Hokkaido Mathematical Journal Vol. 32 (2003) p. 209-276

Invariant CR structures on compact homogeneous manifolds

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(Received March 7, 2001; Revised February 4, 2002)

Abstract. An explicit classification of simply connected compact homogeneous CR manifolds G/L of codimension one, with non-degenerate Levi form, is given. There are three classes of such manifolds:

a) the standard CR homogeneous manifolds which are homogeneous S^1-bundles over a flag manifold F, with CR structure induced by an invariant complex structure on F;

b) the Morimoto-Nagano spaces, i.e. sphere bundles $S(N) \subset TN$ of a compact rank one symmetric space $N = G/H$, with the CR structure induced by the natural complex structure of $TN = G^{\mathbb{C}}/H^{\mathbb{C}}$;

c) the following manifolds: $SU_n/T^1 \cdot SU_{n-2}$, $SU_p \times SU_q/T^1 \cdot U_{p-2} \cdot U_{q-2}$, $SU_n/T^1 \cdot SU_2 \cdot SU_2 \cdot SU_{n-4}$, $SO_{10}/T^1 \cdot SO_6$, $E_6/T^1 \cdot SO_8$; these manifolds admit canonical holomorphic fibrations over a flag manifold (F, J_F) with typical fiber $S(S^k)$, where $k = 2, 3, 5, 7$ or 9, respectively; the CR structure is determined by the invariant complex structure J_F on F and by an invariant CR structure on the typical fiber, depending on one complex parameter.

Key words: homogeneous CR manifolds, real hypersurfaces, contact homogeneous manifolds.

1. Introduction

An *almost CR structure* on a manifold M is a pair $(\mathcal{D}, J)$, where $\mathcal{D} \subset TM$ is a distribution and J is a complex structure on $\mathcal{D}$. The complexification $\mathcal{D}^{\mathbb{C}}$ can be decomposed as $\mathcal{D}^{\mathbb{C}} = \mathcal{D}^{10} + \mathcal{D}^{01}$ into sum of complex eigendistributions of J, with eigenvalues i and $-i$.

An almost CR structure is called *integrable* or, shortly, *CR structure* if the distribution $\mathcal{D}^{01}$ (and hence also $\mathcal{D}^{10}$) is involutive, i.e. with space of sections closed under Lie bracket. This is equivalent to the following conditions:

$$[JX, Y] + [X, JY] \in \mathcal{D},$$

$$[JX, JY] - [X, Y] - J([JX, Y] + [X, JY]) = 0,$$

2000 Mathematics Subject Classification : Primary 32V05; Secondary 53C30, 53C15.

for any two fields X, Y in $\mathcal{D}$.

A map $\varphi : (M, \mathcal{D}, J) \to (M', \mathcal{D}', J')$ between two CR manifolds is called *holomorphic map* if $\varphi_*(\mathcal{D}) \subset \mathcal{D}'$ and $\varphi_*(JX) = J'\varphi_*(X)$.

Two CR structures $(\mathcal{D}, J)$ and $(\mathcal{D}', J')$ are called *equivalent* if there exists a diffeomorphism such that $\phi_*(\mathcal{D}) = \mathcal{D}'$ and $\phi_* J = J'$.

The codimension of $\mathcal{D}$ is called *codimension of the CR structure*. Note that a CR structure of codimension zero is the same as a complex structure.

A codimension one CR structure $(\mathcal{D}, J)$ on a $2n+1$-dimensional manifold M is called *Levi non-degenerate* if $\mathcal{D}$ is a contact distribution. This means that any local (contact) 1-form θ, which defines the distribution (i.e. such that $\ker\theta = \mathcal{D}$) is maximally non-degenerate, that is $(d\theta)^n \wedge \theta \neq 0$.

Note that any real hypersurface M of a complex manifold N has a natural codimension one CR structure $(\mathcal{D}, J_{\mathcal{D}})$ induced by the complex structure J of N, where

$$\mathcal{D} = \{X \in TM,\ JX \in TM\}, \qquad J_{\mathcal{D}} = J|_{\mathcal{D}}.$$

In the following, if the opposite is not stated, by *CR structure* we will mean *integrable codimension one Levi non-degenerate CR structure*. Sometimes, if the contact distribution $\mathcal{D}$ is given, we will identify a CR structure with the associated complex structure J.

A CR manifold, that is a manifold M with a CR structure $(\mathcal{D}, J)$, is called *homogeneous* if it admits a transitive Lie group of holomorphic transformations.

If the opposite is not stated, we will always assume that the homogeneous CR manifold $(M, \mathcal{D}, J)$ is simply connected.

The aim of this paper is to give a complete classification of simply connected homogeneous CR manifolds $M = G/L$ of a compact Lie group G. This gives a classification of all simply connected homogeneous CR manifolds, since any compact homogeneous CR manifold admits a compact transitive Lie group of holomorphic transformations (see [1] and [12]).

The simplest example of compact homogeneous CR manifold is the standard sphere $S^{2n-1} \subset \mathbb{C}^n$ with the induced CR structure.

More elaborated examples are provided by the following construction of A. Morimoto and T. Nagano ([9]). Let $N = G/H$ be a compact rank one symmetric space (shortly 'CROSS'). The tangent space TN can be identified with the homogeneous space $G^{\mathbb{C}}/H^{\mathbb{C}}$. Hence, it admits a natural

$G^{\mathbb{C}}$-invariant complex structure J. Any regular orbit $G \cdot p = S(N) \simeq G/L$ in $TN = G^{\mathbb{C}}/H^{\mathbb{C}}$ is a sphere bundle; in particular it is a real hypersurface with (Levi non-degenerate) G-invariant CR structure.

Moreover, these examples together with the standard sphere $S^{2n-1} \subset \mathbb{C}^n$ exhaust the class of CR structures induced on a codimension one orbit $M = G \cdot x \subset C$ of a compact Lie group G of holomorphic transformations of a Stein manifold C. We call the homogeneous CR manifolds which are equivalent to such orbits $G \cdot p = S(N)$ in the tangent space of a CROSS *Morimoto-Nagano spaces*.

Another important class of examples is obtained as follows. Let $F = G/K$ be a flag manifold of a connected, compact, semisimple Lie group G and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$, with $\mathfrak{k} = \mathrm{Lie}(K)$, be the associated orthogonal decomposition of $\mathfrak{g} = \mathrm{Lie}(G)$, w.r.t. the Cartan-Killing form. Let also J_F be a G-invariant complex structure on $F = G/K$. It can be shown that if Z is an element in the center of $\mathfrak{k} = \mathrm{Lie}(K)$ and if it satisfies some suitable hypothesis (i.e. it is a $\mathfrak{k}$-regular element; see definition in §3.1), then the subalgebra $\mathfrak{l}_Z = \mathfrak{k} \cap (Z)^{\perp}$ generates a closed subgroup $L_Z \subset G$, and the homogeneous manifold $M = G/L_Z$ admits a G-invariant CR structure $(\mathcal{D}, J)$ with the following two properties:

i) $\mathcal{D}$ is the unique G-invariant distribution corresponding to the subspace $\mathfrak{m} = (\mathfrak{k})^{\perp} \subset (\mathfrak{l}_Z)^{\perp} \simeq T_oG/L_Z$, $o = eL_Z$;
ii) $(\mathcal{D}, J)$ is the unique CR structure with distribution $\mathcal{D}$, so that the natural projection $\pi : (M = G/L_Z, \mathcal{D}, J) \to (F = G/K, J_F)$ is holomorphic.

We call any such homogenous CR manifold $(M = G/L_Z, \mathcal{D}, J)$ a *standard CR manifold*, determined by the flag manifold $F = G/K$, the $\mathfrak{k}$-regular element Z and the invariant complex structure J_F on F.

In the fundamental paper [1], H. Azad, A. Huckleberry and W. Richthofer showed that Morimoto-Nagano spaces and standard CR manifolds play a basic role in the description of compact homogeneous CR manifolds (see also [8] and [11]). In fact, they prove that any compact compact homogeneous CR manifold is either a standard CR manifold or a Morimoto-Nagano space or it admits a natural holomorphic fibration onto a flag manifold whose standard fiber is a Morimoto-Nagano space.

In this paper, we carry out the explicit classification of all compact homogeneous manifold G/L of a compact Lie group G, which admit an invariant CR structure, and we determine all invariant CR structures on each of such spaces. As a result of this classification, we are able to determine the exact list of all compact homogeneous CR manifolds, which are neither standard nor Morimoto-Nagano spaces. In particular, we prove that the only manifolds, which may occur as fibers of the Azad-Huckleberry-Richthofer's holomorphic fibration of non-standard, non-Morimoto-Nagano spaces, are the sphere bundles $S(S^k)$ with $k = 2, 3, 5, 7$ and 9.

The first step consists in characterizing all homogeneous manifolds $M = G/L$ of a compact Lie groups G, which admit a G-invariant contact structure (see §3.1). We first prove that, for any such manifold $M = G/L$, the center $Z(G)$ of G is at most one dimensional and that the semisimple part G^{ss} acts transitively on M. By this fact, we may always assume that G is semisimple. Secondly, we observe that, for a given compact semisimple Lie group G, there is a one to one correspondence between simply connected homogeneous contact manifolds $(M = G/L, \mathcal{D})$ and non-zero elements $Z \in \mathfrak{g} = \mathrm{Lie}(G)$ (defined up to scaling), which generate a closed one-parametric subgroup and whose centralizer $C_{\mathfrak{g}}(Z)$ contains no non-trivial ideal of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$.

In fact, any such element Z determines: a) an orthogonal decomposition (w.r.t. the Cartan-Killing form)

$$\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m},$$

where $C_{\mathfrak{g}}(Z) = \mathfrak{l} \oplus \mathbb{R}Z$; b) an associated homogeneous manifold $M = G/L$, with L connected closed subgroup generated by $\mathfrak{l}$, endowed with the invariant contact structure $\mathcal{D}$, which is the G-invariant extension of the Ad_L-invariant subspace $\mathfrak{m} \subset (\mathbb{R}Z + \mathfrak{m}) \simeq T_oM$, $o = eL$. The element Z is called *contact element* of the homogeneous contact manifold $(M = G/L, \mathcal{D})$.

We also prove that if $(M = G/L, \mathcal{D})$ is the contact homogeneous manifold associated with the contact element Z, then there exists a natural principal S^1-fibration onto the flag manifold $F_Z = \mathrm{Ad}_G(Z) = G/C_G(Z)$

$$\pi : M = G/L \longrightarrow F_Z = G/C_G(Z),$$

where $C_G(Z)$ denotes the centralizer of Z in G.

We then determine all distinct invariant contact distributions, which exists on a given homogeneous contact manifold $(G/L, \mathcal{D})$. It turns out that only two possibilities may occur:

α) there exists exactly one invariant contact structure;

β) there exists a family of invariant contact structures, which is parameterized by the point of a two dimensional sphere S^2.

The contact manifolds of case (β) are called *special contact manifolds*. Examples of special manifolds can be constructed as follows. Let G be a simple compact Lie group without center and let $Q = G/Sp_1 \cdot H'$ be the associated Wolf space, that is the homogeneous quaternionic Kähler manifold, where $Sp_1 \cdot H'$ is the normalizer in G of the 3-dimensional subalgebra $\mathfrak{sp}_1(\mu)$ of $\mathfrak{g}$ associated with the maximal root μ. Then the associated 3-Sasakian homogeneous manifold $M = G/H'$ is a special contact manifold. Any $0 \neq Z \in \mathfrak{sp}_1(\mu)$ is a contact element. Furthermore, any two invariant contact structures on M are equivalent under a transformation, which commutes with G, defined by the right action of an element from Sp_1.

We prove that any special contact manifold, with only one exception, is a manifold constructed in this way. In fact (see Theorem 3.6)

Theorem 1.1 *Any special contact manifold $M = G/L$ is either the 3-Sasakian homogeneous manifold G/H' of a simple Lie group G, described above, or $M = G_2/Sp_1$, where Sp_1 is the 3-dimensional subgroup of the exceptional Lie group G_2, with Lie algebra $\mathfrak{sp}_1(\mu)$, where μ is the maximal root of G_2.*

The second step of the classification consists in determining which homogeneous contact manifold $(M = G/L, \mathcal{D})$, with G compact semisimple, admits at least one G-invariant CR structure $(\mathcal{D}, J)$. The answer is simple: *any homogeneous contact manifold of the above kind admits al least one G-invariant CR structure.* In fact, if $\mathcal{D}$ is associated with the contact element Z and F is the corresponding flag manifold $F = \mathrm{Ad}_G(Z) = G/K$, with $K = G_G(Z)$, then it can be checked that Z is a $\mathfrak{k}$-regular element and $\mathfrak{l} = (Z)^{\perp} \cap \mathfrak{k}$. So, for any invariant complex structure J_F on $F = \mathrm{Ad}_G(Z) = G/K$, the manifold $M = G/L$ admits the G-invariant standard CR structure $(\mathcal{D}, J_F)$ associated with Z and J_F.

The last (and longest) step consists in classifying all invariant CR structures for any homogeneous contact manifold $(M = G/L, \mathcal{D})$. For this part of the classification, we need to introduce the concepts of *primitive* and *non-primitive* CR structures (see §4.3). A compact homogeneous CR manifold $(M = G/L, \mathcal{D}, J)$ is called *non-primitive* if there exists a holomorphic

G-equivariant fibration (called *CRF fibration*)

$$\pi : M = G/L \longrightarrow F = G/Q,$$

onto a flag manifold $F = G/Q$, of positive dimension, equipped with an invariant complex structure J_F. We call M *primitive* if it does not admit any CRF fibration.

A fiber of a CRF fibration $\pi : M = G/L \to F = G/Q$ is always either S^1 or a homogeneous compact CR manifold Q/L.

Examples of non-primitive CR manifolds are given by the standard CR manifolds, since, by construction, they always admit a CRF fibration with fiber S^1. Examples of primitive CR manifolds are given by any Morimoto-Nagano space which is not a sphere S^{2n-1} (which is standard CR manifold) nor a 3-dimensional SU_2-orbit in $TS^2 = T(SU_2/T^1)$ (which is a non-standard CR manifold that admits a CRF fibration onto $S^2 = SU_2(T^1)$ with fiber S^1) (see §8).

In §5, we classify all invariant CR structures on the special contact manifolds, we determine which of those CR structures are primitive and, for those which are non-primitive, we exhibit a natural CRF fibration with primitive fibers.

After the result of §5, we have to discuss the non-special contact manifolds. At this regard, we observe that on a non-special contact manifold $(M = G/L, \mathcal{D})$, the invariant distribution $\mathcal{D}$ is uniquely determined and that the standard CR structures on $M = G/L$ are in one-to-one correspondence with the invariant complex structures J_F on the flag manifold $F = \mathrm{Ad}_G(Z)$ associated with the contact element Z of the distribution $\mathcal{D}$. Since the description of invariant complex structures on a flag manifold is known (see e.g. [3], [4], [5], [10]), it remains to classify the non-special contact manifolds which admit non-standard CR structures, together with all their admissible invariant CR structures.

A characterization on non-standard CR structures can be obtained by means of the *anticanonical map* ϕ, which was defined in [1] (see §4.4). Let $(M = G/L, \mathcal{D}_Z, J)$ be a homogeneous CR manifold of a compact Lie group G and $\mathfrak{g}^{\mathbb{C}} = \mathfrak{l}^{\mathbb{C}} + \mathbb{C}Z + \mathfrak{m}^{10} + \mathfrak{m}^{01}$ the corresponding decomposition of $\mathfrak{g}^{\mathbb{C}}$. Then the *anticanonical map* ϕ is the holomorphic map of M into the Grassmanian of k-planes, $k = \dim_{\mathbb{C}}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01})$, given by

$$\phi : M = G/L \longrightarrow Gr_k(\mathfrak{g}^{\mathbb{C}}) \subset \mathbb{C}P^N, \quad \phi : gL \mapsto \mathrm{Ad}_g([\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}]).$$

Note that ϕ is a G-equivariant map onto the orbit $G \cdot p$ of $p = [\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}] \in Gr_k(\mathfrak{g}^{\mathbb{C}})$ under the natural adjoint action of G on $Gr_k(\mathfrak{g}^{\mathbb{C}})$.

The following theorem gives the required characterization (see Theorem 6.3):

Theorem 1.2 *Let $(M = G/L, \mathcal{D}_Z, J)$ be a homogeneous CR manifold.*

(1) *If it is standard, then the image $\phi(M) = G \cdot p$ of the anticanonical map is the flag manifold $F_Z = G/K$, associated with the contact structure $\mathcal{D}_Z$. Hence $\phi : M \to \phi(M) = F_Z$ is the natural S^1-fibration.*

(2) *If it is non-standard, then $\phi : M \to \phi(M) = G \cdot p$ is a finite holomorphic covering, with respect to the CR structure of $G \cdot p \subset Gr_k(\mathfrak{g}^{\mathbb{C}})$ induced by the complex structure of $Gr_k(\mathfrak{g}^{\mathbb{C}})$.*

Using Theorem 1.2 and several algebraic lemmata, we reach the classification of all non-standard CR structures on non-special contact manifolds in §7 (see Propositions 7.3, 7.5 and 7.6). In particular we determine the list of all non-special contact manifolds $(M = G/L, \mathcal{D})$ admitting a non-standard CR structure, together with the explicit description of all invariant non-standard CR structures on such manifolds. We also determine which of them is not primitive and, for any non-primitive CR manifold, we indicate a natural CRF fibration with primitive fibers.

In §8, we give the complete lists of primitive CR manifolds and of non-primitive CR manifolds, simply combining the previous results on special and non-special contact manifolds. From such lists, it follows that the Morimoto-Nagano spaces which are different from the standard spheres S^{2n-1} and from the 3-dimensional orbits of SU_2 in $T(S^2) = T(SU_2/T^1)$ are exactly all primitive CR manifolds.

In §8, we also give a precise description of any non-standard, non-primitive CR structure on a homogeneous manifold $M = G/L$ in terms of some suitable painted Dynkin graph, that is of a Dynkin graph of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ with nodes painted in three colors.

We have to mention that a few important steps towards a classification of non-standard CR manifolds were performed also by Azad, Huckleberry and Richthofer in [1]. There, the authors study in great detail the image of a non-standard CR manifold $M = G/L$ under the anti-canonical map. We recall that such image is a G-orbit $\hat{M} = G/\hat{L} = G \cdot p \subset \mathbb{C}P^N$ in a complex projective space $\mathbb{C}P^N$ and it is finitely covered by $M = G/L$. Notice also that the G-orbit $\hat{M} = G/\hat{L} = G \cdot p \subset \mathbb{C}P^N$ is a real hypersurface in the complex orbit $\Omega = G^{\mathbb{C}} \cdot p = G^C/H$. Azad, Huckleberry and Richthofer prove that there always exists a natural $G^{\mathbb{C}}$-equivariant fibration $\pi : \Omega = G^{\mathbb{C}}/H \to G^{\mathbb{C}}/P$ onto a flag manifold $G^{\mathbb{C}}/P$ (possibly of dimension 0) and they call it *Stein-Rational fibration*. We may observe that a Stein-Rational fibration $\pi : \Omega = G^{\mathbb{C}}/H \to G^{\mathbb{C}}/P$ onto a flag manifold of positive dimension induces always a CRF fibration $\pi : G/\hat{L} \to G^{\mathbb{C}}/P = G/G \cap P$ on the real hypersurface $G/\hat{L} = G \cdot p$. Studying the possibilities for the fiber of a Stein-Rational fibration, Azad, Huckleberry and Richthofer find several limitations on the parabolic subgroup P and on the isotropy subgroup H. It is possible to use such limitations to determine a few corresponding conditions that are satisfied by the subgroups Q and L of a given compact semisimple Lie group G, when there exists an invariant non-standard CR structure on G/L and a CRF fibration $\pi : G/L \to G/Q$. But none of such conditions is sufficient to determine if a given homogeneous manifold G/L does actually admit an invariant, non-standard CR structure. For this reason, it is not possible to infer the exact list of non-standard CR manifolds using only those conditions.

On the other hand, most of the arguments in [1] do not make any use of the condition of Levi non-degeneracy and several results of that paper give very useful information on the structure of compact homogeneous Levi degenerate CR manifolds of a very wide class (see also [11]).

As a final remark, we would like to mention that our classification of compact homogeneous CR manifolds have several important corollaries concerning compact cohomogeneity one Kähler manifolds. In particular, such corollaries are an essential tool towards the classification of Kähler-Einstein manifolds in the above class. They will be discussed in a forthcoming paper.

We heartily thank the referee for his very valuable comments.

2. Basic facts about CR structures

Definition 2.1

(1) A CR structure on a manifold M is a pair $(\mathcal{D}, J)$, where $\mathcal{D} \subset TM$ is a distribution on M and $J \in \operatorname{End} \mathcal{D}$, $J^2 = -1$, is a complex structure on $\mathcal{D}$.

(2) A CR structure $(\mathcal{D}, J)$ is called *integrable* if J satisfies the following integrability condition:

$$[JX, Y] + [X, JY] \in \mathcal{D},$$
$$[JX, JY] - [X, Y] - J([JX, Y] + [X, JY]) = 0 \tag{2.1}$$

for any pair of vector fields X, Y in $\mathcal{D}$.

In the sequel, by CR manifold we will understand a manifold M with *integrable* CR structure.

If $(\mathcal{D}, J)$ is a CR structure then the complexification $\mathcal{D}^{\mathbb{C}} \subset T^{\mathbb{C}}M$ of the distribution $\mathcal{D}$ is decomposed into a sum $\mathcal{D}^{\mathbb{C}} = \mathcal{D}^{10} + \mathcal{D}^{01}$ of two mutually conjugated ($\mathcal{D}^{10} = \bar{\mathcal{D}}^{01}$) J-eigendistributions with eigenvalues i and $-i$. The integrability condition (2.1) means that these eigendistributions are involutive (i.e. closed under the Lie bracket).

The *codimension* of a CR structure $(\mathcal{D}, J)$ is defined as the codimension of the distribution $\mathcal{D}$. Remark that a codimension zero CR structure is the same as a complex structure on a manifold. A codimension one CR structure $(\mathcal{D}, J)$ is also called a CR structure of *hypersurface type*, because such is the structure which is induced on a real hypersurface of a complex manifold. In this case the distribution $\mathcal{D}$ can be described locally as the kernel of a 1-form θ. The form θ defines a J-invariant symmetric bilinear form

$$\mathcal{L}_q^\theta : \mathcal{D}_q \times \mathcal{D}_q \to \mathbb{R}$$

given by

$$\mathcal{L}^\theta(v, w) = (d\theta)(v, Jw)$$

for any $v, w \in \mathcal{D}$. It is the real part of a $\mathbb{C}$-valued Hermitian form and it is called the *Levi form*. Remark that the 1-form θ is defined up to multiplication by a function f everywhere different from zero and that $\mathcal{L}^{f\theta} = f\mathcal{L}^\theta$. In particular, the conformal class of a Levi form depends only on the CR structure.

A CR structure $(\mathcal{D}, J)$ of hypersurface type is called *non-degenerate* if it has non-degenerate Levi form or, in other words, if $\mathcal{D}$ is a contact distribution. In this case a 1-form θ with $\ker \theta = \mathcal{D}$ is called *contact form.*

A smooth map $\varphi : M \to M'$ of one CR manifold $(M, \mathcal{D}, J)$ into another one $(M', \mathcal{D}', J')$ is called *holomorphic map* if

a) $\varphi_*(\mathcal{D}) \subset \mathcal{D}'$;

b) $\varphi_*(Jv) = J'\varphi_*(v)$ for all $v \in \mathcal{D}$.

In particular, we may speak about CR *transformation* of a CR manifold $(M, \mathcal{D}, J)$ as a transformation φ such that φ and φ^{-1} are CR maps. In general, the group of all CR transformations is not a Lie group, but it is a Lie group when $(\mathcal{D}, J)$ is of hypersurface type and it is Levi non-degenerate.

Definition 2.2 A CR manifold $(M, \mathcal{D}, J)$ is called *homogeneous* if it admits a transitive Lie group G of CR transformations.

Our aim is to classify compact homogeneous codimension one non-degenerate CR manifolds. The following theorem, which is indeed a consequence of the results in [1], shows that we may identify any such manifold with a quotient space G/L of a compact Lie group G.

Theorem 2.3 ([1], [12]) *Let $(M, \mathcal{D}, J)$ be a compact non-degenerate CR manifold of hypersurface type. Assume that it is homogeneous, i.e. that there exists a transitive Lie group A of CR transformations. Then a maximal compact connected subgroup G of A acts on M transitively and one may identify M with the quotient space G/L where L is the stabilizer of a point $p \in M$.*

Now we fix some notations. If the opposite is not stated, we will assume that a CR structure is of hypersurface type, integrable and Levi non-degenerate.

The Lie algebra of a Lie group is denoted by the corresponding gothic letter.

For any subset A of a Lie group G or of its Lie algebra $\mathfrak{g}$, we denote by $C_G(A)$ and $C_{\mathfrak{g}}(A)$ its centralizer in G and $\mathfrak{g}$, respectively. $Z(G)$ and $Z(\mathfrak{g})$ denote the center of a Lie group G and a Lie algebra $\mathfrak{g}$. By homogeneous manifold $M = G/L$ we mean a homogeneous manifold of a compact connected Lie group G with *connected stability subgroup* L and such that the action of G on M is *almost effective*, i.e. has a finite kernel of non-effectivity.

3. Compact homogeneous contact manifold

3.1. Homogeneous contact manifolds of a compact Lie group G

Let $M = G/L$ be a homogeneous manifold of a compact Lie group G with connected stabilizer L.

Any G-invariant contact distributions $\mathcal{D}$ on M is uniquely associated with a global G-invariant contact form θ on M (determined up to a multiple) such that $\mathcal{D} = \ker\theta$. On the other hand, any G-invariant contact form θ on M is uniquely associated with an element $\theta \in \mathfrak{g}^*$ of the dual space of $\mathfrak{g} = \mathrm{Lie}(G)$, which satisfies the following four conditions (see e.g. [2]): i) it vanishes on $\mathfrak{l} = \mathrm{Lie}\, L$; ii) it is $\mathrm{Ad}_{\mathfrak{l}}$-invariant; iii) $\mathrm{Ker}\, d\theta \not\subset \mathfrak{l}$; iv) $\mathrm{Ker}\,\theta \cap \mathrm{Ker}\, d\theta \subset \mathfrak{l}$. Any $\theta \in \mathfrak{g}^*$ which satisfies conditions i)–iv) will be called *contact form of* G/L.

Fix now an Ad_G-invariant Euclidean metric $\mathcal{B}$ on $\mathfrak{g}$ and denote by $\mathfrak{l}^\perp$ the orthogonal complement to $\mathfrak{l}$ in $\mathfrak{g}$.

The vector $Z = \mathcal{B}^{-1} \circ \theta$ which corresponds to a contact form θ is called a *contact element* of the manifold $(M = G/L, \mathcal{D})$.

From the fact that θ is a contact form, it follows that $Z = \mathcal{B}^{-1} \circ \theta$ is characterized by the properties that:

(1) $Z \in \mathfrak{l}^\perp$ and

(2) the centralizer $C_{\mathfrak{g}}(Z) = \mathfrak{l} \oplus \mathbb{R}Z$.

Hence, we have the following

Proposition 3.1 *There exists a natural bijection between invariant contact structures on a homogeneous manifold $M = G/L$ and contact elements Z defined up to a scaling.*

We will denote by $\mathcal{D}_Z$ the contact structure on M defined by a contact element Z. A homogeneous manifold $M = G/L$ with an invariant contact structure $\mathcal{D}$ is called *homogeneous contact manifold.*

Proposition 3.1 implies the following

Corollary 3.2 *Let G/L be a homogeneous contact manifold of a compact Lie group G which acts effectively. Then the the center $Z(G)$ of G has dimension* 0 *or* 1.

