Hokkaido Mathematical Journal Vol. 32 (2003) p. 75–83

A sufficient condition on global stability in a logistic equation with piecewise constant arguments[1]

Yoshiaki MUROYA

(Received June 29, 2001)

Abstract. We establish a sufficient condition for the contractivity for solutions and global asymptotic stability for the positive equilibrium N^* of the following logistic equation with piecewise constant arguments:

$$\begin{cases} \dfrac{dN(t)}{dt} = N(t)r(t)\Big\{1 - \sum\limits_{k=0}^{m} b_k N(l-k)\Big\}, & l \le t < l+1, \quad l = 0,1,2,\ldots, \\ N(0) = N_0 > 0, \quad \text{and} \quad N(-k) = N_{-k} \ge 0, & 1 \le k \le m, \end{cases}$$

where $r(t)$ is continuous on $[0,\infty), r(t) \ge 0$, $r(t) \not\equiv 0$, $b_0 > 0$ and $b_k \ge 0$, $1 \le k \le m$. This new condition that $b_0 > \sum_{k=1}^{m} b_k$ and $r_l = \int_l^{l+1} r(s)ds$, $0 < \inf_{l \ge 0} r_l \le r_l \le 1$, $l = 0,1,2,\ldots$, is another type condition than those given by J. W.-H. So and J. S. Yu (1995, *Hokkaido Math. J.* **24**, 269–286) and others.

Key words: contractivity, global stability, logistic equation with piecewise constant delays.

1. Introduction

Consider the following logistic equation with piecewise constant arguments:

$$\frac{dN(t)}{dt} = N(t)r(t)\Big\{1 - \sum_{k=0}^{m} b_k N(l-k)\Big\}, \qquad l \le t < l+1, \quad l = 0,1,2,\ldots, \tag{1.1}$$

with initial conditions of the form

$$N(0) = N_0 > 0 \quad \text{and} \quad N(-k) = N_{-k} \ge 0, \quad 1 \le k \le m, \tag{1.2}$$

where $r(t)$ is continuous on $[0,\infty)$, $r(t) \ge 0$, $r(t) \not\equiv 0$, and

$$b_0 > 0, \quad \text{and} \quad b_k \ge 0, \quad 1 \le k \le m. \tag{1.3}$$

2000 Mathematics Subject Classification : 34D23.

[1]Research partially supported by Waseda University Grant for Special Research Projects 2001A-571.

Concerning the conditions of the global asymptotic stability for the positive equilibrium N^* of Eq. (1.1), Gopalsamy, Kulenovic and Ladas [2] have obtained $r < \frac{\log 2}{m+1}$ for $r(t) \equiv r$, and So and Yu [4] improved this condition to $\int_0^{\infty} r(t)dt = +\infty$ and $\sup_{l\geq 0}\int_{l-m}^{l+1} r(t)dt \leq \frac{3}{2}$.

In this note, using some techniques (see Lemmas 2.1 and 2.3), we obtain a new condition for the contractivity of solutions and the globally asymptotic stability for the positive equilibrium N^* of Eq. (1.1).

This condition that $b_0 > \sum_{k=1}^{m} b_k$ and for

$$r_l = \int_l^{l+1} r(s)ds, \quad 0 < \inf_{l\geq 0} r_l \leq r_l \leq 1, \quad l = 0, 1, 2, \dots$$

(see Theorem 2.1), is related to Theorem 2.1 in Wang and Lu [5] and is another type condition than those of So and Yu [4] and others (cf. Chen and Liu [1], Yu [6] and Zhou and Zhang [7]).

