

On H -separable extensions of QF-3 rings

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Abstract. Let a ring A be an H -separable extension of a subring B of A , that is, $A \otimes_B A$ is an A - A -direct summand of a finite direct sum of copies of A . If furthermore, (a) B is a left B -direct summand of A (or (b) A is left B -finitely generated projective), and if B is a right (resp. left) artinian QF-3 ring, then A is also a right (resp. left) artinian QF-3 ring. Consequently, if A is an H -separable extension of a serial ring B with one of the conditions (a), (b), then A is also a serial ring. In particular H -separable extension of a uni-serial ring is always uni-serial.

Key words: H -separable extension, QF-3 ring, serial ring, uni-serial ring.

Introduction

A ring A is said to be an H -separable extension of a subring B of A , if $A \otimes_B A$ is isomorphic to an A - A -direct summand of a finite direct sum of copies of A as an A - A -module. H -separable extension is a special type of separable extensions. It was introduced by K. Hirata in [4] to generalize the notion of Azumaya algebra. The structure of H -separable extension was researched by Hirata himself in [5], [6] and the author in [10], [11], [12] and so forth. The structure of H -separable extension of a simple ring was completely determined by the author in [13]. That is, in the case where B is a simple ring, A is a left (or right) projective H -separable extension of B , if and only if A is simple, $V_A(B)$ is a finite dimensional simple C -algebra and $B = V_A(V_A(B))$, where $V_A(B)$ is the centralizer of B in A and C is the center of A (Theorem 1 [13]). This theorem gives the generalization of classical inner Galois theory of simple artinian rings researched by Noether, Brauer and Artin and others. The definitions and characterizations of H -separable extension, separable extension and semisimple extension are always concerned with tensor products and Hom functors. Our desire is to find the characterizations of them without using tensor products or Hom functors, but using only inner structure of the ring, in the case where B has some special property. The above theorem is one of the successful cases. On

the other hand in [15] K. Hirata and the author found characterizations of semisimple extension in the case where both A and B are local serial rings using only the lengths of composition series (Theorem 6 [15]). But when we omit the condition that A is local, we can not know anything even in the case where A is H -separable over B . The author expects that these results will be generalized to the case of H -separable extensions of indecomposable uniserial rings.

In this paper the author will show that under some additional conditions an H -separable extension of a left QF-3 ring is also a left QF-3 ring (Theorems 2 and 3). As a consequence we can show that a left projective H -separable extension of a serial ring is also a serial ring (Theorem 7), and that any H -separable extensions of uniserial rings are uniserial (Theorem 8).

H -separable extensions of QF-3 rings.

Throughout this paper A is a ring with the identity element 1, B is a subring of A containing 1 and C is the center of A . Furthermore we write $V_A(X) = \{a \in A \mid xa = ax \text{ for } \forall x \in X\}$ for any subset X of A , and $D = V_A(B)$. First we will introduce some fundamental properties of H -separable extension briefly. We always have the following A - A -homomorphism

$$\begin{aligned} \eta : A \otimes_B A &\longrightarrow \text{Hom}({}_C D, {}_C A) \\ \eta(a \otimes b)(d) &= adb \quad (\forall a, b \in A, \forall d \in D) \end{aligned}$$

A is an H -separable extension of B if and only if D is C -finitely generated projective and η is an isomorphism. We also have the following A - A - and D - D -homomorphism

$$\begin{aligned} \zeta_l : D \otimes_C A &\longrightarrow \text{Hom}({}_B A, {}_B A) \\ \zeta_l(d \otimes a)(x) &= d \otimes a \quad (\forall a, x \in A, \forall d \in D) \end{aligned}$$

If A is H -separable over B , ζ_l is an isomorphism. Therefore in this case $\text{End}({}_B A)$ is left and right A -finitely generated projective, and contains A as an A - A -direct summand. The map ζ_r of $A \otimes_C D$ to $\text{End}({}_B A)$ is similarly defined, and the same results hold. In the case where A is H -separable over B , we consider the following conditions;

- (a) B is a left B -direct summand of A .
- (a-2) B is a left B -direct summand of A , and A is left B -flat.
- (a-3) B is a left B -direct summand of A , and A is right B -flat.
- (b) A is left B -finitely generated projective.
- (b-2) A is left B -finitely generated projective, and $B = V_A(V_A(B))$.