Moreover, if $Z(G)$ is one dimensional, then any contact element Z has nonzero orthogonal projections $Z_{Z(\mathfrak{g})}$, $Z_{\mathfrak{g}'}$ on $Z(\mathfrak{g})$ and $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$, and the stability subalgebra $\mathfrak{l}$ can be written as

$$\mathfrak{l} = [C_{\mathfrak{g}'}(Z_{\mathfrak{g}'})]_\varphi \stackrel{\text{def}}{=} \{X = Y + \varphi(Y),\ Y \in C_{\mathfrak{g}'}(Z_{\mathfrak{g}'})\}$$

where $\varphi : C_{\mathfrak{g}'}(Z_{\mathfrak{g}'}) \to Z(\mathfrak{g}) \approx \mathbb{R}$ is a non-trivial Lie algebra homomorphism.

Proof. Clearly $C_{\mathfrak{g}}(Z) \supset Z(\mathfrak{g})$. If $\dim Z(\mathfrak{g}) \geq 2$ then $\mathfrak{l} \cap Z(\mathfrak{g}) \neq \{0\}$ and this contradicts the fact that G acts effectively. The other claims follow immediately. $\square$

Now we associate with a homogeneous contact manifold $(M = G/L, D_Z)$ a flag manifold

$$F_Z = G/K \stackrel{\text{def}}{=} \mathrm{Ad}_G Z = \mathrm{Ad}_{G'}(Z_{\mathfrak{g}'})$$

where $K = C_G(Z)$ is the centralizer of the contact element Z. We will call F_Z *the flag manifold associated to a contact element* Z.

Note that the contact form $\theta = \mathcal{B} \circ Z$ is a connection (form) in the S^1 bundle $\pi : G/L \to F_Z$ and that the corresponding contact structure $\mathcal{D} = \ker\theta$ is the horizontal distribution of this connection.

We describe now all homogeneous contact manifolds $(G/L, \mathcal{D}_Z)$ with given associated flag manifold $F = G/K$ of a semisimple Lie group G.

Consider the orthogonal reductive decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$$

associated with the flag manifold $F = G/K$.

We say that an element Z of the center $Z(\mathfrak{k})$ is $\mathfrak{k}$*-regular* if it generates a closed 1-parametric subgroup of G and the centralizer $C_G(Z) = K$.

One can check that if Z is $\mathfrak{k}$-regular, then the subalgebra

$$\mathfrak{l}_Z = \mathfrak{k} \cap (Z)^{\perp}$$

generates a closed subgroup, which we denote by L_Z. Therefore

Proposition 3.3 *Let $F = G/K$ be a flag manifold of a compact, semisimple Lie group G. There is a natural* 1-1 *correspondence*

$$Z \Longleftrightarrow (G/L_Z, \mathcal{D}_Z)$$

between the $\mathfrak{k}$-regular elements $Z \in \mathfrak{g}$ (determined up to a scaling) and the homogeneous contact manifolds $(G/L, D)$ with associated flag manifold $F = G/K$.

Proof. The proof is straightforward. □

3.2. Invariant contact structures on a contact manifold $M = G/L$

Now we describe all invariant contact structures on a given homogeneous manifold $M = G/L$. We will show that generically there is no more then one such structure.

Definition 3.4 A homogeneous manifold G/L is called homogeneous contact manifold of *non-special type* (respectively, of *special type* or, shortly, *special*) if it admits a unique (respectively, more then one) invariant contact structure.

3.2.1. Main examples of special homogeneous contact manifolds

Let $\mathfrak{g}$ be a compact semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and R the root system of the pair $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$.

Recall that a root $\alpha \in R$ defines a 3-dimensional regular subalgebra $\mathfrak{g}^{\mathbb{C}}(\alpha)$ with standard basis given by the root vectors E_α, $E_{-\alpha}$ and

$$H_\alpha = [E_\alpha, E_{-\alpha}] = \frac{2}{|\alpha|^2}\mathcal{B}^{-1} \circ \alpha \tag{3.1}$$

satisfying the relation $[H_\alpha, E_{\pm\alpha}] = \pm 2E_{\pm\alpha}$. The intersection of this subalgebra with $\mathfrak{g}$ is a 3-dimensional compact subalgebra denoted by $\mathfrak{g}(\alpha)$. We will call $\mathfrak{g}(\alpha)$ the *subalgebra associated with the root* α and denote by $G(\alpha)$ the 3-dimensional subgroup of the adjoint group $G = \mathrm{Int}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g})^0$ generated by $\mathfrak{g}(\alpha)$.

Note that two such subalgebras are conjugated by an inner automorphism of $\mathfrak{g}$ if and only if the corresponding roots have the same length.

Fix a system R^+ of positive roots of R and put $R^- = -R^+$. The highest root μ of R^+ defines the following gradation of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2, \tag{3.2}$$

where

$$\begin{aligned} &\mathfrak{g}_{-2} = \mathbb{C}E_{-\mu} \quad \mathfrak{g}_2 = \mathbb{C}E_\mu \quad \mathfrak{g}_0 = \mathbb{C}H_\mu + \mathfrak{g}_0' \quad \mathfrak{g}_0' = C_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{g}(\mu)) \\ &\mathfrak{g}_{-1} = \sum_{\beta \in R^- \setminus (\{-\mu\} \cup R_o)} \mathbb{C}E_\beta \qquad \mathfrak{g}_1 = \sum_{\beta \in R^+ \setminus (\{\mu\} \cup R_o)} \mathbb{C}E_\beta \end{aligned} \tag{3.3}$$

and $R_o = \{\alpha \in R,\ \alpha \perp \mu\}$ is the root system of the subalgebra $\mathfrak{g}_0 = C_{\mathfrak{g}}(H_\mu)$.

(3.2) is called the *gradation associated with the highest root*.

The explicit decomposition (3.2) for any simple complex Lie algebra is given in Table 1 of the Appendix.

Denote by $\mathfrak{l} = C_{\mathfrak{g}}(\mathfrak{g}(\mu)) = \mathfrak{g}_0' \cap \mathfrak{g}$ the centralizer of $\mathfrak{g}(\mu)$ in $\mathfrak{g}$ and by L the corresponding connected subgroup of G. It is easy to check that $L = C_G(\mathfrak{g}(\mu))$.

Lemma 3.5 *Let G be a compact simple Lie group without center and let $L = C_G(\mathfrak{g}(\mu))$ be as defined above. Then any non zero vector $Z \in \mathfrak{g}(\mu)$ is a contact element of the manifold G/L. In particular, G/L is a homogeneous contact manifold of special type.*

Proof. Observe that $Z \in \mathfrak{g}(\mu)$ is a contact element if and only if $C_{\mathfrak{g}}(Z) = \mathfrak{l}+\mathbb{R}Z$ and then $g \cdot Z$ is contact for any $g \in G(\mu)$. Since $G(\mu)$ acts transitively on the unit sphere of $\mathfrak{g}(\mu)$, the Lemma follows from the fact that

$$C_{\mathfrak{g}}(iH_\mu) = \mathfrak{g}_0 \cap \mathfrak{g} = \mathfrak{l} + \mathbb{R}(iH_\mu)$$

and hence that iH_μ is a contact element. □

Remark that the contact manifolds $M = G/L = G/C_G(\mathfrak{g}(\mu))$, with G simple, carry invariant 3-Sasakian structure and they exhaust all homogeneous 3-Sasakian manifolds (see [6]).

3.2.2. Classification of special homogeneous contact manifolds

The previous examples almost exhaust the class of special homogeneous contact manifolds. In fact, we have the following classification theorem.

Theorem 3.6 *Let $M = G/L$ be a special homogeneous contact manifold of a compact Lie group G. Then the group G is simple and either L is the centralizer of the subalgebra $\mathfrak{g}(\mu)$ associated with the highest root and M is a homogeneous 3-Sasakian manifold or $G = G_2$ and L is the centralizer of the subalgebra $\mathfrak{g}(\nu)$ associated with a short root ν.*

Proof. We prove first that if G is not semisimple and, hence, $\dim Z(\mathfrak{g}) = 1$, then a contact element Z is unique up to a scaling and M is not special. Indeed, we have the decomposition

$$\mathfrak{k} = C_{\mathfrak{g}}(Z) = \mathfrak{l} \oplus \mathbb{R}Z = \mathfrak{l} \oplus Z(\mathfrak{g})$$

since $Z(\mathfrak{g}) \cap \mathfrak{l} = 0$, by effectivity. The line $\mathbb{R}Z$ is determined uniquely as the orthogonal complement to $\mathfrak{l}$ in $\mathfrak{k} = \mathfrak{l} + Z(\mathfrak{g})$.

Now we may assume that $\mathfrak{g}$ is semisimple. We need the following:

Lemma 3.7 *Let $\mathfrak{g}$ be compact semisimple and let $\mathfrak{l} \subset \mathfrak{g}$ be a subalgebra, which contains no ideal of $\mathfrak{g}$. If there exist two not proportional vectors Z, $Z' \in \mathfrak{l}^\perp$ such that*

$$C_{\mathfrak{g}}(Z) = \mathfrak{l} + \mathbb{R}Z, \quad \mathfrak{l} + \mathbb{R}Z' \subseteq C_{\mathfrak{g}}(Z'),$$

then $\mathfrak{g}$ is simple and there exists a root $\alpha \in R$ such that:

(1) $\mathfrak{l} = C_{\mathfrak{g}}(\mathfrak{g}(\alpha))$;

(2) Z, $Z' \in \mathfrak{g}(\alpha)$ *and* $C_{\mathfrak{g}}(Z') = C_{\mathfrak{g}}(\mathfrak{g}(\alpha)) + \mathbb{R}Z'$;

(3) $C_{\mathfrak{g}}(\mathfrak{l}) = Z(\mathfrak{l}) + \mathfrak{g}(\alpha)$;

(4) *for any root β which is orthogonal to α, $\alpha \pm \beta$ is not a root.*

Proof. We put $\mathfrak{k} = C_{\mathfrak{g}}(Z)$ and consider the orthogonal decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m} = (\mathfrak{l} + \mathbb{R}Z) + \mathfrak{m}.$$

Denote by R the root system of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ with respect to a Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ which is the complexification of a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{k}$. Then the element Z' can be written as

$$Z' = cZ + \sum_{i=1}^{k} c_i E_{\alpha_i}$$

for some root vectors E_{α_i} and constants c, c_i. The condition $[\mathfrak{l}, Z'] = 0$ implies $\alpha_i(\mathfrak{h} \cap \mathfrak{l}) = 0$ if $c_i \neq 0$. Since $\mathfrak{h} \cap \mathfrak{l}$ is of codimension one in $\mathfrak{h}$, there exist exactly two (proportional) roots with this property, say α and $-\alpha$. This shows that $\mathfrak{l} \subset C_{\mathfrak{g}}(\mathfrak{g}(\alpha))$. Moreover, since $Z \in \mathfrak{h} \cap \mathfrak{l}^{\perp}$, we obtain also that Z is proportional to $H_\alpha = [E_\alpha, E_{-\alpha}]$ and (1) follows. In particular, $\mathfrak{g}$ must be simple and now (2) is clear. (3) follows from (2).

To prove (4), assume that there is a root β which is orthogonal to α and such that $\alpha + \beta$ is a root. Then the vector $E_\beta + E_{-\beta} \in \mathfrak{g}^{\mathbb{C}}$ does not belong to $\mathfrak{l}^{\mathbb{C}} = C_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{g}(\alpha))$, but it is orthogonal to Z (since Z is proportional to H_α) and belongs to the centralizer of Z: contradiction. □

Now we conclude the proof of Theorem 3.6. Let G be a compact semisimple Lie group and let Z, Z' two non-proportional contact elements for G/L. By Lemma 3.7, G is simple and $L = C_G(\mathfrak{g}(\alpha))$. By direct inspection of the root systems of simple Lie groups, a root α satisfies the condition (4) of Lemma 3.7 if and only if it is a long root or if it is a short root in the G_2 type system. This concludes the proof. □

3.3. Isotropy representation of a homogeneous contact manifold

Let $M = G/L$ be a homogeneous contact manifold with invariant contact structure $\mathcal{D}$ associated to a contact element Z. Let $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m}$ be the corresponding orthogonal decomposition. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ which belongs to $\mathfrak{k} = \mathfrak{l} + \mathbb{R}Z = Z(\mathfrak{k}) + \mathfrak{k}'$ (where $\mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}]$ is the semisimple part of $\mathfrak{k}$). Then

$$\mathfrak{h} = Z(\mathfrak{k}) + \mathfrak{h}' = Z(\mathfrak{l}) + \mathbb{R}Z + \mathfrak{h}',$$

where we denote by $\mathfrak{h}'$ a Cartan subalgebra of $\mathfrak{k}'$. Remark that $\mathfrak{h}(\mathfrak{l}) = Z(\mathfrak{l}) + \mathfrak{h}'$ is a Cartan subalgebra of $\mathfrak{l}$.

Denote by R (resp. R_o) the root system of $\mathfrak{g}^{\mathbb{C}}$ (resp. $\mathfrak{k}^{\mathbb{C}}$) w.r.t. the Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ and let $R' = R \setminus R_o$. We will denote by $\mathfrak{h}(\mathbb{R})$ the standard real form of $\mathfrak{h}$, spanned by R, that is

$$\mathfrak{h}(\mathbb{R}) = \mathfrak{h} \cap \mathcal{B}^{-1}(\langle R \rangle).$$

We put $\mathfrak{t} = \mathfrak{z}(\mathfrak{k}) \cap \mathfrak{h}(\mathbb{R})$. Then $Z \in i\mathfrak{t}$ and we may identify

$$\vartheta = -i\theta = -i\mathcal{B}(Z, \cdot)$$

with the corresponding element in $\mathfrak{t}^* \subset \mathfrak{h}(\mathbb{R})^* = \mathrm{span}_{\mathbb{R}} R$.

Consider the decomposition of the $\mathfrak{k}^{\mathbb{C}}$-module $\mathfrak{m}^{\mathbb{C}}$ into sum of irreducible $\mathfrak{k}^{\mathbb{C}}$-modules

$$\mathfrak{m}^{\mathbb{C}} = \sum \mathfrak{m}(\gamma). \tag{3.4}$$

Here, $\mathfrak{m}(\gamma)$ stands for the irreducible $\mathfrak{k}^{\mathbb{C}}$-module with highest weight $\gamma \in R'$.

The following Lemma states a well known property of flag manifolds (see e.g. [3] or [4]).

Lemma 3.8 *The $\mathfrak{k}^{\mathbb{C}}$-modules $\mathfrak{m}(\gamma)$ are pairwise not equivalent and, in particular, the decomposition* (3.4) *is unique. The modules $\mathfrak{m}(\gamma)$ are irreducible also as $\mathfrak{l}^{\mathbb{C}}$-modules.*

Proof. We only need to check that a module $\mathfrak{m}(\gamma)$ is irreducible also as an $\mathfrak{l}^{\mathbb{C}}$-module. But it is sufficient to observe that the semisimple parts of $\mathfrak{l}^{\mathbb{C}}$ and of $\mathfrak{k}^{\mathbb{C}}$ coincide and to recall that, whenever $\dim_{\mathbb{C}} \mathfrak{m}(\gamma) > 1$, the semisimple part of $\mathfrak{k}^{\mathbb{C}}$ acts non-trivially and irreducibly on $\mathfrak{m}(\gamma)$. □

From Lemma 3.8 we derive the following technical proposition, which will be useful in the following sections.

Proposition 3.9 *Let $M = G/L$ be a homogeneous contact manifold and let Z be a contact element for M. Assume that $G \neq G_2$ or that $G = G_2$ and $\vartheta = -i\mathcal{B} \circ Z$ is not proportional to a short root of R.*

Then for any irreducible $\mathfrak{k}^{\mathbb{C}}$-module $\mathfrak{m}(\gamma)$ there exists at most one distinct $\mathfrak{k}^{\mathbb{C}}$-module $\mathfrak{m}(\gamma')$ which is isomorphic to $\mathfrak{m}(\gamma)$ as $\mathfrak{l}^{\mathbb{C}}$-module.

This is the case if and only if the highest weights γ and γ' are ϑ-congruent, i.e. $\gamma' = \gamma + \lambda\vartheta$ for some real number λ.

Corollary 3.10 *Let M and Z as in the Proposition* 3.9. *Then*:

a) *if the modules $\mathfrak{m}(\gamma)$, $\mathfrak{m}(\gamma')$ are equivalent as $\mathfrak{l}^{\mathbb{C}}$-modules, then for any weight $\alpha \in R'$ of $\mathfrak{m}(\gamma)$, there exists exactly one weight $\alpha' \in R'$ of $\mathfrak{m}(\gamma')$ which is ϑ-congruent to α;*

b) *for any root $\alpha \in R'$ there exists at most one root $\alpha' \in R'$ which is ϑ-congruent to α, i.e. such that $\alpha' = \alpha + \lambda\vartheta$ for some real number $\lambda \neq 0$.*

Proof of Proposition 3.9. Observe that two irreducible $\mathfrak{l}^{\mathbb{C}}$-modules $\mathfrak{m}(\gamma)$ and $\mathfrak{m}(\gamma')$ are isomorphic if and only if their highest weights $\gamma|_{\mathfrak{h}(\mathfrak{l})}$ and $\gamma'|_{\mathfrak{h}(\mathfrak{l})}$ coincide. This occurs if and only if $\gamma' = \gamma + \lambda\vartheta$ for some $\lambda \in \mathbb{R}$.

Assume now that there exist three distinct isomorphic $\mathfrak{l}^{\mathbb{C}}$-modules $\mathfrak{m}(\gamma)$, $\mathfrak{m}(\gamma')$ and $\mathfrak{m}(\gamma'')$. Then $\tilde{R} = \mathrm{span}_{\mathbb{R}}(\gamma, \gamma', \gamma'') \cap R$ is a 2-dimensional root system and γ, γ' and γ'' belong to the straight line $\gamma + \mathbb{R}\vartheta$. Checking all 2-dimensional root systems, $2A_1$, A_2, B_2, G_2, we conclude that this is possible only if $\tilde{R}$ is of type B_2 or G_2 and ϑ is proportional to a short root. We claim that both these cases cannot occur.

If $\tilde{R}$ has type G_2, then $\tilde{R} = R$ which contradicts to the assumptions.

If $\tilde{R}$ has type B_2, one of the roots γ, γ', γ'' is orthogonal to ϑ and this is impossible because

$$\vartheta^{\perp} \cap R = R_o = R \setminus R'$$

while γ, γ', $\gamma'' \in R'$. □

4. General properties of compact homogeneous CR manifolds

4.1. Infinitesimal description of invariant CR structures

Let $(M = G/L, \mathcal{D}_Z)$ be a homogeneous contact manifold of a connected compact Lie group G with connected stabilizer L and let

$$\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m} \tag{4.1}$$

the associated orthogonal decomposition where $\mathfrak{k} = C_{\mathfrak{g}}(Z) = \mathfrak{l} + \mathbb{R}Z$.

Definition 4.1 A complex subspace $\mathfrak{m}^{10}$ of $\mathfrak{m}^{\mathbb{C}}$ is called *holomorphic* if

i) $\mathfrak{m}^{10} \cap \mathfrak{m}^{01} = \{0\}$, where $\mathfrak{m}^{01} = \overline{\mathfrak{m}^{10}}$ and 'bar' denotes the complex conjugation with respect to the real subspace $\mathfrak{g}$;

ii) $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^{10} + \mathfrak{m}^{01}$;

iii) $\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10}$ is a complex subalgebra of $\mathfrak{g}^{\mathbb{C}}$.

In the following we will refer to condition iii) as the *integrability condition.*

Note that if the integrability condition holds, also $\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}$ is a subalgebra. Furthermore, any holomorphic subspace $\mathfrak{m}^{10}$ defines an $\mathrm{ad}_{\mathfrak{l}}$-invariant complex structure J on $\mathfrak{m}$, whose $(+i)$- and $(-i)$-eigenspaces are exactly $\mathfrak{m}^{10}$ and $\mathfrak{m}^{01}$.

Proposition 4.2 *Let $(M = G/L, \mathcal{D}_Z)$ be a compact homogeneous contact manifold and $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m}$ be the associated decomposition. Then there exists a natural one to one correspondence between the set of invariant CR structures $(\mathcal{D}_Z, J)$ on M and the set of holomorphic subspaces $\mathfrak{m}^{10}$ of $\mathfrak{m}^{\mathbb{C}}$.*

Proof. Recall that, under the natural identification of $\mathbb{R}Z + \mathfrak{m}$ with the tangent space $T_{eL}M$, we have that $\mathfrak{m} = \mathcal{D}_Z|_{eL}$. Moreover, any invariant CR structure $(\mathcal{D}_Z, J)$ defines a decomposition $\mathcal{D}_Z^{\mathbb{C}} = \mathcal{D}^{10} + \mathcal{D}^{01}$ into two mutually conjugated invariant integrable distributions. Then one can easily check that the complex subspace $\mathfrak{m}^{10} = \mathcal{D}^{10}_{eL} \subset \mathfrak{m}^{\mathbb{C}}$ is a holomorphic subspace.

Conversely an holomorphic subspace $\mathfrak{m}^{10}$ and its conjugate subspace $\mathfrak{m}^{01} = \overline{\mathfrak{m}^{10}}$ are $\mathrm{ad}_{\mathfrak{l}}$-invariant and also Ad_L-invariant since L is connected. Then they can be extended to two invariant integrable complex distributions $\mathcal{D}^{10}$ and $\mathcal{D}^{01}$ such that $\mathcal{D}^{\mathbb{C}} = \mathcal{D}^{10} + \mathcal{D}^{01}$ with $\mathcal{D}^{10} \cap \mathcal{D}^{01} = 0$. Hence they may be considered as eigendistributions of an invariant CR structure $(\mathcal{D}_Z, J)$ on M. □

4.2. Standard CR structures

We want to show how to construct an invariant CR structure $(\mathcal{D}_Z, J)$ on a homogeneous contact manifold $(M = G/L, \mathcal{D}_Z)$ starting from an invariant complex structure J on the associated flag manifold F_Z.

Let $F = G/K$ be a flag manifold and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ the associated reductive decomposition. Recall that an invariant complex structure J_F on F is associated with a decomposition $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^{10} + \mathfrak{m}^{01}$ such that

$$a)\ \mathfrak{m}^{01} = \overline{\mathfrak{m}^{10}}; \quad b)\ \mathfrak{p} = \mathfrak{k}^{\mathbb{C}} + \mathfrak{m}^{10} \text{ is a subalgebra of } \mathfrak{g}^{\mathbb{C}}. \tag{4.2}$$

We say that $\mathfrak{m}^{10}$ is *the holomorphic subspace associated with J_F.*

It is known that $\mathfrak{p}$ is a parabolic subalgebra, with reductive part $\mathfrak{k}^{\mathbb{C}}$ and nilradical $\mathfrak{m}^{10}$. Moreover, we can always choose a system of positive roots R^+ for $\mathfrak{g}^{\mathbb{C}}$, such that $\mathfrak{m}^{10}$ is generated by root vectors E_α, with $\alpha \in R^+$. We say that such system R^+ is *compatible with the complex structure* J_F.

Let $(M = G/L, \mathcal{D}_Z)$ be a homogeneous contact manifold, $\mathfrak{g} = (\mathfrak{l} + \mathbb{R}Z) + \mathfrak{m} = \mathfrak{k} + \mathfrak{m}$ the corresponding decomposition and $F_Z = G/K$ the associated flag manifold. Any invariant complex structure J_F on F_Z induces an invariant CR structure $(\mathcal{D}_Z, J)$, which is the one corresponding to the same holomorphic subspace $\mathfrak{m}^{10} \subset \mathfrak{m}^{\mathbb{C}}$ as J_F.

Definition 4.3 An invariant CR structure $(\mathcal{D}, J)$ on a homogeneous contact manifold $(M = G/L, \mathcal{D})$, which is induced by an invariant complex structure J_F on the associated flag manifold $F = G/K$, is called *standard CR structure*.

Remark 4.4 Since any flag manifold admits at least one invariant complex structure, we may conclude that *any homogeneous contact manifold* $(G/L, \mathcal{D})$*, with* G *compact, admits an invariant CR structure* $(\mathcal{D}, J)$.

The following Lemma gives an algebraic characterization of the standard CR structures.

Lemma 4.5 *An invariant CR structure* $(\mathcal{D}, J)$ *on a homogeneous contact manifold* $(M = G/L, \mathcal{D})$ *is standard if and only if the corresponding complex structure* J *on* $\mathfrak{m}$ *is* $\mathrm{Ad}(K)$*-invariant.*

Proof. The proof is straightforward. □

Since the description of all invariant complex structures on flag manifolds is well known (see [3], [4], [5], [10]), the problem of classification of the invariant CR structures on compact homogeneous spaces reduces to the description of *non-standard* invariant CR structures.

The following proposition reduces the problem to the case of G semisimple.

Proposition 4.6 *Let* $(M = G/L, \mathcal{D})$ *be a contact manifold of a compact Lie group* G *with* $\dim Z(G) = 1$*. Then any invariant CR structure with underlying distribution* $\mathcal{D}$ *is standard.*

Proof. It follows immediately from the fact that any $\mathrm{Ad}(L)$-invariant decomposition $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^{10} + \mathfrak{m}^{01}$ is clearly also $\mathrm{Ad}(K)$-invariant, since $K = L \cdot Z(G)$. □

4.3. Holomorphic fibering of homogeneous CR manifolds

Let $(M = G/L, \mathcal{D}, J)$ be a homogeneous CR manifold with a *standard* CR structure J associated to a complex structure J_F on the associated flag manifold $F = G/K$. Then the natural projection

$$\pi : G/L \longrightarrow F = G/K$$

is a G-equivariant holomorphic fibration.

More generally we give the following definition.

Definition 4.7 Let $M = G/L$ be a homogeneous manifold with invariant CR structure $(\mathcal{D}, J)$.

(1) Any G-equivariant holomorphic fibering

$$\pi : M = G/L \longrightarrow F = G/Q$$

of $(M, \mathcal{D}, J)$ over a flag manifold $F = G/Q$ equipped with an invariant complex structure J_F is called *CRF fibration*;

(2) we say that a homogeneous CR manifold $(M = G/L, \mathcal{D}, J)$ is *primitive* if it doesn't admit a non-trivial CRF fibration;

(3) a non-primitive homogeneous CR manifold $(M = G/L, \mathcal{D}, J)$, admitting a CRF fibration with typical fiber S^1, is called *circular*.

Remark that *any standard CR structure is circular* and that *the typical fiber Q/L of a CRF fibration carries a natural invariant CR structure.*

The following Lemma gives a characterization of primitive CR structures.

Lemma 4.8 *A homogeneous CR manifold $(G/L, \mathcal{D}, J)$ admits a non-trivial CRF fibration if and only if there exists a proper parabolic subalgebra $\mathfrak{p} = \mathfrak{r} + \mathfrak{n} \neq \mathfrak{g}^{\mathbb{C}}$ (here $\mathfrak{r}$ is a reductive part and $\mathfrak{n}$ the nilpotent part) such that*

$$a)\ r = (\mathfrak{p} \cap \mathfrak{g})^{\mathbb{C}}; \qquad b)\ \mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10} \subset \mathfrak{p}; \qquad c)\ \mathfrak{l}^{\mathbb{C}} \neq \mathfrak{r}.$$

In this case, G/L admits a CRF fibration with basis G/Q, where Q is the connected subgroup generated by $\mathfrak{q} = \mathfrak{r} \cap \mathfrak{g}$.

Proof. Suppose that $(M = G/L, \mathcal{D}, J)$ is non-primitive and let $\pi : G/L \rightarrow$

G/Q be a CRF fibration over a flag manifold $F = G/Q$ with invariant complex structure J_F. Consider the decompositions associated to J and J_F

$$\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m}, \qquad \mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^{10} + \mathfrak{m}^{01},$$
$$\mathfrak{g} = \mathfrak{q} + \mathfrak{m}', \qquad \mathfrak{m}'^{\mathbb{C}} = \mathfrak{m}'^{10} + \mathfrak{m}'^{01}.$$

Since π is holomorphic and non-trivial, the subalgebra $\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10}$ is properly contained in the parabolic subalgebra $\mathfrak{p} = \mathfrak{q}^{\mathbb{C}} + \mathfrak{m}'^{10}$, with reductive part $\mathfrak{q}^{\mathbb{C}} = (\mathfrak{g} \cap \mathfrak{p})^{\mathbb{C}}$. Furthermore, since the fiber has positive dimension, $\mathfrak{l} \neq \mathfrak{q}$.