2. Contractivity and global stability

Consider the following logistic equation with piecewise constant delays:

$$\frac{dN(t)}{dt} = N(t)r(t)\left\{1 - \sum_{k=0}^{m} b_k N(l-k)\right\},$$
$$l \leq t < l+1, \quad l = 0, 1, 2, \dots, \tag{2.1}$$

with initial conditions of the form

$$N(0) = N_0 > 0 \quad \text{and} \quad N(-k) = N_{-k} \geq 0, \quad 1 \leq k \leq m, \tag{2.2}$$

where we assume that $r(t)$ is continuous on $[0, \infty)$, $r(t) \geq 0$, $r(t) \not\equiv 0$ and,

$$b_0 > 0, \quad \text{and} \quad b_k \geq 0, \quad 1 \leq k \leq m. \tag{2.3}$$

Let

$$r_l^t = \int_l^t r(s)ds, \quad \text{for } l \leq t < l+1, \quad \text{and}$$
$$r_l = \int_l^{l+1} r(s)ds, \quad l = 0, 1, 2, \dots . \tag{2.4}$$

Then, Eq. (2.1) is equivalent to the following difference equation (see Lemma 2.1):

$$N(l+1) = N(l)\exp\left\{r_l\left(1 - \sum_{k=0}^{m} b_k N(l-k)\right)\right\}, \quad l = 0, 1, 2, \dots . \tag{2.5}$$

As similar to Gopalsamy and Liu [3], we see

$$\begin{cases} N(t) > 0, \quad \text{for} \quad t \in [l, l+1), \quad \text{and} \\ \dfrac{d}{dt}\left[\dfrac{1}{N(t)}\exp\left\{r_l^t\left(1 - \sum_{k=0}^{m} b_k N(l-k)\right)\right\}\right] = 0, \end{cases} \tag{2.6}$$

and hence, we have

$$N(t) = N(l)\exp\left\{r_l^t\left(1 - \sum_{k=0}^{m} b_k N(l-k)\right)\right\}, \qquad l \le t < l+1, \quad l = 0, 1, 2, \dots . \tag{2.7}$$

Let

$$N^* = \frac{1}{\sum_{k=0}^{m} b_k}, \quad \text{and} \quad t_l = 1 - \sum_{k=0}^{m} b_k N(l-k), \quad l = 0, 1, 2, \dots . \tag{2.8}$$

Then, we get a basic lemma.

Lemma 2.1

$$N(l+1) = N(l)\exp(r_l t_l), \quad l = 0, 1, 2, \dots, \tag{2.9}$$

and

$$N(l+1) - N^* = \begin{cases} \left\{1 - b_0 N(l)\dfrac{\exp(r_l t_l) - 1}{t_l}\right\}(N(l) - N^*) \\ \qquad - N(l)\dfrac{\exp(r_l t_l) - 1}{t_l}\displaystyle\sum_{k=1}^{m} b_k (N(l-k) - N^*), \quad \textit{if} \ \ t_l \neq 0, \\ \{1 - b_0 N(l) r_l\}(N(l) - N^*) - N(l) r_l \displaystyle\sum_{k=1}^{m} b_k (N(l-k) - N^*), \\ \qquad\qquad \textit{if} \ \ t_l = 0. \end{cases} \tag{2.10}$$

Proof. From Eq. (2.7) and (2.8), we have Eq. (2.9). Then,

$$N(l+1) = \begin{cases} N(l) + \left\{\left(\sum_{k=0}^{m} b_k\right)N^* - \sum_{k=0}^{m} b_k N(l-k)\right\} N(l)[(\exp\{r_l t_l\} - 1)/t_l], & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t_l \neq 0, & \\ N(l) + \left\{\left(\sum_{k=0}^{m} b_k\right)N^* - \sum_{k=0}^{m} b_k N(l-k)\right\} N(l) r_l, \quad t_l = 0, & \end{cases}$$

from which we have Eq. (2.10). □

On the uniform persistence for solutions of Eq. (2.1), we have the following well known result.

Lemma 2.2 (see Theorem 2.3 in So and Yu [4]) *Assume*

$$\sup_{l \geq 0} \int_{l-m}^{l+1} r(s)ds \leq M < +\infty. \tag{2.11}$$

Then, the solution $N(t)$ of Eq. (2.1) is bounded above and is bounded below from 0.

