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Weighted weak type inqualities for maximal commutators of Bochner-Riesz operator

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Abstract. In this paper, we establish endpoint estimates of $L(\log L)$ type for maximal commutators of Bochner-Riesz operators, and the weighted weak type estimates for the commutators are also obained.

Key words: Bochner-Riesz operator, commutator, weak type inequality, weight, sharp function.

1. Introduction

Let $b \in BMO(R^n)$ and T be a Calderon-Zygmumd operator. Consider the commutator defined by

$$[b,T]f(x) = b(x)Tf(x) - T(bf)(x).$$

A classical result of Coifman, Rochberg and Weiss [3] proved that $[b,T]$ is bounded on $L^p(R^n)$ $(1 < p < \infty)$. However, it is observed that $[b,T]$ is not, in general, weak type (1,1). In fact, Perez [10] proved that $[b,T]$ satisfies $L(\log L)$ type inequality. The purpose of this paper is to consider a similar problem: how to establish the weak type inequalities for the maximal commutators of Bochner-Riesz operators. Recently, the boundedness of the commutators on $L^p(R^n)$ and Herz-type Hardy spaces are studied in [7], [9], we go on doing this work. We show that the commutators satisfy $L(\log L)$ type inequalities, and the weighted weak type inequalities for the commutators are also obtained. In Section 2 and 3, we will give some concepts and Theorems of this paper, whose proofs will appear in Section 5, and Section 4 contains some Lemmas.

2. Definition

Let us first introduce some concepts.

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Definition Let b be a locally integrable function and $m \in \mathbb{N}$. The maximal operator $B^{m,a}_{*,b}$ associated with the commutator generated by the Bochner-Riesz operator is defined by

$$B^{m,a}_{*,b}(f)(x) = \sup_{r>0} |B^{m,a}_{r,b}(f)(x)| \tag{1}$$

where

$$B^{m,a}_{r,b}(f)(x) = \int_{R^n} B^a_r(x-y)f(y)(b(x)-b(y))^m dy \tag{2}$$

and $(B^a_r(f))\hat{}(\xi) = (1-r^2|\xi|^2)^a_+ \hat{f}(\xi)$. If $m=1$, we denote simply $B^{1,a}_{*,b} = B^a_{*,b}$ and $B^{1,a}_{r,b}(f) = B^a_{r,b}(f)$. We also define

$$B^a_*(f)(x) = \sup_{r>0} |B^a_r(f)(x)|$$

which is the Bochner-Riesz operator (see [8]).

Let E be the space of bounded functions on $(0,\infty)$, $E = \{h : ||h|| = \sup_{r>0} |h(r)| < \infty\}$, then, for each fixed $x \in R^n$, $B^a_r(f)(x)$ may be viewed as a mapping from $(0,\infty)$ to E, and it is clear that

$$B^a_*(f)(x) = ||B^a_r(f)(x)|| \quad \text{and}$$
$$B^a_{*,b}(f)(x) = ||b(x)B^a_r(f)(x) - B^a_r(bf)(x)||$$

Let Mf be the Hardy-Littlewood maximal operator. For $\delta > 0$, we define

$$M_\delta(f) = [M(|f|^\delta)]^{1/\delta} \quad \text{and} \quad M^{\#}_\delta(f) = [M^{\#}(|f|^\delta)]^{1/\delta},$$

where

$$M^{\#}(f)(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y) - f_Q| dy$$

and

$$f_Q = |Q|^{-1}\int_Q f(y)dy.$$

The corresponding dyadic maximal operators are denoted by M^d_δ and $M^{\#,d}_\delta$, respectively, (see [10])