Conversely, if $\mathfrak{p} = \mathfrak{r} + \mathfrak{n} \subset \mathfrak{g}^{\mathbb{C}}$ is a parabolic subalgebra with reductive subalgebra $\mathfrak{r} = \mathfrak{q}^{\mathbb{C}}$, where $\mathfrak{q} = \mathfrak{p} \cap \mathfrak{g}$, then we may consider the orthogonal decompositions

$$\mathfrak{g} = \mathfrak{q} + \mathfrak{m}', \qquad \mathfrak{g}^{\mathbb{C}} = \mathfrak{r} + \mathfrak{m}'^{\mathbb{C}} = \mathfrak{r} + \mathfrak{n} + \mathfrak{n}',$$

where $\mathfrak{n}' = \mathfrak{n}^{\perp} \cap \mathfrak{m}'^{\mathbb{C}}$. By the remarks at the beginning of §4.2, there exists a unique invariant complex structure J_F with associated holomorphic space $\mathfrak{m}'^{10} = \mathfrak{n}$. Therefore if $\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10} \subset \mathfrak{p}$, $\mathfrak{l} \neq \mathfrak{q}$ and Q is the reductive subgroup generated by $\mathfrak{q}$, it is clear that $\pi : G/L \to G/Q$ is a non-trivial CRF fibration. □

4.4. The anticanonical map of a homogeneous CR manifold

Let $(M = G/L, \mathcal{D}_Z, J)$ be a homogeneous CR manifold of a compact Lie group G and

$$\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m}, \quad \mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^{10} + \mathfrak{m}^{01}$$

the associated decompositions of $\mathfrak{g}$ and of $\mathfrak{m}^{\mathbb{C}}$.

To characterize non-standard invariant CR structures, we need to recall the definition of *anticanonical map* of a homogeneous CR manifold introduced for the first time in [1]. It is a G-equivariant holomorphic map

$$\phi : M = G/L \longrightarrow \mathrm{Gr}_k(\mathfrak{g}^{\mathbb{C}})$$

into the Grassmanian of complex k-planes, $k = \dim_{\mathbb{C}}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01})$, of $\mathfrak{g}^{\mathbb{C}}$ given by

$$\phi : gL \mapsto \mathrm{Ad}_g(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}).$$

Due to the existence of standard holomorphic G-equivariant embedding

$$\imath : \mathrm{Gr}_k(\mathfrak{g}^{\mathbb{C}}) \longrightarrow \mathbb{C}P^N, \quad N = \binom{\dim \mathfrak{g}^{\mathbb{C}}}{k} - 1,$$

$$V = \mathrm{span}(e_1, \ldots, e_k) \overset{\imath}{\mapsto} [V] = \mathbb{C}(e_1 \wedge \cdots \wedge e_k),$$

we may consider ϕ as a G-equivariant map into $\mathbb{C}P^N$. To prove that the map ϕ is holomorphic it is sufficient to check that the linear map

$$\phi_* : \mathcal{D}_0 = \ker \theta|_{T_0 M} = \mathfrak{m} \longrightarrow T_{[\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}]} \mathrm{Gr}_k(\mathfrak{g}^{\mathbb{C}})$$

commutes with the complex structure.

Let $v = X + \bar{X} \in \mathfrak{m}$, where $X \in \mathfrak{m}^{10}$. Then

$$\phi_*(v) = \mathrm{ad}_{(X+\bar{X})}([\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}]) = \mathrm{ad}_X([\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}]).$$

Therefore

$$\begin{aligned} \phi_*(Jv) = \phi_*(iX - i\bar{X}) &= \mathrm{ad}_{iX}([\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}]) \\ &= i\, \mathrm{ad}_X([\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}]) = i\phi_*(v). \end{aligned}$$

This shows that the map ϕ is holomorphic.

Remark that the stabilizer Q of the point $[\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}]$ in $\phi(M) = G/Q$ is the normalizer $Q = N_G(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01})$.

Characterizations of non-standard CR structures by means of the anti-canonical map will be given in §6, after proving some main facts on CR structures on special contact manifolds.

5. Classification of CR structures on special contact manifolds

We describe here all invariant CR structures $(\mathcal{D}_Z, J)$ on a special contact manifold G/L. Recall that in this case G is simple and $L = C_G(\mathfrak{g}(\alpha))$, by Theorem 3.6, where either $\alpha = \mu$ is the highest root or $G = G_2$ and $\alpha = \nu$ is a short root.

We have the following orthogonal decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m} = \mathfrak{l} + \mathfrak{a} + \mathfrak{n}, \tag{5.1}$$

where $\mathfrak{a} = \mathfrak{g}(\alpha)$ is the 3-dimensional subalgebra associated with the root α, $Z = iH_\alpha \in \mathfrak{a}$ and $\mathfrak{l} = C_{\mathfrak{g}}(\mathfrak{a})$ is its centralizer.

Let $(\mathcal{D}, J)$ be an invariant CR structure on G/L which is determined by the contact element $Z = iH_\alpha$ and by the decompositions

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^{10} + \mathfrak{m}^{01} = \mathfrak{a}^{10} + \mathfrak{n}^{10} + \mathfrak{a}^{01} + \mathfrak{n}^{01}, \tag{5.2}$$

where $\mathfrak{a}^{10} = \mathfrak{a}^{\mathbb{C}} \cap \mathfrak{m}^{10}$, $\mathfrak{n}^{10} = \mathfrak{n}^{\mathbb{C}} \cap \mathfrak{m}^{10}$ and $\mathfrak{m}^{01} = \mathfrak{a}^{01} + \mathfrak{n}^{01} = \overline{\mathfrak{m}^{10}}$.

Since $\mathfrak{a}^{\mathbb{C}} \simeq \mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{a}^{10} + \mathfrak{a}^{01}$ is the orthogonal complement to $\mathbb{C}Z$ in $\mathfrak{a}^{\mathbb{C}}$, we can write $\mathfrak{a}^{10} = \mathbb{C}Z'$, for some $Z' \in \mathfrak{m}^{\mathbb{C}} \cap \mathfrak{a}^{\mathbb{C}}$.

Note that a regular element X of $\mathfrak{a}^{\mathbb{C}}$ (up to rescaling) can be always identified with iH_α, where α is a root of $\mathfrak{g}^{\mathbb{C}}$ with respect to some Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}^{\mathbb{C}}$ and such that $\mathfrak{a} = \mathfrak{g}(\alpha)$. In particular, since any contact element Z of $\mathfrak{g}$ is a regular element for $\mathfrak{a}^{\mathbb{C}}$, it can be always identified with iH_α.

If $\alpha = \mu$ is the highest root, the eigenspace decomposition of ad_{H_α} gives the gradation

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2, \tag{5.3}$$

which is described in (3.3) and Table 1. Table 1 shows that for $\mathfrak{g}^{\mathbb{C}} \neq A_\ell$, the $\mathfrak{g}_0$-modules $\mathfrak{g}_{\pm1}$ are irreducible, their dimension is $\dim_{\mathbb{C}} \mathfrak{g}_{\pm1} = 1/2 \dim_{\mathbb{C}} \mathfrak{n}^{\mathbb{C}}$ and

$$[\mathfrak{g}_{\pm1}, \mathfrak{g}_{\pm1}] = \mathfrak{g}_{\pm2}. \tag{5.4}$$

If $\mathfrak{g}^{\mathbb{C}} = A_\ell$, each $\mathfrak{g}_0$-module $\mathfrak{g}_{\pm1}$ decomposes into two not equivalent irreducible $\mathfrak{g}_0$-modules: $\mathfrak{g}_{\pm1} = \mathfrak{g}_{\pm1}^{(1)} + \mathfrak{g}_{\pm1}^{(2)}$. Moreover, the following relations hold:

$$[\mathfrak{g}_1^{(i)}, \mathfrak{g}_1^{(i)}] = \{0\} = [\mathfrak{g}_{-1}^{(i)}, \mathfrak{g}_{-1}^{(i)}], \quad [\mathfrak{g}_1^{(i)}, \mathfrak{g}_1^{(j)}] = \mathfrak{g}_2, \quad [\mathfrak{g}_{-1}^{(i)}, \mathfrak{g}_{-1}^{(j)}] = \mathfrak{g}_{-2}, \tag{5.5}$$

$$[\mathfrak{g}_1^{(i)}, \mathfrak{g}_{-2}] = \mathfrak{g}_{-1}^{(j)}, \quad [\mathfrak{g}_{-1}^{(i)}, \mathfrak{g}_2] = \mathfrak{g}_1^{(j)}, \quad \overline{\mathfrak{g}_1^{(i)}} = \mathfrak{g}_{-1}^{(i)} \quad (i \neq j). \tag{5.6}$$

The modules $\mathfrak{g}_1^{(i)}$ and $\mathfrak{g}_{-1}^{(j)}$ $(i \neq j)$ are isomorphic as $\mathfrak{g}_0'$-modules and, for both values of i, $\dim_{\mathbb{C}} \mathfrak{g}_{\pm1}^{(i)} = 1/4 \dim_{\mathbb{C}} \mathfrak{n}^{\mathbb{C}}$.

When $\mathfrak{g}^{\mathbb{C}} = G_2$ and $\alpha = \nu = \varepsilon_1$ is a short root, the eigenspace decomposition of operator ad_{H_ν} defines the following gradation of $\mathfrak{g}^{\mathbb{C}}$:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_{-3} + \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2 + \mathfrak{g}_3, \tag{5.7}$$

where

$$\mathfrak{g}_0 = \mathfrak{g}_0' + \mathbb{C}H_\nu, \quad \mathfrak{g}_0' = C_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{g}(\nu)) = \langle E_{\pm(\varepsilon_2 - \varepsilon_3)}, H_{\varepsilon_2 - \varepsilon_3} \rangle,$$
$$\mathfrak{g}_2 = \mathbb{C}E_{\varepsilon_1}, \quad \mathfrak{g}_{-2} = \mathbb{C}E_{-\varepsilon_1}, \quad \mathfrak{g}^{\mathbb{C}}(\nu) = \mathfrak{g}_2 + \mathfrak{g}_{-2} + \mathbb{C}H_{\varepsilon_1},$$

$$\begin{aligned} &\mathfrak{g}_1 = \langle E_{-\varepsilon_3}, E_{-\varepsilon_2}\rangle, \qquad \mathfrak{g}_3 = \langle E_{\varepsilon_1-\varepsilon_3}, E_{\varepsilon_1-\varepsilon_2}\rangle, \\ &\mathfrak{g}_{-i} = \overline{\mathfrak{g}_i} \quad \text{for} \quad i = 1, 3 \end{aligned} \tag{5.8}$$

(see Appendix for notation).

Note that all subspaces $\mathfrak{g}_i$ are irreducible $\mathfrak{g}_0'$-modules and that the modules $\mathfrak{g}_j$, $j = \pm 1, \pm 3$, are equivalent $\mathfrak{g}_0'$-modules. Furthermore, $[\mathfrak{g}_{\pm 1}, \mathfrak{g}_{\pm 1}] = \mathfrak{g}_{\pm 2}$ and $[\mathfrak{g}_{\pm 3}, \mathfrak{g}_{\pm 3}] = \{0\}$.

The following Theorem gives the complete classification of the invariant CR structures on special contact manifolds.

Theorem 5.1 *Let $(M = G/L, \mathcal{D}_Z)$ be a special contact manifold. Then:*

a) *if $G \neq SU_{\ell+1}$ and $M \neq G_2/Sp_1$, where Sp_1 denotes the subgroup with Lie algebra $\mathfrak{sp}_1(\mu)$ with μ maximal root, then there exists (up to a sign) a unique invariant CR structure $(\mathcal{D}_Z, J)$, and it is the standard one.*

b) *if $M = G_2/Sp_1$, where Sp_1 denotes the subgroup with Lie algebra $\mathfrak{sp}_1(\mu)$ with μ maximal root, there exists a 1-1 correspondence between the invariant CR structures (determined up to a sign) and the points of the unit disc $D = \{t \in \mathbb{C},\ |t| < 1\}$. Using the notation of the Appendix for the roots of G_2 and under the identification $Z = iH_{\varepsilon_1}$, a point $t \in D$ corresponds to the CR structure $(\mathcal{D}, J_t)$ with the holomorphic subspace*

$$\begin{aligned} \mathfrak{m}^{10} = &\mathbb{C}(E_{\varepsilon_1} + t^2 E_{-\varepsilon_1}) + \mathbb{C}(E_{-\varepsilon_3} + tE_{\varepsilon_2}) + \mathbb{C}(E_{-\varepsilon_2} + tE_{\varepsilon_3}) \\ &+ \mathbb{C}(E_{\varepsilon_1-\varepsilon_2} + t^3 E_{\varepsilon_3-\varepsilon_1}) + \mathbb{C}(E_{\varepsilon_1-\varepsilon_3} + t^3 E_{\varepsilon_2-\varepsilon_1}). \end{aligned} \tag{5.9}$$

The CR structure $(\mathcal{D}_Z, J_t)$ is standard if and only if $t = 0$.

c) *if $G = SU_2$ and hence $M = SU_2$, there exists a 1-1 correspondence between the invariant CR structures (determined up to a sign) and the points of the unit disc D. Under the identification $Z = iH_\alpha$, a point $t \in D$ corresponds to the CR structure $(\mathcal{D}, J_t)$ with the holomorphic subspace*

$$\mathfrak{m}^{10} = \mathbb{C}(E_\alpha + tE_{-\alpha}). \tag{5.10}$$

The CR structure $(\mathcal{D}_Z, J_t)$ is standard if and only if $t = 0$.

d) *if $G = SU_\ell$, $\ell > 2$, and hence $M = SU_\ell/U_{\ell-2}$, the set of all invariant CR structures (determined up to a sign) consists of:*

d.1) *the standard CR structure $(\mathcal{D}_Z, J^{(0)})$, induced by the invariant complex structure $J^{(0)}$ on $F_Z = SU_\ell/T^2 \cdot SU_{\ell-2}$, which is the natural complex structure of the twistor space of the Wolf space $Gr_2(\mathbb{C}^\ell) = SU_\ell/S(U_2 \cdot U_{\ell-2})$;*

d.2) *three families* $(\mathcal{D}_Z, J_t)$, $(\mathcal{D}_Z, J'_t)$ *and* $(\mathcal{D}_Z, J_t^{(0)})$ *of invariant CR structures, parameterized by the points of the unit disc D. Under the identification* $Z = iH_\mu$, *the CR structures* $(\mathcal{D}_Z, J_t)$, $(\mathcal{D}_Z, J'_t)$ *and* $(\mathcal{D}_Z, J_t^{(0)})$ *have the following holomorphic subspaces*

$$(\text{for } J_t) \qquad \mathfrak{m}^{10} = \mathbb{C}(E_\mu + tE_{-\mu}) + \mathfrak{g}_1^{(1)} + \mathfrak{g}_{-1}^{(2)}, \tag{5.11}$$

$$(\text{for } J'_t) \qquad \mathfrak{m}'^{10} = \mathbb{C}(E_\mu + tE_{-\mu}) + \mathfrak{g}_1^{(2)} + \mathfrak{g}_{-1}^{(1)}, \tag{5.12}$$

$$(\text{for } J_t^{(0)}) \quad \mathfrak{m}''^{10} = \mathbb{C}(E_\mu + t^2E_{-\mu}) + (\mathfrak{g}_1^{(1)} + t\mathfrak{g}_{-1}^{(2)}) + (\mathfrak{g}_1^{(2)} + t\mathfrak{g}_{-1}^{(1)}), \tag{5.13}$$

where

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{l}^{\mathbb{C}} + \mathbb{C}Z + \mathfrak{m}^{\mathbb{C}} = \mathfrak{g}'_0 + \mathbb{C}(iH_\mu) + (\mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_1 + \mathfrak{g}_2),$$

and where $(\mathfrak{g}_1^{(i)} + t\mathfrak{g}_{-1}^{(j)})$ *denotes the unique* $\mathfrak{g}'_0$*-invariant subspace of* $\mathfrak{g}_1 + \mathfrak{g}_{-1}$, *with highest weight vector* $E_1^{(i)} + tE_{-1}^{(j)}$, *where* $E_{\pm1}^{(k)}$, $k = 1, 2$, *are highest weight vectors of* $\mathfrak{g}_{\pm1}^{(k)}$.

A CR structure $(\mathcal{D}_Z, J_t)$, $(\mathcal{D}_Z, J'_t)$ *or* $(\mathcal{D}_Z, J_t^{(0)})$ *is standard if and only if* $t = 0$.

Corollary 5.2 *Let* $(M = G/L, \mathcal{D}_Z)$ *be a special contact manifold with* $G = SU_\ell$.

(1) *if* $M = SU_2$, *then* $(M, \mathcal{D}_Z)$ *admits (up to sign) only one standard CR structure and one family of non-standard CR structures, parameterized by the punctured unit disc* $D \setminus \{0\} \subset \mathbb{C}$; *any non-standard CR structure is circular and the anti-canonical map* $\phi : M \longrightarrow \phi(M)$ *is a finite covering*;

(2) *if* $M = SU_\ell/U_{\ell-2}$, $\ell > 2$, *then* $(M, \mathcal{D}_Z)$ *admits (up to a sign) exactly three standard CR structures (namely* $(\mathcal{D}_Z, J^{(0)})$, $(\mathcal{D}_Z, J_0)$ *and* $(\mathcal{D}_Z, J'_0)$*) that are induced by three invariant complex structures of the corresponding flag manifold* $F_Z = SU_\ell/T^2 \cdot SU_{\ell-2}$, *plus three families* $(\mathcal{D}_Z, J_t^{(0)})$, $(\mathcal{D}_Z, J_t)$ *and* $(\mathcal{D}_Z, J'_t)$ *of non-standard CR structures, parameterized by the points of the punctured unit disc* $t \in D \setminus \{0\}$; *any non-standard CR structure* $(\mathcal{D}_Z, J_t^{(0)})$ *is primitive, while the CR structures* $(\mathcal{D}_Z, J_t)$ *and* $(\mathcal{D}_Z, J'_t)$ *are circular*; *furthermore, each CR structure* $(\mathcal{D}_Z, J_t)$ *or* $(\mathcal{D}_Z, J'_t)$ *admits also a CRF fibration*

$$\pi : M = SU_\ell/U_{\ell-2} \longrightarrow Gr_2(\mathbb{C}^\ell) = SU_\ell/S(U_2 \times U_{\ell-2})$$

with fiber SO_3 over the Wolf space $Gr_2(\mathbb{C}^\ell)$ equipped with its (unique up to a sign) complex structure; finally, for any of the non-standard CR structures, the anti-canonical map $\phi : M \longrightarrow \phi(M)$ is a finite covering;

(3) *if $M = G_2/SU_2$ with the subgroup SU_2 as described in Theorem* 5.1, *then $(M, \mathcal{D}_Z)$ admits (up to sign) only one standard CR structure and one family of primitive CR structures, parameterized by the punctured unit disc $D \setminus \{0\} \subset \mathbb{C}$.*

Remark 5.3 The complex structures J_0 and J_0' on F_Z coincide on the fibers of the twistor fibration $\pi : F_Z \to Gr_2(\mathbb{C}^\ell)$ but are projected into two opposite complex structures of $Gr_2(\mathbb{C}^\ell)$.

Proof. The proof of Theorem 5.1 reduces to classification of the decompositions (5.2), which correspond to an integrable CR structure, for each special contact manifold $(G/L, \mathcal{D}_Z)$. For any decomposition (5.2), the subspace $\mathfrak{a}^{10}$ can be expressed as $\mathfrak{a}^{10} = \mathbb{C}Z'$ for some suitable $Z' \in \mathfrak{a}^{\mathbb{C}}$. Therefore we have to cases:

(1) Z' is a regular element of $\mathfrak{a}^{\mathbb{C}}$;

(2) Z' is a non-regular (hence nilpotent) element of $\mathfrak{a}^{\mathbb{C}} \simeq \mathfrak{sl}_2(\mathbb{C})$.

Case (1): Consider first $\mathfrak{a} = \mathfrak{g}(\mu)$, with μ long root of the simple group G. Since Z' is regular, we may assume that $Z' = iH_\mu$ and we may consider the corresponding graded decomposition (5.3). Recall that $\mathfrak{l}^{\mathbb{C}} = C_{\mathfrak{g}}(\mathfrak{g}(\mu)) = \mathfrak{g}_0'$.

Hence the subalgebra $\mathfrak{b} = \mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10}$ is contained in

$$\begin{aligned}\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10} &= \mathfrak{g}_0' + \mathfrak{a}^{10} + \mathfrak{n}^{10} = \mathfrak{g}_0' + \mathbb{C}H_\mu + \mathfrak{n}^{10} \\ &= \mathfrak{g}_0 + \mathfrak{n}^{10} \subset \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_{-1}\end{aligned}$$

since $\mathfrak{n}^{\mathbb{C}} \subset \mathfrak{g}_1 + \mathfrak{g}_{-1}$, being orthogonal to $\mathfrak{a}^{\mathbb{C}} = \mathbb{C}H_\mu + \mathfrak{g}_2 + \mathfrak{g}_{-2}$. In case $\mathfrak{g}^{\mathbb{C}} \neq A_\ell$, $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are irreducible $\mathfrak{g}_0$-modules and hence either $\mathfrak{g}_1$ or $\mathfrak{g}_{-1}$ is included in $\mathfrak{n}^{10}$. However $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2$ and $[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = \mathfrak{g}_{-2}$, and hence there is no subalgebra $\mathfrak{b}$ of $\mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_{-1}$ which contains $\mathfrak{g}_0$ properly. In conclusion, if $\alpha = \mu$ is a long root, then $\mathfrak{g}^{\mathbb{C}} = A_\ell$.

Consider now the case in which $G = G_2$ and $\mathfrak{a} = \mathfrak{g}(\nu)$, with ν short root of $\mathfrak{g}^{\mathbb{C}}$. We assume that $Z' = iH_\nu$ and we consider the corresponding graded decomposition (5.7). Then $\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10}$ is contained in

$$
\begin{aligned}
\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10} &= \mathfrak{l}^{\mathbb{C}} + \mathfrak{a}^{10} + \mathfrak{n}^{10} \\
&= \mathfrak{g}_0' + \mathbb{C}H_\nu + \mathfrak{n}^{10} \subset \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_{-1} + \mathfrak{g}_3 + \mathfrak{g}_{-3}
\end{aligned} \tag{5.14}
$$

because $\mathfrak{n}^{\mathbb{C}}$ is orthogonal to $\mathfrak{a}^{\mathbb{C}} = \mathbb{C}H_\nu + \mathfrak{g}_2 + \mathfrak{g}_{-2}$. Since $\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10} = \mathfrak{g}^0 + \mathfrak{n}^{10}$ is a subalgebra and $\dim_{\mathbb{C}} \mathfrak{g}_{\pm 1} = \dim_{\mathbb{C}} \mathfrak{g}_{\pm 3} = \frac{1}{2} \dim_{\mathbb{C}} \mathfrak{n}^{10}$, $\mathfrak{n}^{10}$ contains two irreducible $\mathfrak{g}_0'$-modules. The only possibility for $\mathfrak{n}^{10}$, so that $\mathfrak{g}_0 + \mathfrak{n}^{10}$ is a subalgebra, is $\mathfrak{n}^{10} = \mathfrak{g}_{-3} + \mathfrak{g}_3$. This means that, for a given Z', there exists at most one CR structure. If we identify the contact element Z (and no longer Z') with iH_ν, $\nu = \varepsilon_1$, then the element Z' can be written in the form

$$
Z' = E_{\varepsilon_1} + sE_{-\varepsilon_1}, \qquad |s| \neq 0, \tag{5.15}
$$

and exchanging $\mathfrak{a}^{10}$ with $\mathfrak{a}^{01} = \overline{\mathfrak{a}^{10}}$ if necessary (which corresponds to changing sign to the complex structure), we may assume that $0 < |s| \leq 1$. Since $\mathfrak{a}^{10} \cap \mathfrak{a}^{01} = \{0\}$ and hence $E_{\varepsilon_1} + sE_{-\varepsilon_1}$ and $\overline{s}E_{\varepsilon_1} + E_{-\varepsilon_1}$ are linearly independent, s satisfies the condition

$$
\det \begin{bmatrix} 1 & s \\ \bar{s} & 1 \end{bmatrix} = 1 - |s|^2 \neq 0 \tag{5.16}
$$

and therefore $s \in D \setminus \{0\} = \{0 < |s| < 1\}$. Now, the reader can check that the subspace $\mathfrak{m}^{10} \subset \mathfrak{g}_1 + \mathfrak{g}_{-1} + \mathfrak{g}_2 + \mathfrak{g}_{-2} + \mathfrak{g}_3 + \mathfrak{g}_{-3}$ described in (5.9) is indeed a holomorphic subspace corresponding to the unique CR structure with $\mathfrak{a}^{10} = \mathbb{C}Z'$, where Z' is of the form (5.14) with $s = t^2$.

Now it remains to classify the invariant CR structures on $(SU_\ell/U_{\ell-2}, \mathcal{D}_Z)$.

For the following part of the proof, it is more convenient to identify the contact element Z (and no longer Z') with iH_μ. We also consider the decomposition (5.3) determined by $Z = iH_\mu$.

Since Z' is a regular element which is orthogonal to $Z = iH_\mu$, it is (up to a factor) of the form $Z' = E_\mu + tE_{-\mu}$ with $|t| \neq 0$. Exchanging $\mathfrak{a}^{10}$ with $\mathfrak{a}^{01} = \overline{\mathfrak{a}^{10}}$ if necessary, we may assume that $0 < |t| \leq 1$ and since $\mathfrak{a}^{10} \cap \mathfrak{a}^{01} = \{0\}$, by the same arguments of before, we get that $t \in D \setminus \{0\} = \{0 < |t| < 1\}$.

We claim that for any point $t \in D \setminus \{0\}$ there exist exactly three invariant CR structures, whose associated subspace $\mathfrak{a}^{10}$ is equal to $\mathbb{C}(E_\mu + tE_{-\mu})$. In fact, one can check that the only $\mathfrak{g}_0'$-invariant subspaces $\mathfrak{m}^{10}$ of $\mathbb{C}(E_\mu + tE_{-\mu}) + \mathfrak{g}_1 + \mathfrak{g}_{-1}$, which satisfy (i) and (ii) of Definition 4.1, are either (5.11), (5.12) or a subspace of the form

$$\mathfrak{m}^{10} = \mathbb{C}(E_\mu + tE_{-\mu}) + (\mathfrak{g}_1^{(1)} + s\mathfrak{g}_{-1}^{(2)}) + (\mathfrak{g}_1^{(2)} + s\mathfrak{g}_{-1}^{(1)}) \tag{5.17}$$

for some coefficient s. One can also check that the subspaces (5.11) and (5.12) satisfy also the integrability condition, while (5.17) satisfies the integrability condition if and only if $t = s^2$. This proves that (5.11), (5.12) and (5.13) are the only holomorphic subspaces of $\mathfrak{m}^\mathbb{C}$ containing $\mathbb{C}(E_\mu + tE_{-\mu})$. In particular, they define three distinct invariant CR structures, which we denote by $(\mathcal{D}_Z, J_t)$, $(\mathcal{D}_Z, J'_t)$ and $(\mathcal{D}_Z, J_t^{(0)})$.

If $\ell = 2$ and hence $M = SU_2$, then $\mathfrak{n} = \{0\}$ and the three CR structures $(\mathcal{D}_Z, J_t)$, $(\mathcal{D}_Z, J'_t)$ and $(\mathcal{D}_Z, J_t^{(0)})$ coincide for any t.

Since for any $t \neq 0$ the holomorphic subspaces $\mathfrak{m}^{10}$ and $\mathfrak{m}'^{01}$ are not ad_Z-invariant, any CR structure $(\mathcal{D}_Z, J_t)$, $(\mathcal{D}_Z, J'_t)$ or $(\mathcal{D}_Z, J_t^{(0)})$ $(t \neq 0)$ is non-standard by Lemma 4.5.