In the case where B is commutative and A is an Azumaya B -algebra all of the above conditions are satisfied. Therefore it is natural to consider these conditions. In fact in the case where A is a B -algebra, A is an Azumaya B -algebra, if and only if A is H -separable over B and (a) is satisfied, if and only if A is H -separable over B and A is B -finitely generated (Corollaries 1.1 and 1.2 [10]).

If A is an H -separable extension of B , and the condition (a) (resp. (b)) is satisfied, then we have $I = A(I \cap B)$ (resp. $I = (I \cap B)A$) for any two-sided ideal I of A by Theorem 4.1 (resp. Theorem 3.1) [11]. Therefore in either case we have $I \cap B \neq 0$ for any non-zero two-sided ideal I of A . Then the following lemmas are useful.

Lemma 1 *Suppose that $I \cap B \neq 0$ holds for each non-zero two-sided ideal I of A . Then for each faithful injective left B -module M , $M^* = \text{Hom}({}_B A, {}_B M)$ is a faithful injective left A -module.*

Proof. Since A is right A -flat and M is B -injective. M^* is left A -injective. Let I be the annihilator ideal of M^* in A , and suppose $I \neq 0$. Then by the assumption we have $I \cap B \neq 0$. Let $0 \neq a \in I \cap B$. Since M is B -faithful, there exists m in M with $am \neq 0$. Then the left B -homomorphism f of B to M defined by $f(x) = xm (x \in B)$ satisfies $f(a) \neq 0$. f is extended to $f^* \in M^*$, since M is B -injective, and $af^* = 0$ since $a \in I$. Then we have $0 = af^*(1) = f^*(a) = f(a) \neq 0$, a contradiction. Thus we have $I = 0$. \square

Lemma 2 *Let A be an H -separable extension of B . Then for each finitely generated projective and injective left B -module M , $M^* = \text{Hom}({}_B A, {}_B M)$ is a finitely generated projective and injective left A -module.*

Proof. Obviously M^* is left A -injective. Since M is B -finitely generated projective, M^* is an A -direct summand of a finite direct sum of copies of $\text{Hom}({}_B A, {}_B B)$, which is contained in $\text{Hom}({}_B A, {}_B A)$. Hence M^* is contained in $P = \bigoplus \text{Hom}({}_B A, {}_B A)$ (finite direct sum), and P is a finitely generated projective left A -module. Since M^* is A -injective, M^* is an A -direct summand of P . Thus M^* is A -finitely generated projective. \square

As an immediate consequence of the above lemmas we have

Theorem 1 *Let A be an H -separable extension of B . Assume that the condition (a) or (b) is satisfied. Then if M is a finitely generated faithful projective and injective left (resp. right) B -module, $M^* = \text{Hom}({}_B A, {}_B M)$*

(resp. $\text{Hom}(A_B, M_B)$) is also a finitely generated faithful projective and injective left (resp. right) A -module.

Proof. By Lemma 2 M^* is a finitely generated projective and injective left (resp. right) A -module. In the case of (a) we have $I = A(I \cap B)$, while in the case of (b) we have $I = (I \cap B)A$ for each two-sided ideal I of A . In either case we can apply Lemma 1 to obtain that M^* is a faithful left (resp. right) A -module. \square

A is said to be a left semisimple extension of B in the case where every left A -module M is (A, B) -projective, that is, the map p of $A \otimes_B M$ to M such that $p(a \otimes m) = am$ ($a \in A$, $m \in M$) is an A -split epimorphism. Separable extensions, and consequently H -separable extensions are always semisimple extensions. We will now consider H -separable extensions of perfect QF-3 rings. Left perfect ring is characterized as a ring whose flat left A -module is always projective (Bass).