Let $\Phi : [0,+\infty) \to [0,+\infty)$ be a Young function, we define the Φ-averge of a function f over a ball Q by means of the following Luxemburg norm

$$||f||_{\Phi,Q} = \inf\left\{\lambda > 0 : |Q|^{-1}\int_Q \Phi(|f(y)|/\lambda)dy \le 1\right\},$$

and the maximal operator M_Φ associated with $||\cdot||_{\Phi,Q}$ by

$$M_\Phi(f)(x) = \sup_{x\in Q} ||f||_{\Phi,Q}.$$

We have the following generalized Hölder's inequality

$$\frac{1}{|Q|}\int_Q |f(y)g(y)|dy \le ||f||_{\Phi,Q}||g||_{\Psi,Q},$$

where Ψ is the complementory Young function of Φ, which is given by (see [10], [12] for details)

$$\Psi(s) = \sup_{0\le t<\infty} (st - \Phi(t)), \quad 0 \le s < \infty.$$

It is obvious that $\Phi_m(t) = t(1+\log^+ t)^m$ is a Young function and its complementary Young function $\Psi_m(t) \sim \exp(t^{1/m})$ (see [10]). Denote $||f||_{\Phi_m,Q}$ and $||f||_{\Psi_m,Q}$ by $||f||_{L(\log L)^m,Q}$ and $||f||_{(\exp L)^{1/m},Q}$, $M_{\Phi_m}(f)$ by $M_{L(\log)^m}(f)$.

3. Theorems

Now we are in the position to state our main results.

Theorem 1 *Let $a > (n-1)/2$ and $b \in BMO(R^n)$. Then there exists a constant $C > 0$ such that for all $\lambda > 0$,*

$$|\{x \in R^n : B^a_{*,b}(f)(x) > \lambda\}|$$
$$\le C||b||_{BMO}(1+\log^+ ||b||_{BMO})\lambda^{-1}\int_{R^n} |f(x)|(1+\log^+(\lambda^{-1}|f(x)|))dx.$$

Theorem 2 *Let $m \in \mathbb{N}$ and $a > (n-1)/2$. Suppose $b \in BMO(R^n)$ and $\omega \in A_1$ (Muckenhouput weight class). Then there exists a constant $C > 0$ such that for all $\lambda > 0$.*

$$\omega(\{x \in R^n : B^{m,a}_{*,b}(f)(x) > \lambda\})$$
$$\le C\left[||b||_{BMO}(1+\log^+ ||b||_{BMO})\right]^m \lambda^{-1}$$
$$\times \int_{R^n} |f(x)|(1+\log^+(\lambda^{-1}|f(x)|))^m\omega(x)dx.$$

Theorem 3 *Let $p \in (1, \infty)$ and $a > n/p - (n+1)/2$, Given a pair of weights (u, v), suppose that for some $r > 1$ and for all balls Q,*

$$\left(\frac{1}{|Q|}\int_Q u^r(x)dx\right)^{1/r}\left(\frac{1}{|Q|}\int_Q v^{-p'/p}(x)dx\right)^{p/p'} \le C < \infty.$$

Then B_*^a satisfies the weak (p,p) inequality

$$u(\{x \in R^n : B_*^a(f)(x) > \lambda\}) \le C\lambda^{-p}\int_{R^n}|f(x)|^p v(x)dx.$$

Theorem 4 *Let $p \in (1, \infty)$ and $a > n/p - (n+1)/2$, $m \in \mathbb{N}$ and suppose $b \in BMO(R^n)$. Given a pair of weight (u, v), suppose that for some $r > 1$ and for all balls Q,*

$$\left(\frac{1}{|Q|}\int_Q u^r(x)dx\right)^{1/r}||v^{-1/p}||^p_{\Phi_m,B} \le C < \infty.$$

where $\Phi_m(t) = t^{p'}(\log(1+t))^{mp'}$.

Then $B_{*,b}^{m,a}$ satisfies the weak (p,p) inequality

$$u(\{x \in R^n : B_{*,b}^{m,a}(f)(x) > \lambda\})$$
$$\le C||b||^{mp}_{BMO}\lambda^{-p}\int_{R^n}|f(x)|^p v(x)dx.$$

4. Some Lemmas

Now, we state some lemmas, which are useful to our theorems in this paper.