Case (2): Since Z' is not regular, it is a nilpotent element of $\mathfrak{a}^\mathbb{C} = \mathfrak{sl}_2(\mathbb{C}) = \mathfrak{g}^\mathbb{C}(\alpha)$. Then we may always choose a Cartan subalgebra $\mathbb{C}H_\alpha$ of $\mathfrak{a}$ so that $Z' \in \mathbb{C}E_\alpha$. Furthermore, since the contact element Z is orthogonal to $\mathfrak{a}^{10} + \mathfrak{a}^{01} = \mathbb{C}E_\alpha + \overline{\mathbb{C}E_\alpha} = \mathbb{C}E_\alpha + \mathbb{C}E_{-\alpha}$, we may assume (after rescaling) that $Z = iH_\alpha$.

Consider first that $\alpha = \mu$ is a long root of G and take the gradation (5.3) of $\mathfrak{g}^\mathbb{C}$ determined with H_μ. Then $\mathfrak{g}_2 = \mathbb{C}Z' = \mathfrak{a}^{10}$ and hence

$$\mathfrak{l}^\mathbb{C} + \mathfrak{m}^{10} = \mathfrak{g}'_0 + \mathfrak{g}_2 + \mathfrak{n}^{10} \subset \mathfrak{g}'_0 + \mathfrak{g}_2 + \mathfrak{g}_1 + \mathfrak{g}_{-1}.$$

Assume that $\mathfrak{g}^\mathbb{C} \neq A_\ell$. Then the $\mathfrak{g}'_0$-modules $\mathfrak{g}_{\pm 1}$ are irreducible and $[\mathfrak{g}_{\pm 1}, \mathfrak{g}_{\pm 1}] = \mathfrak{g}_{\pm 2}$. Hence the only subalgebra of $\mathfrak{g}'_0 + \mathfrak{g}_2 + \mathfrak{g}_1 + \mathfrak{g}_{-1}$, which properly contains $\mathfrak{g}'_0 + \mathfrak{g}_2$, is $\mathfrak{g}'_0 + \mathfrak{g}_1 + \mathfrak{g}_2$. Hence $\mathfrak{m}^{10} = \mathfrak{g}_1 + \mathfrak{g}_2$.

Vice versa, $\mathfrak{m}^{10} = \mathfrak{g}_1 + \mathfrak{g}_2$ is a holomorphic subspace of $\mathfrak{m}^\mathbb{C} = (\mathfrak{l}^\mathbb{C} + \mathbb{C}Z)^\perp = \mathfrak{g}_0^\perp$ and hence it corresponds to an invariant CR structure on $(G/L, \mathcal{D}_Z)$. Since $Z = iH_\mu \in N_\mathfrak{g}(\mathfrak{g}'_0 + \mathfrak{g}_{-1} + \mathfrak{g}_{-2}) = N_\mathfrak{g}(\mathfrak{l}^\mathbb{C} + \mathfrak{m}^{01})$, this CR structure is standard.

Assume now that $\mathfrak{g}^\mathbb{C} = A_\ell$ and again consider the decomposition (5.3) determined by $Z = iH_\mu$. Since $\dim_\mathbb{C} \mathfrak{g}_{\pm 1}^{(i)} = 1/4 \dim_\mathbb{C} \mathfrak{n}^\mathbb{C}$, the $\mathfrak{g}'_0$-module $\mathfrak{n}^{10}$ can be written in one of the following five forms:

$$1)\ \mathfrak{n}^{10} = (\mathfrak{g}_1^{(1)})_\varphi + (\mathfrak{g}_{-1}^{(1)})_\psi, \qquad 2)\ \mathfrak{n}^{10} = \mathfrak{g}_1^{(1)} + \mathfrak{g}_{-1}^{(2)},$$

$$3)\ \mathfrak{n}^{10} = \mathfrak{g}_1^{(2)} + \mathfrak{g}_{-1}^{(1)}, \qquad 4)\ \mathfrak{n}^{10} = \mathfrak{g}_1, \qquad 5)\ \mathfrak{n}^{10} = \mathfrak{g}_{-1},$$

where $\varphi : \mathfrak{g}_1^{(1)} \to \mathfrak{g}_{-1}^{(2)}$ and $\psi : \mathfrak{g}_{-1}^{(1)} \to \mathfrak{g}_1^{(2)}$ are two $\mathfrak{g}_0'$-equivariant homomorphisms and where $(\mathfrak{g}_1^{(1)})_\varphi$ and $(\mathfrak{g}_{-1}^{(1)})_\psi$ denote the subspaces of the form

$$(\mathfrak{g}_1^{(1)})_\varphi = \{X + \varphi(X) : X \in \mathfrak{g}_1^{(1)}\}, \quad (\mathfrak{g}_{-1}^{(1)})_\psi = \{X + \psi(X) : X \in \mathfrak{g}_{-1}^{(1)}\}.$$

Case 5) cannot occur because in that case $[\mathfrak{n}^{10}, \mathfrak{n}^{10}] = \mathfrak{g}_{-2}$ and this contradicts the fact that $\mathfrak{g}_0' + \mathfrak{n}^{10} + \mathfrak{g}_2$ is a subalgebra.

Also case 1) may not occur. In fact, φ is either trivial or an isomorphism. In case φ is an isomorphism, for any $0 \neq X \in \mathfrak{g}_1^{(1)}$, it is possible to find an element $Y \in \mathfrak{g}_{-1}^{(1)}$ so that $[\varphi(X), Y]$ is non-trivial and belongs to $\mathfrak{g}_{-2}$. Hence,

$$[X + \varphi(X), Y + \psi(Y)] \underset{\mathrm{mod}\ \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2}{\equiv} [\varphi(X), Y] \in \mathfrak{g}_{-2}.$$

This contradicts the fact that $\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10}$ is a subalgebra of $\mathfrak{g}_0' + \mathfrak{g}_1 + \mathfrak{g}_2$. We conclude that, if case 1) occurred, $\mathfrak{n}^{10} = \mathfrak{g}_1^{(1)} + (\mathfrak{g}_{-1}^{(1)})_\psi$. Now, for any $X \in \mathfrak{g}_1^{(1)}$ we may consider an element $Y + \psi(Y) \in (\mathfrak{g}_{-1}^{(1)})_\psi$ so that $[X, Y] = \lambda H_\mu$ for some $\lambda \neq 0$. Hence

$$[X, Y + \psi(Y)] = \lambda H_\mu \mod \mathfrak{g}_0' + \mathfrak{g}_2$$

This gives a contradiction with the fact that $\mathfrak{g}_0' + \mathfrak{n}^{10} + \mathfrak{g}_2$ is a subalgebra and the claim is proved.

For the cases 2), 3) and 4), $\mathfrak{m}^{10}$ equals one of the following three subspaces

$$\mathfrak{g}_1^{(1)} + \mathfrak{g}_{-1}^{(2)} + \mathfrak{g}_2,\ \mathfrak{g}_1^{(2)} + \mathfrak{g}_{-1}^{(1)} + \mathfrak{g}_2,\ \mathfrak{g}_1 + \mathfrak{g}_2 \tag{5.18}$$

and one can check that any of them is a holomorphic subspace.

By Proposition 4.2, they determine three distinct CR structures denoted by $(\mathcal{D}, J)$, $(\mathcal{D}, J')$ and $(\mathcal{D}, J^{(0)})$, respectively. For any of the three subspaces (5.18), the normalizer $N_{\mathfrak{g}}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01})$ contains $\mathfrak{g}_0 \cap \mathfrak{g} = \mathfrak{l} + \mathbb{R}Z$ and hence the corresponding CR structures are standard.

Finally, observe that $(\mathcal{D}, J^{(0)})$ is induced by the invariant complex structure J_F on the flag manifold $F_Z = SU_\ell / T^2 \cdot SU_{\ell-2}$ which is associated to the following black-white Dynkin graph and which is the invariant complex structure of the twistor space of the Wolf space $Gr_2(\mathbb{C}^\ell) = SU_\ell/S(U_2 \cdot U_{\ell-2})$; moreover, the subspace of $J^{(0)}$ coincides with the subspace given in (5.13) for $t = 0$; on the other hand, the subspaces of J and J' are the subspaces given in (5.11) and (5.12) for $t = 0$. All corresponding CR structures coincide if $M = SU_2$.

It remains to consider the case in which $G = G_2$ and $\mathfrak{a} = \mathfrak{g}(\nu)$, where ν is a short root. Consider the decomposition (5.7) determined by H_ν so that $\mathbb{C}Z' = \mathbb{C}E_\nu = \mathfrak{g}_2$.

As before, we identify Z with iH_ν. We have

$$\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10} = \mathfrak{g}_0' + \mathfrak{a}^{10} + \mathfrak{n}^{10} \subset \mathfrak{g}_0' + \mathfrak{g}_2 + \mathfrak{g}_{-1} + \mathfrak{g}_1 + \mathfrak{g}_{-3} + \mathfrak{g}_3$$

because $\mathfrak{n}^{\mathbb{C}}$ is orthogonal to $\mathfrak{a}^{\mathbb{C}} = \mathbb{C}H_\nu + \mathfrak{g}_{-2} + \mathfrak{g}_2$. We claim that $\mathfrak{g}_3 \subset \mathfrak{n}^{10}$. In fact, for any element $X \in \mathfrak{n}^{10}$ consider the decomposition

$$X = X_{-3} + X_{-1} + X_1 + X_3, \qquad X_i \in \mathfrak{g}_i.$$

Then, one of the four vectors X, $X' = [E_\nu, X]$, $X'' = [E_\nu, [E_\nu, X]]$, $X''' = [E_\nu, [E_\nu, [E_\nu, X]]]$ is a non-trivial element of $\mathfrak{g}_3$ and it belongs to $\mathfrak{n}^{10}$. Since $\mathfrak{g}_3$ is $\mathfrak{g}_0'$-irreducible, the claim follows.

Similarly, we claim that $\mathfrak{g}_1 \subset \mathfrak{n}^{10}$. To prove this, take any element $X \in \mathfrak{n}^{10}$ which has a decomposition of the form

$$X = X_{-3} + X_{-1} + X_1, \qquad X_i \in \mathfrak{g}_i.$$

Then, either X or $X' = [E_\nu, X]$ or $X'' = [E_\nu, [E_\nu, X]]$ is a non-trivial element of $\mathfrak{g}_1 + \mathfrak{g}_3$, with non-vanishing projection on $\mathfrak{g}_1$. This implies that $\mathfrak{g}_1 \cap \mathfrak{n}^{10} \neq \{0\}$ and hence that $\mathfrak{g}_1 \subset \mathfrak{n}^{10}$. Since $\dim_{\mathbb{C}}(\mathfrak{g}_1 + \mathfrak{g}_3) = \dim_{\mathbb{C}} \mathfrak{n}^{10}$, we conclude that $\mathfrak{n}^{10} = \mathfrak{g}_1 + \mathfrak{g}_3$ and that $\mathfrak{m}^{10} = \mathfrak{g}_1 + \mathfrak{g}_2 + \mathfrak{g}_3$. Indeed, since $\mathfrak{l}^{\mathbb{C}} + \mathfrak{g}_1 + \mathfrak{g}_2 + \mathfrak{g}_3$ is always a subalgebra, there exists an integrable CR structure whose associated holomorphic subspace is $\mathfrak{m}^{10} = \mathfrak{g}_1 + \mathfrak{g}_2 + \mathfrak{g}_3$. Furthermore, $N_{\mathfrak{g}}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01})$ contains $Z = iH_\nu$ and hence this CR structure is standard. □

Proof of Corollary 5.2. (1) By Theorem 5.1, it remains only need to check that any non-standard CR structure on $M = SU_2$ is circular and that the associated anti-canonical map is a finite covering.

By (5.10), the CR structure $(\mathcal{D}_Z, J)$ is non-standard if and only if the corresponding holomorphic subspace is of the form $\mathfrak{m}^{10} = \mathbb{C}(E_\alpha + tE_{-\alpha})$ with $0 < |t| < 1$. Since $\mathfrak{l} = \{0\}$ and the element $E_\alpha + tE_{-\alpha}$ is a regular element of $\mathfrak{sl}_2(\mathbb{C})$, then $\mathfrak{m}^{10}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$ and any parabolic subalgebra $\mathfrak{p}$ which contains $\mathfrak{m}^{10}$ satisfies the conditions a), b) and c) of Lemma 4.8. This implies that $M = SU_2$ admits a CRF fibration over SU_2/T^1, where T^1 is the 1-dimensional subgroup generated by the subspace $\mathfrak{t} = \mathfrak{p} \cap \mathfrak{su}_2$.

On the other hand, when $0 < |t| < 1$,

$$\begin{aligned} &N_{\mathfrak{g}}(\mathbb{C}(E_\alpha + tE_{-\alpha})) \\ &= \{X = a(iH_\alpha) + b(E_\alpha + E_{-\alpha}) + ic(E_\alpha - E_{-\alpha}) \\ &\qquad\qquad \in \mathfrak{su}_2 : [X, E_\alpha + tE_{-\alpha}] \in \mathbb{C}(E_\alpha + tE_{-\alpha})\} = \{0\}. \end{aligned}$$

Then, by the remarks at the end of §4.4, the stabilizer Q of the image of the anti-canonical map $\phi(SU_2) = SU_2/Q$ is 0-dimensional and the anti-canonical map is a covering map.

(2) We first observe that each non-standard CR structure $(\mathcal{D}_Z, J_t^{(0)})$ is primitive. In fact, by Lemma 4.8, if one of such CR structures is non-primitive, then there exists a parabolic subalgebra $\mathfrak{p} \subsetneq \mathfrak{g}$, which satisfies a), b) and c) of Lemma 4.8. On the other hand, one can check that in this case, there is no proper subalgebra of $\mathfrak{g}^{\mathbb{C}}$ which properly contains $\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10}$, with $\mathfrak{l}^{\mathbb{C}} = \mathfrak{g}'_0$ and $\mathfrak{m}^{10}$ as in (5.13).

Now, we want to prove that each non-standard CR structure $(\mathcal{D}_Z, J_t)$ or $(\mathcal{D}_Z, J'_t)$ admits a CRF fibration onto $Gr_2(\mathbb{C}^\ell) = SU_\ell/S(U_2 \cdot U_{\ell-2})$.

Indeed, note that, if we consider the decomposition (5.3) determined by the regular contact element $Z = iH_\mu$, any CR structure $(\mathcal{D}_Z, J_t)$ or $(\mathcal{D}_Z, J'_t)$ corresponding to the holomorphic subspaces defined in (5.11) and (5.12) satisfies

$$\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01} \subset \mathfrak{p} = \mathfrak{g}_0 + \mathfrak{g}_{-1}^{(1)} + \mathfrak{g}_1^{(2)} + \mathfrak{g}_{-2} + \mathfrak{g}_2, \tag{5.19}$$

$$\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}'^{10} \subset \mathfrak{p}' = \mathfrak{g}_0 + \mathfrak{g}_{-1}^{(2)} + \mathfrak{g}_1^{(1)} + \mathfrak{g}_{-2} + \mathfrak{g}_2, \tag{5.20}$$

respectively. A reductive part for both subalgebras $\mathfrak{p}$ and $\mathfrak{p}'$ is $\mathfrak{r} = \mathfrak{r}' = (\mathfrak{l} + \mathfrak{a})^{\mathbb{C}}$. Therefore, by Lemma 4.8, the CR structures $(\mathcal{D}, J_t)$ and $(\mathcal{D}, J'_t)$ are non-primitive and they admit a CRF fibration over the Wolf space $SU_{\ell+1}/S(U_2 \cdot U_{\ell-1})$ with typical fiber $S(U_2 \cdot U_{\ell-1})/U_{\ell-1} = SO_3$.

We now want to prove that any non-standard CR structure $(\mathcal{D}_Z, J_t)$ or $(\mathcal{D}_Z, J'_t)$ admits also a CRF fibration with standard fiber S^1. Let us use the same notation as before and observe that, for any complex holomorphic subspace $\mathfrak{m}^{10}$ or $\mathfrak{m}'^{10}$ defined in (5.11) or (5.12), the element $X = E_\mu + tE_{-\mu} \in \mathfrak{m}^{10} \cap \mathfrak{m}'^{10}$ is a regular element of $\mathfrak{g}^{\mathbb{C}}(\mu) \subset \mathfrak{g}^{\mathbb{C}}$. Hence, if we denote by $\hat{\mathfrak{p}}$ any parabolic subalgebra $\hat{\mathfrak{p}}(\mu) \subset \mathfrak{g}^{\mathbb{C}}(\mu)$, which properly contains $E_\mu + tE_{-\mu}$ or $E_\mu + t^2 E_{-\mu}$, we get that

$$\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01} \subset \mathfrak{p}_\mu = \mathfrak{g}_0' + \mathfrak{g}_{-1}^{(1)} + \mathfrak{g}_1^{(2)} + \hat{\mathfrak{p}}(\mu), \tag{5.21}$$

$$\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}'^{10} \subset \mathfrak{p}_\mu' = \mathfrak{g}_0 + \mathfrak{g}_{-1}^{(2)} + \mathfrak{g}_1^{(1)} + \hat{\mathfrak{p}}(\mu). \tag{5.22}$$

Note that $\mathfrak{p}_\mu$ and $\mathfrak{p}_\mu'$ are two parabolic subalgebras of $\mathfrak{g}^{\mathbb{C}}$ which satisfy a), b) and c) of Lemma 4.8 and hence that the CR structures $(\mathcal{D}, J_t)$ and $(\mathcal{D}, J_t')$ admit CRF fibrations with 1-dimensional fibers.

It remains to check that the anti-canonical map of any non-standard CR structure is a covering map. As in the proof of (1), this reduces to checking that for any holomorphic subspace defined in (5.11) and (5.12), $N_{\mathfrak{g}}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{10}) = N_{\mathfrak{g}}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}'^{10}) = \mathfrak{l}^{\mathbb{C}}$ and hence that the image of the anti-canonical map has the same dimension as G/L.

(3) It is only a matter to check that the non-standard CR structures with holomorphic subspace $\mathfrak{m}^{10}$ given in (5.9) are primitive. This can be done as in (2), using Lemma 4.8. □

6. A characterization of non-standard CR structures

The aim of this section is to give a criterion which distinguishes standard CR structures in the class of circular CR structures and to furnish a complete characterization of standard and non-standard CR structures by means the anti-canonical map.

Let $(\mathcal{D}, J)$ be a circular CR structure on G/L and let $Z_{\mathcal{D}}$ be a contact element associated to $\mathcal{D}$. Let also $\pi : G/L \to G/Q$ be the CRF fibration onto the flag manifold G/Q with fiber $S^1 = Q/L$. Notice that, since $\mathfrak{q}$ is the isotropy subalgebra of a flag manifold, $\mathfrak{q}$ is of the form $\mathfrak{q} = \mathfrak{l} + \mathbb{R}Z_J$ for some $Z_J \in C_{\mathfrak{g}}(\mathfrak{l}) \cap (\mathfrak{l})^\perp$.

Moreover, since π is holomorphic, we also have that $\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01} \subset \mathfrak{q}^{\mathbb{C}} + \mathfrak{m}^{01}$ and that $\mathfrak{q}^{\mathbb{C}} + \mathfrak{m}^{01}$ is a subalgebra with nilradical $\mathfrak{m}^{01}$. This implies that $\mathfrak{q} = \mathfrak{l} + \mathbb{R}Z_J \subset N_{\mathfrak{g}}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01})$.

At this point, we need the following Lemma, which in fact was proved in [1].

Lemma 6.1 *Let $G/Q = \phi(G/L)$ be the image of the anticanonical map. Then* $\dim Q/L \leq 1$.

Proof. We need to prove that $\dim \mathfrak{q}/\mathfrak{l} \leq 1$, where $\mathfrak{q} = N_{\mathfrak{g}}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01})$ is the stability subalgebra of the flag manifold G/Q. Since $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m}$, it is sufficient to check that $\mathfrak{q} \cap \mathfrak{m} = 0$. Let $v \in \mathfrak{q} \cap \mathfrak{m}$. Then

$$\mathcal{B}(Z, [v, \mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}]) \subset \mathcal{B}(Z, \mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}) = \{0\}$$

and in particular

$$\{0\} = \mathcal{B}(Z, [v, \mathfrak{l} + \mathfrak{m}]) = -\mathcal{B}([v, Z], \mathfrak{l} + \mathfrak{m}).$$

This means that $v \in N_{\mathfrak{g}}(Z) = \mathfrak{k} = \mathfrak{l} + \mathbb{R}Z$ and hence that $v \in \mathfrak{k} \cap \mathfrak{m} = \{0\}$. $\square$

By Lemma 6.1, we have that $\dim N_{\mathfrak{g}}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}) \leq \dim \mathfrak{l} + 1$ and therefore that $\mathfrak{q} = N_{\mathfrak{g}}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01})$. In particular, the circular CR structure $(\mathcal{D}, J)$ is standard if and only if $\mathfrak{q} = \mathfrak{k}$, i.e. if and only if $\mathbb{R}Z_J = \mathbb{R}Z_{\mathcal{D}}$.

If G/L is a contact manifold of non-special type, then $\dim C_{\mathfrak{g}}(\mathfrak{l}) \cap (\mathfrak{l})^{\perp} = 1$ and hence $\mathbb{R}Z_J = \mathbb{R}Z_{\mathcal{D}}$. From this we conclude that any circular CR structure on a non-special contact manifold is standard and a circular non-standard CR structure may exist only on a special contact manifold.

Now, the class of all invariant CR structures on special contact manifolds is explicitly classified in Theorem 5.1 and Corollary 5.2. From this classification, the following description of all circular CR structures is immediately obtained.

Theorem 6.2 *Let $M = G/L$ be a homogeneous contact manifold of a compact Lie group G. Then $M = G/L$ admits an invariant non-standard circular CR structure $(\mathcal{D}, J)$ if and only if $M = SU_\ell/U_{\ell-2}$ for $\ell \geq 2$.*

By means of Theorem 6.2, we may finally obtain the following important description of standard and non-standard CR structures.

Theorem 6.3 *Let*

$$\phi : M = G/L \longrightarrow Gr_k(\mathfrak{g}^{\mathbb{C}})$$

be the anticanonical map of a homogeneous CR manifold $(M = G/L, \mathcal{D}_Z, J)$.

(1) *If the CR structure is standard, then the image $\phi(M)$ is G-equivariantly biholomorphic to the associated flag manifold $F_Z = G/K = \mathrm{Ad}_G Z$ en-*

dowed with the complex structure J_F which induces the CR structure $(\mathcal{D}_Z, J)$.

In this case, ϕ is a CRF fibration with fiber S^1 and the normalizer in $\mathfrak{g}$ of $\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}$ is

$$\mathfrak{k} = N_{\mathfrak{g}}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}) = \mathfrak{l} + \mathbb{R}Z$$

and it is equal to the stabilizer of the point $[\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}] \in \phi(M)$ in G.

(2) *If the CR structure is not standard, then the image $\phi(M) = G/Q$ is a homogeneous CR manifold with CR structure induced by the complex structure of $Gr_k(\mathfrak{g}^{\mathbb{C}})$ and $\phi : M \to \phi(M)$ is a finite covering.*

Proof. (1) Notice that, by Lemma 4.5, if $(\mathcal{D}_Z, J)$ is standard then $N_{\mathfrak{g}}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}) \supset \mathfrak{l} + \mathbb{R}Z$. Therefore, from Lemma 6.1, we get that $N_{\mathfrak{g}}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01}) = \mathfrak{l} + \mathbb{R}Z = \mathfrak{k}$ and the image $\phi(G/L)$ of the anticanonical map coincides with the flag manifold $F = G/K$.

To prove (2), we first show that if the CR structure is non-standard, then the anti-canonical map $\varphi : G/L \to \phi(G/L)$ is a finite covering. In fact, if the CR structure is non-circular, the fiber of the anticanonical map is not 1-dimensional (otherwise it would give a CRF fibration with S^1-fiber) and by Lemma 6.1 this implies that $\varphi : G/L \to \phi(G/L)$ is a finite covering. If the CR structure is circular and non-standard, by Theorem 6.2 and Corollary 5.2, $M = SU_\ell/U_{\ell-2}$ and again $\varphi : G/L \to \phi(G/L)$ is a finite covering. The other part of the claim follows immediately by the holomorphicity and the G-equivariance of ϕ. □

7. Classification of non-standard CR structures on non-special homogeneous contact manifolds

7.1. Notation

In all this section,

- $(G/L, \mathcal{D}_Z)$ denotes a simply connected non-special homogeneous contact manifold of a compact Lie group G;
- $\mathfrak{k} = C_{\mathfrak{g}}(Z) = \mathfrak{l} \oplus \mathbb{R}Z$ is the orthogonal decomposition of the centralizer $\mathfrak{k}$ of Z and $\mathfrak{m}$ is the orthogonal complement to $\mathfrak{k}$ in $\mathfrak{g}$;
- $\mathfrak{h} \subset \mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{k}$ and hence of $\mathfrak{g}$;
- $\theta = \mathcal{B} \circ Z|_{\mathfrak{h}}$ is the 1-form on $\mathfrak{h}$ dual to Z and $\vartheta = -i\theta = -i\mathcal{B} \circ Z|_{\mathfrak{h}}$; we will refer to both of them as *contact forms*;
- R (resp. R_o) is the root system of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ (resp. of $(\mathfrak{k}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$) and $R' =$

$R \setminus R_o$;
- E_α is the root vector with root α in the Chevalley normalization (see e.g. [7]);
- a subset $S \subset R$ is called *closed subsystem* if $(S+S) \cap R \subset S$;
- if S is a closed subsystem of roots, then $\mathfrak{g}(S) \subset \mathfrak{g}^{\mathbb{C}}$ is the subalgebra generated by the root vectors E_α, $\alpha \in S$;
- recall that the root vectors E_α, $\alpha \in R'$, span $\mathfrak{m}^{\mathbb{C}}$;
- $\mathfrak{m}(\alpha)$ denotes the irreducible $\mathfrak{k}^{\mathbb{C}}$-submodules of $\mathfrak{m}^{\mathbb{C}}$, with highest weight $\alpha \in R'$;
- if $\mathfrak{m}(\alpha)$ and $\mathfrak{m}(\beta)$ are equivalent as $\mathfrak{l}^{\mathbb{C}}$-modules, we denote by $\mathfrak{m}(\alpha) + t\mathfrak{m}(\beta)$ the irreducible $\mathfrak{l}^{\mathbb{C}}$-module with the highest weight vector $E_\alpha + tE_\beta$, $\alpha, \beta \in R'$, $t \in \mathbb{C}$; note that together with $\mathfrak{m}(\beta)$, these modules exhaust all the irreducible $\mathfrak{l}^{\mathbb{C}}$-submodules of $\mathfrak{m}(\alpha) + \mathfrak{m}(\beta)$ (see Lemma 7.1);
- by *Dynkin graph* Γ we will understand the Dynkin graph associated with a root system R of a compact semisimple Lie algebra $\mathfrak{g}$; we associate with the nodes of Γ the simple roots of R as in [7] (see Table 4 in the Appendix).

7.2. Preliminaries

By the results in §5, the classification of invariant CR structures reduces to the classification of non-standard CR structures on homogeneous contact manifolds of non-special type. This will be the contents of §7.3 and §7.4.

In this section we give two important lemmata that settle the main tools for the classification. The first Lemma is an immediate corollary of Proposition 3.9.

Lemma 7.1 *Let $(M = G/L, \mathcal{D}_Z, J)$ be a homogeneous CR manifold associated with holomorphic subspace $\mathfrak{m}^{10} \subset \mathfrak{m}^{\mathbb{C}}$ and J the associated complex structure on $\mathfrak{m}$. Assume also that $G \neq G_2$ or that $G = G_2$ and that the contact form ϑ is not proportional to a short root of R.*

Then a minimal J-invariant $\mathfrak{k}^{\mathbb{C}}$-submodule $\mathfrak{n}$ of $\mathfrak{m}^{\mathbb{C}}$ is either $\mathfrak{k}^{\mathbb{C}}$-irreducible (and hence $\mathfrak{n} = \mathfrak{m}(\alpha)$ for some $\alpha \in R'$) or it is the sum $\mathfrak{m}(\alpha) + \mathfrak{m}(\beta)$ of two such $\mathfrak{k}^{\mathbb{C}}$-modules, where the roots α and β are ϑ-congruent (i.e. $\beta = \alpha + \lambda\vartheta$, for some $\lambda \in \mathbb{R}$).