The next lemma is a generalization of Proposition 5.3 (g) [6]

Lemma 3 *Let A be a left semisimple extension of B . Assume that A is left B -flat. If B is left perfect, A is also left perfect.*

Proof. Let M be a flat left A -module. Since A is left B -flat, M is B -flat. Then M is B -projective, since B is left perfect. Then M is A -projective, since A is left semisimple over B . Thus every flat left A -module is A -projective. \square

Theorem 2 *Let A be an H -separable extension of B , and assume that the condition (a-2) or (b) is satisfied. If B is a left perfect right QF-3 ring, A is also a left perfect right QF-3 ring.*

Proof. By Lemma 3 we see that A is left perfect. Now by Theorem 1 A has a finitely generated faithful projective and injective right module M^* . Since A is semi-perfect, we have $M^* = (e_1A)^{n_1} \oplus (e_2A)^{n_2} \oplus \cdots \oplus (e_kA)^{n_k}$, where $\{e_i \mid 1 \leq i \leq k\}$ is a subset of the set of basic idempotents of A , and $n_i > 0$. Then eA ($e = e_1 + e_2 + \cdots + e_k$) is also a faithful projective and injective right A -module, and each e_iA is the injective hull of a minimal right ideal I_i of A , since A is left perfect. Obviously $I_i \neq I_j$ for $i \neq j$. Then eA is the minimal faithful right A -module (Lemma 31.5 [1]). \square

Proposition 1 *Let A be an H -separable extension of B with the condition (a). Then if B is a right artinian QF-3 ring, A is also a right artinian*

QF-3 ring.

Proof. By Theorem 4.1 [11] A is right B -finitely generated, and hence is a right artinian ring. The remainder part is proved by the similar way as above. \square

Now we will consider the H -separable extension of semiprimary QF-3 rings. Let us denote the Jacobson radical of A by $J(A)$. In order to know the relation between $J(A)$ and $J(B)$ the following easy remark is very useful.

Remark 1 If $B = V_A(V_A(B))$, then we have $J(A) \cap B \subseteq J(B)$.

Proof. For each x in $J(A) \cap B$, $1 - x$ has the inverse in A . But since $1 - x \in V_A(D)$, its inverse is contained in $V_A(D) = B$. Hence $J(A) \cap B$ is a quasi-regular ideal of B , and is contained in $J(B)$. \square

Proposition 2 Let A be an H -separable extension of B . Assume that one of the conditions (a-2), (a-3), (b-2) is satisfied. Then if B is semiprimary, A is also semiprimary.

Proof. Write $N = J(A)$. In the case of (a) we have $N = A(N \cap B)$ and $B = V_A(D)$. Then by the above remark $N \cap B$ is contained in $J(B)$, which is nilpotent. Hence N is nilpotent. In the case of (b-2) we have $N = (N \cap B)A$ and $B = V_A(D)$, which implies the same result. On the other hand A is a separable extension of a perfect ring B and is left (or right) B -flat. Hence by Lemma 3 A is left (or right) perfect. Therefore A/N is a semisimple ring. \square

By Theorem 2 and Proposition 2 we have

Theorem 3 Let A be an H -separable extension of B , and suppose that the condition (a-2) or (b-2) is satisfied. If B is a semiprimary left (resp. right) QF-3 ring, A is also a semiprimary left (resp. right) QF-3 ring.

Theorem 4 Let A be an H -separable extension of B with the condition (a) or (b). Then if $E({}_B B)$ is B -projective, $E({}_A A)$ is A -projective, where $E({}_B B)$ and $E({}_A A)$ are the injective hulls of the left B -module B and the left A -module A , respectively.

Proof. By the assumption and Lemma 6.1 [16] $M = E({}_B B)$ is finitely generated projective and injective as left B -module. Hence $M^* = \text{Hom}({}_B A, {}_B M)$ is a finitely generated projective and injective left A -module by Lemma 2.

On the other hand in the case where the condition (a) is satisfied $B^* = \text{Hom}({}_B A, {}_B B)$ is a left A -progenerator by Proposition 5 [12]. Therefore we have

$$A \subseteq B^* \oplus B^* \oplus \cdots \oplus B^* \subseteq M^* \oplus M^* \oplus \cdots \oplus M^* = P$$

Since P is left A -injective, $E({}_A A)$ is contained in P . And $E({}_A A)$ is an A -direct summand of P , since $E({}_A A)$ is A -injective. Then $E({}_A A)$ is A -finitely generated projective since P is A -finitely generated projective. In the case where the condition (b) is satisfied we have

$$A \subseteq \text{Hom}({}_B A, {}_B A) \subseteq B^* \oplus B^* \oplus \cdots \oplus B^* \subseteq M^* \oplus M^* \oplus \cdots \oplus M^*$$

Then for the same reason as above we have the results. \square

Theorem 5 *Let A be an H -separable extension of B with the condition (b). If B is a left artinian QF-3 ring, A is also a left artinian QF-3 ring.*

Proof. This is clear by Theorem 4 and Theorem 31.6 [1]. \square

Next we will consider the case where A is torsionless as left B -module, which is a weaker condition than (b).