Lemma 1 (Kolmogorov, [6]) *Let $0 < p < q < \infty$ and for any function $f \ge 0$, define that*

$$||f||_{WL^q} = \sup_{\lambda>0}\lambda|\{x \in R^n : f(x) > \lambda\}|^{1/q},$$
$$N_{p,q}(f) = \sup_E ||f\chi_E||_p/||\chi_E||_r, \ (1/r = 1/p - 1/q)$$

where the supremum is taken for all measurable sets E with $0 < |E| < \infty$. Then

$$||f||_{WL^q} \le N_{p,q}(f) \le (q/(q-p))^{1/p}||f||_{WL^q}.$$

Lemma 2 ([10]) *There exists a constant $C > 0$ such that for any function f and for all $\lambda > 0$,*

$$|\{y \in R^n : M^2 f(y) > \lambda\}| \\ \leq C\lambda^{-1} \int_{R^n} |f(y)|(1 + \log^+(\lambda^{-1}|f(y)|))dy,$$

where $M^2 = M \circ M$.

Lemma 3 ([10]) (1) *There exists a constant $C > 0$ such that for all $\lambda > 0$, $\varepsilon > 0$,*

$$|\{y \in R^n : M^d f(y) > \lambda,\ M^{\#,d} f(y) \leq \lambda\varepsilon\}| \\ \leq C\varepsilon|\{y \in R^n : M^d f(y) > \lambda/2\}|.$$

(2) *Let $\varphi : (0, +\infty) \to (0, +\infty)$ be a doubling function. Then there exists a constant $C > 0$ such that*

$$\sup_{\lambda>0} \varphi(\lambda)|\{y \in R^n : M_\delta f(y) > \lambda\}| \\ \leq C \sup_{\lambda>0} \varphi(\lambda)|\{y \in R^n : M_\delta^{\#} f(y) > \lambda\}|.$$

Lemma 4 *Let $0 < \delta < 1$ and $a > (n-1)/2$. Suppose that f and $B_*^a(f)$ are locally integrable. Then there exists a constant $C > 0$ such that*

$$M_\delta^{\#}(B_*^a(f))(x) \leq CMf(x), \quad \textit{for all} \ \ x \in R^n.$$

Lemma 5 *Let $0 < \delta < \varepsilon < 1$ and $a > (n-1)/2$, suppose that $b \in BMO(R^n)$. Then there exists a constant $C > 0$ such that*

$$M_\delta^{\#}(B_{*,b}^a(f))(x) \leq C||b||_{BMO}[M_\varepsilon(B_*^a(f))(x) + M^2 f(x)]$$

for all smooth functions f.

Lemma 6 *Let $\Phi(t) = t(1 + \log^+ t)$ and $L_\delta(f) = \sup_{t>0} \frac{1}{\Phi(1/t)} \left|\{y \in R^n : M_\delta(B_{*,b}^a(f))(y) > t\}\right|$ and $b \in BMO(R^n)$. Then there exists a constant $C > 0$ such that for all $\varepsilon > 0$ and $0 < \delta < 1$, when $a > (n-1)/2$, we have*

$$L_\delta(f) \leq C\varepsilon L_\delta(f) + C||b||_{BMO}(1 + \log^+ ||b||_{BMO}) \\ \times \sup_{t>0} \frac{1}{\Phi(1/t)}|\{y \in R^n : M^2 f(y) > t\}|.$$

Lemma 7 *Let $\Phi(t) = t(1 + \log^+ t)$ and $a > (n-1)/2$. Suppose that $b \in BMO(R^n)$. Then there exists a constant $C > 0$ such that for any smooth function f with compact support,*

$$\sup_{t>0} \frac{1}{\Phi(1/t)} |\{y \in R^n : B^a_{*,b}(f)(y) > t\}|$$
$$\leq C||b||_{BMO}(1 + \log^+ ||b||_{BMO}) \sup_{t>0} \frac{1}{\Phi(1/t)} |\{y \in R^n : M^2 f(y) > t\}|.$$

Proof of Lemma 4. Given $x \in R^n$, and a ball $Q = Q(x_0, R) \ni x$, let aQ be the ball with the same center as Q and a times radius of Q. Let $f_1 = f\chi_{2Q}$, $f_2 = f\chi_{(2Q)^c}$. Then, for all $z \in Q(x_0, R)$, we have

$$|B^a_*(f)(z) - B^a_*(f_2)(x_0)| \leq ||B^a_r(f)(z) - B^a_r(f_2)(x_0)||$$
$$\leq B^a_*(f_1)(z) + ||B^a_r(f_2)(z) - B^a_r(f_2)(x_0)||.$$