Proof. Consider the decomposition $\mathfrak{m}^{\mathbb{C}} = \sum \mathfrak{m}(\gamma)$ into irreducible $\mathfrak{k}$-submodules as in §3.3. The claim follows immediately from the fact that any

$\mathrm{ad}_{\mathfrak{l}}$-invariant complex structure J on $\mathfrak{m}$ preserves the $\mathfrak{l}^{\mathbb{C}}$-*isotypic components* (i.e. the sum of all mutually equivalent irreducible $\mathfrak{l}^{\mathbb{C}}$-modules) and that, under the hypotheses of Proposition 3.9, the multiplicity of any irreducible $\mathfrak{l}$-module $\mathfrak{m}(\gamma)$ is less or equal to 2. □

Lemma 7.2 *Let $(G/L, \mathcal{D}_Z, J)$ be a homogeneous CR manifold with non-standard CR structure. Then G is either simple or of the form $G = G_1 \times G_2$, where each G_i is simple.*

Moreover, if $G = G_1 \times G_2$ and $R = R_1 \cup R_2$ is the corresponding decomposition of the root system, then there exist two roots $\mu_1 \in R_1$, $\mu_2 \in R_2$, such that the pairs of roots $(\mu_1, -\mu_2)$ and $(-\mu_1, \mu_2)$ are the only ones which are ϑ-congruent; in particular, $\vartheta = \mu_1 + \mu_2$ is not proportional to any root.

Proof. Since the CR structure $(\mathcal{D}_Z, J)$ is non-standard, the associated complex structure J on $\mathfrak{m}$ is not $\mathrm{ad}_{\mathfrak{k}}$-invariant; in particular there exists some minimal J-invariant $\mathfrak{k}^{\mathbb{C}}$-module in $\mathfrak{m}^{\mathbb{C}}$, which is not $\mathfrak{k}^{\mathbb{C}}$-irreducible. By Lemma 7.1, there exist at least two roots α, β, which are ϑ-congruent. Without loss of generality, we may assume that $\vartheta = \alpha - \beta$.

If ϑ is proportional to some root γ, then this root belongs to some summand $\mathfrak{g}_i$ of $\mathfrak{g}$, $i = 1, \ldots, r$. Hence, $\mathfrak{k} = C_{\mathfrak{g}}(Z)$ contains all other simple summands of $\mathfrak{g}$ and the same holds for $\mathfrak{l}$. By effectivity, this implies that $\mathfrak{g} = \mathfrak{g}_1$.

If $\vartheta = \alpha - \beta$ is not proportional to any root and α and β belong to the same summand $\mathfrak{g}_1$, then $\mathfrak{g} = \mathfrak{g}_1$ as before. Assume that they belong to two different summands $\mathfrak{g}_1$ and $\mathfrak{g}_2$. The same arguments of before show that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and that $\pm(\alpha, \beta)$ are the only pairs of roots which are ϑ-congruent. □

We will perform the classification by considering separately two cases: when the contact form ϑ is proportional to a root and when it is not proportional to any root. Note that by Lemma 7.2, the first case may occur only when G is simple.

7.3. Case when the contact form is proportional to a root

Recall that the Weyl group of a simple Lie group acts transitively on the set of roots of the same length. In particular any long root can be considered as a maximal root. Since we assume that the contact manifold $(M = G/L, \mathcal{D}_Z)$ is non-special and G is simple, we may suppose that ϑ is proportional to a short root (i.e. strictly shorter then a long root) and hence G equals either SO_{2n+1}, Sp_n or F_4. Note that if $G = G_2$ then any contact manifold $(G_2/L, \mathcal{D}_Z)$, with contact form ϑ proportional to a short root, is special (see §3.2.2).

Proposition 7.3 *Let $(G/L, \mathcal{D}_Z)$ be a homogeneous non-special contact manifold of a simple group G, such that the contact form ϑ is proportional to a root. Then:*

(1) *G/L is SO_{2n+1}/SO_{2n-1}, $Sp_n/Sp_1 \times Sp_{n-2}$ or F_4/SO_7 and ϑ is proportional to a short root of G;*

(2) *there exists a 1-1 correspondence between the invariant CR structures on $(G/L, \mathcal{D}_Z)$ (determined up to a sign) and the points of the unit disc $D \subset \mathbb{C}$;*

(3) *more precisely, any point $t \in D$ corresponds to the CR structure $(\mathcal{D}_Z, J_t)$ whose holomorphic subspace $\mathfrak{m}^{10}$ is listed in the following table (see §7.1 for notation) :*

G/L	ϑ	$\mathfrak{m}^{10}$
$\frac{SO_{2n+1}}{SO_{2n-1}} = S(S^{2n})$	ε_1	$\mathfrak{m}(\varepsilon_1+\varepsilon_2) + t\mathfrak{m}(-\varepsilon_1+\varepsilon_2)$
$\frac{Sp_n}{Sp_1 \times Sp_{n-2}} = S(\mathbb{H}P^{n-1})$	$\varepsilon_1+\varepsilon_2$	$(\mathfrak{m}(2\varepsilon_1) + t^2\mathfrak{m}(-2\varepsilon_2)) \oplus (\mathfrak{m}(\varepsilon_1+\varepsilon_3) + t\mathfrak{m}(-\varepsilon_2+\varepsilon_3))$
$\frac{F_4}{\mathrm{Spin}_7} = S(\mathbb{O}P^2)$	ε_1	$(\mathfrak{m}(\varepsilon_1+\varepsilon_2) + t^2\mathfrak{m}(-\varepsilon_1+\varepsilon_2))$ $\oplus\, (\mathfrak{m}(1/2(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4))$ $+\, t\mathfrak{m}(1/2(-\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)))$

(4) *a CR structure $(\mathcal{D}_Z, J_t)$ is standard if and only if $t = 0$; in all other cases it is primitive.*

Proof. For each group G equal to $SO_{2\ell+1}$, Sp_ℓ or F_4 we may assume that ϑ is the short root $\vartheta = \varepsilon_1$, $\varepsilon_1 + \varepsilon_2$ or ε_1, respectively. The associated decomposition $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m}$ is given in Table 2 of the Appendix. It is not difficult to determine the decomposition of $\mathfrak{m}^{\mathbb{C}}$ into irreducible submodules. The result is given in Table 2. Then one has to find all decompositions $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^{10} + \mathfrak{m}^{01}$ into two $\mathfrak{l}^{\mathbb{C}}$-modules which satisfy the following conditions: a) $\mathfrak{m}^{01} = \overline{\mathfrak{m}^{10}}$; b) $[\mathfrak{m}^{10}, \mathfrak{m}^{10}] \subset \mathfrak{m}^{10} + \mathfrak{l}^{\mathbb{C}}$. The modules $\mathfrak{m}^{10}$ which satisfy condition a) are of the following form:

$$\begin{aligned}
G = SO_{2\ell+1} : \ & \mathfrak{m}^{10} = \mathfrak{m}^{10}_t = \mathfrak{m}(\varepsilon_1+\varepsilon_2) + t\mathfrak{m}(-\varepsilon_1+\varepsilon_2);\\
G = Sp_\ell : \ & \mathfrak{m}^{10} = \mathfrak{m}^{10}_{t,s} = (\mathfrak{m}(2\varepsilon_1) + s\mathfrak{m}(-2\varepsilon_2))\\
& \quad \oplus (\mathfrak{m}(\varepsilon_1+\varepsilon_3) + t\mathfrak{m}(-\varepsilon_2+\varepsilon_3));\\
G = F_4 : \ & \mathfrak{m}^{10} = \mathfrak{m}^{10\prime}_{t,s} = (\mathfrak{m}(\varepsilon_1+\varepsilon_2) + s\mathfrak{m}(-\varepsilon_1+\varepsilon_2))\\
& \quad \oplus (\mathfrak{m}(1/2(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)) + t\mathfrak{m}(1/2(-\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)))
\end{aligned}$$

for some $s, t \neq 0$. One can easily check that $\mathfrak{m}^{10}_t$ satisfies condition b) for every t. The module $\mathfrak{m}^{10}_{t,s}$ satisfies condition b) if and only if $s = t^2$. To prove it one should observe that the only brackets between $\mathfrak{l}^{\mathbb{C}}$-weight vectors in $[\mathfrak{m}^{10}_{t,s}, \mathfrak{m}^{10}_{t,s}]$, which are non-trivial modulo $\mathfrak{l}^{\mathbb{C}}$, are

$$\begin{aligned}
&[E_{\varepsilon_1+\varepsilon_i} + tE_{-\varepsilon_2+\varepsilon_i}, E_{\varepsilon_1-\varepsilon_i} + tE_{-\varepsilon_2-\varepsilon_i}]\\
&\qquad \equiv N_{\varepsilon_1+\varepsilon_i,\varepsilon_1-\varepsilon_i} E_{2\varepsilon_1} + t^2 N_{-\varepsilon_2+\varepsilon_i,-\varepsilon_2+\varepsilon_i} E_{-2\varepsilon_2} \qquad \mod \mathfrak{l}^{\mathbb{C}}\\
&[E_{\varepsilon_1+\varepsilon_i} + tE_{-\varepsilon_2+\varepsilon_i}, E_{\varepsilon_2-\varepsilon_i} + tE_{-\varepsilon_1-\varepsilon_i}]\\
&\qquad \equiv N_{\varepsilon_1+\varepsilon_i,\varepsilon_2-\varepsilon_i} E_{\varepsilon_1+\varepsilon_2} + t^2 N_{-\varepsilon_2+\varepsilon_i,-\varepsilon_1+\varepsilon_i} E_{-\varepsilon_1-\varepsilon_2} \quad \mod \mathfrak{l}^{\mathbb{C}}
\end{aligned}$$

By a straightforward computation, it follows that these vectors are in $\mathfrak{m}^{10}_{t,s}$ if and only if $s = t^2$.

A similar argument shows that also $\mathfrak{m}^{10\prime}_{t,s}$ satisfies condition b) if and only if $s = t^2$.

Observe that up to an exchange between $\mathfrak{m}^{10}$ and $\mathfrak{m}^{01}$ (which corresponds to changing the sign of complex structure J), we may always assume that $|t| \leq 1$. It remains to check the condition $\mathfrak{m}^{01} \cap \mathfrak{m}^{10} = \{0\}$: in all cases, this implies $\det \begin{bmatrix} 1 & t \\ \bar{t} & 1 \end{bmatrix} \neq 0$ and hence that $|t| < 1$.

To prove (4), note that, in all cases listed in the table above, $N_{\mathfrak{g}}(\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}^{01})$ contains Z only if $t = 0$ and hence, by Theorems 6.2 and 6.3, this is the only case when the CR structure is standard. Moreover, in all cases, if $t \neq 0$ there exists no proper parabolic subalgebra $\mathfrak{p} \supset \mathfrak{l}^{\mathbb{C}}$ which satisfies the conditions of Lemma 4.8. □

7.4. Painted Dynkin graphs and CR-graphs

In this subsection we introduce the concepts of painted Dynkin graphs and of CR-graphs. They will be necessary to state the classification of non-standard CR structures corresponding to contact forms not proportional to any root.

A *painted Dynkin graphs of* $\mathfrak{g} = \mathrm{Lie}(G)$ is a Dynkin graph of the Lie algebra $\mathfrak{g}$ with nodes painted in three colors: white ($\circ$), black ($\bullet$) and 'grey' ($\otimes$).

Recall that any flag manifold $F = G/Q$ with an invariant complex structure J_F is defined (up to equivalence) by a black-white Dynkin graph, where the subalgebra $\mathfrak{q} = \mathrm{Lie}(Q)$ is generated by the Cartan subalgebra and the root vectors associated with the white nodes. The complex structure J_F is determined by the decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{q}^{\mathbb{C}} + \mathfrak{m}^{10} + \mathfrak{m}^{01}$$

where $\mathfrak{m}^{10}$ is the nilpotent subalgebra generated by the root vectors associated to black nodes (see e.g. [3], [4]).

With a painted Dynkin graph Γ (equipped by simple roots in a standard way), we associate two flag manifolds $F_1(\Gamma) = G/K$ and $F_2(\Gamma) = G/Q$ and two invariant complex structure $J_1(\Gamma)$ and $J_2(\Gamma)$ on $F_1(\Gamma)$ and $F_2(\Gamma)$, respectively, as follows. The pairs $(F_1(\Gamma) = G/K, J_1(\Gamma))$ and $(F_2(\Gamma) = G/Q, J_2(\Gamma))$ are the flag manifolds with invariant complex structures defined by the black-white graphs obtained from Γ by considering the grey nodes as black and, respectively, white.

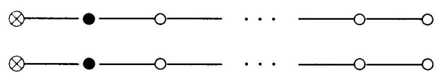
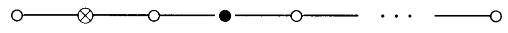
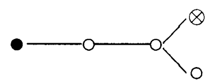
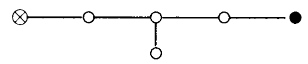
Note that Q contains K and that the natural fibration

$$\varpi : F_1(\Gamma) = G/K \to F_2(\Gamma) = G/Q$$

is holomorphic and a fiber Q/K is a flag manifold with an induced invariant complex structure J'. Moreover, $J_1(\Gamma)$ is canonically defined by $J_2(\Gamma)$ and J'.

Conversely, if $F_1 = G/K$ and $F_2 = G/Q$ are two flag manifolds with invariant complex structures J_1 and J_2 such that $Q \supset K$ and the equivariant fibration $\varpi : F_1 \to F_2$ is holomorphic, then we may associate with F_1 and F_2 a painted Dynkin graph in an obvious way.

Definition 7.4 A *CR-graph* is a pair $(\Gamma, \vartheta(\Gamma))$, formed by a painted Dynkin graph Γ and a linear combination $\vartheta(\Gamma)$ of simple roots, given in the following table:

type	$\mathfrak{g}$	Γ	$\vartheta(\Gamma)$
I	$A_n \quad (n > 1)$		$\varepsilon_1 - \varepsilon_2$
II	$A_p + A'_q$ $(p + q > 2)$		$(\varepsilon_1 - \varepsilon_2) - (\varepsilon'_1 - \varepsilon'_2)$
III	$A_n \quad (n > 3)$		$\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4$
IV	D_5		$\varepsilon_{n-3} + \varepsilon_{n-2} + \varepsilon_{n-1} + \varepsilon_n$
V	E_6		$2\varepsilon_1 + \varepsilon_6 + \varepsilon$

The correspondence between nodes and simple roots is as in Table 4 of the Appendix.

The CR-graphs of type I are called *special CR-graph.* All the others are called *non-special CR-graphs.*

Let $(\Gamma, \vartheta(\Gamma))$ be a CR-graph. We fix a Cartan subalgebra $\mathfrak{h}$ of the associated compact Lie algebra $\mathfrak{g}$ and define the element $Z(\Gamma) = i\mathcal{B}^{-1} \circ \vartheta(\Gamma) \in \mathfrak{h}$. Then $Z(\Gamma)$ is a contact element and we call the corresponding contact manifold $(M(\Gamma) = G/L, \mathcal{D}_{Z(\Gamma)})$ *the contact manifold associated with the CR-graph* $(\Gamma, \vartheta(\Gamma))$. Note that $M(\Gamma)$ is special if and only if the CR-graph is special.

7.5. Case when the contact form is not proportional to any root

In this case we obtain the following classification.

Proposition 7.5 *Let $(M = G/L, \mathcal{D}_Z)$ be a contact manifold with contact form ϑ not proportional to any root. If it admits a primitive invariant CR structure $(\mathcal{D}_Z, J)$, then it is one of the following.*

If G is simple then

a) *$G/L = SO_{2n}/SO_{2n-2}$, $n > 2$, and ϑ is either ε_1 or, when $n = 4$, $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 \pm \varepsilon_4$; moreover the holomorphic subspace of the CR structure $(\mathcal{D}_Z, J)$ is given by*

$$\mathfrak{m}^{10} = \mathfrak{m}(\varepsilon_1 + \varepsilon_2) + t\mathfrak{m}(\beta) \tag{7.1}$$

where $\beta = -\varepsilon_1 + \varepsilon_2$, $-\varepsilon_3 - \varepsilon_4$ or $-\varepsilon_3 + \varepsilon_4$ (the last two cases occur only for $n = 4$) and t belongs to the punctured unit disc $D \setminus \{0\} \subset \mathbb{C}$;

b) $G/L = \mathrm{Spin}_7/SU_3 = S(S^7) = S^7 \times S^6$, $\vartheta = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$ *and the holomorphic subspace of* $(\mathcal{D}_Z, J)$ *is given by*

$$\mathfrak{m}^{10} = \mathfrak{m}(\varepsilon_1 + \varepsilon_2) + t\mathfrak{m}(-\varepsilon_3) + \overline{\mathfrak{m}(\varepsilon_1 + \varepsilon_2)} + \frac{1}{t}\overline{\mathfrak{m}(-\varepsilon_3)} \tag{7.2}$$

for some $t \in D \setminus \{0\}$.

If G *is not simple then*

c) $G/L = SU_2 \times SU_2/T^1 = S(S^3) = S^3 \times S^2$, $\vartheta = (\varepsilon_1 - \varepsilon_2) - (\varepsilon_1' - \varepsilon_2')$ *and the holomorphic subspace of* $(\mathcal{D}_Z, J)$ *is*

$$\mathfrak{m}^{10} = \mathbb{C}(E_{\varepsilon_1 - \varepsilon_2} + tE_{\varepsilon_1' - \varepsilon_2'}) + \mathbb{C}\Big(E_{-(\varepsilon_1 - \varepsilon_2)} + \frac{1}{t}E_{-(\varepsilon_1' - \varepsilon_2')}\Big). \tag{7.3}$$

In all cases we consider ϑ *up to a factor and up to a transformation from the Weyl group* $W(R)$, *and* J *up to a sign.*

Proposition 7.6 *A homogeneous contact manifold* $(G/L, \mathcal{D}_Z)$ *with contact form* ϑ *not proportional to any root, admits a non-standard non-primitive CR structure if and only if it is G-contact diffeomorphic to the contact manifold* $(M(\Gamma) = G/L, \mathcal{D}_{Z(\Gamma)})$ *associated with a non-special CR-graph* $(\Gamma, \vartheta(\Gamma))$ *(see Definition* 7.4*).*

For any invariant CR structure $(\mathcal{D}_{Z(\Gamma)}, J)$ *on* $M(\Gamma) = G/L$ *the natural projection* $\pi : M(\Gamma) = G/L \to F_2(\Gamma) = G/Q$ *is holomorphic w.r.t. the complex structure* $J_2(\Gamma)$ *or* $-J_2(\Gamma)$.

The CR structures for which π *is holomorphic w.r.t.* $J_2(\Gamma)$ *are in* 1-1 *correspondence with the invariant CR structures on the fiber* $C = Q/L$ *subordinated to the induced contact structure* $\mathcal{D}_{Z(\Gamma)} \cap TC$.

More precisely, if

$$\mathfrak{q}^{\mathbb{C}} = \mathfrak{l}^{\mathbb{C}} + \mathbb{C}Z + \mathfrak{m}_C^{10} + \mathfrak{m}_C^{01}, \quad \mathfrak{g}^{\mathbb{C}} = \mathfrak{q}^{\mathbb{C}} + \mathfrak{m}_{J_2}^{10} + \mathfrak{m}_{J_2}^{01}$$

are the two decompositions of $\mathfrak{q}^{\mathbb{C}}$ *and* $\mathfrak{g}^{\mathbb{C}}$ *associated with an invariant CR structure on the fiber* $C = Q/L$ *and with the complex structure* $J_2(\Gamma)$ *on* $F_2(\Gamma)$, *then*

$$\mathfrak{m}^{10} = \mathfrak{m}_C^{10} + \mathfrak{m}_{J_2}^{10} \tag{7.4}$$

is the holomorphic subspace of the corresponding CR structure on $M(\Gamma)$. *Moreover, this CR structure is non-standard if and only if the CR structure on* C *is primitive.*

The rest part of the section is devoted to the proof of Propositions 7.5

and 7.6. We need some additional notations.

For a fixed CR structure $(\mathcal{D}_Z, J)$, we set

$$R_J^{\pm} = \{\alpha \in R' : J(E_\alpha) = \pm i E_\alpha\},$$
$$R_J = R_J^+ \cup R_J^-, \quad R_{\mathfrak{e}} \stackrel{\text{def}}{=} R' \setminus R_J \tag{7.5}$$

and we define the subspaces

$$\mathfrak{m}_J^{\pm} = \sum_{\beta \in R_J^{\pm}} \mathbb{C} E_\beta, \quad \mathfrak{m}_J = \mathfrak{m}_J^+ + \mathfrak{m}_J^-, \quad \mathfrak{e} \stackrel{\text{def}}{=} \sum_{\beta \in R_{\mathfrak{e}}} \mathbb{C} E_\beta \subset \mathfrak{m}^{\mathbb{C}}. \tag{7.6}$$

Note that J is standard if and only if $R_J = R'$. We define also the closed subsystem

$$\tilde{R}_{\mathfrak{e}} = [R_{\mathfrak{e}}] \stackrel{\text{def}}{=} R \cap \operatorname{span}_{\mathbb{R}}(R_{\mathfrak{e}}), \quad \tilde{R}_o = R_o \cap \tilde{R}_{\mathfrak{e}},$$

and we set $R'_o = R_o \setminus \tilde{R}_o$.

The following Lemma collects some basic properties of these objects.

Lemma 7.7

(1) *$R_J = -R_J$ and $R_{\mathfrak{e}} = -R_{\mathfrak{e}}$;*

(2) *for any $\alpha \in R_{\mathfrak{e}}$ there exists exactly one root $\beta \in R_{\mathfrak{e}}$ which is ϑ-congruent to α;*

(3) *for any pair $\alpha, \beta \in R_{\mathfrak{e}}$ of ϑ-congruent roots, there exist two uniquely determined complex numbers $\lambda, \mu \neq 0$ such that*

$$e_{\alpha,\beta} = E_\alpha + \lambda E_\beta \in \mathfrak{m}^{10}, \qquad f_{\alpha,\beta} = E_\alpha + \mu E_\beta \in \mathfrak{m}^{01}. \tag{7.7}$$

(4) *$(R_J^{\pm} + R_o) \cap R \subset R_J^{\pm}$ and $(R_{\mathfrak{e}} + R_o) \cap R \subset R_{\mathfrak{e}}$;*

(5) *$(R_J^{\pm} + R_{\mathfrak{e}}) \cap R \subset R_J^{\pm} \cup R_{\mathfrak{e}} \cup R_o$.*

Proof. (1) is clear. To see (2), (3) and (4), observe that $\alpha \in R_J$ if and only if E_α belongs to an irreducible $\mathfrak{k}^{\mathbb{C}}$-module which is also J-invariant; hence (2), (3) and (4) follow from Lemma 7.1 and Corollary 3.10.

The proof of (5) is the following. Let $\gamma \in R_J^+$ and $\alpha, \beta \in R_{\mathfrak{e}}$ a pair of two ϑ-congruent roots. If $\gamma + \alpha \in R_J^-$, consider the element $f_{-\alpha,-\beta} \in \mathfrak{m}^{01}$ as defined in (7.7). Since $E_{\gamma+\alpha} \in \mathfrak{m}^{01}$, by the integrability condition

$$[E_{\gamma+\alpha}, f_{-\alpha,-\beta}] = C E_\gamma + X \in \mathfrak{m}^{01} + \mathfrak{l}^{\mathbb{C}}$$

for some $C \neq 0$ and $X \notin \mathbb{C} E_\gamma$. This implies that $\gamma \in R_J^-$: contradiction. □

For any $\alpha \in R$, a root $\beta \in R$, which is ϑ-congruent to α, is said to be *ϑ-dual to α* and we say that (α, β) is a *ϑ-dual pair*. By Corollary 3.10 any root admits *at most one* ϑ-dual root; by Lemma 7.7 (3), *any root in $R_{\mathfrak{k}}$* has *exactly one* ϑ-dual root.

Lemma 7.8 *Let (α, α') be a ϑ-dual pair in $R_{\mathfrak{k}}$. Then the root subsystem $\tilde{R} = R \cap \operatorname{span}_{\mathbb{R}}\{\alpha, \alpha'\}$ is of type $A_1 + A_1$. In particular $\alpha \perp \alpha'$ and $\alpha \pm \alpha' \notin R$.*

Proof. Assume that $\tilde{R} \neq A_1 + A_1$. Then $\tilde{R}$ is a root system of type A_2, B_2 or G_2. Since by assumptions $\vartheta = \alpha - \alpha'$ is proportional to no root, looking at the corresponding root systems, we find that up to a transformation from the Weyl group there are the following possibilities:

$$\begin{aligned} R = A_2 : &\quad \alpha = \varepsilon_0 - \varepsilon_2, \quad \alpha' = \varepsilon_2 - \varepsilon_1; \\ \tilde{R} = B_2 : &\quad \alpha = \varepsilon_1, \quad \alpha' = -\varepsilon_1 + \varepsilon_2; \\ \tilde{R} = G_2 : &\quad \alpha = -\varepsilon_2, \quad \alpha' = -\varepsilon_1 + \varepsilon_2. \end{aligned}$$

Note that in each of these three cases, $\alpha + \alpha' = \beta \in R$.

Case $\tilde{R} = A_2$: In this case $\vartheta = (\varepsilon_0 - \varepsilon_2) - (\varepsilon_2 - \varepsilon_1) = \varepsilon_0 + \varepsilon_1 - 2\varepsilon_2$ and $\beta = \alpha + \alpha'$ is orthogonal to ϑ and hence it belongs to R_o. Moreover $\mathfrak{l}^{\mathbb{C}} = C_{\mathfrak{g}^{\mathbb{C}}}(Z)$ contains the subalgebra

$$\mathfrak{l}' = \mathbb{C}H_{\varepsilon_0 - \varepsilon_1} + \mathbb{C}E_{\varepsilon_0 - \varepsilon_1} + \mathbb{C}E_{\varepsilon_1 - \varepsilon_0}.$$

At the same time, by Lemma 7.7 (3), $\mathfrak{m}^{01}$ contains the element $f_{\varepsilon_0 - \varepsilon_2, \varepsilon_2 - \varepsilon_1} = E_{\varepsilon_0 - \varepsilon_2} + \mu E_{\varepsilon_2 - \varepsilon_1}$, with some fixed $\mu \neq 0$. Since $\mathfrak{m}^{01}$ is $\mathfrak{l}^{\mathbb{C}}$-invariant, $\mathfrak{m}^{01}$ contains also the subspace

$$[E_{\varepsilon_1 - \varepsilon_0}, \mathbb{C}f_{\varepsilon_0 - \varepsilon_2, \varepsilon_2 - \varepsilon_1}] = \mathbb{C}(E_{\varepsilon_1 - \varepsilon_2} - \mu E_{\varepsilon_2 - \varepsilon_0}).$$

By integrability condition, this implies that

$$\begin{aligned} &[E_{\varepsilon_0 - \varepsilon_2} + \mu E_{\varepsilon_2 - \varepsilon_1}, E_{\varepsilon_1 - \varepsilon_2} - \mu E_{\varepsilon_2 - \varepsilon_0}] \\ &\qquad = \mu(-H_{\varepsilon_0 - \varepsilon_2} + H_{\varepsilon_2 - \varepsilon_1}) \in \mathfrak{m}^{01} + \mathfrak{l}^{\mathbb{C}} \end{aligned}$$

and hence we conclude that $-H_{\varepsilon_0 - \varepsilon_2} + H_{\varepsilon_2 - \varepsilon_1} \in \mathfrak{l}^{\mathbb{C}}$. But this cannot be because $-H_{\varepsilon_0 - \varepsilon_2} + H_{\varepsilon_2 - \varepsilon_1}$ is not orthogonal to $Z = i\mathcal{B}^{-1} \circ \vartheta$.