Let M be a left B -module and $a \in A$ with $aM^* = 0$, where $M^* = \text{Hom}({}_B A, {}_B M)$. Then for each f in M^* we have $0 = (af)(1) = f(a)$. Thus if A is M -torsionless (i.e., $A \subseteq \Pi M$), M^* is faithful as left A -module. If A is left B -torsionless, and M is B -faithful, we have $A \subseteq \Pi B$, $B \subseteq \Pi M$ and consequently $A \subseteq \Pi M$. Thus M^* is A -faithful. Then by Lemma 2 we have

Proposition 3 *Let A be an H -separable extension of B . Assume that A is left B -torsionless. If B has a finitely generated faithful projective and injective left B -module M . A has also a finitely generated faithful projective and injective left A -module.*

Proof. If M is a left B -module satisfying the condition of the theorem, M^* satisfies the same condition as A -module. \square

We use the same notation as [16] concerning with the notion of a left (right) quotient ring and the maximal left (right) quotient ring. So see [16] for detail. The next remark, which is a generalization of the remark stated in page 46 [16], may be already well-known. But it is very useful. Therefore we will state it as a remark.

Remark 2 Let M be a faithful torsionless right A -module. Then the double centralizer T of M_A is a left quotient ring of A .

Proof. Let $S = \text{End}(M_A)$. Then $T = \text{End}({}_S M)$. Since M is A -faithful, A is a subring of T . For any g in $\text{Hom}(M_A, A_A)$ and z in M , we define the map $g_{(z)}$ of M to M by $g_{(z)}(x) = zg(x)$ for each x in M . Clearly $g_{(z)} \in S$, and we have $g_{(z)}(xt) = (g_{(z)}(x))t$, hence $zg(xt) = (zg(x))t = z(g(x)t)$, where we regard $g(x)$ as an element of T . This means $g(xt) = g(x)t$ for each x in M and t in T . Now let $t_1, t_2 \in T$ with $t_2 \neq 0$. Then there exists an m in M such that $mt_2 \neq 0$. Since M is A -torsionless there exists an f in $\text{Hom}(M_A, A_A)$ such that $f(mt_2) \neq 0$. But $f(mt_2) = f(m)t_2 \neq 0$, and $A \ni f(mt_1) = f(m)t_1$. Hence there exists an $r (= f(m))$ in A with $rt_1 \in A$, $rt_2 \neq 0$. \square

Theorem 6 Let A be an H -separable extension of B , and assume that A is left B -torsionless. Then we have;

- (1) $B'' = V_A(V_A(B))$ is a right quotient ring of B .
- (2) If B is right QF-3, B'' is a left and right quotient ring of B and right QF-3.
- (3) If B is QF-3, B'' is also QF-3.

Proof. (1) Since the map ζ_l of $D \otimes_C A$ to $\text{Hom}({}_B A, {}_B A)$ is an isomorphism, the double centralizer of the left B -module ${}_B A$ is isomorphic to $\text{Hom}({}_D A_A, {}_D A_A) = V_A(V_A(B))$. Hence we have (1) by the above remark. (2) follows from (4.3) and (4.5) [16] and (1). (3) is also immediate by (4.3) [16] and (2). \square

Lastly we will apply the above results to serial or uniserial rings. A left artinian ring is a serial ring if and only if all of its factor rings are QF-3. Let A be an H -separable extension of B with the condition (a) and ψ a ring homomorphism of A onto the other ring \overline{A} , and $\overline{B} = \psi(B)$. Then by Proposition 3.2 [11] \overline{A} is also an H -separable extension of \overline{B} with the condition (a). Therefore by Theorem 2 and Proposition 1 we have

Theorem 7 Let A be an H -separable extension of B , and assume that the condition (a) or (b) is satisfied. If B is a serial ring, A is also a serial ring.

