $|s_i - \alpha \log|x|| \leq \frac{M}{|x|}$ when $|x| > r$. Hence $|h(x)| \leq \frac{2M}{|x|}$ when $|x| > r$; consequently $h \in L^p$ on $|x| > r$, if $p > 2$. □

Proof of Theorem 3.8. 1) Suppose $h \in L^p(\omega)$ for some finite $p \geq 1$. Then by Lemma 3.12, h tends to 0 at infinity and consequently Corollary 3.7 and the remarks that follow can be used to assert the existence of a bounded biharmonic function b near infinity such that $\Delta b = h$.

2) Conversely, let h be harmonic outside a compact set such that for a bounded biharmonic function b, $\Delta b = h$. Then by Corollary 3.7, h tends to 0 at infinity. Consequently by Lemma 3.12, $h \in L^p(\omega)$ for $p > \frac{n}{n-2}$ and some neighbourhood ω of the point at infinity. □

We conclude this section with a remark on the set of nonremovable singularities for bounded biharmonic functions.

Theorem 3.13 *Let R be a biharmonic-extension space. Suppose K is compact, w is open, $K \subset w \subset R$ and b is a bounded biharmonic funtion on $w \backslash K$ which does not extend as a biharmonic function on w. Then given any open set $\Omega \supset K$, there exists a bounded biharmonic function on $\Omega \backslash K$ which does not extend biharmonically on Ω.*

Proof. It is enough to prove the theorem assuming $\Omega = R$. Since b is bounded biharmonic on $w \backslash K$, (easily modifying the proof of Theorem 3.1 [4] given for $\mathbb{R}^n$ to suit the Riemannian space R), we can write $b = s - v$ on $w\backslash K$ where s is biharmonic on $R\backslash K$ and v is biharmonic on w.

Since R is a biharmonic-extension space, there exists a biharmonic function B on R such that $u = s - B$ is bounded near infinity; hence we can assume that for a relatively compact open set ω_0, $K \subset \omega_0 \subset \bar{\omega}_0 \subset \omega$, u is bounded on $R\backslash\omega_0$. But $b = u-(v-B)$ on $\omega\backslash K$ and b and $v-B$ are bounded on $\omega\backslash K$; hence u is bounded on $\omega\backslash K$. Thus u is a bounded biharmonic fucntion on $R\backslash K$. Moreover u cannot be extended as a biharmonic function on R; for if u extends biharmonically on R, ($v - B$ being biharmonic on ω), b also extends biharmonically on ω, a contradiction. □

4. Flux condition

In this section also, R stands for a hyperbolic Riemann surface or a hyperbolic Riemannian manifold of dimension ≥ 2.

Definition 4.1 R is said to satisfy the *flux condition* if given any harmonic function h outside a compact set with flux h at infinity 0, there exist a biharmonic fucntion B on R and a bounded biharmonic function u outside a compact set such that $L(B + u) = -h$ near infinity.

Remark 4.1 1) $\mathbb{R}^n$, $n \geq 2$, satisfy the flux condition. (See Theorem 12 [6]).

2) If R is a biharmonic-extension space, this definition is redundant. For, in this case, for any harmonic h outside a compact set, there always exist B and u as in the definition such that $L(B + u) = -h$ near infinity.

Theorem 4.1 *Let R be hyperbolic but not a biharmonic-extension space, satisfying the flux condition. Then given a biharmonic function b outside a compact set, there exists a biharmonic funtion B_0 on R such that $b - B_0$ is bounded near infinity if and only if flux Lb at infinity is* 0.

Proof. 1) Let $h = -Lb$ and flux h at infinity 0. Since R satisfies the flux condition, $L(B + u) = -h$. Hence $b = B + u + h_0$ near infinity where h_0 is harmonic outside a compact set. Since R is hyperbolic, there exist a harmonic function H on R and a bounded harmonic function v outside a compact set such that $h_0 = H + v$. Hence if $B_0 = B + H, B_0$ is biharmonic on R and $b - B_0$ is bounded near infinity.