Case $\tilde{R} = B_2$ or G_2: Then $\beta = \alpha + \alpha'$ is not orthogonal to $\vartheta = \alpha - \alpha'$ and, moreover,

$$(\beta + \mathbb{R}\vartheta) \cap R = \emptyset.$$

These two facts show that $\beta \in R \setminus (R_{\mathfrak{k}} \cup R_o) = R_J$. Changing the sign of α and α', if necessary, we may assume that $\beta \in R_J^+$.

Consider the vector $f_{\alpha,\alpha'} = E_\alpha + \mu E_{\alpha'} \in \mathfrak{m}^{01}$ which is defined by (7.7). Then $\overline{E_\alpha + \mu E_{\alpha'}} = E_{-\alpha} + \bar{\mu} E_{-\alpha'} \in \mathfrak{m}^{10}$ and by integrability condition its commutator with E_β is also in $\mathfrak{m}^{10} + \mathfrak{l}^{\mathbb{C}}$. Therefore

$$[E_{-\alpha} + \bar{\mu} E_{-\alpha'}, E_\beta] = N_{-\alpha,\beta} E_{\alpha'} + \bar{\mu} N_{-\alpha',\beta} E_\alpha \in \mathfrak{m}^{10}.$$

Hence the coefficient λ of the vector $e_{\alpha,\alpha'}$ defined by (7.7) is

$$\lambda = \frac{N_{-\alpha,\beta}}{\bar{\mu} N_{-\alpha',\beta}}. \tag{7.8}$$

Since we use the Chevalley normalization (see §7.1), $N_{-\alpha,\beta} = \pm(p+1)$ for any two roots α, β, where $p \geq 0$ is the maximal integer such that $\beta + p\alpha \in \tilde{R}$ (see e.g. [7]). Using this formula, we obtain from (7.8) that if $\tilde{R} = B_2$, $\lambda\bar{\mu} = \pm 2$, while if $\tilde{R} = G_2$, $\lambda\bar{\mu} = \pm 3$.

On the other hand, by integrability condition

$$[e_{\alpha,\alpha'}, \overline{f_{\alpha,\alpha'}}] = [E_\alpha + \lambda E'_\alpha, E_{-\alpha} + \bar{\mu} E_{-\alpha'}] = H_\alpha + \lambda\bar{\mu} H_{\alpha'} \in \mathfrak{l}^{\mathbb{C}}.$$

This means that $\vartheta(H_\alpha + \lambda\bar{\mu} H_{\alpha'}) = 0$, i.e. that

$$\langle \vartheta | \alpha \rangle + \lambda\bar{\mu} \langle \vartheta | \alpha' \rangle = 0,$$

where $\langle \vartheta | \alpha \rangle = 2(\vartheta, \alpha)/(\alpha, \alpha)$. Hence for $\vartheta = \alpha - \alpha'$, we obtain

$$2 - \langle \alpha' | \alpha \rangle + \lambda\bar{\mu}[-2 \ + \langle \alpha | \alpha' \rangle] = 0.$$

In case $\tilde{R} = B_2$, $\langle \alpha' | \alpha \rangle = -2$ and $\langle \alpha | \alpha' \rangle = -1$ so that $\lambda\bar{\mu} = 4/3$; in case $\tilde{R} = G_2$, $\langle \alpha' | \alpha \rangle = -3$ and $\langle \alpha | \alpha' \rangle = -1$ so that $\lambda\bar{\mu} = 5/3$. In both cases we get a contradiction with the previously determined values for $\lambda\bar{\mu}$. □

Now we determine the possible types of the root subsystem $\tilde{R}_{\mathfrak{k}} = R \cap \operatorname{span}_{\mathbb{R}}(R_{\mathfrak{k}})$.

Lemma 7.9 *If $\tilde{R}_{\mathfrak{k}}$ is not of the form $A_1 \cup A_1$, then $\tilde{R}_{\mathfrak{k}}$ and R are both indecomposable root systems.*

Proof. By Lemma 7.8, we may assume that rank $\tilde{R}_{\mathfrak{e}} > 2$. Suppose that $\tilde{R}_{\mathfrak{e}}$ is decomposable into two mutually orthogonal subsystems R_1 and R_2. Let $\alpha \in R_1 \cap R_{\mathfrak{e}}$, $\alpha' \in R_2 \cap R_{\mathfrak{e}}$ and β, β' the ϑ-dual roots of α and α', respectively. Since ϑ cannot be in the span of R_1, it is clear that $\beta \in R_2$ and that $\beta' \in R_1$. Then the identity

$$\mathbb{R}\vartheta = \mathbb{R}(\alpha - \beta) = \mathbb{R}(\alpha' - \beta')$$

implies that $\alpha + \rho\beta' = \rho\alpha' + \beta = 0$ for some $\rho \neq 0$. From this follows that $\beta' = -\alpha$, $\beta = -\alpha'$ and that rank $\tilde{R}_{\mathfrak{e}} = 2$: contradiction.

A similar contradiction arises if we replace $\tilde{R}_{\mathfrak{e}}$ by R. □

Note that by Lemma 7.9, if $G = G_1 \times G_2$, then the only possibility for $\tilde{R}_{\mathfrak{e}}$ is $A_1 \cup A_1$.

The following Lemma gives a more detailed description of the root subsystem $\tilde{R}_{\mathfrak{e}}$.

Lemma 7.10 *The root subsystem $\tilde{R}_{\mathfrak{e}}$ has type D_ℓ, $\ell > 1$ or B_3 and, up to a factor and a transformation from the Weyl group $W = W(R)$, the contact form ϑ is one of the following:*

(1) *if $\tilde{R}_{\mathfrak{e}} = D_2 = A_1 + A_1'$ and α, α' are roots of the summands A_1 and A_1', then $\vartheta = \alpha - \alpha'$;*
(2) *if $\tilde{R}_{\mathfrak{e}} = D_3$ or D_ℓ, with $\ell > 4$, then $\vartheta = 2\varepsilon_1$;*
(3) *if $\tilde{R}_{\mathfrak{e}} = D_4$ then $\vartheta = 2\varepsilon_1$ or $\vartheta = \varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4$ or $\vartheta = \varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4$;*
(4) *if $\tilde{R}_{\mathfrak{e}} = B_3$ then $\vartheta = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$.*

Note that in case $\tilde{R}_{\mathfrak{e}} = D_4$, all three contact forms ϑ in (3) are equivalent with respect to automorphisms of the root system.

Proof. From Lemma 7.9, it is sufficient to consider the case when rank $\tilde{R}_{\mathfrak{e}} > 2$ and $\tilde{R}_{\mathfrak{e}}$ is indecomposable. For each indecomposable root system $\tilde{R}_{\mathfrak{e}}$ we describe, up to a transformation from the Weyl group, all pairs of roots (α, α'), which are orthogonal and such that $\alpha \pm \alpha' \notin R$. By Lemma 7.8 such pairs are the only candidates for ϑ-dual pairs in $R_{\mathfrak{e}}$. For each case, we consider the corresponding form $\vartheta = \alpha - \alpha'$, and describe all ϑ-dual pairs in $\tilde{R}_{\mathfrak{e}}$. Then, assuming that $\alpha, \alpha' \in R_{\mathfrak{e}}$, we check if the case is possible looking if the ϑ-dual pairs in $R_{\mathfrak{e}}$ may generate $\tilde{R}_{\mathfrak{e}}$.

Case (A): $\tilde{R}_{\mathfrak{e}} = A_\ell$.

Up to a transformation from the Weyl group, the pair (α, α') is equal to $(\varepsilon_1-\varepsilon_2, \varepsilon_3-\varepsilon_4)$. Then $\vartheta=(\varepsilon_1-\varepsilon_2)-(\varepsilon_3-\varepsilon_4)$ and the ϑ-dual pairs are (up to sign)

$$(\varepsilon_1-\varepsilon_2, \varepsilon_3-\varepsilon_4); \quad (\varepsilon_1-\varepsilon_3, \varepsilon_2-\varepsilon_4).$$

Since $\beta=\varepsilon_2-\varepsilon_3 \in R_o=(\vartheta)^{\perp} \cap R$, then $\varepsilon_1-\varepsilon_3=\alpha+\beta \in R_{\mathfrak{e}}$ and hence also the second ϑ-dual pair is in $R_{\mathfrak{e}}$. In particular $\operatorname{rank} \tilde{R}_{\mathfrak{e}}=3$ and $\tilde{R}_{\mathfrak{e}}=A_3=D_3$.

Case (B): $\tilde{R}_{\mathfrak{e}}=B_{\ell}$.

We have three possibilities for (α, α') according to their lengths:

i) $(\alpha, \alpha')=(\varepsilon_1+\varepsilon_2,-(\varepsilon_3+\varepsilon_4))$;
ii) $(\alpha, \alpha')=(\varepsilon_1+\varepsilon_2,-\varepsilon_3)$;
iii) $(\alpha, \alpha')=(\varepsilon_1,-\varepsilon_2)$.

The last case is not possible, since we assume that $\vartheta=\alpha-\alpha'$ is proportional to no root.

i) $\vartheta=\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4$ and the ϑ-dual pairs are (up to sign)

$$\begin{aligned}&(\varepsilon_1+\varepsilon_2,-(\varepsilon_3+\varepsilon_4)); \quad(\varepsilon_1+\varepsilon_3,-(\varepsilon_2+\varepsilon_4)); \\ &(\varepsilon_1+\varepsilon_4,-(\varepsilon_2+\varepsilon_3))\end{aligned}. \tag{7.9}$$

As in case (A), one can check that all these ϑ-dual pairs are in $R_{\mathfrak{e}}$ and that they span a space of dimension 4. Since the ϑ-dual pairs consist of long roots, they cannot generate the root system B_{ℓ} and hence this case is impossible.

ii) $\vartheta=\varepsilon_1+\varepsilon_2+\varepsilon_3$ and the ϑ-dual pairs are (up to sign)

$$\begin{aligned}&(\alpha=\varepsilon_1+\varepsilon_2, \alpha'=-\varepsilon_3); \quad(\beta=\varepsilon_2+\varepsilon_3, \beta'=-\varepsilon_1); \\ &(\gamma=\varepsilon_3+\varepsilon_1, \gamma'=-\varepsilon_2)\end{aligned}. \tag{7.10}$$

Again all pairs in (7.10) consist of roots in $R_{\mathfrak{e}}$. This implies that $\operatorname{rank} \tilde{R}_{\mathfrak{e}}=3$.

Case (C): $\tilde{R}_{\mathfrak{e}}=C_{\ell}$.

As in case (B), we have three possibilities.

i) $(\alpha, \alpha')=(\varepsilon_1+\varepsilon_2,-(\varepsilon_3+\varepsilon_4))$;
ii) $(\alpha, \alpha')=(\varepsilon_1+\varepsilon_2,-2\varepsilon_3)$;
iii) $(\alpha, \alpha')=(2\varepsilon_1,-2\varepsilon_2)$.

As in (B), the last case is not possible.

i) $\vartheta = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4$ and the ϑ-dual pairs are given (up to sign) in (7.9). This implies that $\pm 2\varepsilon_i \in R_J$, $i = 1, \dots, 4$, because it has no ϑ-dual root and it is not orthogonal to ϑ. Note also that the roots $\varepsilon_i - \varepsilon_j$, $i, j = 1, \dots, 4$, belong to R_o, because they are orthogonal to ϑ. Therefore

$$R_{\mathfrak{e}} \subset \{\pm(\varepsilon_i + \varepsilon_j),\ 1 \leq i, j \leq 4\} \subset (R_o + \{\pm 2\varepsilon_i\}) \cap R \subset R_J$$

and this is a contradiction.

ii) $\vartheta = \varepsilon_1 + \varepsilon_2 + 2\varepsilon_3$.

In this case, up to sign, there is only one ϑ-dual pair, that is $(\varepsilon_1 + \varepsilon_2, -2\varepsilon_3)$. On the other hand, $\varepsilon_1 - \varepsilon_2 \in R_o$ and hence $2\varepsilon_1 = (\varepsilon_1 + \varepsilon_2) + (\varepsilon_1 - \varepsilon_2) \in R_{\mathfrak{e}}$: contradiction.

Case (D): $\tilde{R}_{\mathfrak{e}} = D_\ell$.

Since $D_3 = A_3$, we may assume that $\ell \geq 4$. Then we have three possibilities:

i) $(\alpha, \alpha') = (\varepsilon_1 + \varepsilon_2, -(\varepsilon_3 + \varepsilon_4))$;
ii) $(\alpha, \alpha') = (\varepsilon_1 + \varepsilon_2, -(\varepsilon_3 - \varepsilon_4)$;
iii) $(\alpha, \alpha') = (\varepsilon_1 + \varepsilon_2, -(\varepsilon_1 - \varepsilon_2))$.

i) $\vartheta = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4$ and the ϑ-dual pairs are given (up to sign) in (7.9) and they all belong to $R_{\mathfrak{e}}$. Hence the rank of $\tilde{R}_{\mathfrak{e}}$ is 4.

A similar argument shows that rank $\tilde{R}_{\mathfrak{e}} = 4$ in case ii), where $\vartheta = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4$.

iii) $\vartheta = 2\varepsilon_1$ and the ϑ-dual pairs are $(\varepsilon_1 + \varepsilon_i, \varepsilon_1 - \varepsilon_i)$, with $i = 2, \dots \ell$, they are all in $R_{\mathfrak{e}}$ and they span the whole system D_ℓ.

Case (E): $\tilde{R}_{\mathfrak{e}} = E_6$, E_7 or E_8.

Let $\alpha, \alpha' \in R_{\mathfrak{e}}$ be a ϑ-dual pair. Since α and α' are orthogonal, we may included them into a subsystem Π of simple roots. According to the type of $\tilde{R}_{\mathfrak{e}}$, without loss of generality, we may assume that α' is one of the following:

$$\begin{aligned}
\tilde{R}_{\mathfrak{e}} = E_6 :&\quad \alpha' = \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon;\\
\tilde{R}_{\mathfrak{e}} = E_7 :&\quad \alpha' = \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + \varepsilon_8;\\
\tilde{R}_{\mathfrak{e}} = E_8 :&\quad \alpha' = \varepsilon_6 + \varepsilon_7 + \varepsilon_8.
\end{aligned}$$

For each case, it follows that $\alpha = \varepsilon_i - \varepsilon_{i+1}$ for some $i \neq \ell - 3$ where $\ell = \operatorname{rank} \tilde{R}_{\mathfrak{e}}$. It can be easily checked that, using permutations of the vectors ε_i which belong to the Weyl group of E_ℓ and which preserve α', we may

assume that either $\alpha = \varepsilon_1 - \varepsilon_2$ or $\alpha = \varepsilon_{\ell-1} - \varepsilon_\ell = -\sum_{i=1}^{\ell-2} \varepsilon_i - 2\varepsilon_\ell$. Therefore we have the following possibilities:
if $\tilde{R}_\mathfrak{e} = E_6$:
i) $\alpha = \varepsilon_1 - \varepsilon_2$ and $\vartheta = \alpha' - \alpha = -\varepsilon_1 + \varepsilon_2 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon$;
ii) $\alpha = \varepsilon_5 - \varepsilon_6$ and $\vartheta = \varepsilon_4 + 2\varepsilon_6 + \varepsilon = \varepsilon_6 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_5 + \varepsilon$;
if $\tilde{R}_\mathfrak{e} = E_7$:
iii) $\alpha = \varepsilon_1 - \varepsilon_2$ and $\vartheta = -\varepsilon_1 + \varepsilon_2 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + \varepsilon_8$;
iv) $\alpha = \varepsilon_6 - \varepsilon_7$ and $\vartheta = \varepsilon_5 + 2\varepsilon_7 + \varepsilon_8 = \varepsilon_7 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_6$;
if $\tilde{R}_\mathfrak{e} = E_8$:
v) $\alpha = \varepsilon_1 - \varepsilon_2$ and $\vartheta = -\varepsilon_1 + \varepsilon_2 + \varepsilon_6 + \varepsilon_7 + \varepsilon_8$;
vi) $\alpha = \varepsilon_7 - \varepsilon_8$ and $\vartheta = \varepsilon_6 + 2\varepsilon_8 = \varepsilon_8 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 - \varepsilon_7$.

We claim that all ϑ-dual pairs belong to $R_\mathfrak{e}$ and that the space they generate has dimension 5 for the cases i), ii) and v); it has dimension 6 for the cases iii) and iv) and dimension 7 for the case vi). Since in all cases the dimension is strictly less then $\operatorname{rank} \tilde{R}_\mathfrak{e} = \ell$, we conclude that the case $R_\mathfrak{e} = E_\ell$ is impossible.

We prove the claim in the cases v) and vi) which occur when $\tilde{R}_\mathfrak{e} = E_8$; in all other cases the proof is similar.

For case v), the ϑ-dual pairs are (up to sign) $(-\varepsilon_1 + \varepsilon_i, \vartheta + \varepsilon_1 - \varepsilon_i)$, where $i = 2, 6, 7, 8$ and they all belong to $R_\mathfrak{e}$. These vectors generate a 5-dimensional vector space. In case vi) the ϑ-dual pairs are $(-\varepsilon_8 + \varepsilon_i, \vartheta + \varepsilon_8 - \varepsilon_i)$, where $i = 1, \ldots, 5$ or 7, and again they are all in $R_\mathfrak{e}$. These vectors generate a 7-dimensional vector space.

Case (F): $\tilde{R}_\mathfrak{e} = F_4$.

We have the following possibilities:
i) $(\alpha, \alpha') = (\varepsilon_1 + \varepsilon_2, -(\varepsilon_3 + \varepsilon_4))$;
ii) $(\alpha, \alpha') = (\varepsilon_1 + \varepsilon_2, -\varepsilon_3)$;
iii) $(\alpha, \alpha') = (\varepsilon_1 + \varepsilon_2, -(1/2(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4))$;
iv) $(\alpha, \alpha') = (\varepsilon_1, -\varepsilon_2)$.

Cases i) and iv) are impossible because $\vartheta = \alpha - \alpha'$ should be proportional to no root. The admissible ϑ-dual pairs for case ii) are given by (7.10) and they all belong to $R_\mathfrak{e}$. They generate a 3-dimensional subspace and this is impossible because $\operatorname{rank} \tilde{R}_\mathfrak{e} = \operatorname{rank} F_4 = 4$. A similar argument is applied for case iii). $\square$

Corollary 7.11 *If G is simple, then the only possibilities for the pair $(R, \tilde{R}_\mathfrak{e})$ are*

$$(A_n, A_3), \quad (A_n, B_3), \quad (B_n, A_3), \quad (B_n, B_3),$$
$$(B_n, D_4), \quad (D_n, D_4), \quad (D_n, D_n), \quad (E_6, D_5),$$
$$(E_7, D_6), \quad (E_8, D_5), \quad (E_8, D_7), \quad (F_4, A_3), \quad (F_4, B_3).$$

Proof. If R is the root system of the simple Lie group G and (α, α') is a ϑ-dual pair in $R_{\mathfrak{e}}$, then the arguments used in the proof of Lemma 7.10 give the result. □

Lemma 7.12 *Let $\tilde{R}_o = R_o \cap \tilde{R}_{\mathfrak{e}}$, $R'_o = R_o \setminus \tilde{R}_o$ and $\alpha, \alpha' \in R_{\mathfrak{e}}$ be a ϑ-dual pair. Then*

a) $\tilde{R}_{\mathfrak{e}} = [(\{\pm\alpha, \pm\alpha'\} + \tilde{R}_o) \cap R] \cup \tilde{R}_o$;

b) $R_{\mathfrak{e}} = (\{\pm\alpha, \pm\alpha'\} + \tilde{R}_o) \cap R$ *and* $R_J \cap \tilde{R}_{\mathfrak{e}} = \emptyset$;

c) $R_Q = R_o \cup R_{\mathfrak{e}}$ *and* $R_P = R_o \cup R_{\mathfrak{e}} \cup R_J^+$ *are closed subsystem of* R; R_Q *is the maximal symmetric subset in* R_P *(i.e. the biggest subset such that* $-R_Q = R_Q$*), and* R_P *is parabolic (i.e. for any root* α*, either* α *or* $-\alpha$ *belongs to it)*;

d) $(R_Q + \tilde{R}_{\mathfrak{e}}) \cap R \subset \tilde{R}_{\mathfrak{e}}$ *and hence* $R_Q = R'_o \cup \tilde{R}_{\mathfrak{e}}$ *is an orthogonal decomposition*;

e) *for any ϑ-dual pair (α, α') let $R_o(\alpha) = (R_o + \{\alpha\}) \cap R$ and $R_o(-\alpha') = (R_o + \{-\alpha'\}) \cap R)$; then the set of roots*

$$S(\alpha, \alpha') = R_o \cup R_o(\alpha) \cup R_o(-\alpha') \cup R_J^+ \tag{7.11}$$

is a closed parabolic subsystem of R.

Proof. a) When $\operatorname{rank} \tilde{R}_{\mathfrak{e}} = 2$ the claim is trivial.

If $\tilde{R}_{\mathfrak{e}} = B_3$, we may assume that $\alpha = \varepsilon_1 + \varepsilon_2$, $\alpha' = -\varepsilon_3$ and $\vartheta = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$. Hence

$$\tilde{R}_o = \tilde{R}_{\mathfrak{e}} \cap (\vartheta)^\perp = \{\varepsilon_i - \varepsilon_j,\ i, j = 1, \ldots, 3\}.$$

By Lemma 7.7 (4),

$$(\{\pm\alpha, \pm\alpha'\} + \tilde{R}_o) \cap R = \{\pm(\varepsilon_i + \varepsilon_j), \pm\varepsilon_i,\ i, j = 1, \ldots, 3\} \subset R_{\mathfrak{e}}.$$

Since $\tilde{R}_{\mathfrak{e}} = \tilde{R}_o \cup \{\pm(\varepsilon_i + \varepsilon_j), \pm\varepsilon_i,\ i, j = 1, \ldots, 3\}$, the claim is proved for this case.

If $\tilde{R}_{\mathfrak{e}} = D_\ell$, the argument is similar. In particular, if $\vartheta = 2\varepsilon_1$, one obtains that $\tilde{R}_o = D_{\ell-1} = \{\pm\varepsilon_i \pm \varepsilon_j,\ i, j > 1\}$ and $R_{\mathfrak{e}} = \{\pm\varepsilon_1 \pm \varepsilon_i\}$.

b) follows directly from a).

c) The closeness of R_Q and R_P follows from Lemma 7.7 (4) and (5) and from point b). The last statement is obvious.

d) The first claim follows from the facts that $\tilde{R}_{\mathfrak{e}} = \mathrm{span}_{\mathbb{R}}(R_{\mathfrak{e}}) \cap R$ and $(R_o + R_{\mathfrak{e}}) \cap R \subset R_{\mathfrak{e}}$. This implies that $\mathfrak{g}(\tilde{R}_{\mathfrak{e}})$ is an ideal of the semisimple Lie algebra $\mathfrak{g}(R_Q)$ and from this also the second claim follows.

e) By point b), $R_o(\alpha) \cup R_o(-\alpha') \subset R_{\mathfrak{e}}$ and hence $R_o \cup R_o(\alpha) \cup R_o(-\alpha') \subset R_Q$ and $S(\alpha, \alpha') \subset R_P = R_Q \cup R_J^+$. Since R_Q corresponds to a reductive part of the parabolic subalgebra $\mathfrak{g}(R_P)$ and R_J^+ corresponds to the nilradical, it follows that $(S(\alpha, \alpha') + R_J^+) \cap R \subset R_J^+$. By d), it remains to check that $R_o(\alpha) \cup R_o(-\alpha)$ is a closed subsystem.

In case $\tilde{R}_{\mathfrak{e}} = 2A_1 = D_2$, we have that $R_o(\alpha) \cup R_o(-\alpha) = \{\alpha, -\alpha'\}$ and hence the claim is trivial.

In case $\tilde{R}_{\mathfrak{e}} = D_\ell$, $\ell > 2$, we may assume that $\vartheta = 2\varepsilon_1$, $\alpha = \varepsilon_1 + \varepsilon_2$, $\alpha' = -(\varepsilon_1 - \varepsilon_2)$. Then $\tilde{R}_o = \{\pm\varepsilon_i \pm \varepsilon_j,\ 1 < i, j\}$ and

$$R_o(\alpha) = \{\varepsilon_1 \pm \varepsilon_i,\ 1 < i\} = R_o(-\alpha') \tag{7.12}$$

and the conclusion follows. In case $\tilde{R}_{\mathfrak{e}} = B_3$, then $\vartheta = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$, $\alpha = \varepsilon_1 + \varepsilon_2$ and $\alpha' = -\varepsilon_3$. Then $\tilde{R}_o = \{\pm(\varepsilon_i - \varepsilon_j)\}$ and

$$R_o(\alpha) = \{\varepsilon_i + \varepsilon_j\}, \qquad R_o(-\alpha') = \{\varepsilon_i\} \tag{7.13}$$

and again the conclusion follows. □

Since $\mathfrak{g}(R_J^+)$ is the nilradical of the parabolic subalgebra $\mathfrak{g}(R_P)$, we may choose an ordering of the roots such that the positive root system R^+ contains R_J^+. In the following α *denotes the maximal root in* $R_{\mathfrak{e}}$ *w.r.t. this ordering* and α' is its associated ϑ-dual root.

Proposition 7.13 *The orthogonal complement* $\mathfrak{m}^{\mathbb{C}}$ *to* $\mathfrak{k}^{\mathbb{C}}$ *in* $\mathfrak{g}^{\mathbb{C}}$ *admits the following* $\mathfrak{k}^{\mathbb{C}}$*-invariant decomposition:*

(1) *if* $\tilde{R}_{\mathfrak{e}} = B_3$ *or* $D_2 = A_1 + A_1$*, then*

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{e} + \mathfrak{m}_J^+ + \mathfrak{m}_J^- = (\mathfrak{m}(\alpha) + \mathfrak{m}(\alpha') + \overline{\mathfrak{m}(\alpha)} + \overline{\mathfrak{m}(\alpha')}) + \mathfrak{m}_J^+ + \mathfrak{m}_J^-, \tag{7.14}$$

(2) *if* $\tilde{R}_{\mathfrak{e}} = D_\ell$*, then*

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{e} + \mathfrak{m}_J^+ + \mathfrak{m}_J^- = (\mathfrak{m}(\alpha) + \mathfrak{m}(\alpha')) + \mathfrak{m}_J^+ + \mathfrak{m}_J^- \tag{7.15}$$

where $\mathfrak{m}(\alpha)$ *and* $\mathfrak{m}(\alpha')$ *are irreducible* $\mathfrak{k}^{\mathbb{C}}$*-modules with highest weights* α*,* α'*, which are equivalent and irreducible as* $\mathfrak{l}^{\mathbb{C}}$*-modules.*

In terms of this decomposition, the holomorphic subspace $\mathfrak{m}^{10}$ *of the CR structure* $(\mathcal{D}_Z, J)$ *(up to sign) is of the form*

(1) *if* $\tilde{R}_{\mathfrak{e}} = B_3$ *or* $D_2 = A_1 + A_1'$

$$\mathfrak{m}^{10} = (\mathfrak{m}(\alpha) + t\mathfrak{m}(\alpha')) + \left(\overline{\mathfrak{m}(\alpha)} + \frac{1}{t}\overline{\mathfrak{m}(\alpha')}\right) + \mathfrak{m}_J^+, \tag{7.16}$$

(2) *if* $\tilde{R}_{\mathfrak{e}} = D_\ell$

$$\mathfrak{m}^{10} = (\mathfrak{m}(\alpha) + t\mathfrak{m}(\alpha')) + \mathfrak{m}_J^+ \tag{7.17}$$

for some $t \in \{x \in \mathbb{C} : 0 < |x| < 1\} = D \setminus \{0\}$.