In order to prove the global asymptotic stability of N^*, we only need to prove the global attractivity of N^*.

Let

$$\underline{r} = \inf_{l \geq 0} r_l,$$

$$\widetilde{f}(t;r) = \begin{cases} \dfrac{e^{rt}-1}{t}, & t \neq 0, \\ r, & t = 0, \end{cases} \quad \text{and} \quad k_l = N(l)\widetilde{f}(t_l; r_l), \quad l = 0, 1, 2, \ldots, \tag{2.12}$$

and we prepare the following important lemma (cf. the proof of Theorem 2 in Wang and Lu [5]).

Lemma 2.3 *Assume that*

$$0 < r_l \leq 1, \quad l = 0, 1, 2, \ldots. \tag{2.13}$$

Then, for any positive integer l, it holds that

$$N(l) \leq \frac{1}{r_l b_0}, \quad \textit{and} \quad 1 - b_0 k_l \geq 0. \tag{2.14}$$

Moreover, if $\underline{r} > 0$, then there exists a positive constant $\underline{k}$ such that for any sufficiently large positive integer l,

$$\underline{k} \leq k_l \leq \frac{1}{b_0}, \tag{2.15}$$

and

$$\limsup_{l \to \infty} N(l) \leq \frac{1}{b_0} \quad \textit{and} \quad \liminf_{l \to \infty} N(l) \geq \frac{1}{b_0}\left(b_0 - \sum_{k=1}^{m} b_k\right)\frac{1}{b_0}. \tag{2.16}$$

Proof. We easily see that (2.13) implies (2.11) and hence, by Lemma 2.2, $N(l)$ is bounded above and is bounded below from 0.

For $f(x) = xe^{r-ax}$, where r and a are positive constants, we have

$$\max_{0 \leq x < +\infty} f(x) = \frac{e^{r-1}}{a}.$$

Thus, by Eqs. (2.9) and (2.13), we see for $l \geq 0$,

$$N(l+1) \leq N(l)\exp\{r_l(1 - b_0 N(l))\} \leq \frac{\exp(r_l - 1)}{r_l b_0} \leq \frac{1}{r_l b_0},$$

and hence,

$$\bar{t}_l = 1 - r_l b_0 N(l) \geq 0, \quad \text{for any } \; l \geq 1. \tag{2.17}$$

Consider the function

$$g(x) = \begin{cases} \dfrac{e^x - 1}{x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Then,

$$g'(x) = \begin{cases} \dfrac{1}{x^2}\{(x-1)e^x + 1\}, & x \neq 0, \\ \dfrac{1}{2}, & x = 0, \end{cases}$$

and for $h(x) = (x-1)e^x + 1$, $h'(x) = xe^x$. Then, $h(x) \geq h(0) = 0$, $g'(x) \geq 0$ and hence, $g(x)$ is a strictly monotone increasing function of x on $(-\infty, +\infty)$.

Hence, by Eqs. (2.8), (2.13) and (2.17), we have that $r_l t_l \leq \bar{t}_l$ and

$$b_0 k_l \leq r_l b_0 N(l) g(r_l t_l) \leq (1-\bar{t}_l) g(\bar{t}_l), \quad l = 1, 2, \dots .$$

We easily see that

$$(1-x)\frac{e^x - 1}{x} \leq 1, \quad \text{for any } \ 0 < x \leq 1.$$

Hence, we have Eq. (2.14).