By Lemma 1 and weak type (1,1) of B^a_* (see [8]), we have

$$\left(\frac{1}{|Q|}\int_Q (B^a_*(f_1)(z))^\delta dz\right)^{1/\delta} = |Q|^{-1} \frac{||B^a_*(f_1)\chi_Q||_\delta}{|Q|^{1/\delta - 1}}$$
$$\leq C|B|^{-1}||B^a_*(f_1)||_{WL^1} \leq C|Q|^{-1}\int_{2Q} |f(z)|dz$$
$$\leq CMf(x).$$

Next, we estimate $||B^a_r(f_2)(z) - B^a_r(f_2)(x_0)||$. To do this, we write

$$B^a_r(f_2)(x) = r^{-n}\int f_2(y)K^a((x-y)/r)dy$$

and K^a satisfies the following (see [8], p. 121).

$$|\nabla^\beta K^a(z)| \leq C(1 + |z|)^{-(a+(n+1)/2)} \quad \text{for } |\beta| \leq 1,$$

where $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ and $\nabla^\beta = \frac{\partial^{|\beta|}}{\partial^{\beta_1}\partial^{\beta_2}\cdot\partial^{\beta_n}}$. We choose

$$(n-1)/2 < a_0 < \min(a, (n+1)/2).$$

Since $|x_0 - y| \sim |z - y|$ when $z \in Q$ and $y \in (2Q)^c$, we consider the following two cases:

Case 1: $0 < r \le R$. In this case, we have, for $x \in Q$,

$$\begin{aligned}
&|B_r^a(f_2)(x)| \\
&\le Cr^{-n} \int_{(2Q)^c} |f(y)|(1+|x-y|/r)^{-(a+(n+1)/2)} dy \\
&= Cr^{-n} \sum_{k=1}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} |f(y)|(1+|x-y|/r)^{-(a+(n+1)/2)} dy \\
&\le Cr^{-n+a+(n+1)/2} \sum_{k=1}^{\infty} \frac{|2^{k+1}Q|}{(2^k R)^{a+(n+1)/2}} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f(y)| dy \right) \\
&= C \left(\frac{r}{R} \right)^{a-(n-1)/2} \sum_{k=1}^{\infty} 2^{k((n-1)/2-a)} \cdot Mf(x) \\
&\le CMf(x).
\end{aligned}$$

It follows that

$$||B_r^a(f_2)(z) - B_r^a(f_2)(x_0)|| \le CMf(x);$$

Case 2: $r > R$. In this case, we have

$$\begin{aligned}
&|B_r^a(f_2)(z) - B_r^a(f_2)(x_0)| \\
&\le r^{-n} \int |f_2(y)||B^a((z-y)/r) - B^a((x_0-y)/r)| dy \\
&\le Cr^{-n-1} \int |f_2(y)||z-x_0|(1+|y-x_0|/r)^{-(a+(n+1)/2)} dy \\
&= Cr^{-n-1} \sum_{k=1}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} |f(y)||z-x_0|(1+|y-x_0|/r)^{-(a+(n+1)/2)} dy \\
&\le Cr^{a_0-(n+1)/2} \sum_{k=1}^{\infty} \frac{R(2^{k+1}R)^n}{(2^k R)^{a_0+(n+1)/2}} \cdot \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f(y)| dy \right) \\
&\le CR^{a_0-(n+1)/2} R^{1+n} \cdot R^{-(a_0+(n+1)/2)} \sum_{k=1}^{\infty} 2^{k((n-1)/2-a_0)} Mf(x) \\
&\le CMf(x).
\end{aligned}$$

It follows that

$$||B_r^a(f_2)(z) - B_r^a(f_2)(x_0)|| \le CMf(x);$$

Thus, by Hölder's inequality,

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q ||B_r^a(f_2)(z)-B_r^a(f_2)(x_0)||^\delta dz\right)^{1/\delta}\\&\quad\le \frac{1}{|Q|}\int_Q ||B_r^a(f_2)(z)-B_r^a(f_2)(x_0)||dz\\&\quad\le CMf(x),\end{aligned}$$

so that

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q |B_*^a(f)(z)-B_*^a(f_2)(x_0)|^\delta dz\right)^{1/\delta}\\&\quad\le C\left(\frac{1}{|Q|}\int_Q |B_*^a(f_1)(z)|^\delta dz\right)^{1/\delta}\\&\qquad+C\left(\frac{1}{|Q|}\int_Q ||B_r^a(f_2)(z)-B_r^a(f_2)(x_0)||^\delta dz\right)^{1/\delta}\\&\quad\le CMf(x),\end{aligned}$$

notice that $||a|^\delta - |b|^\delta| \le |a-b|^\delta$ for $a,b\in R$. Then the conclusion of Lemma 4 follows from the inequality above. □