Proof. All what we need is to prove the next lemma. \square

Lemma 4 *Let A be an H -separable extension of B with the condition (b). Then for any ring homomorphism ψ of A onto \overline{A} , \overline{A} is an H -separable extension of $\overline{B} = \psi(B)$ with the condition (b).*

Proof. Let $\{f_i, z_i\}$ be a dual basis of ${}_B A$. Then there exists $\sum e_{ij} \otimes a_{ij}$ in $D \otimes_C A$ such that $f_i(x) = \sum e_{ij} x a_{ij}$ for each x in A , since ζ_l is an isomorphism. Write $\psi(x) = \overline{x}$ for each x in A . Since $\sum e_{ij} A a_{ij} = f_i(A) \subseteq B$, we see $\sum \overline{e}_{ij} \overline{A} \overline{a}_{ij} \subseteq \overline{B}$. Then since $\psi(D) \subseteq V_{\overline{A}}(\overline{B})$, if we define $\overline{f}_i(\overline{x}) = \sum \overline{e}_{ij} \overline{x} \overline{a}_{ij}$ for each x in A , \overline{f}_i is a left \overline{B} -homomorphism of \overline{A} to \overline{B} . And we have $\sum \overline{f}_i(\overline{x}) \overline{z}_i = \sum \overline{e}_{ij} \overline{x} \overline{a}_{ij} \overline{z}_i = \overline{\sum f_i(x) z_i} = \overline{x}$. Thus $\{\overline{f}_i, \overline{z}_i\}$ is a dual basis for ${}_{\overline{B}} \overline{A}$. \square

Uniserial ring is characterized as a ring all of whose factor rings are quasi-Frobenius (QF) rings. A ring is uniserial if and only if it is a finite direct product of matrix rings over local uniserial rings. On the other hand by Theorem 4.2 [11] an H -separable extension of a QF ring is also a QF ring. In this case B is a left (as well as right) B -direct summand of A , since each QF ring is left (and right) self injective. Therefore we have

Theorem 8 *Let B be a uniserial ring, and $B = B_1 \oplus B_2 \oplus \cdots \oplus B_n$ be the ring decomposition with each B_i a matrix ring over a local serial ring R_i . Let A be an H -separable extension of B . Then we have*

- (1) *A is a uniserial ring, and we have $B = V_A(V_A(B))$. A is finitely generated as a left (as well as right) B -module.*
- (2) *If $N \cap B_i \neq 0$ for each i where N is the Jacobson radical of A , A is left (as well as right) B -finitely generated projective.*
- (3) *If B is indecomposable as a ring, A is also indecomposable, and A is left (and right) B -finitely generated projective.*

Proof. (1) is clear by the above remark and Theorem 4.1 [11]. Each central idempotent of A is contained in the center of B , since $B = V_A(D) \supseteq C$. Therefore in order to prove (2) we may assume that A is indecomposable as a ring. Then A is a finite direct sum of mutually isomorphic indecomposable left ideals, and each left A -direct summand of A is faithful. Let $R = R_1 \oplus R_2 \oplus \cdots \oplus R_m$, and $J(R_i) = J_i$. Since each B_i is H -separable over R_i , B is H -separable over R . Hence A is H -separable over R . Moreover we have $(N \cap R_i) B_i = N \cap B_i \neq 0$, since B_i is a matrix ring over R_i . Thus we have $N \cap R_i = J_i^{r_i} \neq 0$, for some r_i . Each indecomposable left R -module is isomorphic to some R_i/J_i^l , and each left R -module is a direct sum of

indecomposable left R -modules. Therefore we can write $A = \oplus (R_i/j_i^k)^{n_{ik}}$. Suppose there exist some i, k such that $J_i^k \neq 0$, and let, for example, R_1/J_1^l is a member of the above indecomposable decomposition with $J_1^l \neq 0$. Then we have

$$\begin{array}{ccccccc}
 A \otimes_R J_1^l \oplus A \otimes_R R_2 \oplus \cdots \oplus A \otimes_R R_m & \rightarrow & A \otimes_R R & \rightarrow & A \otimes_R (R_1/J_1^l) & \rightarrow & 0 \quad (\text{exact}) \\
 \downarrow \text{epi.} & & \downarrow \text{iso.} & & \downarrow \text{iso.} & & \\
 0 \rightarrow AJ_1^l \oplus AR_2 \oplus \cdots \oplus AR_m & \rightarrow & A & \rightarrow & AR_1/AJ_1^l & \rightarrow & 0 \quad (\text{exact})
 \end{array}$$