In (2.8), we see

$$t_l \geq \underline{t}_l \equiv 1 - \sum_{k=0}^{m} b_k \Big(\limsup_{l \to \infty} N(l) \Big) > -\infty.$$

Put

$$\underline{k} \equiv \begin{cases} \dfrac{1}{2}\Big(\liminf_{l\to\infty} N(l) \Big) \dfrac{\exp(\underline{rt}_l) - 1}{\underline{t}_l} > 0, & \underline{t}_l \neq 0, \\ \dfrac{1}{2}\Big(\liminf_{l\to\infty} N(l) \Big) \underline{r} > 0, & \underline{t}_l = 0. \end{cases}$$

Then, for any sufficiently large positive integer l,

$$0 < \underline{k} \leq k_l \leq \frac{1}{b_0}.$$

Next, let a sequence $\{l_p\}_{p=1}^{\infty}$ satisfy the following condition:

$$0 < \lim_{l\to\infty} N(l_p) = \limsup_{l\to\infty} N(l) < +\infty.$$

Then, by (2.10) and (2.15), we have that for any sufficiently large positive integer p,

$$N(l_{p+1}) - N^* \leq (1 - b_0 k_{l_{p+1}-1})(N(l_{p+1}-1) - N^*) \\ - k_{l_{p+1}-1} \Big(\sum_{k=1}^{m} b_k \Big) \Big(\min_{1 \leq k \leq m+1} N(l_{p+1} - k) - N^* \Big).$$

Let $p \to +\infty$, in the above equation. Then,

$$\lim_{p\to\infty} \max_{1\leq k \leq m+1} N(l_{p+1} - k) \leq \limsup_{l\to\infty} N(l) \quad \text{and}$$

$$\lim_{p\to\infty} \min_{1\le k\le m+1} N(l_{p+1}-k) \ge \liminf_{l\to\infty} N(l),$$

and by (2.15), we have

$$b_0\Big(\limsup_{l\to\infty} N(l) - N^*\Big) + \Big(\sum_{k=0}^{m} b_k\Big)\Big(\liminf_{l\to\infty} N(l) - N^*\Big) \le 0.$$

Then, we get

$$\begin{aligned}\limsup_{l\to\infty} N(l) - N^* &\le -\Big\{\Big(\sum_{k=1}^{m} b_k\Big)/b_0\Big\}\Big(\liminf_{l\to\infty} N(l) - N^*\Big)\\ &\le \Big\{\Big(\sum_{k=1}^{m} b_k\Big)/b_0\Big\}N^*,\end{aligned}$$

and hence, we have

$$\limsup_{l\to\infty} N(l) \le \Big\{1 + \Big(\sum_{k=1}^{m} b_k\Big)/b_0\Big\}N^* = \frac{1}{b_0}.$$

Similarly, let a sequence $\{l_p\}_{p=1}^{\infty}$ satisfy the following condition:

$$\lim_{p\to\infty} N(l_p) = \liminf_{l\to\infty} N(l) > 0.$$

By (2.10) and (2.15), we have that for any sufficiently large positive integer l,

$$\begin{aligned}N(l_{p+1}) - N^* \ge{}& (1 - b_0 k_{l_{p+1}-1})(N(l_{p+1}-1) - N^*)\\ &- k_{l_{p+1}-1}\Big(\sum_{k=1}^{m} b_k\Big)\Big(\max_{1\le k\le m+1} N(l_{p+1}-k) - N^*\Big).\end{aligned}$$

Therefore, as similar to the above discussions, we get

$$b_0\Big(\liminf_{l\to\infty} N(l) - N^*\Big) + \Big(\sum_{k=1}^{m} b_k\Big)\Big(\limsup_{l\to\infty} N(l) - N^*\Big) \ge 0.$$

Then,

$$\liminf_{l\to\infty} N(l) - N^* \ge -\Big\{\Big(\sum_{k=1}^{m} b_k\Big)/b_0\Big\}\Big(\limsup_{l\to\infty} N(l) - N^*\Big),$$

and

$$\liminf_{l\to\infty} N(l) \geq \left\{1+\left(\sum_{k=1}^{m} b_k\right)/b_0\right\}N^* - \left\{\left(\sum_{k=1}^{m} b_k\right)/b_0\right\}\limsup_{l\to\infty} N(l)$$
$$= \frac{1}{b_0} - \left\{\left(\sum_{k=1}^{m} b_k\right)/b_0\right\}\limsup_{l\to\infty} N(l).$$