Remark From Lemma 4, we get

$$M^{\#}(B_*^a(f))(x)\le CM_pf(x),\qquad 1<p<\infty.$$

Proof of Lemma 5. Given $x\in R^n$ and a ball $Q=Q(x_0,R)\ni x$. We write, for $y\in Q$,

$$\begin{aligned}&b(y)B_r^a(f)(y)-B_r^a(bf)(y)\\&\quad=(b(y)-b_{2Q})B_r^a(f)(y)-B_r^a((b-b_{2Q})f\chi_{2Q})(y)\\&\qquad-B_r^a((b-b_{2Q})f\chi_{(2Q)^c})(y)\end{aligned}$$

so that

$$\begin{aligned}&|B_{*,b}^a(f)(y)-B_*^a((b-b_{2Q})f\chi_{(2Q)^c})(x_0)|\\&\le ||(b(y)-b_{2Q})B_r^a(f)(y)||+||B_r^a((b-b_{2Q})f\chi_{2Q})(y)||\\&\quad+||B_r^a((b-b_{2Q})f\chi_{(2Q)^c})(y)-B_r^a((b-b_{2Q})f\chi_{(2Q)^c})(x_0)||\\&=I_1(y)+I_2(y)+I_3(y).\end{aligned}$$

For $I_1(y)$, by Hölder's inequality with exponent p and p', where $1<p<\varepsilon/\delta$, and $1/p+1/p'=1$ we have

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q (I_1(y))^\delta dy\right)^{1/\delta} &= \left(\frac{1}{|Q|}\int_Q |b(y)-b_{2Q}|^\delta (B_*^a(f)(y))^\delta dy\right)^{1/\delta}\\ &\le \left(\frac{1}{|Q|}\int_Q |b(y)-b_{2Q}|^{\delta p'}\right)^{1/(\delta p')}\left(\frac{1}{|Q|}\int_Q (B_*^a(f)(y))^{\delta r} dy\right)^{1/(\delta p)}\\ &\le C||b||_{BMO} M_{\delta p}(B_*^a(f))(x)\\ &\le C||b||_{BMO} M_\varepsilon(B_*^a(f))(x). \qquad (3)\end{aligned}$$

For $I_2(y)$, by Lemma 1 and the weak type (1,1) of B_*^a (see [8]), we have

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q (I_2(y))^\delta dy\right)^{1/\delta} &= \left(\frac{1}{|Q|}\int_Q (B_*^a((b-b_{2Q})f\chi_{2Q})(y))^\delta dy\right)^{1/\delta}\\ &\le C|2Q|^{-1}\frac{||B_*^a((b-b_{2Q})f\chi_{2Q})||_\delta}{|2Q|^{1/\delta-1}}\\ &\le C|2Q|^{-1}||B_*^a((b-b_{2Q})f\chi_{2Q})||_{WL^1}\\ &\le C|2Q|^{-1}\int_{2Q}|b(y)-b_{2Q}|\;|f(y)|dy.\end{aligned}$$

By the generalized Hölder's inequality and the fact (see [10]).

$$||b-b_Q||_{\exp L,Q}\le C||b||_{BMO},$$

we obtain

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q (I_2(y))^\delta dy\right)^{1/\delta} &\le C||b-b_{2Q}||_{\exp L,2Q}||f||_{L\log L,2Q}\\ &\le C||b||_{BMO}M_{L\log L}f(x). \qquad (4)\end{aligned}$$

For $I_3(y)$, we proceed to do it as in the proof of Lemma 4, and by the properties of $BMO(R^n)$ functions (see [14]), we have