But $A \otimes_R (R/J_1^l)$ is left A -projective, since it is an A -direct summand of $A \otimes_R A$ which is isomorphic to $\text{Hom}({}_C V_A(R), {}_C A)$ and $V_A(R)$ is C -projective. Therefore the above exact sequences split as A -map, and we can write $A = AJ_1^l \oplus AR_l/AJ_1^l \oplus I$ as left A -module. On the other hand since R is a left (as well as right) R -direct summand of A , we have $N = A(N \cap R) = (N \cap R)A = A(J_1^{r_1} \oplus J_2^{r_2} \oplus \cdots \oplus J_m^{r_m})$ with $J_i^{r_i} \neq 0$. If $r = r_1 \leq l$, $J_1^l \subseteq J_1^r$ and AJ_1^l is nilpotent. But it is impossible because AJ_1^l is generated by an idempotent. If $l < r$, we have

$$J_1^r AR_1 \subseteq (N \cap R)AR_1 = A(N \cap R)R_1 \subseteq AJ_1^r \subseteq AJ_1^l.$$

Thus AR_1/AJ_1^l is annihilated by J_1^r . But since it is an A -direct summand of A , it must be faithful, a contradiction. This means that each J_i^k in the above indecomposable decomposition must be 0, and A is left R -projective. Then since B is separable over R , A is B -projective. (3) is obvious by (2), and the fact that each central idempotent of A is contained in B . \square

References

- [1] Anderson F. and Fuller K., *Rings and Categories of Modules*. Springer G.T.M. 13.
- [2] DeMeyer F. and Ingraham, *Separable Algebras over Commutative Rings*. Springer Lecture Notes in Math. **181**, 1971.
- [3] Elliger S., *Über automorphismen und derivationen von ringen*. J. reine. angew. Math. **277** (1975), 155–177.
- [4] Hirata K. and Sugano K., *On semisimple extensions and separable extensions over non commutative rings*. J. Math. Soc. Japan, **18** (1966), 360–373.
- [5] Hirata K., *Some types of separable extensions*. Nagoya Math. J. **33** (1968), 107–115.
- [6] Hirata K., *Separable extensions and centralizers of rings*, Nagoya Math. J. **35** (1969), 31–45.
- [7] Hirata K. and Sugano K., *On semisimple extensions of serial rings*. Hokkaido Math. J. **29** (2000), 1–10.

- [8] Kadison L., *New examples of Frobenius extensions*. University Lecture Series, **14** A.M.S., 1999.
- [9] McMahon E. and Mewborn A., *Separable extensions of non commutative rings*. Hokkaido Math. J. **13** (1984), 74–88.
- [10] Sugano K., *Note on semisimple extensions and separable extensions*. Osaka J. Math. **4** (1967), 265–270.
- [11] Sugano K., *On projective H -separable extensions*. Hokkaido Math. J. **5** (1976), 44–54.
- [12] Sugano K., *Note on automorphisms in separable extension of non commutative rings*. Hokkaido Math. J. **9** (1980), 268–274.
- [13] Sugano K., *On H -separable extensions of two-sided simple rings*. Hokkaido Math. J. **11** (1982), 246–252.
- [14] Sugano K., *On H -separable extensions of primitive rings II*. Hokkaido Math. J. **19** (1990), 35–44.
- [15] Sugano K., *On strongly separable Frobenius extensions*. Hokkaido Math J. **25** (1996), 657–666.
- [16] Tachikawa H., *Quasi-Frobenius Rings and Generalizations*. Springer Lecture Notes in Math. **351**, 1973.
- [17] Yamashiro Y., *On strongly separable extensions*. Hokkaido Math. J. **19** (1990), 281–287.

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