Proof. From $(R_o + \alpha) \cap R \subset R_{\mathfrak{e}}$ and the definition of α, the root α is the maximal weight of the $\mathfrak{k}^{\mathbb{C}}$-module in $\mathfrak{m}^{\mathbb{C}}$ which contains E_α. Moreover since α' is ϑ-congruent to α, then also α' is the maximal weight of an $\mathfrak{l}^{\mathbb{C}}$- and hence $\mathfrak{k}^{\mathbb{C}}$-module, and the $\mathfrak{l}^{\mathbb{C}}$-modules $\mathfrak{m}(\alpha)$ and $\mathfrak{m}(\alpha')$ are equivalent. By Lemma 7.12 b), it follows that the subspace $\mathfrak{e}$, spanned by the root vectors E_γ, $\gamma \in R_{\mathfrak{e}}$, is given by

$$\mathfrak{e} = \mathfrak{m}(\alpha) + \mathfrak{m}(\alpha') + \overline{\mathfrak{m}(\alpha)} + \overline{\mathfrak{m}(\alpha')}.$$

Moreover if $\tilde{R}_{\mathfrak{e}} = D_\ell$, $\ell > 2$, $R_o(\alpha) = R_o(-\alpha')$ (see (7.12)) and hence $\mathfrak{m}(\alpha) = \overline{\mathfrak{m}(\alpha')}$ (see also Table 3 in the Appendix).

From Lemma 7.1 and the remark in the second to the last point of §7.1, we obtain that the holomorphic subspace $\mathfrak{m}^{10}$ is of the form

$$\mathfrak{m}^{10} = (\mathfrak{m}(\alpha) + t\mathfrak{m}(\alpha')) + \mathfrak{m}_J^+$$

when $\tilde{R}_{\mathfrak{e}} = D_\ell$, $\ell > 2$, and of the form

$$\mathfrak{m}^{10} = (\mathfrak{m}(\alpha) + t\mathfrak{m}(\alpha')) + (\overline{\mathfrak{m}(\alpha)} + s\overline{\mathfrak{m}(\alpha')}) + \mathfrak{m}_J^+$$

when $\tilde{R}_{\mathfrak{e}} = B_3$ or $D_2 = A_1 + A_1$, for some $t, s \neq 0$. By exchanging $\mathfrak{m}^{10}$ with $\mathfrak{m}^{01}$ (which corresponds to changing the sign of J) we may assume that $|t| \leq 1$. Using the integrability condition and the assumption that $\vartheta = \alpha - \alpha' \notin R$, we have

$$[E_\alpha + tE_{\alpha'}, E_{-\alpha} + sE_{-\alpha'}] = H_\alpha + tsH_{\alpha'} \in \mathfrak{m}^{10} + \mathfrak{l}^{\mathbb{C}}$$

and therefore $H_\alpha + tsH_{\alpha'} \in \mathfrak{l}^{\mathbb{C}}$. Using (3.1) we get

$$0 = \vartheta(H_\alpha + tsH_{\alpha'}) = \langle\vartheta|\alpha\rangle + ts\langle\vartheta|\alpha'\rangle.$$

So

$$s = -\frac{1}{t}\frac{\langle\vartheta|\alpha\rangle}{\langle\vartheta|\alpha'\rangle}.$$

If $\tilde{R}_{\mathfrak{e}} = 2A_1$, it is immediate to check that $\langle\vartheta|\alpha\rangle = 2 = -\langle\vartheta|\alpha'\rangle$. In case $\tilde{R}_{\mathfrak{e}} = B_3$, we may assume that $\vartheta = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$, $\alpha = \varepsilon_1 + \varepsilon_2$ and $\alpha' = -\varepsilon_3$. Hence again $\langle\vartheta|\alpha\rangle = -\langle\vartheta|\alpha'\rangle$ and this shows that in both cases $s = 1/t$.

Finally, the condition $\mathfrak{m}^{10} \cap \mathfrak{m}^{01} = \{0\}$ implies that the vectors $E_\alpha + tE_{\alpha'}$ and $\overline{E_{-\alpha} + \frac{1}{t}E_{-\alpha'}} = E_\alpha + \frac{1}{t}E_{\alpha'}$ are linearly independent, and hence $|t| \neq 1$. □

Lemma 7.14

(1) *Let $\mathfrak{q}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} + \mathfrak{e}$ and $\mathfrak{p} = \mathfrak{q}^{\mathbb{C}} + \mathfrak{m}_J^+$. Then $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}^{\mathbb{C}}$, with reductive part $\mathfrak{q}^{\mathbb{C}}$ and nilradical $\mathfrak{m}_J^+$. Moreover, if Q is the connected subgroup of G with Lie algebra $\mathfrak{q} = \mathfrak{q}^{\mathbb{C}} \cap \mathfrak{g}$, then $F_2 = G/Q$ is a flag manifold and $\mathfrak{m}_J^+$ is the holomorphic subspace of an invariant complex structure J_2 on $F_2 = G/Q$.*

(2) *The subspace $\mathfrak{m}_{J_1}^{10} = \mathfrak{m}(\alpha) + \mathfrak{m}(-\alpha') + \mathfrak{m}_J^+$ is the holomorphic subspace of an invariant complex structure J_1 of $F_Z = G/K$.*

(3) *The natural G-equivariant projections*

$$\pi : G/L \longrightarrow G/Q, \qquad \pi' : G/K \longrightarrow G/Q$$

are holomorphic fibrations w.r.t. the CR structure $(\mathcal{D}_Z, J)$ on G/L, the complex structure J_1 on $F_Z = G/K$ and the complex structure J_2 on $F_2 = G/Q$, respectively. Moreover, the typical fiber $C = Q/L$ of π is either $\mathrm{Spin}_7/SU_3 = S^7 \times S^6$ or $SO_{2\ell}/SO_{2\ell-2}$, $\ell > 1$ and the induced invariant CR structure is primitive.

(4) *The typical fiber $C = Q/L$ of π may be equal $SO_4/SO_2 = S^3 \times S^2$ only if $G = G_1 \times G_2$, with each G_i simple.*

Proof. (1) The proof follows from Lemma 7.12 c) and the remark that $\mathfrak{p} = \mathfrak{g}(R_P) + \mathfrak{h}^{\mathbb{C}}$ and $\mathfrak{q}^{\mathbb{C}} = \mathfrak{g}(R_Q) + \mathfrak{h}^{\mathbb{C}}$.

(2) We have to check the conditions a) and b) of (4.2). Condition a) is obvious. Condition b) means that $\mathfrak{k}^{\mathbb{C}} + \mathfrak{m}(\alpha) + \mathfrak{m}(-\alpha') = \mathfrak{g}(S(\alpha, \alpha')) + h^{\mathbb{C}}$ is a subalgebra. This follows from Lemma 7.12 e).

(3) The first claim follows from Lemma 4.8.

For the second claim, we recall that we have the following decompositions of the Lie algebras $\mathfrak{q}^{\mathbb{C}}$ and $\mathfrak{l}^{\mathbb{C}}$:

$$\mathfrak{l}^{\mathbb{C}} = \mathfrak{g}(R'_o) \oplus \big(\mathfrak{g}(\tilde{R}_o) + Z(\mathfrak{l}^{\mathbb{C}})\big),$$
$$\mathfrak{q}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} + \mathfrak{e} = \mathfrak{g}(R'_o) \oplus \big(\mathfrak{g}(\tilde{R}_{\mathfrak{e}}) + Z(\mathfrak{q}^{\mathbb{C}})\big).$$

Since the fiber Q/L has a non-standard CR structure, the group $Q' = Q/N$, where N is its kernel of non-effectivity, is semisimple by Corollary 3.2 and Proposition 4.6. Therefore it has Lie algebra $\mathfrak{q}'^{\mathbb{C}} = \mathfrak{g}(\tilde{R}_{\mathfrak{e}}) = B_3$ or D_ℓ. The corresponding stability subalgebra $\mathfrak{l}'^{\mathbb{C}} = \mathfrak{l}^{\mathbb{C}}/\mathfrak{n}^{\mathbb{C}}$ has rank equal to $\operatorname{rank}(\mathfrak{q}'^{\mathbb{C}}) - 1$ and his semisimple part is $\mathfrak{g}(\tilde{R}_o) = A_2$ or $D_{\ell-1}$. Hence the fiber $Q/L = Q'/L'$, considered as homogeneous manifold of the effective group Q', is either Spin_7/SU_3 or $SO_{2\ell}/SO_{2\ell-2}$ (note that SO_7 does not contains SU_3). The manifold Spin_7/SU_3 can be identified with the unit sphere bundle $S(\mathrm{Spin}_7/G_2) = S(S^7) = S^7 \times S^6$.

The holomorphic subspace $\mathfrak{m}^{10}(Q/L)$ of the CR structure of the fiber Q/L is of the form $(\mathfrak{m}(\alpha) + t\mathfrak{m}(\alpha')) + (\overline{\mathfrak{m}(\alpha)} + 1/t\overline{\mathfrak{m}(\alpha')})$ for some $t \neq 0$ and the minimal $\mathfrak{k}^{\mathbb{C}}$-module generated by $\mathfrak{m}^{10}(Q/L)$ is $\mathfrak{e}$. By Lemma 4.8, this implies that the CR structure on Q/L is primitive.

(4) It is sufficient to observe that if G is simple, the case $\tilde{R}_{\mathfrak{e}} = A_1 \cup A_1$ cannot occur by Corollary 7.11. $\square$

Lemma 7.14 (3) and Proposition 7.13 directly imply Proposition 7.5.

Now it remains to prove Proposition 7.6. Let $(M = G/L, \mathcal{D}_Z, J)$ be a non-standard non-primitive CR manifold with contact form ϑ not proportional to any root. We recall that in Lemma 7.14 (3) we defined a complex structure J_1 on the flag manifold $F_Z = G/K$, associated with the decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} + \mathfrak{m}^{10}_{J_1} + \mathfrak{m}^{01}_{J_1}$. We also defined another flag manifold $F_2 = G/Q$, with $\mathfrak{q}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} + \mathfrak{e}$, with invariant complex structure J_2 associated with the decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{q}^{\mathbb{C}} + \mathfrak{m}^+_J + \mathfrak{m}^-_J$ and such that the projection $\pi : (F_Z = G/K, J_1) \to (F_2 = G/Q, J_2)$ is holomorphic. Moreover the CR structure $(\mathcal{D}_Z, J)$ on G/L has the holomorphic subspace defined in (7.16) and (7.17).

The subalgebra $\mathfrak{k}^{\mathbb{C}}$ corresponds to the root subsystem R_o, which has the orthogonal decomposition $R_o = R'_o \cup \tilde{R}_o$, and $\mathfrak{q}^{\mathbb{C}}$ corresponds to the root subsystem with the orthogonal decomposition $R_Q = R'_o \cup \tilde{R}_{\mathfrak{e}} = R'_o \cup (\tilde{R}_o \cup R_{\mathfrak{e}})$ (see Lemma 7.12). Moreover there are only three possibilities for the pair of subsystems $(\tilde{R}_{\mathfrak{e}}, \tilde{R}_o)$, namely $(D_2 = 2A_1, \emptyset)$, $(D_\ell, D_{\ell-1})$, $\ell > 2$, or (B_3, A_2). However, the following lemma shows that this last case cannot occur.

Lemma 7.15 *If $R_J \neq \emptyset$, then $\tilde{R}_{\mathfrak{e}} \neq B_3$.*

In other words, the fiber $C = Q/L$ of the CRF fibration $\pi : G/L \to G/Q$ described in Lemma 7.14 (3) cannot be Spin_7/SU_3 if the base is not trivial.

Proof. Assume that $\tilde{R}_{\mathfrak{e}} = B_3$. Then G is simple and R is indecomposable by Lemma 7.9. So R has type either B_n or F_4, because these are the only connected Dynkin graphs which contain a subgraph of type B_3.

If $R = F_4$, using the notation of the Appendix, we may assume that $(\alpha = \varepsilon_2 + \varepsilon_3, \alpha' = -\varepsilon_4)$ is a ϑ-dual pair in $R_{\mathfrak{e}}$. Since $\vartheta = \varepsilon_2 + \varepsilon_3 + \varepsilon_4$, then $-\varepsilon_4 + \varepsilon_1 \in R_J$, because it is not orthogonal to ϑ nor has a ϑ-dual root; moreover $-\varepsilon_1 \in R_o = R \cap (\vartheta)^{\perp}$ and hence $-\varepsilon_4 = (-\varepsilon_4 + \varepsilon_1) - \varepsilon_1 \in R_J$, by Lemma 7.7 (4): contradiction.

Assume now that $R = B_n$, $n > 3$. Then we may assume that $(\alpha, \alpha') = (\varepsilon_1 + \varepsilon_2, -\varepsilon_3)$ is a ϑ-dual pair in $R_{\mathfrak{e}}$ and hence that $\vartheta = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$. Then, as before, we get that $-\varepsilon_3 + \varepsilon_4 \in R_J$, $-\varepsilon_4 \in R_o$ and hence that $-\varepsilon_3 = (-\varepsilon_3 + \varepsilon_4) + (-\varepsilon_4) \in R_J$: contradiction. □

Now we construct some special basis Π for R, which we will call *good*. For any basis Π let

$$\Pi_o = \Pi \cap R_o, \quad \tilde{\Pi}_o = \Pi_o \cap \tilde{R}_{\mathfrak{e}}, \quad \tilde{\Pi}_{\mathfrak{e}} = \Pi \cap \tilde{R}_{\mathfrak{e}}, \quad \Pi_{\mathfrak{e}} = \Pi \cap R_{\mathfrak{e}}.$$

Then

$$\tilde{\Pi}_{\mathfrak{e}} = \Pi_{\mathfrak{e}} \cup \tilde{\Pi}_o.$$

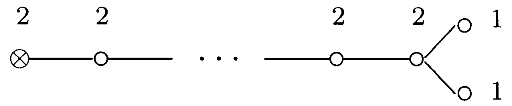
A basis Π is called *good* if

$$\tilde{R}_o = [\tilde{\Pi}_o], \quad \tilde{R}_{\mathfrak{e}} = [\tilde{\Pi}_{\mathfrak{e}}], \quad R_o = [\Pi_o],$$

where for any subset $A \subset \Pi$ we denote $[A] = \mathrm{span}(A) \cap R$.

A good basis exists because $R_o \cup \tilde{R}_{\mathfrak{e}} = R'_o \cup \tilde{R}_{\mathfrak{e}}$ is a closed subset of roots, R'_o is orthogonal to $\tilde{R}_{\mathfrak{e}}$ and $R_o = R'_o \cup (R_o \cap \tilde{R}_{\mathfrak{e}}) = R'_o \cup \tilde{R}_o$. In fact, we may take a basis $\tilde{\Pi}_o$ for $\tilde{R}_o$, extend it to a basis $\tilde{\Pi}_{\mathfrak{e}}$ for $\tilde{R}_{\mathfrak{e}}$, add to it a basis for R'_o and finally extend everything to a basis Π for R.

By the remarks before Lemma 7.15, the pair $(\tilde{\Pi}_{\mathfrak{e}}, \tilde{\Pi}_o)$ is of type $(D_\ell, D_{\ell-1})$, $\ell > 2$, or $(2A_1, \emptyset)$ and it can be represented by the following two graphs



(7.18)

(7.19)

where the subdiagram $\tilde{\Pi}_o$ is obtained by deleting the grey nodes. Moreover, by Lemma 7.10, the contact form ϑ is the linear combination of the simple roots associated with the nodes of (7.18) and (7.19) with the indicated coefficients. For example, if $(\tilde{\Pi}_{\mathfrak{e}}, \tilde{\Pi}_o) = (D_\ell, D_{\ell-1})$ and if we use the standard correspondence between nodes and roots, we get

$$\begin{aligned}\vartheta &= 2(\varepsilon_1 - \varepsilon_2) + \cdots + 2(\varepsilon_{\ell-2} - \varepsilon_{\ell-1}) + (\varepsilon_{\ell-1} - \varepsilon_\ell) + (\varepsilon_{\ell-1} + \varepsilon_\ell) \\ &= 2\varepsilon_1.\end{aligned}$$

Note that if $\ell = 4$, using two permutations of the simple roots corresponding to the end nodes, one gets the other two possible contact forms, namely $\vartheta = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4$ and $\vartheta = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4$.

Remark that a good basis Π together with the subsets Π_o and $\Pi_{\mathfrak{e}}$ completely determines the homogeneous CR manifold $M = G/L$ and the flag manifolds $(F_Z = G/K, J_1)$ and $(F_2 = G/Q, J_2)$. In fact the root systems $R_o = R(K)$ of K and $R(Q)$ of Q are given by $R_o = [\Pi_o]$ and $R(Q) = [\tilde{\Pi}_e = \Pi_o \cup \Pi_{\mathfrak{e}}]$ and $\mathfrak{l} = \mathfrak{k} \cap (\ker \vartheta)$, where ϑ is defined by (7.18)–(7.19).

Notice also that by definition of good basis

$$R \cap (\vartheta)^{\perp} = R_o = [\Pi_o] \tag{7.20}$$

and hence that

$$\Pi_o = \Pi \cap (\vartheta)^{\perp}. \tag{7.21}$$

Any good basis Π together with the subsets Π_o and $\Pi_{\mathfrak{e}}$ can be represented by a painted Dynkin graph $\Gamma = \Gamma(\Pi)$ if we paint the nodes corresponding to the roots of $\Pi_{\mathfrak{e}}$ in grey, the nodes of Π_o in white and all others in black.

We call such graph Γ *a painted Dynkin graph associated with the CR manifold* $(M = G/L, \mathcal{D}_Z, J)$.

Any associated painted Dynkin graph has the following two properties.

(1) It contains a unique proper subgraph $\Gamma_{\mathfrak{e}}$ of type (7.18), if it is connected, or of type (7.19), if it is not connected; moreover in this second

case, $\Gamma = \Gamma_1 \cup \Gamma_2$ has two connected components and each of them contains exactly one grey node.

(2) The black nodes are exactly the nodes which are linked to $\Gamma_{\mathfrak{e}}$.

Indeed, (1) follows from definition of good basis, Lemma 7.9 and Lemma 7.2. (2) follows from (7.21).

A painted Dynkin graph which satisfies (1) and (2) is called *admissible graph*.

Let Γ be an admissible graph and $\Gamma_{\mathfrak{e}}$ the corresponding subgraph of type (7.18) or (7.19). We denote by $\vartheta(\Gamma)$ the linear combination of roots associated with the nodes of $\Gamma_{\mathfrak{e}}$ as prescribed in (7.18)–(7.19).

An admissible graph Γ is called *good* if

$$[\Pi_o] = R \cap (\vartheta)^{\perp}, \tag{7.22}$$

where Π_o is the set of simple roots associated with the white nodes of Γ. Remark that by (7.20) any graph associated with $(M = G/L, \mathcal{D}_Z, J)$ is a good graph. The converse of this statement is also true.

Lemma 7.16 *Any good graph is a painted Dynkin graph associated to a homogeneous CR manifolds $(G/L, \mathcal{D}_Z, J)$, which have a contact form ϑ parallel to no roots and where $(\mathcal{D}_Z, J)$ is non-standard and non-primitive.*

Proof. Let Γ be a good graph and $\vartheta(\Gamma)$ the corresponding contact form. As described in §7.4, Γ defines two flag manifolds $F_1(\Gamma) = G/K$ and $F_2(\Gamma) = G/Q$, with invariant complex structures $J_1(\Gamma)$ and $J_2(\Gamma)$, respectively. Denote by

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} + \mathfrak{m}^{10}_{J_1} + \mathfrak{m}^{01}_{J_1}, \qquad \mathfrak{g}^{\mathbb{C}} = \mathfrak{q}^{\mathbb{C}} + \mathfrak{m}^{10}_{J_2} + \mathfrak{m}^{01}_{J_2}$$

the corresponding associated decompositions. Consider also the element $Z = i\mathcal{B}^{-1} \circ \vartheta(\Gamma)$. Since the 1-parametric subgroup generated by Z is closed, by Proposition 3.3 it defines a contact manifold $(M = G/L, \mathcal{D}_Z)$ with $\mathfrak{l} = \mathfrak{k} \cap (Z)^{\perp}$. Moreover the fiber $C = Q/L$ of the fibration $\pi : G/L \to G/Q$, together with the contact structure induced on C by Z, is one of the contact manifolds described in Proposition 7.5 admitting a primitive CR structure.

If $\mathfrak{m}^{10}_C$ is the holomorphic subspace of such CR structure, then $\mathfrak{m}^{10} = \mathfrak{m}^{10}_C + \mathfrak{m}^{10}_{J_2}$ is the holomorphic subspace of a non-standard CR structure $(\mathcal{D}_Z, J)$ on G/L and the associated painted Dynkin graph is exactly $(\Gamma, \vartheta(\Gamma))$. In fact, the conditions i) and ii) of Definition 4.1 are immediate.

The integrability condition follows from the fact that $\mathfrak{m}_C^{10}$ is a holomorphic subspace for a CR structure on $C = Q/L$ (and hence that $\mathfrak{l}^{\mathbb{C}} + \mathfrak{m}_C^{10}$ is a subalgebra), that $\mathfrak{m}_{J_2}^{10}$ is the nilradical of the parabolic subalgebra $\mathfrak{q}^{\mathbb{C}} + \mathfrak{m}_{J_2}^{10}$, and that $\mathfrak{m}_C^{10} \subset \mathfrak{q}^{\mathbb{C}}$. □

Now the classification of homogeneous CR manifolds of the considered type reduces to the classification of good graphs Γ.

Case 1: Γ is not connected.

In this case $\Gamma_{\mathfrak{e}} = A_1 \cup A_1$, $\Gamma = \Gamma_1 \cup \Gamma_2$, where each Γ_i is a connected component which corresponds to a root system R_i, and $R = R_1 \cup R_2$. Moreover $\vartheta = \alpha_1 - \alpha_2$, where $\alpha_i \in R_i$.

We prove that if Γ is good then $R = A_p \cup A_q$, with $p + q > 1$ and that Γ is a CR-graph of type II.

First of all, one can easily check that if one of the connected components Γ_i is not of type A_q, then Γ is not good, that is that there exists a root $\beta \in R \cap (\vartheta)^\perp$ which is not in $[\Pi_o]$. For example, if $R_1 = D_q$, we may assume that $\vartheta = \alpha_1 - \alpha_2$, where $\alpha_1 = \varepsilon_1 - \varepsilon_2$. Then $\beta = \varepsilon_1 + \varepsilon_2 \in R \cap (\vartheta)^\perp$ but it is not in $[\Pi_o]$.

Assume now that $R = A_p \cup A_q$. Without loss of generality we may assume that $\alpha_1 = \varepsilon_k - \varepsilon_{k+1}$, $\alpha_2 = \varepsilon'_r - \varepsilon'_{r+1}$ are the roots associated with the grey nodes of Γ_1 and Γ_2, respectively. Then $R \cap (\vartheta)^\perp = A_{p-2} \cup A_{q-2}$ and it coincides with $[\Pi_o]$ if and only if the nodes of the roots α_i are end nodes. This proves that Γ is good if and only if it is a CR-graph of type II (see Definition 7.4).

Case 2: Γ is connected .

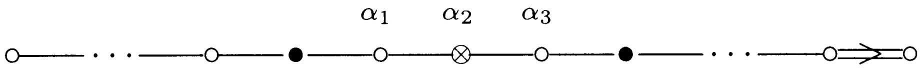
In this case, Γ is a good graph only if the type of the pair $(\Gamma, \Gamma_{\mathfrak{e}})$ is one of the following

$$\begin{aligned} &(A_n, A_3), \quad (B_n, A_3), \quad (D_n, D_4), \quad (E_6, D_5), \\ &(E_7, D_6), \quad (E_8, D_5), \quad (E_8, D_7). \end{aligned}$$

This follows from Corollary 7.11 and the fact that (A_n, A_3), (B_n, B_3), (B_n, D_4), (D_n, D_n), (F_4, A_3) and (F_4, B_3) do not correspond to any admissible graph.

We first prove that the cases (B_n, A_3), (E_7, D_6), (E_8, D_5) and (E_8, D_7) are not possible.

i) $(\Gamma, \Gamma_{\mathfrak{e}}) = (B_n, A_3)$. In this case Γ is of the form

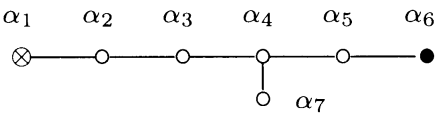


where $\alpha_1 = \varepsilon_k - \varepsilon_{k+1}$, $\alpha_2 = \varepsilon_{k+1} - \varepsilon_{k+2}$ and $\alpha_3 = \varepsilon_{k+2} - \varepsilon_{k+3}$. Then $\vartheta(\Gamma) = \alpha_1 + 2\alpha_2 + \alpha_3 = \varepsilon_k + \varepsilon_{k+1} - \varepsilon_{k+2} - \varepsilon_{k+3}$ and

$$\Pi_o = \{\varepsilon_i - \varepsilon_{i+1},\ i = 1, \dots, k-2;\ k;\ k+2;\ k+4, \dots, n-1;\ \varepsilon_n\}.$$

However the root $\beta = \varepsilon_{k+1} + \varepsilon_{k+2} \in (\vartheta(\Gamma))^{\perp} \cap R$ but it does not belong to $[\Pi_o]$: contradiction.

ii) $(\Gamma, \Gamma_{\mathfrak{e}}) = (E_7, D_6)$. In this case Γ and $\vartheta(\Gamma)$ are

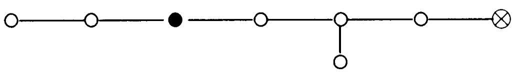


, $\vartheta(\Gamma) = 2\varepsilon_1 + \varepsilon_7 + \varepsilon_8$.

However, this situation corresponds to no good graph, because the root $\beta = \varepsilon_7 - \varepsilon_8$ is in $\vartheta(\Gamma)^{\perp} \cap R$, but it does not belong to

$$\begin{aligned}[\Pi_o] &= [\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_7\}] \\ &= \{\varepsilon_a - \varepsilon_b,\ \pm(\varepsilon_a + \varepsilon_b + \varepsilon_7 + \varepsilon_8),\ 1 \le a,\, b \le 6\}.\end{aligned}$$

iii) $(\Gamma, \Gamma_{\mathfrak{e}}) = (E_8, D_5)$. Then Γ and $\vartheta(\Gamma)$ are



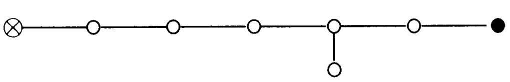
, $\vartheta(\Gamma) = \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 - \varepsilon_8$.

One can easily check that the root $\beta = \varepsilon_1 + \varepsilon_2 + \varepsilon_4$ is orthogonal to $\vartheta(\Gamma)$, but it doesn't belong to the subsystem

$$[\Pi_o] = [\{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_8\}]$$

generated by white roots: contradiction.

iv) $(\Gamma, \Gamma_{\mathfrak{e}}) = (E_8, D_7)$. Then Γ and $\vartheta(\Gamma)$ are



, $\vartheta(\Gamma) = 2\varepsilon_1 + \varepsilon_8$.

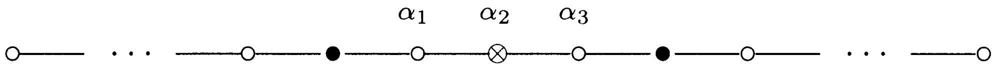
Also this case is not possible because $\varepsilon_7 - \varepsilon_9 \in R \cap (\vartheta(\Gamma))^{\perp}$ but it is not in $[\Pi_o] = [\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_8\}]$.

It remains to describe the good graphs of the following types

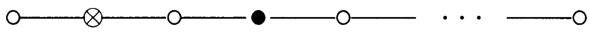
1) (A_n, A_3), 2) (D_n, D_4), 3) (E_6, D_5).

(1) $(\Gamma, \Gamma_{\mathfrak{e}}) = (A_n, A_3)$.