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q (I_3(y))^\delta dy\right)^{1/\delta}\\ &\le \frac{1}{|Q|}\int_Q ||B_r^a((b-b_{2Q})f\chi_{(2Q)^c})(y)-B_r^a((b-b_{2Q})f\chi_{(2Q)^c})(x_0)||dy\\ &\le C\sum_{k=1}^{\infty}2^{k((n-1)/2-a_0)}\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(y)|\;|b(y)-b_{2Q}|dy\end{aligned}$$

$$\begin{aligned}
&\le C\sum_{k=1}^{\infty} 2^{k((n-1)/2-a_0)}\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(y)|\,|b(y)-b_{2^{k+1}Q}|dy \\
&\quad + C\sum_{k=1}^{\infty} 2^{k((n-1)/2-a_0)}\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|b_{2^{k+1}Q}-b_{2Q}|\,|f(y)|dy \\
&\le C\sum_{k=1}^{\infty} 2^{k((n-1)/2-a_0)}||b-b_{2^{k+1}Q}||_{\exp L,2^{k+1}Q}||f||_{L\log L,2^{k+1}Q} \\
&\quad + C\sum_{k=1}^{\infty} 2^{k((n-1)/2-a_0)}k||b||_{BMO}Mf(x) \\
&\le C||b||_{BMO}M_{L\log L}f(x) + C||b||_{BMO}Mf(x) \\
&\le C||b||_{BMO}M_{L\log L}f(x). \qquad (5)
\end{aligned}$$

Note that $M^2 f \approx M_{L\log L}f$ (cf. [10]) and by (3) $\sim$ (5). We obtain the desired result. □

Proof of Lemma 6. Because $|\{Mf(y) > t\}| \sim |\{M^d f(y) > t\}|$, from Lemma 3, we have

$$\begin{aligned}
&|\{y \in R^n : M_\delta(B^a_{*,b}(f))(y) > t\}| \\
&\quad \le C\varepsilon|\{y \in R^n : M((B^a_{*,b}(f))^\delta)(y) > t^\delta/2\}| \\
&\qquad + C|\{y \in R^n : M^{\#}((B^a_{*,b}(f))^\delta)(y) > \varepsilon t^\delta\}| \\
&\quad \equiv I + II.
\end{aligned}$$

For II, by Lemma 5, with $\varepsilon = p\delta$, $1 < p < 1/\delta$, we have

$$\begin{aligned}
II \le\ & |\{y \in R^n : M_{p\delta}(B^a_*(f))(y) > \varepsilon^{1/\delta}t/(2C||b||_{BMO})\}| \\
& + |\{y \in R^n : M^2 f(y) > \varepsilon^{1/\delta}t/(2C||b||_{BMO})\}|,
\end{aligned}$$

where C is the constant in Lemma 5.

Let $\alpha = \varepsilon^{1/\delta}/(2C||b||_{BMO})$, we obtain

$$\begin{aligned}
&\frac{1}{\Phi(1/t)}|\{y \in R^n : M_\delta(B^a_{*,b}(f))(y) > t\}| \\
&\le \frac{C\varepsilon}{\Phi(1/t)}|\{y \in R^n : M_\delta(B^a_{*,b}(f))(y) > t/2^{1/\delta}\}| \\
&\quad + \frac{1}{\Phi(1/t)}|\{y \in R^n : M_{p\delta}(B^a_*(f))(y) > \alpha t\}|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Phi(1/t)}|\{y\in R^n : M^2f(y)>\alpha t\}| \\
&\le \frac{C\varepsilon}{\Phi(2^{1/\delta}/t)}|\{y\in R^n : M_\delta(B^a_{*,b}(f))(y)>t/2^{1/\delta}\}| \\
&+\frac{C||b||_{BMO}(1+\log^+||b||_{BMO})}{\Phi(1/\alpha t)}|\{y\in R^n : M_{p\delta}(B^a_*(f))(y)>\alpha t\}| \\
&+\frac{C||b||_{BMO}(1+\log^+||b||_{BMO})}{\Phi(1/\alpha t)}|\{y\in R^n : M^2f(y)>\alpha t\}| \\
&\le C\varepsilon L_\delta(f)+C||b||_{BMO}(1+\log^+||b||_{BMO}) \\
&\times\sup_{t>0}\frac{1}{\Phi(1/t)}\Big(|\{y\in R^n : M_{p\delta}(B^a_*f)(y)>t\}| \\
&\quad+\frac{C||b||_{BMO}(1+\log^+||b||_{BMO})}{\Phi(1/ct)}|\{f\in R^n : M^2f(y)>t\}|\Big),
\end{aligned}
$$