2) Conversely, let b be a biharmonic function outside a compact set such that for a biharmonic function B_0 in R, $b - B_0$ is bounded near infinity.

Suppose flux Lb at infinity is $\alpha \neq 0$.

Let G be the Green potential on R with pole at some $\{y\}$ and let $LQ = -G$. Take flux G at infinity as β. Then $\beta \neq 0$ and flux $L\big(Q + \frac{\beta}{\alpha}b\big)$ at infinity is 0. Hence from what we have proved in (1) above, there exists a biharmonic function B_1 on R such that $Q + \frac{\beta}{\alpha}b + B_1$ is bounded near infinity. Since we know that $b - B_0$ is bounded near infinity, we conclude $Q_1 = Q + \frac{\beta}{\alpha}B_0 + B_1$ is bounded near infinity. We shall show that this conclusion leads to the contradiction that R is a biharmonic-extension space.

For, let s be any biharmonic function outside a compact set in R. Let flux Ls at infinity is λ. Then flux $L\big(s + \frac{\lambda}{\beta}Q_1\big) = 0$. Hence by the flux condition there exists a biharmonic function B_2 on R such that $s + \frac{\lambda}{\beta}Q_1 - B_2$ is bounded near infinity. Since Q_1 is bounded near infinity, $s - B_2$ is bounded near infinity. This implies that R is a biharmonic-extension space, a contradiction.

This contradiction arises out of the assumption that $\alpha \neq 0$. Thus, we have proved that flux Lb at infinity is 0. □

5. Biharmonic extension in parabolic spaces

In this section, R stands for a parabolic Riemann surface or a parabolic Riemannian manifold of dimension ≥ 2.

Let $E = E_y(x)$ be the Evans function on R with pole at some point $\{y\}$. Let us denote Q by $LQ = -E$ on R. Since $|E_y(x) - E_{y_1}(x)|$ is bounded near infinity for two different poles y and y_1, in the context of biharmonic extension the pole y can be fixed conveniently.

Definition 5.1 A parabolic Riemannian space R is said to be a biharmonic-extension space if given a biharmonic function b outside a compact set there exist a biharmonic function B on R and a constant α such that $b - B - \alpha E$ is bounded near infinity.

Theorem 5.1 *A parabolic Riemannian space R is a biharmonic-extension space if and only if for any finite continuous logarithmic potential p with compact harmonic support on R, if $Lg = -p$, there exist a biharmonic function B on R and a constant α such that $g - B - \alpha E$ is bounded near infinity.*

Proof. The proof is similar to that of Theorem 3.1 with the proviso that any harmonic function h outside a compact set in R can be written as $h =$

$p_1 - p_2 + H$ near infinity where p_1 and p_2 are finite continuous logarithmic potentials with compact harmonic support on R and H is harmonic on R; or in another representation $h = f + \alpha E + v$ near infinity where f is harmonic on R and v is bounded harmonic near infinity. $\square$

If the parabolic space R satisfies the flux condition (the same as in Definition 4.1), then as in Section 4 we can prove the following:

Lemma 5.2 *Let the parabolic space R satisfy the flux condition. Then any biharmonic function b outside a compact set is of the form $b = B + \alpha Q + \beta E + u$ near infinity where B is biharmonic on R, u is bounded biharmonic near infinity, α and β are constants and $LQ = -E$.*

Lemma 5.3 *If the parabolic space R satisfies the flux condition and if $Q + B + \beta E$ is bounded near infinity for some biharmonic function B on R, then R is a biharmonic-extension space.*

Theorem 5.4 *Let R be a parabolic but not a biharmonic-extension space, satisfying the flux condition. Then given a biharmonic function b outside a compact set, there exist a biharmonic function B_0 on R and a constant α such that $b - B_0 - \alpha E$ is bounded near infinity if and only if flux Lb at infinity is 0.*

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