Assume that $\Gamma_{\mathfrak{e}}$ is not at an end of Γ, that is



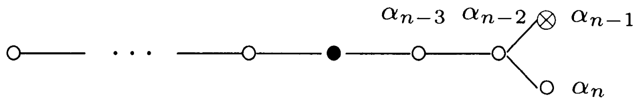
Then we may assume that $\alpha_i = \varepsilon_{p+i} - \varepsilon_{p+i+1}$. Then $\vartheta(\Gamma) = \alpha_1 + 2\alpha_2 + \alpha_3 = \varepsilon_{p+1} + \varepsilon_{p+2} - \varepsilon_{p+3} - \varepsilon_{p+4}$ and the root $\beta = \varepsilon_p - \varepsilon_{p+5} \in R \cap (\vartheta(\Gamma))^{\perp}$ but it is not in the span of Π_o; hence the graph is not good. On the other hand one can easily check that the graph



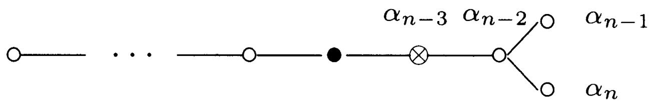
is good.

(2) $(\Gamma, \Gamma_{\mathfrak{e}}) = (D_n, D_4)$.

In this case we have two admissible graphs:



(7.23)



(7.24)

Using the standard equipment, we have that if Γ is given by (7.23), then

$$\vartheta(\Gamma) = \alpha_{n-3} + 2\alpha_{n-2} + \alpha_{n-1} + 2\alpha_n = \varepsilon_{n-3} + \varepsilon_{n-2} + \varepsilon_{n-1} + \varepsilon_n$$

and $R \cap (\vartheta(\Gamma))^{\perp} = D_{n-4} \cup A_3$. If Γ is given by (7.24), then

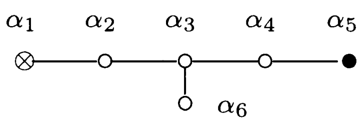
$$\vartheta(\Gamma) = 2\alpha_{n-3} + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n = 2\varepsilon_{n-3}$$

and $R \cap (\vartheta(\Gamma))^{\perp} = D_{n-1}$.

Since in both cases $[\Pi_o] = A_{n-5} \cup A_3$, the graph (7.24) is not good, while the graph (7.23) is good only when $n = 5$.

(3) $(\Gamma, \Gamma_{\mathfrak{e}}) = (E_6, D_5)$.

Up to isomorphism, we have only one admissible graph



Using the standard equipment, we get

$$\vartheta(\Gamma) = 2(\alpha_1 + \alpha_2 + \alpha_3) + \alpha_4 + \alpha_6 = 2\varepsilon_1 + \varepsilon_6 + \varepsilon.$$

Then

$$R \cap (\vartheta(\Gamma))^{\perp} = \{\varepsilon_a - \varepsilon_b,\ \pm(\varepsilon_a + \varepsilon_b + \varepsilon_6 + \varepsilon),\ a, b = 2, 3, 4, 5\}$$
$$= [\Pi_o] = D_4$$

and hence the graph is good.

This concludes the classification of good graphs. We proved that the pairs $(\Gamma, \vartheta(\gamma))$ given by all good graphs are exactly the non-special CR-graphs of Definition 7.4. By the remarks before Lemma 7.16 and the Lemmata 7.14 and 7.16, Proposition 7.6 follows.

8. Primitive and non-standard non-primitive CR manifolds

In this conclusive section, we collect the information of Theorem 5.1, Corollary 5.2 and Propositions 7.3, 7.5 and 7.6 to enumerate all primitive CR manifolds and non-standard non-primitive CR manifolds.

The list of primitive CR manifolds is easily obtained taking in account the primitive CR manifolds described in Corollary 5.2 and the primitive CR manifolds described in Propositions 7.3 and 7.5. Moreover, we have the following.

Lemma 8.1 *Any primitive CR manifold $(M = G/L, \mathcal{D}, J)$ is a Morimoto-Nagano space.*

Proof. By Theorem 6.3, the anti-canonical map

$$\phi : G/L \longrightarrow \mathrm{Gr}_k(\mathfrak{g}^C) \subset \mathbb{C}P^N$$

is a finite holomorphic covering of the G-orbit $G/\hat{L} = G \cdot p = \phi(G/L) \subset \mathrm{Gr}_k(\mathfrak{g})$, $p = \phi(eL)$. Observe also that $\phi(G/L) = G/\hat{L}$ does not admit any CRF fibration: in fact, if $G \cdot p = G/\hat{L}$ admitted a CRF fibration, then there would exists a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}^C$, which satisfies the conditions of Lemma 4.8 and hence also $M = G/L$ would admit a CRF fibration.

By [1], the orbit $G/\hat{L} = G \cdot p$ is a real hypersurface in the complex orbit $G^{\mathbb{C}} \cdot p = G^{\mathbb{C}}/H$ of the complexified group $G^{\mathbb{C}}$ and the complex orbit $G^{\mathbb{C}} \cdot p = G^{\mathbb{C}}/H$ is either Stein or it admits a holomorphic fibration $\pi : G^{\mathbb{C}}/H \to F = G^C/P$ onto a flag manifold $F = G^C/P$. This second case cannot occur, because otherwise π would induce a CRF fibration $\pi : G/\hat{L} \to F = G^C/P$. Since G is a compact Lie group acting holomorphically on the Stein manifold $G^{\mathbb{C}} \cdot p = G^{\mathbb{C}}/H$ with orbits of codimension one, by [9], it follows immediately that the orbit $G/\hat{L} = G \cdot p \subset G^{\mathbb{C}}/H$ is a Morimoto-Nagano space. By a direct inspection of all cases, it can also be checked that any Morimoto-Nagano space is either simply connected or it admits a holomorphic covering by a simply connected Morimoto-Nagano space. Hence, also the simply connected primitive CR manifold $M = G/L$ is a Morimoto-Nagano space. □

Collecting these information, we have the following theorem.

Theorem 8.2 *Let $(M = G/L, \mathcal{D}_Z, J)$ be a simply connected, primitive, homogeneous CR manifold and let $\theta = \mathcal{B} \circ Z|_{\mathfrak{t}}$ be the dual form of the contact element Z restricted to a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k} = C_{\mathfrak{g}}(Z) = \mathfrak{l} + \mathbb{R}Z$. Then G/L is isomorphic to the universal covering space of a sphere bundle $S(N) \subset T(N)$ of a CROSS N. The groups G, $K = C_G(Z)$, the form $\vartheta = -i\theta$ and the CROSS N are listed in the following table.*

n^o	G	$K = C_G(Z)$	ϑ	$N = G/H$
1	$SU_2 \times SU_2'$	$T^1 \times T^{1'}$	$(\varepsilon_1 - \varepsilon_2) + (\varepsilon_1' - \varepsilon_2')$	$S^3 = \frac{SO_4}{SO_3}$
2	Spin_7	$T^1 \cdot SU_3$	$\varepsilon_1 + \varepsilon_2 + \varepsilon_3$	$S^7 = \frac{\mathrm{Spin}_7}{G_2}$
3	G_2	$T^1 \cdot SU_2$	ε_1	$S^6 = \frac{G_2}{SU_3}$
4	F_4	$T^1 \cdot SO_7$	ε_1	$\mathbb{O}P^2 = \frac{F_4}{\mathrm{Spin}_9}$
5	$SO_{2n+1} \quad n > 1$	$T^1 \cdot SO_{2n-1}$	ε_1	$S^{2n} = \frac{SO_{2n+1}}{SO_{2n}}$
6	$SO_{2n} \quad n > 2$	$T^1 \cdot SO_{2n-2}$	ε_1	$S^{2n-1} = \frac{SO_{2n}}{SO_{2n-1}}$
7	$SU_{n+1} \quad n > 1$	$T^1 \cdot U_{n-1}$	$\varepsilon_1 - \varepsilon_2$	$\mathbb{C}P^n = \frac{SU_{n+1}}{U_n}$
8	Sp_n	$T^1 \cdot Sp_1 \cdot Sp_{n-2}$	$\varepsilon_1 + \varepsilon_2$	$\mathbb{H}P^{n-1} = \frac{Sp_n}{Sp_1 \cdot Sp_{n-1}}$

Putting together the lists of non-primitive CR manifolds in Corollary 5.2 and those in Proposition 7.6, we also obtain the complete table of non-standard non-primitive CR manifolds, which give in the following theorem.

Theorem 8.3 *Let $(M = G/L, \mathcal{D}_Z, J)$ be a simply connected homogeneous CR manifold with a non-standard non-primitive CR structure.*

Then, either $M = SU_2$ or there exists a unique CRF fibration

$$\pi : M = G/L \longrightarrow F = G/Q$$

over a flag manifold F with an invariant complex structure J_F, such that the fiber $C = Q/L$ is either a primitive CR manifold or is equal to $SO_3 = S(S^2)$. Moreover the groups G, L, the primitive fiber $C = Q/L$ and the flag manifold $F = G/Q$ are as in the following table (in n.2, the subgroups U_{p-2} and U'_{q-2} of L are subgroups of the factors SU_p and SU'_q of G, respectively) :

n^o	G	L	$C = Q/L$	$F = G/Q$
1	$SU_n \quad n > 2$	$T^1 \cdot SU_{n-2}$	$SO_3 = S(S^2)$	$\frac{SU_n}{S(U_2 \cdot U_{n-2})}$
2	$SU_p \times SU'_q$ $p+q > 4$	$T^1 \cdot U_{p-2} \cdot U'_{q-2}$	$\frac{SO_4}{SO_2} = S(S^3)$	$\frac{SU_p}{S(U_2 \cdot U_{p-2})} \times \frac{SU_q}{S(U_2 \cdot U_{q-2})}$
3	$SU_n \quad n > 4$	$T^1 \cdot (SU_2 \times SU_2) \cdot SU_{n-4}$	$\frac{SO_6}{SO_4} = S(S^5)$	$\frac{SU_n}{S(U_4 \times U_{n-4})}$
4	SO_{10}	$T^1 \cdot SO_6$	$\frac{SO_8}{SO_6} = S(S^7)$	$\frac{SO_{10}}{T^1 \cdot SO_8}$
5	E_6	$T^1 \cdot SO_8$	$\frac{SO_{10}}{SO_8} = S(S^9)$	$\frac{E_6}{T^1 \cdot SO_{10}}$

In particular, the fiber C is a sphere bundle $S(S^r) \subset TS^r$ where $r = 2, 3, 5, 7$ or 9. The CR manifolds in n.1 admit also a CRF fibration with fiber S^1.

We conclude with the next Theorem 8.4, where it is indicated how a non-standard, non-primitive CR structure can be totally recovered from a CR-graph (see the definition and basic properties of CR-graphs in §7.4).

We recall that the explicit classification of non-standard, non-primitive CR structures on non-special CR manifolds using non-special CR-graphs is already given in Proposition 7.6. On the other hand, the explicit description given in Theorem 5.1 and Corollary 5.2 of non-standard, non-primitive CR structures on special CR manifolds can be easily restated using special CR-graphs. Putting these results together, one obtains the following description in term of CR-graphs of any non-standard, non-primitive CR structure.

Theorem 8.4 *Let $M = G/L$ be a simply connected, homogeneous CR manifold with a non-primitive, non-standard CR structure $(\mathcal{D}_Z, J)$. Suppose also that $M \neq SU_2$.*

Denote by $\pi : G/L \to F_Z = G/K$ the natural (non-holomorphic) fibration associated with the contact structure $\mathcal{D}_Z$ and by $\pi' : G/L \to F_2 = G/Q$ the unique CRF fibration over a flag manifold $F_2 = G/Q$ with invariant complex structure J_2, with non-standard fiber Q/L of minimal dimension, which is either primitive or admitting a CRF fibration with fiber S^1.

Then $Q \supset K$ and the sequence of fiberings

$$M = G/L \longrightarrow F_Z = G/K \longrightarrow F_2 = G/Q$$

is holomorphic with respect to the standard CR structure $(\mathcal{D}, J_s)$ on M, associated to $(\mathcal{D}, J)$, the corresponding complex structure J_s on F_Z and the complex structure J_2 on F_2.

Moreover, the painted Dynkin graph Γ associated to the flag manifolds $F_1 = F_Z$, F_2 with complex structures $J_1 = J_s$ and J_2, respectively, is a CR graph and (up to a transformation from the Weyl group) Z is proportional to $Z(\Gamma) = i\mathcal{B}^{-1} \circ \vartheta(\Gamma)$.

Conversely, if Γ is a CR-graph, then there exists a unique homogeneous contact manifold $(M = G/L, \mathcal{D}_Z)$ such that $Z = i\mathcal{B}^{-1} \circ \vartheta(\Gamma)$ and $F_Z = F_1(\Gamma) = G/K$. The complex structure $J_1(\Gamma)$ defines the unique standard CR structure $(\mathcal{D}_Z, J_1(\Gamma))$ on M such that the sequence of fibrations

$$M = G/L \longrightarrow F_Z = F_1(\Gamma) = G/K \longrightarrow F_2(\Gamma) = G/Q$$

is holomorphic w.r.t. $(\mathcal{D}_Z, J_1(\Gamma))$, $J_1(\Gamma)$ and $J_2(\Gamma)$. The space of the invariant CR structures $(\mathcal{D}_Z, J)$ on M such that the projection $\pi' : M \to F_2(\Gamma)$ is holomorphic, is parameterized by the points of the unit disc $D \in \mathbb{R}^2$. The center of D corresponds to the CR structure $(\mathcal{D}_Z, J_1(\Gamma))$ and the other points correspond to the non-standard CR structures. Moreover a CR structure is non-standard if and only if it induces a non-standard CR structure on the fiber Q/L; such induced CR structure is always primitive, with the exceptions of the cases in which Γ is a special CR-graph.

Appendix

The notation used in the following Tables is the same of [7]. We recall that the weights of the groups B_ℓ, C_ℓ, D_ℓ and F_4 are expressed in terms of an orthonormal basis $(\varepsilon_1, \dots, \varepsilon_\ell)$ of $\mathfrak{h}(\mathbb{Q})^*$. The weights of the groups A_ℓ, E_7, E_8 and G_2 are expressed using vectors $\varepsilon_1, \dots, \varepsilon_{\ell+1} \in \mathfrak{h}(\mathbb{Q})^*$ such that

$$\sum \varepsilon_i = 0, \qquad (\varepsilon_i, \varepsilon_j) = \begin{cases} \dfrac{\ell}{\ell+1} & i = j \\ -\dfrac{1}{\ell+1} & i \neq j \end{cases} \tag{A.1}$$

It is useful to recall that if $\sum a_i = 0$, then $(\sum a_i \varepsilon_i, \sum b_j \varepsilon_j) = \sum a_i b_i$. For E_6, the weights are expressed by vectors $\varepsilon_1, \dots, \varepsilon_6$, which satisfy (A.1) with $\ell = 5$, and by an auxiliary vector ε which is orthogonal to all ε_i and satisfies $(\varepsilon, \varepsilon) = 1/2$.

In Table 1, for any simple complex Lie group $\mathfrak{g}^{\mathbb{C}}$, we give the corresponding root system R, the longest root μ (unique up to inner automorphisms), the subalgebra $\mathfrak{g}_0' = C_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{g}(\mu))$, the subsystem of roots R_o corresponding to $\mathfrak{g}_0'$, the decomposition into irreducible submodules of the $\mathfrak{g}_0$-module $\mathfrak{g}_1$ which appear in the decomposition (3.2), and the set of roots $R_1 = R^+ \setminus (\mu \cup R_o)$.

For a set of simple roots of $\mathfrak{g}_0'$, we denote by $\{\pi_1, \dots, \pi_\ell\}$ the corresponding system of fundamental weights and, for any weight $\lambda = \sum a_i \pi_i$, we denote by $V(\lambda)$ the irreducible $\mathfrak{g}_0'$-module with highest weight λ.

In Table 2, we give the information needed to determine the holomorphic subspaces $\mathfrak{m}^{10}$ when $\mathfrak{g}^{\mathbb{C}}$ is a simple Lie algebra and the contact form $\vartheta = -i\mathcal{B} \circ Z|_{\mathfrak{h}}$ is parallel to a short root.

In Table 3 we give the same information for the cases $\mathfrak{g}^{\mathbb{C}} = B_3$ or D_ℓ and ϑ proportional to no root and associated with a primitive CR structure.

In both tables we give the root systems R, the contact form ϑ, the subalgebra $\mathfrak{l}^{\mathbb{C}} = C_{\mathfrak{g}^{\mathbb{C}}}(Z) \cap (Z)^{\perp}$, the root subsystem R_o of $\mathfrak{l}^{\mathbb{C}}$ and the list of the highest weights for the irreducible $\mathfrak{k}^{\mathbb{C}}$-modules in $\mathfrak{m}^{\mathbb{C}}$ ($\mathfrak{k}^{\mathbb{C}} = C_{\mathfrak{g}^{\mathbb{C}}}(Z)$). We group the highest weights corresponding to equivalent $\mathfrak{l}^{\mathbb{C}}$-modules with curly brackets.

In Table 4 we recall the Dynkin graphs associated with indecomposable root systems and the correspondence used in [7] between nodes and simple roots.

Table 1

$\mathfrak{g}$	R	μ	$\mathfrak{g}_0'$	R_o	$\mathfrak{g}_1$	R_1
A_ℓ	$\varepsilon_i-\varepsilon_j$ $1\le i,j\le \ell+1$	$\varepsilon_1-\varepsilon_{\ell+1}$	$A_{\ell-2}+\mathbb{R}$	$\varepsilon_a-\varepsilon_b$ $2\le a,b\le\ell$	$V(\pi_1)+$ $V(\pi_{\ell-2})$	$\varepsilon_1-\varepsilon_a,\ \varepsilon_a-\varepsilon_{\ell+1}$ $2\le a\le\ell$
B_ℓ	$\pm\varepsilon_i\pm\varepsilon_j,\ \pm\varepsilon_i$ $1\le i,j\le\ell$	$\varepsilon_1+\varepsilon_2$	$A_1+B_{\ell-2}$	$\pm(\varepsilon_1-\varepsilon_2),\ \pm\varepsilon_a\pm\varepsilon_b$ $\pm\varepsilon_a$ $3\le a,b\le\ell$	$V(\pi_1)\otimes$ $V(\pi_1')$	$\varepsilon_1,\ \varepsilon_2$ $\varepsilon_1\pm\varepsilon_a,\ \varepsilon_2\pm\varepsilon_a$ $3\le a\le\ell$
C_ℓ	$\pm\varepsilon_i\pm\varepsilon_j,\ \pm2\varepsilon_i$ $1\le i,j\le\ell$	$2\varepsilon_1$	$C_{\ell-1}$	$\pm\varepsilon_a\pm\varepsilon_b,\ \pm2\varepsilon_a$ $2\le a,b\le\ell$	$V(\pi_1)$	$\varepsilon_1\pm\varepsilon_a$ $2\le a\le\ell$
D_ℓ	$\pm\varepsilon_i\pm\varepsilon_j$ $1\le i,j\le\ell$	$\varepsilon_1+\varepsilon_2$	$A_1+D_{\ell-2}$	$\pm(\varepsilon_1-\varepsilon_2),\ \pm\varepsilon_a\pm\varepsilon_b$ $3\le a,b\le\ell$	$V(\pi_1)\otimes$ $V(\pi_1')$	$\varepsilon_1\pm\varepsilon_a,\ \varepsilon_2\pm\varepsilon_a$ $3\le a\le\ell$
E_6	$\varepsilon_i-\varepsilon_j,\ \pm2\varepsilon$ $\varepsilon_i+\varepsilon_j+\varepsilon_k\pm\varepsilon$ $1\le i,j,k\le6$	2ε	A_5	$\varepsilon_i-\varepsilon_j$	$V(\pi_1)$	$\varepsilon_i+\varepsilon_j+\varepsilon_k\pm\varepsilon$
E_7	$\varepsilon_i-\varepsilon_j$ $\varepsilon_i+\varepsilon_j+\varepsilon_k+\varepsilon_\ell$ $1\le i,j,k,\ell\le8$	$-\varepsilon_7+\varepsilon_8$	D_6	$\varepsilon_a-\varepsilon_b$ $\varepsilon_7+\varepsilon_8+\varepsilon_a+\varepsilon_b$ $\varepsilon_a+\varepsilon_b+\varepsilon_c+\varepsilon_d$ $1\le a,b,c,d\le6$	$V(\pi_1)$	$-\varepsilon_7+\varepsilon_a,\ \varepsilon_8-\varepsilon_a$ $\varepsilon_8+\varepsilon_a+\varepsilon_b+\varepsilon_c$ $1\le a,b,c\le6$
E_8	$\varepsilon_i-\varepsilon_j$ $\pm(\varepsilon_i+\varepsilon_j+\varepsilon_k)$ $1\le i,j,k\le9$	$\varepsilon_1-\varepsilon_9$	E_7	$\varepsilon_a-\varepsilon_b$ $\pm(\varepsilon_1+\varepsilon_9+\varepsilon_a)$ $\pm(\varepsilon_a+\varepsilon_b+\varepsilon_c)$ $2\le a,b,c\le8$	$V(\pi_1)$	$\varepsilon_1-\varepsilon_a,\ -\varepsilon_9+\varepsilon_a$ $\varepsilon_1+\varepsilon_a+\varepsilon_b$ $2\le a,b\le8$
F_4	$\frac{\pm\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4}{2}$ $\pm\varepsilon_i\pm\varepsilon_j,\ \pm\varepsilon_i$ $1\le i,j\le4$	$\varepsilon_1+\varepsilon_2$	C_3	$\pm(\varepsilon_1-\varepsilon_2)$ $\pm\frac{\varepsilon_1-\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4}{2}$ $\pm\varepsilon_a,\ \pm\varepsilon_a\pm\varepsilon_b$ $3\le a,b\le4$	$V(\pi_1)$	$\varepsilon_1,\ \varepsilon_2$ $\frac{\varepsilon_1+\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4}{2}$
G_2	$\varepsilon_i-\varepsilon_j,\ \pm\varepsilon_i$ $1\le i,j\le3$	$\varepsilon_1-\varepsilon_2$	A_1	$\pm\varepsilon_3$	$V(\pi_1)$	$\varepsilon_1-\varepsilon_3,\ \varepsilon_1-\varepsilon_2$ $\varepsilon_3-\varepsilon_2$

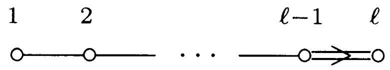
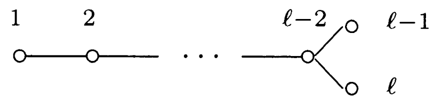
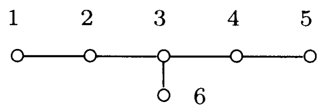
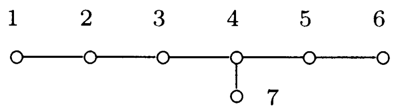
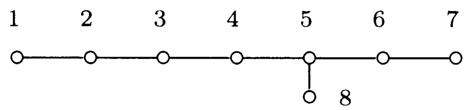
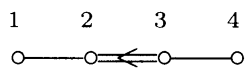
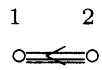
Table 2

$\mathfrak{g}$	R	$\vartheta =$ $i\mathcal{B}\circ Z$	$A\mathfrak{l}^{\mathbb{C}} =$ $C_{\mathfrak{g}^{\mathbb{C}}}(Z)\cap(Z)^{\perp}$	R_o	highest weights for $\mathfrak{m}^{\mathbb{C}}$ grouped into sets of equivalent $\mathfrak{l}^{\mathbb{C}}$-modules
B_ℓ	$\pm\varepsilon_i\pm\varepsilon_j,\ \pm\varepsilon_i$ $1\le i,j\le\ell$	ε_1	$B_{\ell-1}$	$\pm\varepsilon_a\pm\varepsilon_b,\ \pm\varepsilon_a$ $2\le a,b\le\ell$	$\{\varepsilon_1+\varepsilon_2, -\varepsilon_1+\varepsilon_2\}$
C_ℓ	$\pm\varepsilon_i\pm\varepsilon_j,\ \pm2\varepsilon_i$ $1\le i,j\le\ell$	$\varepsilon_1+\varepsilon_2$	$A_1+C_{\ell-2}$	$\pm(\varepsilon_1-\varepsilon_2),\ \pm2\varepsilon_a$ $\pm\varepsilon_a\pm\varepsilon_b$ $3\le a,b\le\ell$	$\{2\varepsilon_1,-2\varepsilon_2\}$ $\{\varepsilon_1+\varepsilon_3,\ -\varepsilon_2+\varepsilon_3\}$
F_4	$\frac{\pm\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4}{2}$ $\pm\varepsilon_i\pm\varepsilon_j,\ \pm\varepsilon_i$ $1\le i,j\le4$	ε_1	B_3	$\pm\varepsilon_a\pm\varepsilon_b,\ \pm\varepsilon_a$ $2\le a,b\le4$	$\{\varepsilon_1+\varepsilon_2,-\varepsilon_1+\varepsilon_2\}$ $\left\{\frac{\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4}{2},\frac{-\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4}{2}\right\}$
G_2	$\varepsilon_i-\varepsilon_j,\ \pm\varepsilon_i$ $1\le i,j\le3$	ε_1	A_1	$\pm(\varepsilon_2-\varepsilon_3)$	$\{\varepsilon_1,-\varepsilon_1\}$ $\{\varepsilon_3-\varepsilon_1,\ \varepsilon_3,\ -\varepsilon_2,\ \varepsilon_1-\varepsilon_2\}$

Table 3

$\mathfrak{g}$	R	$\vartheta = i\mathcal{B}\circ Z$	$\mathfrak{l}^{\mathbb{C}} = C_{\mathfrak{g}^{\mathbb{C}}}(Z)\cap(Z)^{\perp}$	R_o	highest weights for $\mathfrak{m}^{\mathbb{C}}$ grouped into sets of equivalent $\mathfrak{l}^{\mathbb{C}}$-modules
B_3	$\pm\varepsilon_i\pm\varepsilon_j,\ \pm\varepsilon_i$ $1\leq i,j\leq 3$	$\varepsilon_1+\varepsilon_2+\varepsilon_3$	A_2	$\pm(\varepsilon_a-\varepsilon_b)$, $1\leq a,b\leq 3$	$\{\varepsilon_1+\varepsilon_2, -\varepsilon_3\}$ $\{-\varepsilon_2-\varepsilon_3,\ \varepsilon_1\}$
D_ℓ	$\pm\varepsilon_i\pm\varepsilon_j$ $1\leq i,j\leq \ell$	ε_1	$D_{\ell-1}$	$\pm\varepsilon_i\pm\varepsilon_j$ $2\leq i,j\leq\ell$	$\{\varepsilon_1+\varepsilon_2, -\varepsilon_1+\varepsilon_2\}$

Table 4

Type of G	Dynkin graphs	Simple roots
A_ℓ		$\alpha_i=\varepsilon_i-\varepsilon_{i+1}$
B_ℓ		$\alpha_i=\varepsilon_i-\varepsilon_{i+1}\ (i<\ell)$, $\alpha_\ell=\varepsilon_\ell$
C_ℓ		$\alpha_i=\varepsilon_i-\varepsilon_{i+1}\ (i<\ell)$, $\alpha_\ell=2\varepsilon_\ell$
D_ℓ		$\alpha_i=\varepsilon_i-\varepsilon_{i+1}\ (i<\ell)$, $\alpha_\ell=\varepsilon_{\ell-1}+\varepsilon_\ell$
E_6		$\alpha_i=\varepsilon_i-\varepsilon_{i+1}\ (i<6)$, $\alpha_6=\varepsilon_4+\varepsilon_5+\varepsilon_6+\varepsilon$
E_7		$\alpha_i=\varepsilon_i-\varepsilon_{i+1}\ (i<7)$, $\alpha_7=\varepsilon_5+\varepsilon_6+\varepsilon_7+\varepsilon_8$
E_8		$\alpha_i=\varepsilon_i-\varepsilon_{i+1}\ (i<8)$, $\alpha_8=\varepsilon_6+\varepsilon_7+\varepsilon_8$
F_4		$\alpha_1=(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4)/2$, $\alpha_2=\varepsilon_4,\ \alpha_3=\varepsilon_3-\varepsilon_4$, $\alpha_4=\varepsilon_2-\varepsilon_3$
G_2		$\alpha_1=-\varepsilon_2,\ \alpha_2=\varepsilon_2-\varepsilon_3$

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