thus, by Lemma 3 and 4, we have

$$
\begin{aligned}
L_\delta(f)&\le C\varepsilon L_\delta(f)+C||b||_{BMO}(1+\log^+||b||_{BMO}) \\
&\quad\times\sup_{t>0}\frac{1}{\Phi(1/t)}|\{y\in R^n : Mf(y)>t\}| \\
&\qquad+C||b||_{BMO}(1+\log^+||b||_{BMO}) \\
&\quad\times\sup_{t>0}\frac{1}{\Phi(1/t)}|\{y\in R^n : M^2f(y)>t\}| \\
&\le C\varepsilon L_\delta(f)+C||b||_{BMO}(1+\log^+||b||_{BMO}) \\
&\quad\times\sup_{t>0}\frac{1}{\Phi(1/t)}|\{y\in R^n : M^2f(y)>t\}|.
\end{aligned}
$$

This complete the proof of Lemma 6. □

Proof of Lemma 7. By Lemma 6, we only need to show that $L_\delta(f)$ is finite, which is similar to the proof of [10] (see [10], p. 173), we omit the details. □

5. Proof of Theorems

Proof of Theorem 1. By homogeneity, it sufice to show the case $\lambda = 1$. Without loss of generality, we assume that f is a smooth function with compact support. By Lemma 7 and 2, we have

$$\begin{aligned}
&|\{x \in R^n : B^a_{*,b}(f)(x) > 1\}| \\
&\leq \sup_{t>0} \frac{1}{\Phi(1/t)} |\{x \in R^n : B^a_{*,b}(f)(x) > t\}| \\
&\leq C||b||_{BMO}(1 + \log^+ ||b||_{BMO}) \sup_{t>0} \frac{1}{\Phi(1/t)} |\{x \in R^n : M^2 f(x) > t\}| \\
&\leq C||b||_{BMO}(1 + \log^+ ||b||_{BMO}) \sup_{t>0} \frac{1}{\Phi(1/t)} \int_{R^n} \Phi(|f(x)|/t) dx \\
&\leq C||b||_{BMO}(1 + \log^+ ||b||_{BMO}) \sup_{t>0} \frac{1}{\Phi(1/t)} \int_{R^n} \Phi(|f(x)|)\Phi(1/t) dx \\
&\leq C||b||_{BMO}(1 + \log^+ ||b||_{BMO}) \int_{R^n} |f(x)|(1 + \log^+ |f(x)|) dx.
\end{aligned}$$

This finishes the proof of Theorem 1. □

The proof of Theorem 2 goes along the same line as that of $m = 1$ and the unweighted case once we give the following lemmas.

Lemma 8 *Let $a > (n-1)/2$, $0 < \delta < \varepsilon < 1$ and suppose $b \in BMO(R^n)$. Then there exists a constant $C > 0$ such that for each smooth function f,*

$$\begin{aligned}
&M^{\#}_\delta(B^{m,a}_{*,b}(f))(x) \\
&\leq C\left(\sum_{j=0}^{m-1} ||b||^{m-j}_{BMO} M_\varepsilon(B^{j,a}_{*,b}(f))(x) + ||b||^m_{BMO} M^{m+1} f(x) \right).
\end{aligned}$$

Lemma 9 *Let $a > (n-1)/2$. Suppose $b \in BMO(R^n)$ and $\omega \in A_1$. That $\Phi_m(t) = t(1 + \log^+ t)^m$. Then there exists a constant $C > 0$ such that for any smooth function f with compact support*

$$\begin{aligned}
&\sup_{t>0} \frac{1}{\Phi_m(1/t)} \omega(\{x \in R^n : B^{m,a}_{*,b}(f)(x) > t\}) \\
&\quad \leq C \left[||b||_{BMO}(1 + \log^+ ||b||_{BMO})\right]^m \\
&\qquad \times \sup_{t>b} \frac{1}{\Phi_m(1/t)} \omega(\{x \in R^n : M^{m+1} f(x) > t\})
\end{aligned}$$

where $\Phi_m(t) = t^{p'}(\log(1 + t))^{mp'}$.