Thus, by the first part of (2.16),

$$\liminf_{l\to\infty} N(l) \geq \frac{1}{b_0} - \left\{\left(\sum_{k=1}^{m} b_k\right)/b_0\right\}\frac{1}{b_0} = \frac{1}{b_0}\left(b_0 - \sum_{k=1}^{m} b_k\right)\frac{1}{b_0}.$$

Hence, we complete the proof. □

From Lemmas 2.1 and 2.3, we obtain a sufficient condition of the contractivity for solutions and the global asymptotic stability for the positive equilibrium N^* of Eq. (2.1).

Theorem 2.1 *Assume that*

$$b_0 > \sum_{k=1}^{m} b_k, \quad \textit{and} \quad 0 < r_l \leq 1, \quad l = 0, 1, 2, \dots . \tag{2.18}$$

Then,

$$|N(l+1) - N^*| \leq \max_{0\leq k\leq m} |N(l-k) - N^*|, \quad l = 0, 1, 2, \dots, \tag{2.19}$$

which implies that solutions of Eq. (2.1) have the contractivity.

Moreover, if $\underline{r} = \inf_{l\geq 0} r_l > 0$, then the positive equilibrium N^* of Eq. (2.1) is globally asymptotically stable.

Proof. By Lemmas 2.1 and 2.3,

$$|N(l+1) - N^*|$$
$$\leq (1 - b_0 k_l)|N(l) - N^*| + k_l\left(\sum_{k=1}^{m} b_k\right)\left(\max_{1\leq k\leq m} |N(l-k) - N^*|\right)$$
$$\leq \left\{1 - k_l\left(b_0 - \sum_{k=1}^{m} b_k\right)\right\}\left(\max_{0\leq k\leq m} |N(l-k) - N^*|\right),$$
$$l = 0, 1, 2, \dots,$$

from which by Eq. (2.18), we get Eq. (2.19).

Moreover, if $\underline{r} > 0$, then by Lemma 2.3, $1 - k_l\big(b_0 - \sum_{k=1}^m b_k\big) < 1 - \underline{k}\big(b_0 - \sum_{k=1}^m b_k\big) < 1$, for any large positive integer l, and hence, $\lim_{l\to\infty} N(l) = N^*$. Thus, we get the conclusion. $\square$

Acknowledgement This work was partially done during the stay of the author at the University of Trieste. The autor wishes to express his gratitude to Professors Alfredo Bellen and Marino Zennaro for their hospitality during his visit.

References

[1] Chen M.P. and Liu B., *Asymptotic behavior of solutions of first order nonlinear delay difference equations*. Computers Math. Appl. **32** (1996), 9–13.

[2] Gopalsamy K., Kulenovic M.R.S. and Ladas G., *On a logistic equation with piecewise constant arguments*. Diff. Int. Equ. **4** (1991), 215–223.

[3] Gopalsamy K. and Liu P., *Persistence and global stability in a population model*. J. Math. Anal. Appl. **224** (1998), 59–80.

[4] So J.W.-H. and Yu J.S., *Global stability in a logistic equation with piecewise constant arguments*. Hokkaido Math. J. **24** (1995), 269–286.

[5] Wang W. and Lu Z., *Global stability of discrete models of Lotka-Volterra tpye*. Nonlinear Analysis **35** (1999), 1019–1030.

[6] Yu J.S., *Asymptotic stability for a linear difference equation with variable delay*. Computers Math. Appl. **36** (1998), 203–210.

[7] Zhou Z. and Zhang Q., *Uniform stability of nonlinear difference systems*. J. Math. Anal. Appl. **225** (1998), 486–500.

Department of Mathematical Sciences
Waseda University
3-4-1 Ohkubo, Shinjuku-ku
Tokyo 169-8555, Japan
E-mail: ymuroya@mn.waseda.ac.jp