Proof of Lemma 8. Following the idea of [5] (also see [10]), for any constant c, we have

$$\begin{aligned}
&B^{m,a}_{r,b} f(x)\\
&= \int_{R^n} [(b(x)-c)-(b(y)-c)]^m \frac{1}{r^n} K^a\Big(\frac{x-y}{r}\Big) f(y)dy\\
&= \sum_{j=1}^{m} C_{j,m}(b(x)-c)^j \int_{R^n} (b(y)-c)^{m-j} \frac{1}{r^n} K^a\Big(\frac{x-y}{r}\Big) f(y)dy\\
&\qquad + B^a_r((b-c)^m f)(x)\\
&= \sum_{j=1}^{m} C_{j,m}(b(x)-c)^j \int_{R^n} [(b(y)-b(x))+(b(x)-c)]^{m-j}\\
&\qquad \times \frac{1}{r^n} K^a\Big(\frac{x-y}{r}\Big) f(y)dy + B^a_r((b-c)^m f)(x)\\
&= \sum_{j=1}^{m}\sum_{k=0}^{m-j} C_{j,k,m}(b(x)-c)^{j+k} \int_{R^n} (b(y)-b(x))^{m-j-k}\\
&\qquad \times \frac{1}{r^n} K^a\Big(\frac{x-y}{r}\Big) f(y)dy + B^a_r((b-c)^m f)(x)\\
&= \sum_{j=0}^{m-1} C_{j,m}(b(x)-c)^{m-j} B^{j,a}_{r,b} f(x) + B^a_r((b-c)^m f)(x).
\end{aligned}$$

Thus, using the same method as in the proof of Lemma 5 and the proof of Lemma 7.1 in [7], and noting that $M^{m+1} \approx M_{L(\log L)^m}$, we obtain the desired estimate. □

Proof of Lemma 9. It suffice to show that the lemma holds for $||b||_{BMO} \le 1$. Since we have the weighted version of Lemma 3 (see [7]):

$$\begin{aligned}
&\omega(\{x \in R^n : Mf(x) > \lambda,\ M^{\#} f(x) \le \lambda\varepsilon\})\\
&\quad \le C\varepsilon\omega(\{x \in R^n : Mf(x) > \lambda/2\}),
\end{aligned}$$

similar to the proof of Lemma 6, we have

$$\begin{aligned}
&\sup_{t>0} \frac{1}{\Phi_m(1/t)} \omega(\{x \in R^n : M_\delta(B^{m,a}_{*,b}(f))(x) > t\})\\
&\quad \le C\left[||b||_{BMO}(1+\log^+ ||b||_{BMO})\right]^m\\
&\qquad\quad \times \sup_{t>0} \frac{1}{\Phi_m(1/t)} \omega(\{x \in R^n : M^{m+1} f(y) > t\}).
\end{aligned}$$

Thus, by the iterating argument, similar to the proof of Lemma 7, we gain the estimate of the Lemma 9. □

Now, Theorem 2 follows from Lemma 8 and 9, we omit the details.

The proof of Theorem 3 is based on Lemma 4 and a version of the Calderon-Zygmund decomposition (see Theorem 3.4 in [4]), which can obtained by the same way as that of Theorem 1.2 in [4], almost without changing any words.

The proof of Theorem 4 depends on Lemma 8 and Theorem 3. A similar argument as in the proof of Theorem 1.6 in [4] will give us the desired inequality. We omit the details.

Corollary *Let* $1 < p < \infty$, $a > n/p - (n+1)/2$ *and* $m \in \mathbb{N}$. *Suppose* $b \in BMO(R^n)$ *and* $\omega \in A_p$. *Then,*

$$||B^{m,a}_{*,b}(f)||_{L^p(\omega)} \le C||b||^m_{BMO}||f||_{L^p(\omega)},$$

for all $f \in L^p(R^n, \omega)$.

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